Leverage and Financing in Distress

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Abstract

Very few firms issue equity to refinance their debt in distress. This simple observation has an important effect on the predictions of capital structure models. A model in which highly-levered firms needing external finance must issue debt explains the overall underleverage puzzle, fully replicates the 'fat' right tail of cross-sectional leverage distribution, and produces realistic default probabilities across firms with different leverage values. The model succeeds even if bankruptcy costs are only 10% of firm's assets, whereas the model that allows for equity issuance requires bankruptcy costs to exceed 60% in order to generate plausible average leverage.

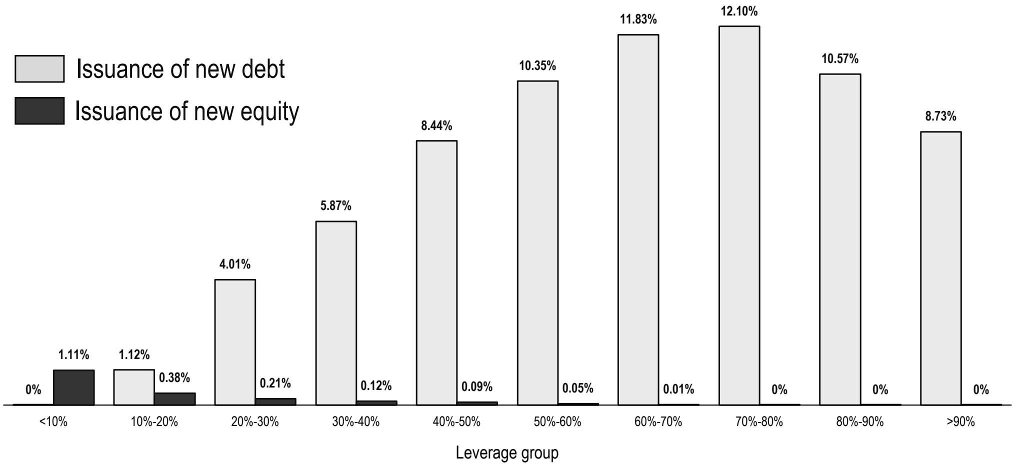
1 Introduction

Interest rates to issue debt are high for firms that already have a lot of debt. This makes debt financing expensive for highly-levered firms. One would assume that firms should, therefore, substitute debt financing with equity financing when leverage becomes higher. Yet, as Figure **??** shows, highly-levered firms issue debt rather than equity.[1] This implies that equity financing costs increase with firm's leverage even faster that debt financing costs - or otherwise firms would not increase their already high leverage. This paper shows that the fact that highly levered firms are effectively cut out from equity capital markets, can explain many capital structure patterns.

First, a model that assumes that equity issuance costs are unaffected by leverage will lead to significant counterfactual implications. To show this, the paper solves a model similar to Goldstein, Ju, and Leland (2001), in which 1) firms in distress can always issue equity, and 2) costs of issuing equity do not change with leverage (either zero or proportional to the amount). In a simulated

[1] Empirical literature on equity issuance strongly supports this: Senber and Senber (1995) report complete absence of equity issuance by firms in distress, while Eckbo, Masulis, and Norli (2007) find that most equity issuances take a form of payments in mergers or employee stock compensations, and that equity issuances in which firms raise cash for business operations are rare. See Literature Review subsection A and Appendix A for a more detailed analysis of empirical research on equity issuance by firms in distress.

Figure 1
Amount of new equity and debt (scaled by assets) issued by a median firm across firms with different leverage



New debt (new equity) is the amount of long-term debt (equity) issued by a firm in a given year, scaled by the book value of asset. Median rather than average values are reported because distributions of debt and equity issuances are skewed, and more so for firms with high values of leverage. Formal regression of new debt (new equity) on firm's leverage results in the coefficient 0.35 (-0.16), both statistically significant at 1% level. Data is taken from Compustat for the period 1987-2017; see Section 3.2 for the details of data analysis. Leverage is the ratio of firm's debt (sum of long-term and short-term) to the book value of assets.

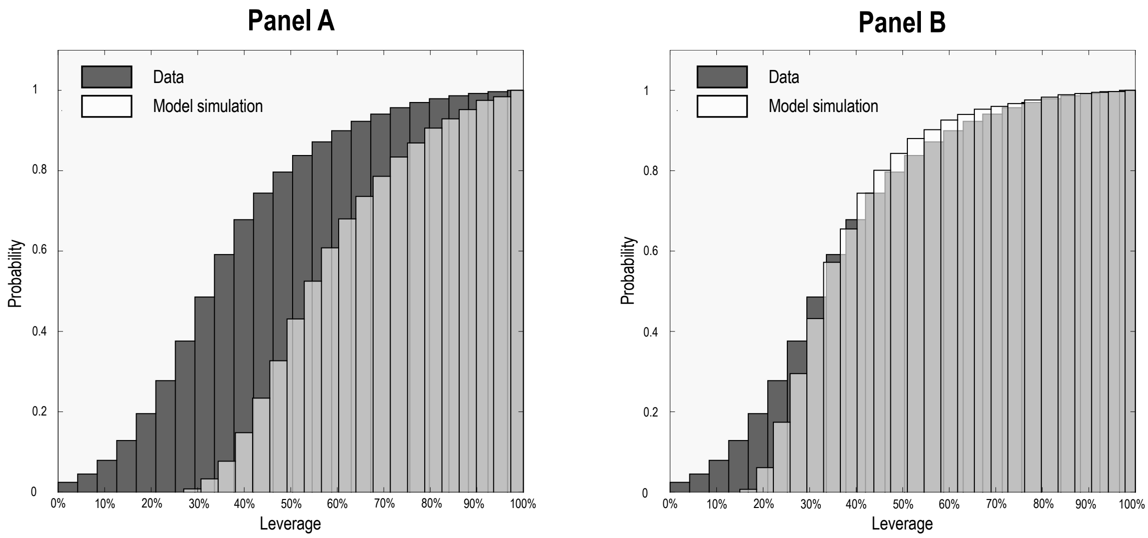
steady-state cross section, 57% of firms *optimally* choose to be in distress, having so much debt that their cash flow is not sufficient to pay interest expenses (and so they issue equity). This is wrong both quantitatively and qualitatively:[2] firms that issue equity in the data are low-levered, and the absolute majority of firms produce enough cash to pay interest expenses. A number of other known capital structure puzzles arise: for instance, the average leverage in the model is too high (58% in the model vs. 36% in the data), or default probabilities are understated (a firm that has 20% default probability within 10 years has 50% leverage in the data and 75% in the model).[3]

While the above-mentioned results are based on the framework of Goldstein, Ju, and Leland (1994), it is important to emphasize that the results are not specific to this particular framework, but come from the assumption that highly-levered firms have a cheap way of external financing (equity in this case). For instance, Hennessy and Whited (2005) use a different model to study

[2]This is not the result of inadequate calibration of parameters: I took parameter values from Hennessy and Whited (2012) paper, in which they consider the same model. A similar result appears in other papers; for instance, in Bharma, Kuehn, and Strebulaev (2011) firms that optimally choose their leverage at time zero issue so much debt that they immediately put themselves in distress, with interest expenses exceeding their cash flow.

[3]This again happens because firms will operate too long before defaulting by issuing equity and paying interests to debtholders

Figure 2
Cummulative distribution function (cdf) of leverage distribution in the data and in the model



For each value of leverage on the horizontal axis, the bars show what is the fraction of firms in the data and in the model with leverage that does not exceed this value. Panel A shows results for the model in which firms can always issue equity, and Panel B shows results for the model in which firms whose profits are not sufficient to pay interest expenses on their debt have to issue more debt to cover the gap. Data is taken from Compustat for the period 1987-2017 for firms that have access to public bond markets (proxied by whether a firm in a given year has S&P long-term credit rating). The model is solved for the benchmark parameter values shown in Table ??. See Section 3 for details of data analysis and model simulations.

firm's capital structure, but assume that equity issuance incurs only flotation costs. They also find that in their model "equity issuers are the most highly indebted". In short, the assumption that highly-levered firms firms can easily issue equity 1) is not supported by the data or empirical literature, and 2) leads to wrong implications, both quantitatively and qualitatively.

As the next step, this paper modifies the model by assuming that firms in distress cannot issue equity, and have to issue more debt every time they do not have enough cash to pay interest expenses. Here is what happens then: when firm's cash flow falls below the level of interest expenses, firm's leverage is already high; by issuing more debt to cover the gap, the firm increases its leverage further, and moves closer to default. Moreover, as debt is fairly priced, interest rates to issue new debt grow exponentially for firms that are deep in distress. In essence, a firm in distress risks falling in a continuous debt spiral, when interest expenses grow faster than the cash flow, and it has to continuously issue new debt to cover payments on previously issued debt. Such mechanism makes distress very costly, and affects expectations of firms that are outside of distress. Even healthy firms now choose significantly smaller leverage ex-ante, even if bankruptcy costs are low and tax-shield benefits are high.

The modified model has much better quantitative results. Firms that fall in distress either quickly recover or quickly default. As the result, only 14% of firms are in distress (have interest expenses that exceed their cash flow), and default probabilities match empirically observed numbers. Expectations about distress reduce firm's target leverage by 20 percentage points (compared to the case when firms could issue equity in distress), and so the model correctly matches the average leverage among all firms in the data, and also the average leverage in subsets of firms grouped by their credit ratings. Moreover, the model closely matches every quantile of leverage distribution between 35% and 100%, as shown on Figure **??** Panel B.

This last result illustrated by Figure **??** that the model correctly matches the whole right-tail distribution of leverage is particularly important for this paper. Firms in the model make capital structure choices based on what they expect will happen to them in distress. While it is impossible to directly test firms' expectations, it is possible to study what actually happens to firms in distress vs. what the model says the expectations are. Because the match between the model-predicted and empirically-observed distributions of leverage of firms in distress is so close, it provides a strong support for the theoretical mechanism used in the model. This is the main contribution of the paper: it uncovers the mechanism that both explains the distribution of leverage of highly-leveraged firms and shows how it affects firm's initial capital structure choice. It is also the first paper that examines this channel and shows its importance.

One of the main advantages of Leland-type models is that they are easy to solve and results are generally available in a closed form. The modification of the model that says that firms in distress cannot issue equity does not come at the cost of tractability. Tractability is preserved in the model that this paper derives, and some results become even simpler, as is the case with, for instance, the default boundary.

The paper shows that large bankruptcy costs are not necessary to match the average value of leverage in a cross-section. In the benchmark calibration, the value of bankruptcy costs is only 10%. Even with this value, the cross-sectional leverage is only 38% for the environment in which firms in distress finance the gap between the cash flow and interest expenses by issuing debt. In contrast, to match the average leverage in a cross-section within the framework in which firms in distress can also issue equity, bankruptcy costs should be at least 60%, and even higher to match other moments.

In the model, firm's default probability depends on its current leverage, and also on how this firm will finance its interest payments in distress. Naturally, keeping firm's leverage constant, default probability is higher if in distress the firm will have to issue debt at exponentially increasing rates, as opposed to cheap equity. A striking result, though, is that the overall default rate is higher in the model in which firms in distress can issue equity. The reason why this happens is because firms in both models endogenously choose how much debt to issue, and in the model, in which firms in distress can only issue debt, they are much more conservative in their leverage policy ex-ante, and so default less often ex-post.

Literature review

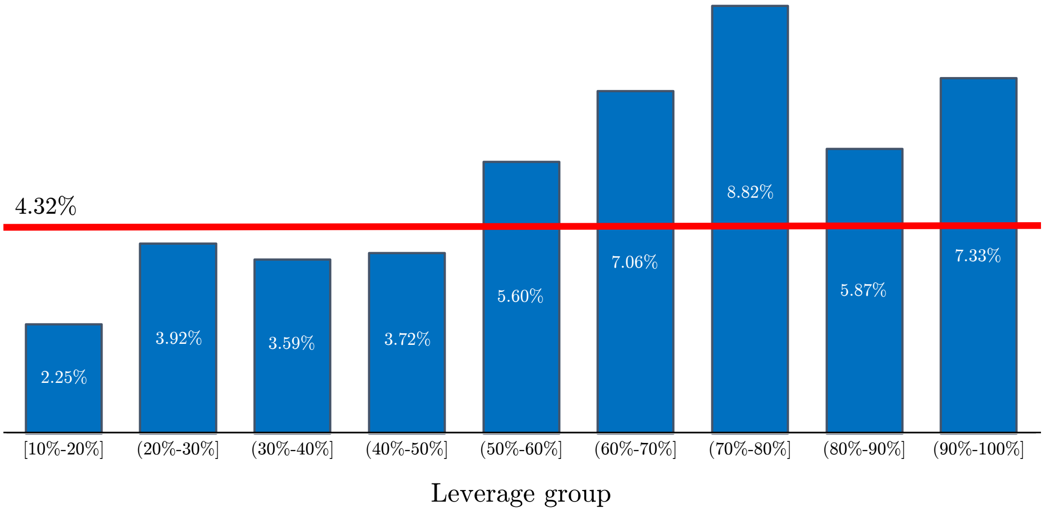
A. Equity issuance by firms in distress

The underlying assumption of the model in this paper is that firms in distress do not issue equity and instead finance the shortfall between the cash flow and interest expenses by issuing more debt. This is a strong assumption that is imposed to achieve a closed-form solution for the function that connects firm's leverage and external financing costs; similar results would hold in the model if distressed firms could issue equity, but equity issuance costs increased with leverage.

Empirical evidence on equity issuance by distressed firms is scant, but also mixed. Below is a review of papers that consider this question. The overall conclusion that follows is that most firms issue equity when their leverage is low; there are instances of equity issuances by financially distressed firms, but the costs are high, and such firms use equity financing because they cannot raise debt.

The first group of papers argues that most firms issue equity when their performance is good. For instance, Senber and Senber (1995) report a complete absence of equity issuance by distressed firms. Fama and French (2005) show that equity issuances are frequent, but most firms issue equity when their leverage is low. Similarly, Mikkelson and Partch (1986) and Eckbo, Masulis, and Norli (2007) find that equity issuances for cash are rare - both in absolute level and relative to public debt issuances. Some other studies provide indirect evidence that firms in distress do not issue equity. Korajczyk, Lucas, and McDonal (1990) find that firm's leverage does not increase significantly two

Figure 3
Stock price discount following equity issuance announcement



Data is based on SDC/Platinum database that tracks equity issuances, and shows the average stock price reaction following an equity issuance announcement. Each bar represents the average discount for a group of companies with a given leverage; for instance, among companies, whose leverage was between 30% and 40% at the time when they announced an equity issuance, the stock price fell by 3.59% on average. Negative 4.32% (red line) is the average stock price reaction following an equity issuance announcement.

years before an SEO; should firms issue equity to make required debt payments when internally generated cash flow is insufficient, one would observe an increase in leverage prior to an SEO. DeAngelo, DeAngelo and Stulz (2008) find that the average leverage of a firm before an SEO is only 27%. Denis and McKeon (2012) show that firms, whose leverage is above the target, tend to cover financial deficit by issuing new debt and increasing leverage further.

Other authors argue in contrary that a sizable number of distressed firms issue equity, but they sell new shares at a large discount, and do so because debt financing is unavailable. Park (2017) finds that public equity offerings decrease for firms in distress, but private placements increase. Walker and Wu (2017) find that a third of all SEOs are conducted by financially distressed firms. Both of these papers, however, use the distress measure from Campbell, Hilsher, and Szilagyi (2008), which is only partially related to firm's leverage. Indeed, the average leverage in the subsample of distressed firms in Walker and Wu is 32%, which implies that these firms are in distress for reasons other than their indebtedness, and they likely have very limited access to debt financing. This conclusion is further reinforced by Lim and Schwert (2017) who study all private placement of equity (PIPEs) by U.S. firms. They find that most firms issuing PIPEs are small distressed firms without access to debt markets: the median leverage of firms issuing PIPEs is only 7.2%, and 93%

of all firms do not have credit rating. When such firms issue PIPEs, they offer shares to the market at a large discount.

Appendix A provides further empirical analysis of the correlation between the frequency of equity issuance and firm's leverage based on Thomson Reuters data. Results show that the amount of equity issuance decreases with firm's leverage, and the discount at which newly issued shares are offered to investors increases with leverage; this conclusion holds for all firms and also for the subsample of firms that have access to public debt markets. Similar conclusion follows from Figure **??** which is based on Compustat data. [4]

The model derived in this paper assumes that firms always have access to debt capital markets.[5] Therefore, the assumption that such firms do not issue equity to pay required debt payments in distress is consistent with empirical evidence discussed above.

The question why costs to issue equity grow for firms in distress is beyond the scope of this paper. To theoretically microfound this assumption, Belyakov (2018) considers a model similar[6] to the one used in this paper, but adds information asymmetry between firm's manager and outside shareholders. He shows that leverage amplifies information asymmetry, and that costs to issue equity escalate with leverage as the result. Capital structure-wise, he obtains results that quantitatively similar to those in this paper, but the model itself is less tractable. Appendix B provides a simplified two-period version of that model that shows how presence of leverage amplifies information asymmetry. Such explanation is consistent with empirical findings of Hertzel and Smith (1993) and Lim and Schwer (2017) who argue that distressed firms are characterized by severe information asymmetry.

B. Underleverage puzzle

This paper is related to the line of literature that discusses the underleverage puzzle. Miller (1977) shows that present value of expected default losses seems disproportionally small compared to tax

[4]The paper does not combine the two databases because they use different definitions of equity issuance. In particular, Thomson Reuters mostly considers SEOs, while Compustat partially considers private placements as well. Assuming that Compustat data is internally consistent, Figure **??** shows the relative scale of debt and equity issuance as firms leverage increases, which would not necessarily be consistent if Compustat and Thomson Reuters data for equity issuance is pooled. On the other hand, Thomson Reuters has data for the discount/premium paid for newly issued shares, which is not available in Compustat

[5]The empirical sample of firms that the paper quantitatively explains also consists of firms with access to debt capital markets (firms with S&P long-term credit rating).

[6]In fact, an identical model

benefits of debt, implying that firms consistently issue less debt than what would be optimal to maximize the value of their shareholders. Graham (2000) estimates that tax benefits of debt add up to 5% of firm value, and also concludes that firms are on average underlevered from the point of view of a trade-off theory.

Faulkender and Petersen (2006) show that at least part of the underleverage puzzle is explained by the fact that not all firms have access to debt capital markets. They find that the difference in average leverage between firms that do and do not have S&P long-term credit rating is almost 20 percentage points. This paper assumes that all firms have access to debt capital markets; nevertheless, the underleverage puzzle remains in the model in which firms can always issue equity.

Chen (2012) provides an alternative explanation to the underleverage puzzle: he focuses on a model in which bankruptcy costs rise in bad states of the economy. He shows that firm's target leverage at refinancing points is significantly smaller relative to a model in which bankruptcy costs do not vary with states of the economy. Nevertheless, when he simulates a cohort of firms, the average simulated leverage exceeds 40%, which is greater than the average leverage observed in the data.

Morellec, Nikolov, and Schuroff (2012) provide an alternative explanation to the underleverage puzzle that relies on the presence of a conflict of interests between shareholders and managers.[7] They use the Goldstein-Ju-Leland framework in which firms in distress can always issue equity, but they assume that firm's manager can divert a small fraction of the cash flow and, therefore, has incentives to keep the firm alive for longer. They show that small agency costs help resolve the underleverage puzzle. Their paper, however, only addresses the mean and median values of leverage distribution and, as authors acknowledge, the "model is statistically rejected for higher-order leverage moments and dispersion measures". Indeed, kurtosis of the simulated leverage distribution in their model is much greater than in the data, implying that the simulated leverage distribution is centered around the mean value. The higher-order moments of leverage distribution, in particular the right-tail distribution of leverage, is the main focus of this paper.

Papers that resolve the underleverage puzzle often assume high values of bankruptcy costs; this increases ex-ante costs of debt issuance and reduces firm's incentives to issue debt. For instance, among the two papers mentioned above, Chen (2010) assumes bankruptcy costs between 40% and

[7] See also Jensen and Meckling (1975)

80%, and Morellec, Nikolov, and Schurhoff (2012) assume bankruptcy costs of 50%[8]. These values are substantially greater than empirically observed among defaulted firms (between 1% and 30%).[9] Glover (2016) points that there is a selection bias among firms that default; specifically, firms with low bankruptcy costs choose leverage values that are high, thus, defaulting more often and creating a bias in empirical estimates of bankruptcy costs. He uses a structural estimation approach and finds average default costs of around 40%. While Glover's argument is valid, the underlying assumption of his structural model is that firms in distress issue equity, which, as noted earlier, is not the case in the data. This paper shows that expectations about increasing costs of external financing in distress affect firm's initial leverage choice significantly even if actual bankruptcy costs are small.

The explanation to the underleverage puzzle provided in this paper has one advantage compared to alternative explanations: it is almost trivial. In essence, the logic follows from the backward induction: once a firm knows that tomorrow it will only be able to refinance its debt at a high rate, it wants to have less debt today. This explanation can't be ignored simply because it relies on an assumption that does not have a solid theoretical foundation - that all forms of external financing - including equity - become more expensive as firm's leverage increases. It is particularly important given that empirical evidence confirms that highly levered firms either only issue debt or issue equity at a high discount, as the previous subsection shows.

C. Goldstein-Ju-Leland framework

This paper has an important theoretical contribution to the class of dynamic capital structure models similar to Leland (1994), Goldstein, Ju, and Leland (2001), or Strebulaev[10] (2007): it derives in a closed form the optimal default boundary for a firm that finances the gap between the cash flow and interest expenses by issuing debt. In the seminal 1994 paper, Leland shows how to derive the optimal default boundary for the case when the firm can always issue equity. The author introduces the smooth-pasting condition for the firm's optimal default boundary: at the time when

[8]It should be noted that Morellec, Nikolov, and Schurhoff assume that default may lead to renegotiation rather than liquidation with substantially lower costs

[9]See, for instance, Gruber and Warner (1977), Andrade and Kaplan (1998), Hennessy and Whited (2007), Davydenko, Srebulaev, and Zhao (2012), or Reindel, Stoughton, and Zechner (2017)

[10]Bhamra, Kuehn, Strebulaev (2009, 2010), Chen (2010), and Glover (2016) consider more recent versions of this framework that allow for switching macroeconomic regimes. While this paper assumes that macroeconomic conditions are stable, the debt refinancing mechanism that this paper develops can be easily implemented in those models.

the firm optimally defaults, both the value of equity and its derivative with respect to the value of cash flow are zero. In this paper, firms in distress issue new debt. Issuance of debt increases firm's interest expenses going forward, and default happens when no further debt issuances are possible. This is achieved by imposing the smooth-pasting condition on the value of debt: the derivative of the value of debt with respect to the coupon payment is zero at the time when default happens. It follows from the model, however, that the smooth-pasting condition for the value of equity at default is preserved.

Tractability of the Goldstein-Ju-Leland framework is one of the main reasons why it is often used in structural estimations literature. Nevertheless, this paper shows that quantitative results of a model that assumes that firms in distress can always issue equity are unrealistic. This can be a significant concern for structural estimation papers, for which the plausibility of the underlying model is of the first-order importance. Modification of the original framework that this paper derives maintains all the tractability features of the original model, but produces much better fit to the data. This suggests that the modified framework can contribute to future structural estimation papers.

The rest of the paper is organized as follows. Section 2 presents mathematical formulation of the model. Section 3 presents model calibration and quantitative results. Section 4 concludes.

2 The model

I derive a model of firm's optimal capital structure in two different environments, to which I refer as D-type and E-type. The ultimate goal is to compare firm's predicted capital structure in these two environments against each other and the data. E-type and D-type environments are different by the assumption of how firms in distress finance required interest payments, with distress defined as the situation, in which firm's time-t cash flow being lower than the required interest payment that this firm has to make at time t. In the E-type environment, firms in distress issue equity to finance the difference between the cash flow and the interest payment (therefore, "E" in the name of the model). In the D-type model, firms in distress issue new debt (therefore, "D" in the name).

For both E-type and D-type environments, I consider two versions of the model; versions are different by the assumption of how many times firms in the economy can issue debt to exploit benefits of tax-shield. Version one maintains the set-up of Leland 1994 paper, where it is assumed that debt can only be issued at t_0[11]. Version two has assumptions of Goldstein, Ju, and Leland (2001). In this version, firms are allowed to issue debt at t_0, and increase its amount subsequently if the financial performance is good.

I start with the description of the economy that is the same for both versions of the model, and define objective functions of agents in each version in the following subsections.

2.1 The economy

I consider a partial-equilibrium model of an economy with a constant risk-free rate r. Firms in the economy are endowed with assets that produce an exogenous stream of cash flows $\{X\}_t$. Evolution of X_t under the risk-neutral probability measure $\mathbb{Q}$ follows Geometric Brownian motion:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t^Q \tag{1}$$

where μ is the risk-neutral drift, and σ is the volatility of firm's cash flows.

Profits in the economy are taxed at the corporate tax-rate τ. I define the value of assets as the expected value of future discounted profits that these assets will produce:

$$E^U(X_t) = \mathbf{E}\left[\int_t^\infty e^{-r(s-t)}(1-\tau)X_s ds\right] = (1-\tau)\frac{X_t}{r-\mu} \tag{2}$$

Firms in the economy can issue debt, and interest payments on debt are tax-deductible. Debt in the model takes the form of a perpetuity that pays a constant coupon rate c per unit of time. At each point in time, a firm produces cash flow X_t and has to pay c_t to its debtholders. $(X_t - c_t)$ is the taxable base of the firm; for simplicity, and similar to other authors, I assume that if firm's taxable base is negative, the firm pays negative taxes, meaning that it receives money from the government. By taking greater amount of debt, firms can reduce the amount of taxes they pay. On the other hand, high amount of debt increases the probability that the firm will not be able to service its debt obligations. When making a decision about timing and amount of debt to issue, firms trade off these benefits of debt (lower tax-payments to the government) against costs of debt

[11] Note that D-type version one will have many debt issuances, but only the first will be done to exploit benefits of tax shield, and all other debt issuances will happen when the firm does not generate enough money to make required interest payments

(higher chances of default). Firms act in the interest of equityholders, and, therefore, I use words "firms" and "equityhodlers" interchangeably later in the text.

2.1.1 Equityholders

A firm in the economy is fully characterized by its current cash flow level X_t, and the coupon payment it has to make c_t; for simplicity, I drop subscripts t in what follows.

In case $X \geq c$, the firm produces enough money to service its debt obligations, and immediate dividends to equityholders are $(1-\tau)(X-c)$. The HJB equation for this case then takes the form[12]:

$$rE(X,c) = (1-\tau)(X-c) + \mu X E'_x + \frac{\sigma^2 X^2}{2} E''_{xx}, \quad \text{if} \quad X \geq c \tag{3}$$

where $E = E(X,c)$ is the value of firm's equity.

If a firm experiences a series of negative shocks, its cash flow may become insufficient to make required debt payments c. In this case, the firm has to finance the shortfall by raising money externally. In the E-type environment, the firm raises equity, and mathematically it is equivalent to negative dividends. Therefore, the HJB equation for the E-type environment for the case $X < c$ is the same as (3).

Equity issuances are not allowed for firms in distress in the D-type environment, and so additional debt issuance is the only source of external financing. It is assumed that there are no transaction costs of debt issuance, and newly issued debt has the same seniority level as the old debt. Denote $D(X,c)$ the value of firm's debt, and dD the value of newly issued debt. The following formula then connects changes in promised future coupon payments with the required newly debt issuances:

$$\Big(c - \big(X + (c-X)\tau\big)\Big)dt = (c-X)(1-\tau)dt = dD = dc\frac{\partial D}{\partial c} \tag{4}$$

The very left-hand side of equation (4) is the difference between the required coupon payment c, and the amount of money the firm has on hands - its profits X, and tax return from the government $\tau(c-X)$. This difference is the shortfall that equityholders must but can not pay to debtholders; this difference should equal to change in debt value dD, which is in turn achieved by promising a

[12]This equation is the same for both versions of the model in both E- and D-type environments

higher coupon payment in the future. It is clear from equation (4) that it is only possible to issue new debt if $\frac{\partial D}{\partial c} > 0$, that is, if value of debt increases when the firm promises to pay more in the future. For now, consider the case when this condition is satisfied.

Equation (4) allows to derive the dynamics of dc for the case when $X < c$:

$$dc = \frac{(c - X)(1 - \tau)}{\frac{\partial D}{\partial c}} dt \tag{5}$$

HJB equation for the equity value on the region $X < c$ in the D-type environment should take into account that both X and c change:

$$rE(X, c) = \frac{(c - X)(1 - \tau)}{\frac{\partial D}{\partial c}} E'_c + \mu X E'_x + \frac{\sigma^2 X^2}{2} E''_{xx}, \quad \text{if} \quad X < c \tag{6}$$

Boundary conditions for the equity value will differ between versions one and two of the model, and are discussed them in the corresponding sections of the paper.

2.1.2 Debtholders

HJB equation for debtholders will differ from the HJB equation for equityholders by only the part that catches instantaneous profit. In the E-type environment, instantaneous profit to debtholders is always cdt. In the D-type environment, it is cdt on the interval $X \geq c$, and $\big((X+(c-X)\tau)dt+dD\big)$ on the interval $X < c$. Note, however, that it follows from equation (4) that

$$(X + (c - X)\tau)dt + dD = (X + (c - X)\tau)dt + \big(c - \big(X + (c - X)\tau\big)\big)dt = cdt \tag{7}$$

Therefore, instantaneous profit to debtholders on both intervals is cdt in both E-type and D-type environments. This result should not come as a surprise: the way changes in debt were modeled in equation (4), debtholders should always get the required payment cdt - either in a form of money (on the interval of $X \geq c$) or a combination of money and promises of greater future payments (on the interval of $X < c$). Hence, there is one single HJB equation for debtholders on both intervals $X \geq c$ and $X < c$ for both E- and D-type environments:

$$rD(X, c) = c + \mu X D'_x + \frac{\sigma^2 X^2}{2} D''_{xx} \tag{8}$$

Boundary conditions will depend on the version one or two of the model.

2.1.3 Default

I assume that at default assets are liquidated at their price (defined in equation (2)), but a fraction α is lost during the liquidation process. All proceeds from assets liquidation go to debtholders, and equityholders receive nothing. Define $X_{def}(c)$ as the default boundary of a firm:

$$X_{def}(c) = \{X : \text{firm in state } (X_t, c) \text{ defaults if and only if } X_t = X\} \tag{9}$$

Values of debt and equity of a firm at default then take the following form:

$$D\big(X_{def}(c), c\big) = X_{def}(c)\frac{(1-\tau)(1-\alpha)}{r-\mu} \tag{10}$$

$$E\big(X_{def}(c), c\big) = 0 \tag{11}$$

Equations (10) and (11) hold for both E- and D-type environments. However, conditions that determine default are going to be different between the two. In the E-type environment, timing of default is chosen by equityholders and is determined by the smooth-pasting condition for the equity value at default:

$$\frac{\partial E}{\partial X}\big(X_{def}(c), c\big) = 0 \tag{12}$$

Economic intuition for equation (12) is as follows. When firm's performance is weak, the firm continuously raises equity to finance coupon payments, meaning that immediate dividends to equityholders are negative. Equityholders agree to receive negative dividends with the hope that financial performance of the firm improves in the future, and dividends become positive. The future discounted value of dividends (both positive and negative) is summarized by firm's market capitalization $E(X, c)$, and equityholders want to choose the default boundary $X_{def}(c)$ as low as possible so that $E(X, c)$ always remains positive; therefore, equation (12)

In the D-type environment, equity issuances are not allowed, and, therefore, dividends to equityholders are always non-negative (positive on the interval $X > c$ and zero on the interval $X \leq c$). Hence, equityholders will never voluntarily choose to default the firm. However, even though equityholders do not want to default the firm, the firm may be in a situation, when internally generated cash is not sufficient to make required coupon payments, and further debt issuances are not possible because debtholders do not believe that the firm will manage to service its debt obligations. As equation (4) shows, the firm is able to issue at least some amount of new debt as long as $\frac{\partial D}{\partial c} > 0$. Hence, the following equation determines default in the D-type model:

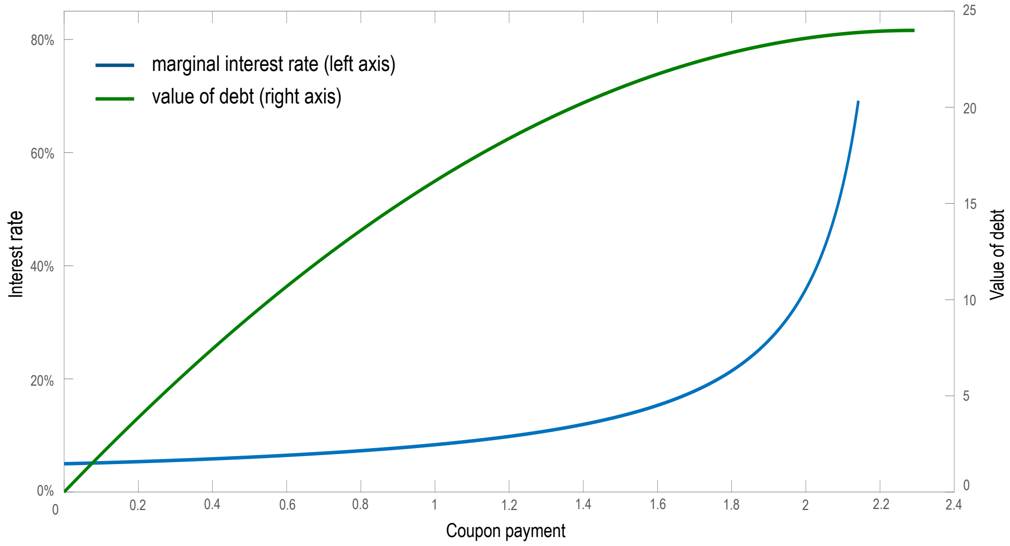
$$\frac{\partial D}{\partial c}\big(X_{def}(c), c\big) = 0 \tag{13}$$

Figure **??** visualizes default in the D-type environment. The green curve on the graph shows how the value of firm's debt changes when the firm offers a higher coupon payment to its debtholders; the blue curve shows the marginal interest rates, at which the next dollar of debt can be raised. Firm with no debt ($c = 0$) can issue the first dollar of debt at $r = 5\%$, which is used as the risk-free rate to solve the model. Marginal interest rates stay low and close to the risk-free rate for firms that have sufficiently low interest payments. However, as the coupon payment becomes very high, the green curve becomes flatter, which means that the firm has to promise to increase future interest payments by a lot to raise an additional dollar of debt. As coupon-to-cash flow ratio approaches its default value, the green curve becomes completely flat, which means that future promises of higher coupon payments do not increase debt value, or, equivalently, next unit of debt can only be issued at the infinite rate. At that point, the firm can not issue new debt, and can not pay interests on its debt out of the operating cash flow either, and so default happens.

Importantly, as Figure **??** shows, the model does not produce unrealistically high marginal interest rates for new debt. While it is true that interest rates for new debt exponentially increase up to infinity with firm's leverage, quantitatively the mt interest rates exceed 20% only when firm's leverage is above 80%, and interest rates exceed 40% when firm's leverage is above 96%. These numbers are empirically-plausible.

Note one very important observation: at the time of default in the D-type environment, equation (13) holds. However, the term $\big(\frac{\partial D}{\partial c}\big)^{-1}$ is the term that multiplies E'_c in equation (6), which is the HJB equation for the equity value in the D-type environment on the region $X < c$. Because LHS of equation (6) is finite, this implies that the term that multiplies $\big(\frac{\partial D}{\partial c}\big)^{-1}$ must approach zero as

Figure 4
Value of debt and marginal interest rates for a firm with current productivity $X = 1$ and varying coupon payments



The graph shows the value of firm's debt (green curve, right axis), and marginal interest rates at which next unit of debt can be raised (blue curve, left axis). When coupon-to-cash flow ratio attains the value 2.27, the firm can't issue new debt, and default happens. Model was solved for the benchmark set of parameters shown in Table **??**.

(X, c) approaches $(X_{def}(c), c)$, and so $E'_c\big(X_{def}(c), c\big) = 0$. Furthermore, note that

$$0 = \frac{\partial E}{\partial c}\big(X_{def}(c), c\big) = \frac{\partial E}{\partial X}\big(X_{def}(c), c\big)\frac{\partial c_{def}(X)}{\partial X} \tag{14}$$

where $c_{def}(X)$ is the inverse function of $X_{def}(c)$[13]. Equation (14) implies that as long as $c_{def}(X)$ is not a constant, $E'_x\big(X_{def}(c), c\big) = 0$, which is exactly the same as equation (12). Hence, *D-type model also features smooth-pasting condition for the value of equity, even though in this model it is a consequence rather than the assumption.*

2.2 Version one

In version one of the model, equityholders choose how much debt to issue at t_0; after t_0, firms in the E-type environment are not allowed to issue anymore debt, and firms in the D-type environment can only issue debt when they are in distress ($X < c$). After debt is issued at t_0, proceeds are distributed to equityholders in a form of immediate dividends. Equityholders choose the initial

[13] As will be clear from the closed-form solution derived below, $X_{def}(c)$ is indeed a function, and the inverse always exists

coupon payment to maximize the value of proceeds from debt issuance plus the value of equity after debt is issued.

2.2.1 D-type

Mathematically, firms in the D-type environment solve the following problem at t_0:

$$\max_{c_0} \Big(D(X_0, c_0) + E(X_0, c_0)\Big) \tag{15}$$

where $D(X, c)$ is the function that satisfies equation (8) on the interval $\{X, c : X \geq X_{def}(c)\}$, and has specific boundary conditions discussed below; $E(X, c)$ is the function that satisfies equation (3) on the interval $\{X, c : X \geq c\}$ and satisfies equation (6) on the interval $\{X, c : c > X \geq X_{def}(c)\}$, is continuous and smooth (derivative is continuous) along the line $X = c$, and has specific boundary conditions discussed below. Because $D(X, c)$ and $E(X, c)$ satisfy second-order PDEs, two conditions for each should be imposed to have the unique solution.

Note that equation (8) has a closed form solution of the following form:

$$D(X, c) = \frac{c}{r} + BX^{\beta}c^{1-\beta} + B_2X^{\beta_2}c^{1-\beta_2} \tag{16}$$

where B and B_2 are constants to be determined, and β and β_2 are respectively the negative and the positive roots of the quadratic growth equation (17)

$$\frac{\sigma^2}{2}\beta^2 + \Big(\mu - \frac{\sigma^2}{2}\Big)\beta - r = 0 \tag{17}$$

The first term in equation (16) is the value of the risk-free bond with constant coupon payment c. The second term converges to $-\infty$ as X converges to 0. Economically, this term catches the effect of an increasing probability of default (and associated default losses), when firm's performance deteriorates. The value of debt at default is known and is given by equation (10), which is one of the two boundary conditions for the value of debt. The third term in equation (16) converges to ∞ as X converges to ∞. Normally, value of a risky debt can not exceed the value of a riskless debt, and so $B_2 = 0$. It then follows from equations (10) and (13) that solutions for B and $X_{def}(c)$ take the following form[14]:

$$X_{def}(c) = -c\frac{\beta}{1-\beta}\frac{r-\mu}{r}\frac{1}{(1-\tau)(1-\alpha)} \tag{18}$$

$$B = -\frac{1}{r(1-\beta)}\left(\frac{c}{X_{def}(c)}\right)^{\beta} \tag{19}$$

Equations (18) and (19) have two important implications. First, $\frac{c}{X_{def}(c)}$ is a constant, and so B is also a constant, which verifies the conjecture for the debt value. Second, $B \neq 0$, which means that default part of equation (16) is not zero. Therefore, *debt is not risk-free in the D-type environment, even though debtholders can trigger the default of a firm quite early.*

The HJB equation for the value of equity on the interval $X > c$ (equation (3)), also has a closed-form solution:

$$E^{X \geq c}(X,c) = \frac{X(1-\tau)}{r-\mu} - \frac{c(1-\tau)}{r} + AX^{\beta}c^{1-\beta} + A_2X^{\beta_2}c^{1-\beta_2} \tag{20}$$

Note that the third term in equation (20) converges to ∞ at a very high rate as X grows to infinity[15]. Assuming that there are no speculative bubbles on the market, $A_2 = 0$, which gives one of the boundary conditions for the value of equity.

Equation (6) does not have a closed-form solution, and should be solved numerically. Denote $E^{X<c}(X,c)$ the solution to equation (6). In order for it to be the solution for the value of equity on the interval $X < c$, it must satisfy the following conditions[16]:

$$X\left(\frac{1-\tau}{r-\mu} - \frac{1-\tau}{r} + A\right) = E^{X<c}(X,X) \tag{21}$$

$$\frac{1-\tau}{r} + (1-\beta)A = -\frac{\partial E^{X<c}}{\partial c}(X,X) \tag{22}$$

[14]Note that for sufficiently large bankruptcy costs α, RHS of equation (18) may become larger than c, implying that default happens when $X > c$ (when the firm is not in distress). Of course, the firm can't be forced to liquidate its assets as long as it is able to make required coupon payments. Essentially, $X_{def}(c)$ should be the minimum between c and RHS of (18), but for a reasonable set of parameters (i.e. $\alpha < 70\%$) (18) will be the solution. Note also that (18) is always positive because β is negative

[15]$\lim_{X\to\infty}\left(\frac{X^{\beta_2}c^{1-\beta_2}}{X}\right) = \infty$

[16]Equation (21) can be written as the equality of partial derivatives with respect to X instead - the idea is that the derivatives from the left and from the right with respect to each variable should be continuous along the line $X = c$

$$E^{X<c}\big(X_{def}(c), c\big) = 0 \tag{23}$$

where equations (21) and (22) are the value-matching and smooth-pasting conditions along the line $X = c$, and equation (23) is the value of equity at default, which is the second boundary condition for the value of equity.

2.2.2 E-type

Firms in the E-type environment solve the same problem as firms in the D-type environment at t_0:

$$\max_{c_0} \Big(D(X_0, c_0) + E(X_0, c_0) \Big) \tag{24}$$

where $D(X, c)$ is the function that satisfies equation (8) on the interval $\{X, c : X \geq X_{def}(c)\}$, and has specific boundary conditions discussed below; $E(X, c)$ is the function that satisfies equation (3) on the interval $\{X, c : X \geq X_{def}(c)\}$, and also has specific boundary conditions discussed below. In addition to that, firms in the E-type environment choose the timing of default, which is defined by equation (12). Conditions at the boundaries for values of both debt and equity are the same as in the D-type environment.

First, note that both $D(X, c)$ and $E(X, c)$ have closed form solutions:

$$D(X, c) = \frac{c}{r} + BX^{\beta}c^{1-\beta} + B_2X^{\beta_2}c^{1-\beta_2} \tag{25}$$

$$E(X, c) = \frac{X(1-\tau)}{r} - \frac{(1-\tau)c}{r} + AX^{\beta}c^{1-\beta} + A_2X^{\beta_2}c^{1-\beta_2} \tag{26}$$

From the boundary conditions at $X >> c$, it can be concluded that $B_2 = 0$ (because value of debt should converge to the value of risk-free debt), and $A_2 = 0$ (to exclude speculative bubbles as $X \to \infty$). The remaining boundary conditions are the values of debt and equity when the firm defaults, and are given by equations (10) and (11). These equations allow to explicitly solve for values of A and B (see Appendix B), and $X_{def}(c)$:

$$X_{def}(c) = -c\frac{\beta}{1-\beta}\frac{r-\mu}{r} \tag{27}$$

Note how default rules are different for firms in the E-type and D-type environments (equations (27) and (18) respectively). Because the timing of default is affected by decisions of debtholders in the D-type environment, bankruptcy costs α appear explicitly in the equation, as opposed to the solution for the default rule for firms in the E-type environment, where bankruptcy costs are only implicitly internalized by equityholders through interest rates at which debt is issued at t_0. However, *even though for each value of c the default boundary* $X_{def}(c)$ *in the E-type environment is lower than the default boundary in the D-type environment (by the factor* $(1-\alpha)(1-\tau)$*), firms in the D-type environment do not necessarily default earlier than firms in the E-type environment*, and the reverse is most likely true.[17] It is explained by the fact that firms in the D-type environment choose a more conservative debt policy: they realize that marginal interest rates to issue new debt in distress grow quickly, and so ex-ante choose a much lower leverage. In contrast, firms in the E-type environment can choose a greater leverage without a fear of going bankrupt soon.

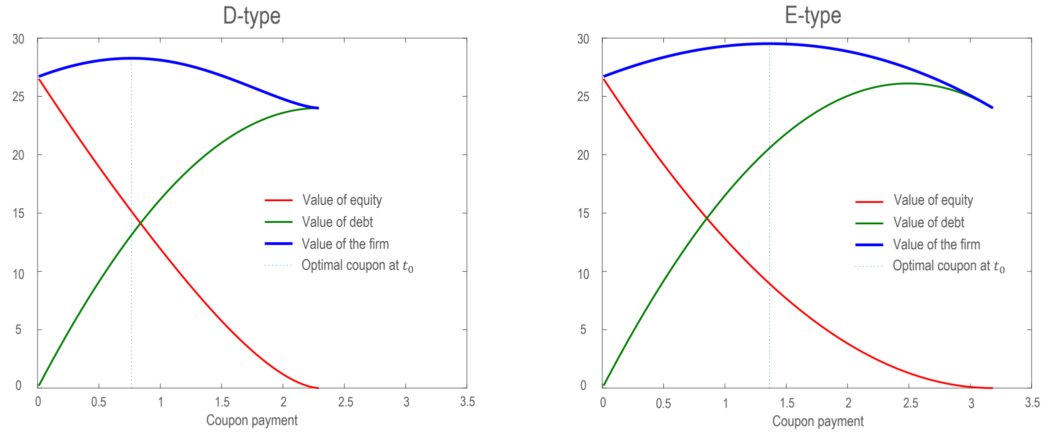
Figure **??** shows the value of debt and equity for a firm with current productivity $X = 1$ and different values of coupon payment in the D-type and E-type environments. For the chosen calibration of parameters, firms in the D-type environment default when coupon-to-cash flow ratio exceeds the value 2.27, at which point the value of equity (red curve) falls to zero, and no further debt issuances are possible (green curve is flat). For the same set of parameters, firms in the E-type environment avoid bankruptcy for longer because they can raise external finance at a much cheaper rate (the default coupon-to-cash flow ratio is 3.23). In both environments, firms that start with $X_0 = 1$ choose the initial coupon level c_0 to maximize the value of the blue curve, which is the sum of the value of equity after debt is issued plus proceeds from debt issuance; the vertical line denotes the optimal initial coupon value. Note that firms in the E-type environment choose to issue debt with coupon payment that significantly exceeds their cash flow.

2.3 Version two

Version two of the model is different from version one of the model by the assumption that firms with strong financial performance can issue new debt after t_0 to further exploit benefits of tax shield. These debt issuances are modeled differently from debt issuances by firms in distress in the D-type environment, and I discuss implications of the difference later in the text. Debt issuances

[17] This depends on the values of the parameters, but this is usually the case as Table **??** in Section 3 shows

Figure 5
Value of debt and equity in version one of the model for a firm with $X = 1$



These two graphs show the value of equity and debt in D-type and E-type environments for a firm with current productivity $X = 1$ and different coupon payments c. At t_0, the firm that starts with $X_0 = 1$ chooses c_0 that maximizes the value of the blue curve, which is the sum of proceeds from debt issuance (green curve on the graphs) plus the value of equity after debt is issued (red curve on the graphs). Model was solved for the benchmark set of parameters shown in Table **??**.

by firms in distress in the D-type environment do not change and are still characterized by equation (5) (firms in the E-type environment do not issue debt in distress at all).

Debt issuances outside of distress are modeled the same way as in Goldstein, Ju, and Leland (2001). To issue new debt to better exploit benefits of tax shield, a firm has to redeem all its debt outstanding first, and then issue new debt with a greater coupon payment. These issuances are costly, and costs are proportional to the amount of new debt issued: for each \$1 of new debt raised, the firm only gets $\$(1 - q)$. If firm's current cash flow level is X, and it pays coupon c on its current debt, and the firm decides to issue new debt with the coupon level c_{new}, proceeds from debt issuance equal to $\Big((1 - q)D(X, c_{new}) - D(X, c)\Big)$; these proceeds are distributed to equityholders as dividends. Every time the firm issues new debt, it chooses c_{new} to maximize the amount of proceeds from debt issuance plus the value of equity after debt is issued. Equation below connects the value of equity before and after debt is issued:

$$E(X, c) = \max_{c_{new}} \Big((1 - q)D(X, c_{new}) - D(X, c) + E(X, c_{new}))\Big) \tag{28}$$

The LHS of equation (28) shows the value of equity right before debt is issued, and the RHS of equation (28) shows the value of equity right after debt is issued. LHS and RHS are equal because equity value function is continuous; economically, equityholders have rational expectations about when the firm issues debt, and the share price of the firm adjusts accordingly.

The scaling property of the model allows to solve for c_{new} in equation (28) easily. To understand the intuition of the scaling property, consider two firms at t_0 with $X_0^1 = 1$ and $X_0^2 = 2$, that is, the second firm is two times larger than the first firm. Because the model features constant return to scale, the coupon payment that the second firm chooses optimally should be two times greater than the coupon payment that the first firm chooses. As the result, values of debt and equity of the second firm should be two times greater than values of debt and equity of the first firm. Effectively, in this set-up the second firm is a greater replica of the first firm, and so all values are proportional. Now consider just one firm but at the time when it chooses to restructure its debt upward. At the short moment when it has repurchased its outstanding debt but before it issued new debt, it has zero debt outstanding, and it is similar to a larger replica of itself at t_0. Hence, the new coupon payment, and the values of debt and equity of this firm after debt is issued should increase by the factor $\frac{X_t}{X_0}$ relative to values of coupon payment, debt and equity at t_0. Let $X_{res}(c)$ be the optimally chosen restructuring boundary of a firm:

$$X_{res}(c) = \{X > c : \text{firm in state } (X_t, c) \text{ issues new debt if and only if } X_t = X\} \tag{29}$$

The assumption that costs to issue new debt q are greater than zero guarantees that firms do not adjust their capital structure continuously. Denote c_0 the coupon payment that a firm chooses at t_0, when its cash flow is X_0. Equation (28) can then be rewritten in the following form:

$$E\big(X_{res}(c), c\big) = (1-q)\frac{X_{res}(c)}{X_0} D(X_0, c_0) - D\big(X_{res}(c), c\big) + \frac{X_{res}(c)}{X_0} E(X_0, c_0) \tag{30}$$

It only remains to show how firms choose the restructuring boundary $X_{res}(c)$. Using the constant return to scale argument as before, it can be shown that $X_{res}(c)$ is proportional to c; denote $A_{res} = \frac{X_{res}}{c}$. I assume that firms choose A_{res} to maximize the value of equity and proceeds from debt issuance at t_0.

2.3.1 D-type

Note that the assumption about costly debt issuance creates some internal inconsistency in the model. Effectively, the model assumes that whenever debt is issued by a firm in distress, debt issuance is costless; however, when new debt is issued by a strongly performing firm to further exploit benefits of tax-shield, debt issuance is costly. It should be noted that the assumption of costly debt issuance was introduced by Goldstein, Ju, and Leland (2001) and is shared by most papers in this class of literature. The assumption guarantees that firms do not adjust their capital structure continuously. By maintaining this assumption, I can better compare results of E-type and D-type models, and also preserve the feature that debt value is not adjusted continuously (debt changes in distress are not done to better exploit benefits of tax-shield, but rather to avoid bankruptcy).

Firms in the D-type environment solve the following problem at t_0:

$$V(X_0) = \max_{A_{res}, c_0} \Big(E(X_0, c_0; A_{res}) + (1-q) D(X_0, c_0) \Big) \tag{31}$$

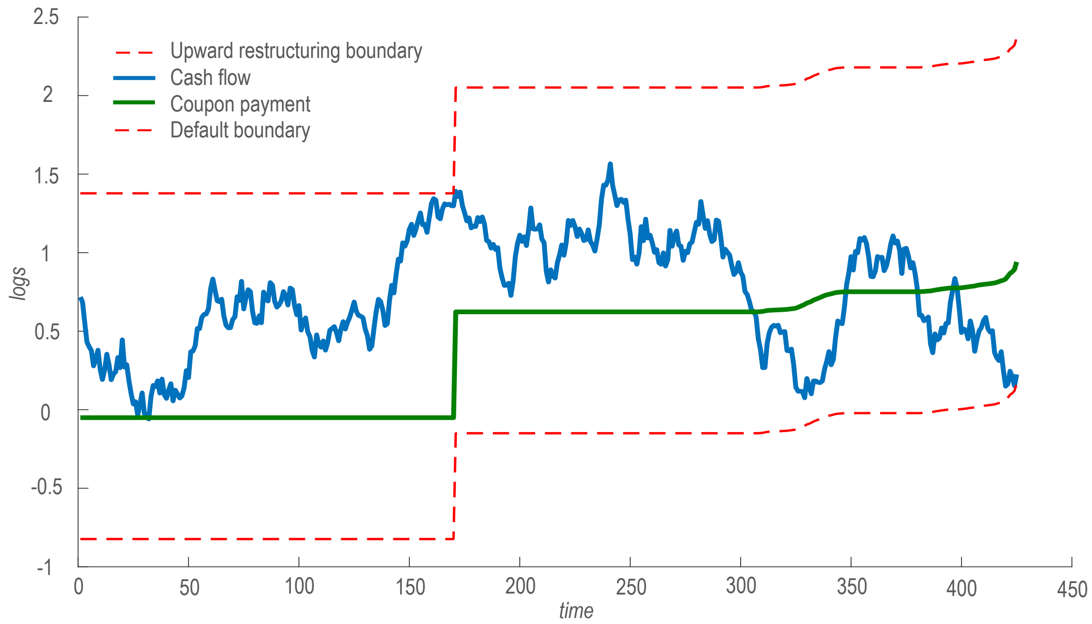
where $D(X, c)$ is the function that satisfies equation (8) on the interval $\{X, c : X \geq X_{def}(c)\}$; $E(X, c; A_{res})$ is the function that satisfies equation (3) on the interval $\{X, c : X \geq c\}$ and satisfies equation (6) on the interval $\{X, c : c > X \geq X_{def}(c)\}$, is continuous and smooth (derivative is continuous) along the line $X = c$. As before, two boundary conditions for both $D(X, c)$ and $E(X, c)$ need to be imposed to have the unique solution.

Boundary conditions for debt value do not change from version one of the model: value of debt at default is still characterized by equation (10), and another boundary condition is equation $B_2 = 0$, which implies that the value of debt can not exceed the value of risk-free debt at any time. Because neither the HJB equation for debt value, nor boundary conditions have changed, the solution to the debt value function is still the same. Moreover, because default rule in the D-type environment is fully characterized by the debt value function, $X_{def}(c)$ is also the same:

$$D(X, c) = \frac{c}{r} + BX^{\beta} c^{1-\beta} \tag{32}$$

$$X_{def}(c) = -c \frac{\beta}{1-\beta} \frac{r-\mu}{r} \frac{1}{(1-\tau)(1-\alpha)} \tag{33}$$

Figure 6
Dynamic of a typical firm in the D-type environment in version two of the model



The firm takes no action as long as its cash flow (blue curve) stays above its coupon payment (green curve) and below the upward restructuring boundary X_{res} (upper red curve). When cash flow level reaches the upward restructuring boundary, the firm issues more debt, and future coupon payments increase. When firm's cash flow is lower than its coupon payment but above the default boundary X_{def} (bottom red curve), the firm slowly raises new debt to cover the shortfall between the cash flow and required coupon payments. The firm defaults when cash flow level reaches the default boundary. Values on the graph are in logs.

As for the equity value function $E(X, c)$, equations (21) and (22) should hold to guarantee that the function is continuous and smooth along the line $X = c$. Furthermore, the value of equity at default is zero, as equityholders receive nothing when the firm defaults. The remaining boundary condition is the value of equity when the firm restructures its debt upward, and is given by the recursive equation (30). There is no closed-form solution for this function, and I use numerical methods to compute its value.

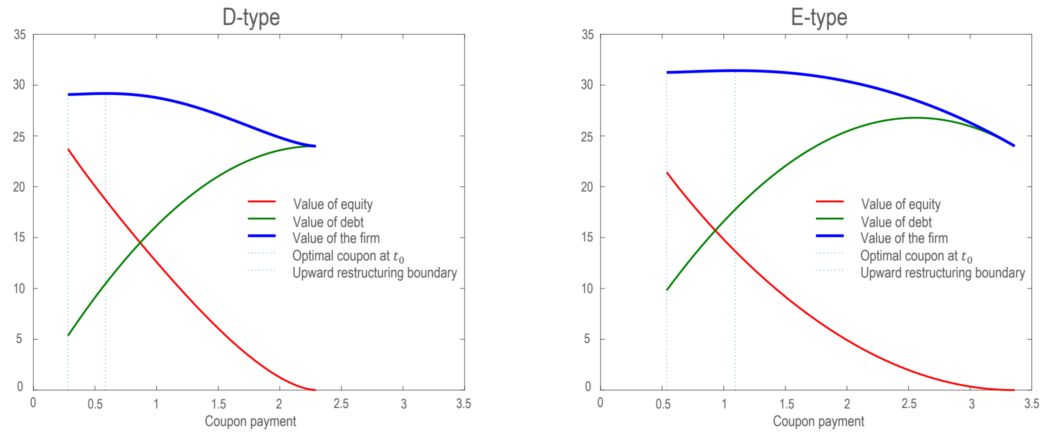
Figure **??** visualizes the behavior of a typical firm in the D-type environment in version two of the model. On the graph, firm's cash flow grows steadily in the beginning, and at time $t = 167$ the firm issues more debt to further exploit benefits of tax-shield. The firm enters distress for the first time at $t = 311$, and it starts issuing debt to cover the gap between the cash flow and required interest payments. As the result, the coupon payment keeps growing all the way before firm's performance improves sufficiently at $t = 362$; because coupon payment grows, the default boundary and the upward restructuring boundary (which are linear in c) also grow. The firm enters distress for the second time at $t = 375$, and, with the exception of a short moment at $t = 394$, stays in distress until it eventually defaults. Note that the rate at which coupon payment grows increases as firm's cash flow approaches the default boundary. This is because debt issuances are endogenously more expensive for firms that are close to default.

2.3.2 E-type

Formulation of the problem for the E-type environment is the same as in the D-type environment with two exceptions: 1) equity value function $E(X, c)$ satisfies equation (3) on both intervals $X \geq c$ and $X < c$, and 2) default rule is determined by equation (11).

Figure **??** shows the value of debt and equity in the version two of the model for a firm with current productivity $X = 1$ and different values of coupon payment.

Figure 7
Value of debt and equity in version two of the model for a firm with $X = 1$



These two graphs show the value of equity and debt in version two of the model in D-type and E-type environments for a firm with current productivity $X = 1$ and different coupon payments c. At t_0, the firm chooses c_0 that maximizes the value of the blue curve, which is the sum of proceeds from debt issuance (green curve on the graphs) plus the value of equity after debt is issued (red curve on the graphs). Every time firm's coupon level is too low relative to its cash flow level, the firm issues more debt. The grey dotted lines on each graph denote the values of c at which the firm issues more debt (the left dotted line) and the value of c that it chooses every time it issues more debt (the right dotted line). Model was solved for the benchmark set of parameters shown in Table **??**.

Table 1
Benchmark parameter values used to solve the model

Variable		Value
risk-free rate	r	5%
growth rate of the cash flow process	μ	2%
volatility of the cash flow process	σ	25%
effective corporate tax rate	τ	20%
bankruptcy costs	α	10%
risk premium	rp	5%
debt issuance costs	q	1%

3 Model solution

3.1 Parameter values

Table **??** shows parameter values that are used to solve the model. These values are taken from Strebulaev and Whited (2012) who provide the review of literature on dynamic capital structure and simulate a number of models similar to the E-type model discussed in this paper. Most parameters are hard to estimate directly in the data; moreover, there is a substantial cross-sectional heterogeneity. Therefore, some assumptions should be made. Specifically mentioned, estimates of bankruptcy costs α, which in this paper is assumed to be 10%, vary from very low to very high. For example, Gruber and Warner (1977) finds that direct bankruptcy costs are about 1% of the assets value, and Andrade and Kaplan (1998) report the value of about 20%. Some authors used a structural estimation approach to infer the bankruptcy costs from firms' observed decisions. In particular, Davydenko, Strebulaev and Zhao (2012) find that default costs are in the range of 10% and 30%, Hennessy and Whited (2007) report values between 8.4% and 15.1%, and Glover (2016) finds the value of about 45%. Glover's estimates are well-above estimates of other authors, but as argued by Reindl, Stoughton, and Zechner (2017), it is because Glover assumes that all firms follow optimal leverage policy, while it is not necessarily the case in the data. The authors estimate a similar model without imposing optimal capital structure and using firms stock prices instead, and find substantially lower values of bankruptcy costs (20%).

Cash flow volatility parameter σ also does not have a precise estimate in the literature. Faulkender and Petersen (2005) report that the average implied asset volatility of firms that have access to public debt is 19%, and Schaefer and Strebulaev (2008) find 23% (also among firms that issue bonds); Reindl, Stoughton, and Zechner (2017) take a structural estimation approach with

Table 2
Results of the model solution

	Benchmark	$\mu = 3.5\%$	$\sigma = 15\%$	$\sigma = 35\%$	$\alpha = 50\%$	$q = 0$
D-type, version 1						
Initial coupon	0.77	1.32	0.80	0.80	0.19	-
Default	0.44	0.24	0.60	0.32	0.78	-
Leverage, t_0	46.44%	42.60%	53.05%	42.17%	13.40%	-
E-type, version 1						
Initial coupon	1.36	2.65	1.28	1.58	0.84	-
Default	0.31	0.17	0.43	0.23	0.31	-
Leverage, t_0	69.72%	70.86%	75.29%	66.95%	48.48%	-
D-type, version 2						
Initial coupon	0.54	0.80	0.65	0.50	0.14	0.43
Upward restructuring	3.56	2.44	2.52	4.38	13.56	-
Default	0.44	0.24	0.60	0.32	0.78	0.44
Leverage, t_0	33.56%	25.15%	42.44%	27.89%	10.03%	26.68%
E-type, version 2						
Initial coupon	1.03	1.94	1.09	1.07	0.65	0.81
Upward restructuring	1.86	0.99	1.50	2.00	2.97	-
Default	0.30	0.14	0.41	0.22	0.30	0.29
Leverage, t_0	53.99%	48.49%	63.39%	48.26%	37.86%	43.94%

This table shows optimal decisions that firms make in the model. Column 2 reports model solutions for the benchmark set of parameters (Table **??**); columns 3-7 report solutions of the model, in which all but one parameters are as in the benchmark set. Initial coupon refers the value c_0 that firms with $X_0 = 1$ optimally choose at t_0. Default is the ratio $X_{def}(c)/c$, and shows the cash flow level X at which firms with $c = 1$ default. Upward restructuring is the ratio $X_{res}(c)/c$, and shows the cash flow level X at which firms with $c = 1$ optimally choose to issue more debt to further exploit benefits of tax-shield; these additional debt issuances are not allowed in version one of the model, and so upward restructuring is not reported for version one of the model. Leverage, t_0 is leverage that firms in the model have right after they issue debt at t_0

Leland-type environment and find asset volatility between 25% and 42%.

The effective corporate tax-rate τ that this paper uses implicitly aggregates the effect of corporate and personal taxes on dividends and interest payments; the resulting value $\tau = 20\%$ is based on the estimates of Graham (2000). In a model, similar to mine, Chen (2010) considers different taxes explicitly, and the resulting effective corporate tax rate in his model is around 18%.

While the model is solved under the risk neutral probability measure $\mathbb{Q}$, actual shock realizations happen under the physical probability measure $\mathbb{P}$. Therefore, in the simulation procedures discussed below, risk-premium $rp = 5\%$ is added to the risk-neutral growth rate μ.

Table **??** presents results of the model for the benchmark set of parameters, and for some variation of the parameters. As expected, firms in the E-type environment start with a greater coupon payment, operate longer before defaulting for the same value of coupon payment, and restructure their debt upward earlier, also for the same value of coupon payment. As discussed in the previous Section, even though the default boundary is lower for firms in the E-type environment, firms in the D-type environment do not necessarily default earlier because they choose lower coupon payment at t_0. Note also that firms in the E-type environment often choose initial coupon payment that is greater than the cash flow level. This partially illustrates the problem that arises when equity financing is always freely available: firms optimally prefer to be in distress every time they readjust their leverage.

As Table **??** suggests, firms always issue more debt at t_0, and restructure debt upward more often when the expected growth rate of the log cash flow, which is $\mu - \frac{\sigma^2}{2}$, is higher. Firms also operate longer before bankruptcy when μ is high: high expected growth rate increases the value of firm's assets. Interestingly, even though high value of σ lowers the expected growth rate of log cash flows, firms with greater value of σ postpone the default decision. High value of σ increases firm's profits in good states, and losses in bad states are bounded (value of equity is always non-negative). Therefore, even though high value of σ reduces the value of equity when the firm is far from distress, it increases the value of the firm for firms deep in distress, and so firms wait longer before defaulting. This logic is straight for firms in the E-type environment (because firms in the E-type environment choose the timing of default), and goes through formulas implicitly in the D-type environment.[18]

Note that consistent with equation (27), default boundary in the E-type environment is independent of bankruptcy costs α. This is because equityholders do not consider interests of debtholders when they choose the timing of default. Bankruptcy costs are only implicitly internalized by equityholders in the E-type environment through interest rates at which they issue debt at t_0, and that is why initial coupon payments vary with α. In contrast, debholders affect the timing of default in the D-type environment, and so the default boundary is greater when α is high in the D-type environment.

An important observation follows from Table **??**: firm's leverage in the model is not a perfect indicator of its indebtedness. For example, once firm's growth rate increases from $\mu = 2\%$ to $\mu = 3.5\%$, firms optimally choose to issue significantly more debt (as indicated by a much higher initial coupon), but leverage falls. This happens because an increase in μ has three effects: 1) value of firm's equity increases, 2) the firm wants to issue debt with a higher coupon payment to better exploit benefits of tax-shield, and 3) value of firm's debt increase because coupon payments are

[18]Specifically, it affects the value of β so that the default boundary becomes lower

higher, and probability of default is lower. Due to the second effect, firm's chosen coupon payment unambiguously increases; however, the effect on leverage is not clear, as both value of debt and equity go up. Changes in other parameters, even if they are small, may have a similar effect. This means that a model may correctly explain empirically observed leverage values because it provides a good representation of the data, or because the parameter values that it uses are estimated with an error. Therefore, in assessing the quality of a model, it is important to examine how well the model matches both market-based as well as non-market based indicators.

3.2 Default probabilities

As Table **??** implies, among two firms that have the same coupon payments c, the firm that operates in the D-type environment will default earlier than the firm that operates in the E-type environment. This happens for two reasons: 1) default threshold is higher, and 2) interest expenses grow exponentially for firms in distress in the model with the D-type environment. On the other hand, firms in the model with the D-type environment choose lower initial leverage. Therefore, it is not straightforward which model produces a higher default probability.

To answer this question, the paper uses pairwise simulations[19]. For each version of the model (version one and version two), I generate two firms that are exposes to the same realization of shocks. The first firm behaves as if it lives in the E-type environment (it can issue both debt and equity), and the second firm behaves as if it lives in the D-type environment (in distress it can only issue debt); both firms make their financing decisions optimally. Each simulation is performed on the monthly basis and continues unless both firms default. Note, however, that firms in version one of the model grow on average, but they issue debt only once, and so their leverage continuously attenuates if they experience a series of positive shocks. Therefore, many simulations of version one of the model should result in no default. For this reason, simulations are additionally terminated after 60000 periods in version one of the model if at least one firm has survived this long.

Simulations are repeated 10000 times for each version of the model, and Table **??** reports the fraction of simulations, in which the firm in the E-type environment defaults before the firm in the

[19]Not formally reported here, I also estimate the annual default rate in the steady-state cross-section of firms in both types of economies in version two of the model. The default rate is higher in the E-type economy than in the D-type economy (1.25% vs. 1.19%). However, the 90% confidence intervals overlap, which does not allow to formally conclude that E-type economy has a higher default rate. Pairwise comparisons avoid this problem. Section 3.3 describes details of the simulation of a steady-state cross-section

Table 3
Default probabilities and firm lifetime in the model

	Version one	Version two
firm in E-type economy defaults earlier	76.8%	68.0%
firm in D-type economy defaults earlier	14.3%	24.3%
firms in both economies default simultaneously	8.9%	7.7%
average lifetime of a firm in E-type economy (months)	275.4	967.1
median lifetime of a firm in E-type economy (months)	151	705
average lifetime of a firm in D-type economy (months)	284.7	1030.3
median lifetime of a firm in d-type economy (months)	163	760

Numbers are estimated using simulations. Simulations are run on a monthly basis independently for each version of the model. Each simulation has two firms with one behaving as if it operates in the E-type environment, and one as if it operates in the D-type environment; firms in E-type model can always issue equity, and firms in distress in D-type model have to issue debt to cover the gap between interest expenses and the cash flow. Both firms in each simulation are exposed to the same realization of shocks. Simulations are continued as long as at least one firm has not defaulted. Firms in version one of the model grow on average, and their leverage attenuates, and so some simulations should result in no default; for this reason, simulations for version one are terminated after 60000 periods if at least one firm has survived. Simulations are repeated 10000 times, and numbers in columns 2 and 3 are averaged among all simulations (version two) or simulations in which both firms defaulted before $t = 60000$ (version one). Rows $1 - 3$ report the fraction of simulations in which one firm defaults before the other or both firms default in the same period. Rows $5 - 7$ report the average and median number of periods that the firm in each environment operated before default

D-type environment or vice versa. As follows from the table, there are paths of shock realizations such that each firm can outlive the other or that both firms default in the same period. Nevertheless, for the majority of cases the firm that operates in the D-type environment operates longer than the firm that operates in the E-type environment, and the difference in average lifetime is one year for version one of the model and five years for version two of the model. Even thought it is harder to pay interest expenses for firms in distress in the model with the D-type environment, they are more conservative in their initial leverage policy, and so the resulting default rate is higher in the E-type economy.

3.3 Leverage distribution

I first collect data on firms' profits, interest expenses, and leverage. The sample of firms comes from Compustat for years 1981-2017.[20] Firms in the financial sector (6000s SICs) and the public sector (9000s SICs) are excluded from the analysis; observations with the book value of assets that is less than $1 million are also excluded.

As Faulkender and Petersen (2005) find, firm's capital structure depends a lot on whether the firm has access to public bond markets. The assumption that firms can issue debt easily is crucial

[20] 1981 is the first year that has data on S&P long-term credit ratings

in this model, and for this reason, the paper only consider firms that have S&P long-term credit rating, which is used as a proxy for whether the firm can issue public debt. Data on the S&P long-term ratings is available on monthly basis, but financial data is annual. To match the datasets, it is assumed that a firm has S&P long-term rating in a given year if it has S&P long-term rating in at least one month of that year. Data on S&P long-term ratings is available between years 1981 and 2017, and there is, on average, 1500 observations in each year. However, years 1981-1984 have only five observations combined, and year 2017 has only 146 observations.

In what follows, leverage is measured as the ratio of firm's total debt (sum of long-term and short-term debt) to the the book value of assets. Some observations have leverage value that exceeds one, and these observations are excluded from the analysis.

This paper also considers the distribution of inverse coverage ratios (ratio of firm's interest expenses to cash flow). There are two main reasons why the inverse coverage ratio and not the coverage ratio is chosen as a target moment. First, some firms in the data have either no debt, or very small values of debt, and, therefore, interest expenses of these firms are small compared to their cash flow. These observations significantly affect the average value of coverage ratios in the data and make it sensitive to how they are treated. For instance, unwisorized average coverage ratio among firms with positive interest expenses is 9.8; it is 7.1 if coverage ratios are additionally winsorized at 0.1-99.9 percentiles, 5.8 if winsorized at 1-99 percentiles, and 5.1 if winsorized at 3-97 percentiles. The inverse coverage ratio avoids the problem of division by zero because all firms in the data have non-zero cash flow, and is, therefore, less sensitive to how outliers are treated: unwinsorized average inverse coverage ratio in a cross-section is 0.21; it is 0.38 if observations are winsorized at 0.1-99.9 percentiles, 0.36 if winsorized at 1-99 percentiles, and 0.36 if winsorized at 3-97 percentiles. Second, the inverse coverage ratio is a more natural parameter for the model discussed in this paper (both in E-type and D-type environments). As argued in Section 3.2, the model features scaling property, and so firm's equity value can be rewritten[21] as $E(X, c) = Xe(\frac{c}{X})$, which is correctly specified for all values of X and c.[22] At the same time, writing $E(X, c) = c\tilde{e}(\frac{X}{c})$ would be inconsistent (and can not be easily extended) for firms with $c = 0$.[23]

[21] Note that debt value can also be rewritten this way

[22] Note that because X_t follows Geometric Brownian motion, it is always positive

[23] It is true that all firms in the model optimally have positive values of c; nevertheless, to solve the model, it is necessary to correctly specify the equity value function for all values of c

Table 4
Numbers in the data

	Period		
	all years	1981-2000	2001-2017
leverage	36.6%	37.6%	35.8%
# of observations	46 648	21 237	25 411
inverse coverage ratio	0.38	0.41	0.36
# of observations	45 723	20 535	25 188
inverse coverage ratio, truncated	0.69	0.78	0.61
# of observations	41 778	18 639	23 139
inverse coverage ratio, censored	0.63	0.70	0.56
# of observations	45 723	20 535	25 188
fraction of firms in distress, all firms			
among all firms	18.5%	20.9%	16.5%
among firms with inverse coverage ratio ≥ 0	10.8%	12.9%	9.1%

This table reports summary statistics on firms' leverage and inverse coverage ratios; only firms that have access to the public debt markets are considered, and the access is proxied by whether a firm has S&P long-term credit rating in a particular year. Average values are reported (except for the number of observations). Leverage is the ratio of total debt (sum of short-term and long-term debt) to the book value of assets. Inverse coverage ratio is the ratio of firm's interest payment to the value of its EBIT. Inverse coverage ratio is winsorized at 0.1% and 99.9% values. The inverse coverage ratio is not always positive; therefore, the average values of truncated and censored inverse coverage ratios are separately considered. The average truncated inverse coverage ratio ignores firm-year observations with negative inverse coverage ratios; to compute the average censored inverse coverage ratio, value of the inverse coverage ratio is reset to zero for firm-year observations with negative inverse coverage ratio. Fraction of firms in distress refers to the fraction of firms with inverse coverage ratio negative or greater than one among all firms (second-to-last row) or the fraction of firms with inverse coverage ratio greater than one among firms with positive inverse coverage ratio (last row).

The paper measures inverse coverage ratio as the ratio of firm's total interest expenses to the value of its EBIT; values are winsorized at the 0.1% and 99.9% levels. While inverse coverage ratio has advantages over the coverage ratio, it also has one drawback: observations with negative inverse coverage ratios are somewhat misleading: these observations drive the average value of the inverse coverage ratio down, thus, creating an impression that the average inverse coverage ratio is low. Low positive inverse coverage ratio usually implies that firms in the population earn significantly more profits than they spend to pay interest expenses, which is not the case for firms with negative EBIT. There are 8.6% of observations with negative inverse coverage ratios. To account for this problem, the paper separately computes the truncated inverse coverage ratio, and the censored inverse coverage ratio. To compute the censored inverse coverage ratio, values of inverse coverage ratio are reset to zero for observations that have negative inverse coverage ratio. To compute the truncated inverse coverage ratio, observations with negative inverse coverage ratio are excluded.

Table **??** presents the summary statistics for the whole period of data, and separately for the first and second halves. The average value of leverage is 36.6%, and it does not change much before and after 2000. The average values of truncated and censored inverse coverage ratios are similar (0.69 and 0.63), implying that its value is close to zero for most firms with negative inverse coverage ratios. Figure **??** additionally shows the whole distribution of leverage and inverse coverage ratios in the data.

The next step is to study how well the model addresses the moments of distribution of leverage and inverse coverage ratios. While static results reported in Table **??** indicate that the version two of the model with the D-type environment comes close to matching empirical values (optimal leverage is 33.56%, and inverse coverage ratio is 0.56), it is misleading to study static results that firms optimally choose. As argued by Strebulaev (2007), average values in a cross-section may differ significantly from what firms optimally choose at t_0. Therefore, before claiming success of the model, it is necessary to generate a steady-state cross-section of firms and examine its average values. The cross-sectional values are only meaningful for version two of the model: in version one, firms are not allowed to increase their leverage after t_0; because on average firms grow, the economy has a single trivial steady-state, in which all firms have zero leverage. Therefore, version one of the model (with both types of environments) has a trivial steady-state distribution, in which all firms have zero leverage and zero inverse coverage ratios.

Simulation approach is used to generate a cross-section of firms in version two of the model. For each type of the environment (E- and D-), I generate an economy populated by $N = 3000$ firms that operate for $T = 3600$ months (300 years). Firms start at t_0 with $X_0 = 1$ and make financing decisions optimally; if a firm defaults, it is replaced by another firm with $X = 1$, thus, maintaining a balanced panel. At the end of the period $t = 3600$, the following simulated moments are computed: the fraction of firms in distress (firms whose interest expenses c are higher than the cash flow X), average inverse coverage ratio among all firms (ratio of interest expenses to the cash flow c/X), average and median leverage values, and the fraction of firms with high leverage (firms whose leverage exceeds certain values: 30%, 40%, 50%, 60%, 70%, 80%, and 90%). Results are reported in Table **??**, and Figure **??** additionally shows the distribution of leverage and inverse coverage ratios.

As simulation results show, model with the D-type environment matches data moments well. Average value of leverage is only slightly higher than in the data (38.4% vs. 36.6%), but its 90%

Table 5
Empirical and simulated moments of cross-sectional distribution

	E-type, version 2	D-type, version 2	Data
fraction of firms in distress	57.9%	13.9%	18.5%
inverse coverage ratio (truncated in the data)	1.20	0.66	0.69
average leverage	58.26%	38.44%	36.64%
median leverage	55.89%	35.11%	34.20%
fraction of firms with			
leverage $\geq$ 30%	99.87%	67.27%	61.04%
leverage $\geq$ 40%	89.47%	35.70%	36.00%
leverage $\geq$ 50%	64.90%	19.30%	20.68%
leverage $\geq$ 60%	39.60%	10.90%	12.00%
leverage $\geq$ 70%	22.47%	6.70%	6.45%
leverage $\geq$ 80%	11.40%	2.90%	3.01%
leverage $\geq$ 90%	4.13%	1.10%	1.09%

This table reports the fraction of firms in distress (firms whose interest expenses exceed the cash flow), the average inverse coverage (ratio of interest expenses to cash flow), and moments of leverage distribution in a simulated cross-section of firms and in the data. Firms in E-type model can always issue equity, and firms in distress in D-type model have to issue debt to cover the gap between interest expenses and the cash flow. Data is taken from Compustat for the period 1981-2017, and only firms with access to public debt markets are considered, which is proxied by having S&P long-term credit rating. To compute values in the model, a balanced panel of $N = 3000$ firms was simulated over $T = 3600$ months, and numbers were averaged over the last period of the simulation.

confidence interval[24] spans between 37.1% and 39.7%, which covers 37.6% average leverage for the period before 2000 and 37.2% reported by Faulkender and Petersen (2005)[25]. More importantly, the model with the D-type environment correctly explains the fraction of firms in the right-tail of leverage distribution, independent of how the right tail is defined. In fact, 90% confidence intervals cover empirical counterparts for almost all threshold values of leverage considered[26]. Furthermore, the fraction of firms in distress is 13.9%, and its 90% confidence intervals is between 10.8% and 16.8% - close to values in the data. The average inverse coverage ratio (0.66) falls in the range between the average truncated and censored inverse coverage ratio in the data, and its 90% confidence interval 0.63-0.70 covers both values. As Figure **??** shows, the model reproduces the overall shape of the distribution of inverse coverage ratios in the data, even though quantitatively, its kurtosis is greater (i.e. the distribution is narrower in the model).

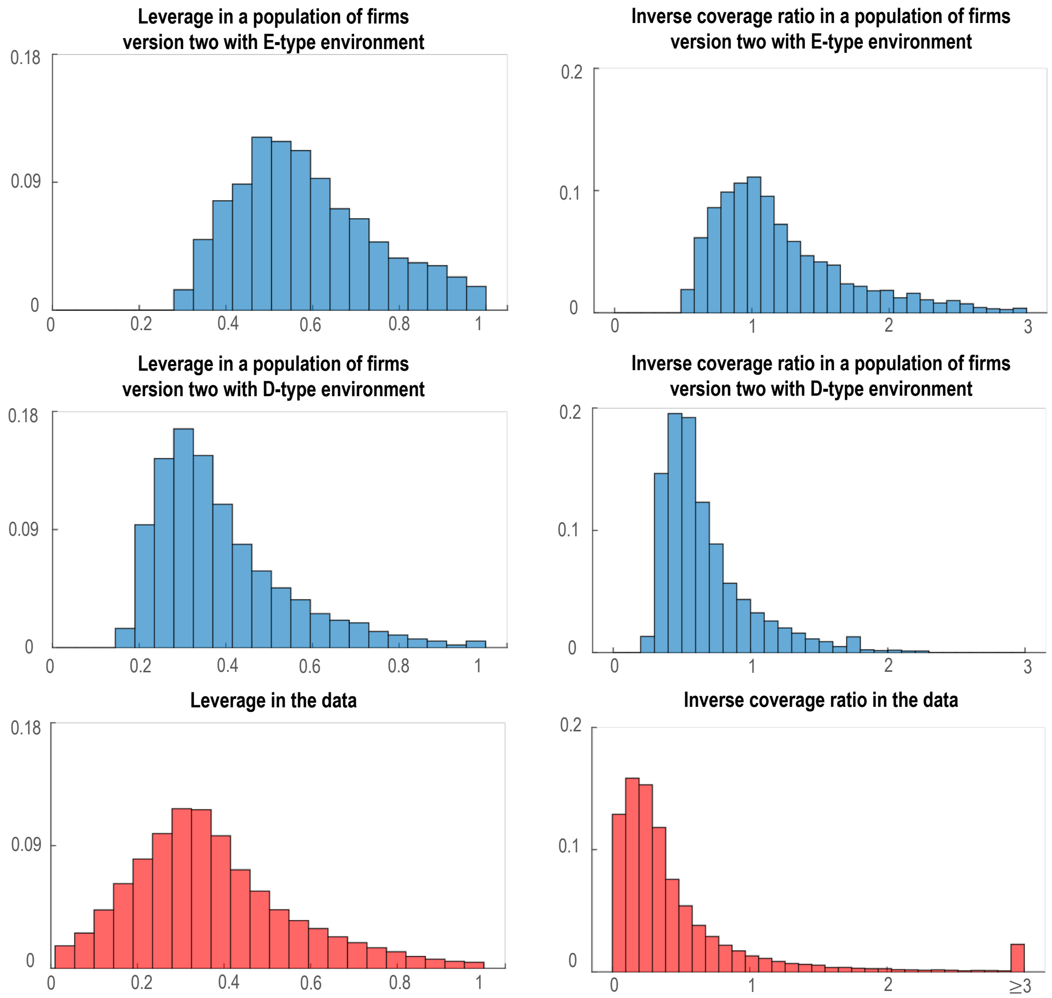
On the other hand, model with the E-type environment does not explain the distribution of the right tail of firms: cross-sectional leverage is 58%, 57% of firms do not produce enough cash to service their interest expenses, and almost all firms (99.9%) have leverage value above 30%.

[24]To compute 90% confidence interval in the model, simulations were repeated 400 times

[25]While not reported in the table, the average cross-sectional leverage is 35.1% if bankruptcy costs $\alpha = 15\%$ and 30.9% if $\alpha = 20\%$

[26]The only case that is not covered by 90% confidence interval is when the right tail of the distribution is defines as firms whose leverage exceeds 30%

Figure 8
Distribution of leverage and inverse coverage ratio



The figure shows distributions of leverage and inverse coverage ratios in a simulated economy of version two of the model and in the data. Firms in E-type model can always issue equity, and firms in distress in D-type model have to issue debt to cover the gap between interest expenses and the cash flow. Data is taken from Compustat for the period 1981-2017, and only firms with access to public debt markets are considered, which is proxied by having S&P long-term credit rating. To produce a distribution in the model, a balanced panel of $N = 3000$ firms was simulated for $T = 3600$ months . The distribution is shown for the last period of the simulation. For the distribution of inverse coverage ratios in the data, only firm-year observations with positive inverse coverage ratio are considered.

I further study to what extent the models can explain distribution of leverage among firms with different credit ratings in the data. Towards this end, firms in the model are matched with firms in the data according to credit ratings based on their default probabilities. Data on default probabilities for firms across credit ratings comes from Moody's report "Measuring Corporate Default Rates" (2006); 10-year default probabilities adjusted for issuer rating withdrawal are used. Average leverage values for firms with different credit rating are calculated using Compustat data.

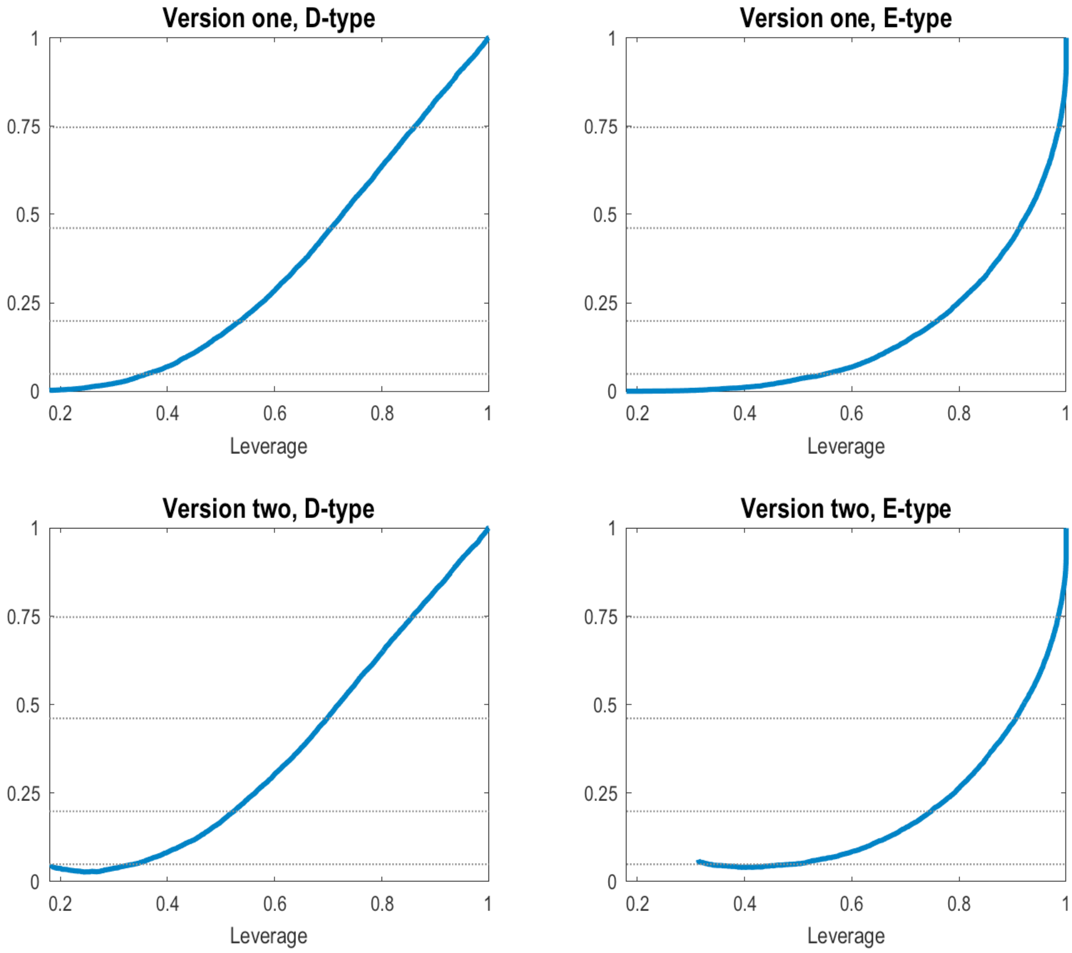
These numbers are shown in the first two rows of Table ??; for comparison, the table also shows leverage values across credit ratings reported by Schaefer and Strebulaev (2008) and Huang and Huang (2012). It should be noted that leverage values that I compute are smaller than those reported by both Schaefer and Strebulaev and Huang and Huang. The difference is likely caused by the fact that Huang and Huang's sample and Schaefer and Strebulaev's sample end before year 2004, while my sample goes up to 2017. As Table ?? suggests, leverage values are indeed lower for firm-year observations after 2003.

I then compute model-implied 10-year default probabilities for firms with different leverage values. Specifically, for each value of leverage L, I generate a firm that starts with this leverage at t_0 and operates optimally for $T = 120$ months or until it defaults. The procedure is repeated $N = 10000$ times, and default probability is measured as the fraction of simulations, in which the firm does not survive until the last period. Figure ?? shows results of the simulations. Because firms in version two of the model optimally issue more debt when their leverage falls to a very low level, there is a lower bound on the leverage that firms can have.

Note the difference in shapes of the default probability curves in D-type and E-type environments. Default probability as a function of firm's leverage is concave around $L = 1$ in the D-type environment, and is convex in the E-type environment. This result comes from the difference in the assumption of how firms in E-type and D-type environments finance the shortfall between required interest payments and the cash flow in distress. A firm in the D-type environment has to issue more debt, and marginal interest rates are very high for firms with high leverage. Moreover, because a firm with high leverage is already deep in distress, it will have to issue debt for many periods before it potentially recovers, and so it needs many consecutive positive shocks to avoid bankruptcy, which is rare. In contrast, firms in the E-type environment can issue equity cheaply, and even a highly levered firm that experiences a few positive shocks will recover quickly (its debt value does not grow). The shape of the default probability curve as a function of leverage is hard to estimate in the data, especially for values of leverage close to one. Implicit empirical evidence, however, argues in favor of the predictions of the D-type model; for example, Gilson (1997) shows that leverage of financially distressed firms remains high before Chapter 11.[27]

[27]It should be noted that the author attributes this finding to the transaction costs of the debt reduction, while this paper explains it by high interest rates on newly issued debt for firms with high leverage

Figure 9
Leverage and 10-year default probability



This figure shows the implied 10-year default probability for firms with different values of leverage. Grey dotted lines on each graph show the 10-year default probability of bonds with different credit ratings in the data: (from bottom to top) Baa, Ba, B, Caa-C. Note that the default probability is not monotone in leverage for version two of the model: this is because firms with very low leverage issue more debt and increase their leverage, therefore, increasing their probability of default.

For each credit rating category, I find a firm in the model that has the same 10-year default probability, and its leverage is reported in the corresponding row of Table **??**. It follows from the table that version two of the model with the D-type environment matches data well, while the model with the E-type environment significantly overestimates average leverage values. This result is not surprising: keeping firm's leverage constant, firms that can only issue debt in distress naturally have greater default probabilities over any finite time-horizon as compared to firms that can also issue equity. Therefore, to match any given default probability, a firm in the E-type environment should have a greater leverage value as opposed to a firm in the D-type environment.

Table 6
Default rates, and leverage for firms with different credit ratings

	Credit rating			
	Baa	Ba	B	Caa-C
Data				
10-year default probability	4.89%	19.86%	46.12%	74.72%
leverage	31.4%	40.5%	55.5%	62.2%
leverage, reported by H&H	43.3%	53.5%	65.7%	-
leverage, reported by S&S	37%	50%	66%	-
D-type, version 1				
leverage	36.1%	53.5%	70.6%	86.1%
E-type, version 1				
leverage	55.0%	75.9%	91.4%	98.7%
D-type, version 2				
leverage	33.8%	52.4%	69.8%	85.6%
E-type, version 2				
leverage	48.8%	74.8%	90.7%	98.5%

This table shows the average values of default probability and leverage for firms with different credit ratings in the data (rows 1-4) and in the model (rows 5-8). Leverage in the data is averaged for firm-year observations with specific S&P long-term credit rating for years 1981-2017. Leverage, reported by H&H is taken from the paper of Huang and Huang (2012), Table 1. Leverage, reported by S&S is taken from the paper of Schaefer and Strebulaev (2008), Table 7. Default probabilities in the data are adjusted for issuer rating withdrawal. To classify firms to specific credit ratings in the model, I match their implied 10-year default probability with the default rate for each credit rating in the data (row 1) using Figure **??**.

3.4 Bankruptcy costs and equity issuance costs

The previous section shows that the model with the E-type environment fails to explain the right tail of the leverage distribution, and a reasonable question is if it is possible to modify the model to improve its empirical predictions. This section examines two potential approaches that may help reconcile the E-type model with the data.

The first approach is to increase initial costs of debt issuance by increasing firm's bankruptcy costs. When bankruptcy costs are high, debtholders do not expect to recover a large fraction of their debt in default, and charge higher rates at the time when debt is issued. This should incentivize firms to issue less debt. Following this logic, I recompute the model for different values of bankruptcy costs[28] and estimate moments from the previous Section: the average and median leverage, the average inverse coverage ratio, the fraction of firms in default, and the fraction of firms with leverage above 75%; results are reported in Table **??** Panel A. To examine the plausibility of different values of bankruptcy costs, Table **??** additionally shows debt recovery rates, which are computed as the ratio of debt value at default relative to the face value of debt.

[28]Other parameter values are the same as in the Table **??**

Table 7
Model with greater bankruptcy costs/costs of equity issuance

	Panel A: bankruptcy costs							
	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 15\%$	$\alpha = 20\%$	$\alpha = 30\%$	$\alpha = 40\%$	$\alpha = 60\%$	$\alpha = 70\%$
avg. leverage	60.3%	58.9%	55.5%	54.0%	50.9%	47.3%	40.6%	39.7%
med. leverage	57.9%	56.4%	52.7%	50.3%	48.2%	43.4%	37.4%	35.3%
avg. invc.	1.26	1.23	1.12	1.08	1.00	0.91	0.78	0.74
distress	65.0%	59.2%	48.9%	43.6%	37.2%	28.6%	22.3%	18.9%
leverage $\geq 75\%$	18.5%	18.1%	14.1%	13.1%	9.1%	8.2%	4.6%	4.5%
recovery rate	37.5%	35.7%	33.9%	32.0%	28.2%	24.3%	16.3%	12.3%
	Panel B: equity issuance costs							
	$\lambda = 5\%$	$\lambda = 10\%$	$\lambda = 15\%$	$\lambda = 20\%$	$\lambda = 40\%$	$\lambda = 60\%$	$\lambda = 80\%$	$\lambda = 100\%$
avg. leverage	55.7%	54.1%	53.0%	51.4%	46.7%	43.9%	41.8%	39.0%
med. leverage	52.0%	51.3%	50.0%	47.9%	42.6%	39.4%	36.6%	34.3%
avg. invc.	1.13	1.07	1.03	0.99	0.85	0.78	0.73	0.66
distress	47.2%	44.1%	40.8%	36.3%	25.6%	21.3%	18.2%	13.7%
leverage $\geq 75\%$	15.1%	13.2%	11.7%	10.7%	7.4%	7.17%	6.9%	3.8%

The table compares results of the version two of the model with E-type environment for different values of bankruptcy costs α and equity issuance costs λ. When equity issuance costs are present, the firm has to raise $1 + \lambda$ dollars of equity to get one dollar. All values are estimated using a simulated steady-state cross-section of firms. To simulate an economy, a balanced panel of $N = 1200$ firms is generated over $T = 3600$ months, and values are averaged for the final period of simulation. Avg. and med. leverage refers to the average and median value of leverage (36.6% and 34.2% in the data); avg. invc. is the average ratio of interest expenses to firm's cash flow (0.63-0.69 in the data); distress is the fraction of firms whose interest expenses exceed the cash flow (16.5%-20.9% in the data); leverage $\geq 75\%$ is the fraction of firms whose leverage is above 75% (4.5% in the data); recovery rate is the value of firm's debt at default divided by the face value of debt.

As follows from the table, if bankruptcy costs are lower than 60%, the fraction of firms with leverage above 75% is four-to-two times greater than empirically observed (4.5%). Bankruptcy costs should go as high as $\alpha = 60\%$ to match this moment and the average leverage; however, even then the average inverse coverage ratio is 0.78 vs. 0.63-0.69 in the data. Moreover, for bankruptcy costs these large, debt recovery rates predicted by the model are too small. According to Moody's report "Moody's Ultimate Recovery Database" (2007), average debt recovery rates are 65% for senior secured bonds and 38% for senior unsecured bonds, and they are less than 20% in the model with bankruptcy costs $\alpha = 60\%$.

While not reported in Table **??**, debt recovery rate is 52.3% in the model with the D-type environment and is independent of bankruptcy costs α. The fact that debt recovery rates do not vary with bankruptcy costs level may sound counter-intuitive at first, but it happens because firm's timing of default is endogenous in the model, and firms that have greater bankruptcy costs default earlier, when the asset value is higher. Mathematically, it follows from how the default rule is determined in the model (equations (18) and (19)). This does not mean, though, that the D-type model predicts that debt recovery rates should be constant in the data: recovery rates in the model depend on firm's growth rate μ and volatility σ, which vary between firms.

The second approach to align the E-type model with the data is to add transaction costs of equity issuance to the model.The way the model is written, it assumes that firms that want to issue equity can do it at no cost. Empirical estimates, however, suggest that costs of equity issuance are positive and significant. For example, Clifford W. Smith Jr. (1977) studies all equity issuances registered between 1971 and 1975, and he finds that costs vary between 2% and 15% of the procceds amount; in a structural estimation paper, Hennessy and Whited (2007) find marginal costs of equity issuance between 5% and 10%, depending on firm's size.

To add equity issuance costs into the model with the E-type environment, it is assumed that immediate dividends to equityholders are $(X - c)(1 - \tau)(1 + \lambda)$, when $X < c$ in equation (3). Economically, this means that firms whose cash flow is lower than required interest expenses have to raise more money than what is necessary to just pay debtholders because a fraction of proceeds is lost. I compute the model for different values of λ between 5% and 100%, and results are shown in Table **??** Panel B.

As the table shows, equity issuance costs can potentially explain empirically observed moments of the right tail distribution of firms, but they should be very large: only when $\lambda = 100\%$ the model with the E-type environment matches every moment. However, this value of λ implies that the firm only gets 50 cents for each value of equity raised. As discussed above, empirical estimates of equity issuance costs are smaller.

Results of this section suggest that plausible values of costs of debt or equity issuances can not address the right tail of the distribution of leverage or inverse coverage ratios, if these costs are uncorrelated with firm's leverage. In choosing the optimal amount of debt, firms are less concerned about the absolute value of external financing costs, and are much more concerned about the relative value of external financing costs in distress, when firms need external money the most.

4 Concluding remarks

The main insight of this paper is that in order to explain why firms in the data choose seemingly low values of leverage, a model does not necessarily need to impose large costs of external financing, but these costs should be relatively higher when firms need money the most - in distress. Firms are much more conservative in their leverage policy ex-ante if they know that rates to refinance their debt will grow exponentially after several negative shocks. To have external financing costs grow endogenously with the leverage, this paper assumes that firms in distress can only issue debt. However, any model in which the cost of equity grow with firm's leverage will have similar results. Most importantly, the data supports this prediction - firms substitute equity financing with debt financing when their leverage is higher, implying that costs to issue equity grow faster than costs to issue debt.

The Leland-type framework is used extensively in the capital structure literature, but the assumption that firms have constant costs to issue equity is both inconsistent with the data and leads to unrealistic predictions about firm's financing behavior. By deriving the new optimal default boundary, which establishes that firms in distress issue debt, this paper shows how both concerns can be addressed while maintaining the tractable framework of the original paper. A model with similar features can be applied more broadly to study related issues in corporate finance and asset pricing.

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5 Appendix

5.1 Appendix A

Table 8
Firm's leverage and quity issuance

Panel A: Amount of equity issuance						
	All firms			Firms with access to public debt		
leverage	-0.49^{**}	-0.15^{**}	-0.16^{**}	-0.02^{*}	-0.06^{**}	-0.07^{**}
log(revenue)			-0.05^{**}			-0.02^{**}
profitability			−0.002			−0.002
market-to-book			0.005^{**}			0.003^{**}
Firm FE	No	Yes	Yes	No	Yes	Yes
Year FE	No	Yes	Yes	No	Yes	Yes
# of observations	8340	8340	7938	2187	2187	2172
R^2	11.98%	87.52%	89.37%	0.10%	85.51%	86.79%

Panel B: Issue discount						
	All firms			Firms with access to public debt		
leverage	−0.13	-6.11^{**}	-5.36^{**}	-6.39^{**}	-8.43^{**}	-8.38^{**}
log(revenue)			−0.11			-1.43^{*}
profitability			0.88^{**}			2.43^{**}
market-to-book			0.05			−0.04
Firm FE	No	Yes	Yes	No	Yes	Yes
Year FE	No	Yes	Yes	No	Yes	Yes
# of observations	6440	6440	6240	2078	2078	2064
R^2	0.01%	70.11%	70.41%	0.92%	65.63%	65.66%

The table reports correlation between firm's leverage and the amount of equity it issues (Panel A) or the discount it offers (Panel B). Panel A LHS variable is the amount of equity a firm issues scaled by the value of assets; Panel B LHS variable is the share price discount the firm offers as reported in SDC platinum and multiplied by negative one (so that the negative number means discount and positive number means premium). Only non-IPO equity issuances are considered (results do not change if IPOs are added). A firm is considered to have access to public debt if it has S&P long-term credit rating in a given year. Leverage is the value of firm's debt (both long-term and short-term) scaled by the value of assets. Profitability is the ratio of EBIT to revenues. Market-to-book ratio is winsorized at 1% and 99% levels, and profitability and equity issuance discount are winsorized at 5% and 95% levels. * and ** denote values significant at 5% and 1% respectively

5.2 Appendix B

Example developed in this Appendix shows that presence of leverage amplifies the problem of information asymmetry, and make equity issuance costs grow with leverage. In the example, for the same uncertainty structure, a firm with zero leverage optimally chooses to issue equity, while costs of equity issuance are preventive for a levered firm. The example is based on the seminal Myers and Majluf (1984) paper.

There is a firm that has assets in place, and an investment opportunity. Firm's quality is not known, but what is known is that it can either be good or bad with probabilities $p = (1-p) = 0.5$. Good firm's asset value is 190, and bad firm's asset value is 110. Required investment for the project is 100 for both types of firms, but the return is higher for the good firm: 120 vs. 110. The project has to be financed now or never, and the firm can only do it by issuing equity. For simplicity, investors are risk-neutral, and there is no discounting between periods, in which the investment takes place and return is realized. Figure **??** summarizes the example.

Figure 10

	Good $p = 0.5$	**Bad** $1 - p = 0.5$
Value of assets in place	190	110
Project required investment	100	100
Project return (gross)	120	100

Assume there is an equilibrium, in which both firms issue equity and finance the project. The value that outsiders assign to the firm is then the following:

$$V^{outs.} = \frac{1}{2}(190 + 120 + 110 + 110) = 265$$

Because the firm has to issue 100 of equity, the share that outside investors will require in return is $\frac{100}{265}$. In order for this to be an equilibrium, investors' beliefs should coincide with the actual behavior of firms. Therefore, both firms should be willing to take the investment opportunity. The good firm knows its type, and it will invest if the value that is left to current shareholders is higher with the investment:

$$V^G = \left(1 - \frac{100}{265}\right)(190 + 120) = 193 > 190$$

where 190 is the value of the good firm if the investment is not made, and no equity is issued. Equilibrium, in which both firms finance the project indeed exists.

Now consider a small modification of this example: assume that the firm, whose type is still unknown, issued debt in the past. Face value of debt is 100, and it has to be repaid next period. Figure **??** summarizes the modified example. Notice that firm's type cannot be revealed by the amount of debt it has, and no firm defaults next period, independent of whether the project is taken or not.

Figure 11

	Good $p = 0.5$	**Bad** $1 - p = 0.5$
Value of assets in place	190	110
Project required investment	100	100
Project return (gross)	120	100
Debt to be repaid	100	100

Assume again that there is an equilibrium, in which both firms issue equity and finance the project. The value that outsiders assign to the firm is then the following:

$$V^{outs.} = \frac{1}{2}(190 + 120 + 110 + 110) - 100 = 165$$

Naturally, firm's equity value is now smaller, as debtholders also have a claim on firm's assets.

In order for such equilibrium to exist, the good firm should be willing to pool with the bad firm. The good firm still knows its type, and in choosing whether to invest or not, it compares the value to its current equityholders with and without the investment:

$$V^G = \left(1 - \frac{100}{165}\right)(90 + 120) = 82.7 < 90$$

Therefore, the good firm chooses not to invest, which means there is no equilibrium in which both firms issue equity. The only difference between the two examples is that the firm is levered in the second example. This indeed shows that firm's equity issuance costs increase with leverage.

5.3 Appendix C

Below values show solution for the value of debt and equity for version one of the model with the E-type environment.

$$E(X,c) = \frac{X(1-\tau)}{r-\mu} - \frac{c(1-\tau)}{r} + AX^{\beta}c^{1-\beta} \tag{34}$$

$$D(X,c) = \frac{c}{r} + BX^{\beta}c^{1-\beta} \tag{35}$$

$$A = \left(\frac{(1-\tau)}{r} z_{def} - \frac{1-\tau}{r-\mu}\right) z_{def}^{\beta-1} \tag{36}$$

$$B = \left(\frac{(1-\tau)(1-\alpha)}{r-\mu} - \frac{z_{def}}{r}\right) z_{def}^{\beta-1} \tag{37}$$

$$z_{def} = \frac{c}{X_{def}(c)} = -\frac{r}{r-\mu}\frac{1-\beta}{\beta} \tag{38}$$

5.4 Appendix C

Table 9
Main formulas and the difference between the model in the E-type and D-type environments

	D-type environment	E-type environment
Process for X	$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t$	$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t$
Process for c_t on the interval $\{X, c : X \geq c\}$	$dc = 0$	$dc = 0$
Process for c_t on the interval $\{X, c : X < c\}$	$dc = \frac{(c - X)(1 - \tau)}{D'_c}$	$dc = 0$
HJB for equityholders on the interval $\{X, c : X \geq c\}$	$rE = (X - c)(1 - \tau) + \mu X E'_x + \frac{\sigma^2 X^2}{2} E''_{xx}$	$rE = (X - c)(1 - \tau) + \mu X E'_x + \frac{\sigma^2 X^2}{2} E''_{xx}$
HJB for equityholders on the interval $\{X, c : X < c\}$	$rE = \mu X E'_x + \frac{\sigma^2 X^2}{2} E''_{xx} + \frac{(c - X)(1 - \tau)}{D'_c} E'_c$	$rE = (X - c)(1 - \tau) + \mu X E'_x + \frac{\sigma^2 X^2}{2} E''_{xx}$
HJB for debtholders	$rD = c + \mu X D'_x + \frac{\sigma^2 X^2}{2} D''_{xx}$	$rD = c + \mu X D'_x + \frac{\sigma^2 X^2}{2} D''_{xx}$
Value of equity at default	$E\big(X_{def}(c), c\big) = 0$	$E\big(X_{def}(c), c\big) = 0$
Value of debt at default	$D\big(X_{def}(c), c\big) = \frac{(1 - \alpha)(1 - \tau) X_{def}(c)}{r - \mu}$	$D\big(X_{def}(c), c\big) = \frac{(1 - \alpha)(1 - \tau) X_{def}(c)}{r - \mu}$
Condition that determines default boundary	$\frac{\partial D}{\partial c}\big(X_{def}(c), c\big) = 0$	$\frac{\partial E}{\partial X}\big(X_{def}(c), c\big) = 0$

This table combines the main formulas for the model in the D-type and E-type environments. The following variables stand for: X - firm's cash flow, c - firm's interest payments, $E = E(x, c)$ - value of firm's equity, $D = D(x, c)$ - value of firm's debt, r - risk-free rate, μ - growth rate of firm's cash flows, σ - volatility of firm's cash flows, τ - corporate tax rate, α - bankruptcy costs, $X_{def}(c)$ - firm's default boundary (firm defaults if $X \leq X_{def}(c)$ for a given value of c), β - negative solution to the quadratic growth equation (17) in the text.