Omitted Variable in Capital Structure Regressions

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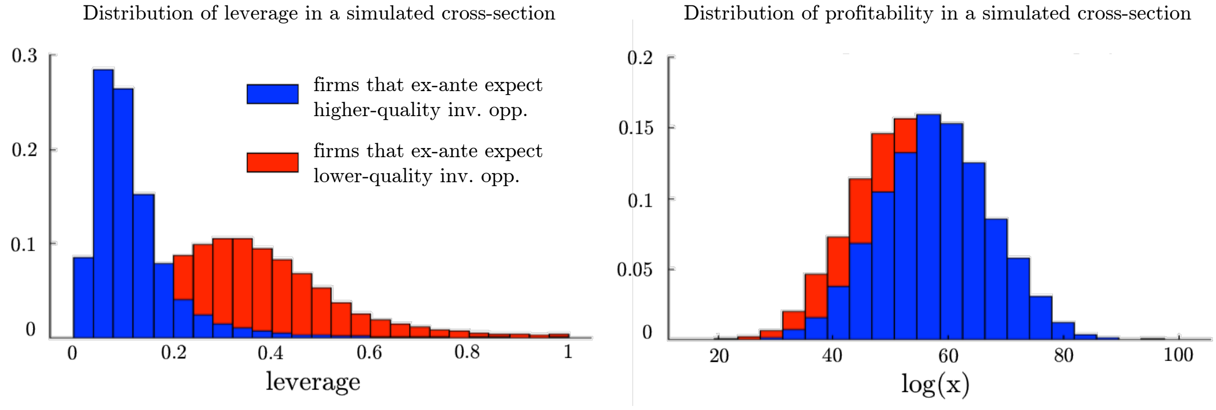
Abstract

This paper addresses two puzzling empirical results in the capital structure literature: why leverage and profitability are negatively correlated, and why investments are explained by the cash flow in a regression controlled for the market-to-book ratio. The paper derives a model, in which firms are heterogeneous in the quality of their investment opportunities. Firms with ex-ante better investment opportunities 1) issue less debt that is not dedicated to finance investments, and 2) are able to raise debt at a smaller rate conditioning on the leverage, which results in a smaller leverage over time in this group of firms. Firms with ex-ante better investment opportunities also invest more, have greater ex-post profitability and greater cash flows. The model presented in the paper is simple and tractable, yet it gives a very good quantitative fit to the data.

1 Introduction

This paper shows how ex-ante difference in the quality of investment opportunities of different firms can 1) translate into ex-post negative relationship between firm's leverage and profitability, and 2) explain why cash flow has positive explanatory power in a regression with investments amount even after market-to-book ratio is controlled for. A simple observation is at the core of this paper. If a firm has to finance some of its future investments with debt, then by issuing debt to pay dividends the firm risks to not be able to raise debt at acceptable rates in the future, and will have to forgo many otherwise profitable investments. Moreover, incentives to delay debt issuance are the greatest for firms whose investment opportunities are expected to be the most profitable. In extreme cases, a firm that expects its future investments to be the most profitable would not be issuing debt that is not dedicated to finance investments at all. The quality of firm's investment opportunities is the omitted variable from the title of this paper.

Figure 1
Leverage and profitability in two groups of firms



This figure shows results of a model simulation. Firms in the model are different by the quality of the investment opportunities they expect, with some firms consistently getting better investment opportunities than others. As evident from the graph, firms with better investment opportunities on average end up with smaller leverage and higher profitability.

To formalize the logic of the previous paragraph, this paper derives a model that captures the trade-off between issuing debt to exploit benefits of tax shield and delaying debt issuance to finance future investments. In the model, firm's profits are taxed at a corporate level, but the firm can shield future profits from taxation by issuing debt. Occasionally, the firm finds investment opportunities, which require fixed initial investments, and allow the firm to get an extra flow of profits going forward. The NPV - or the profitability - of investment opportunities differ between firms, and firms finance these investments with debt. A firm can always issue debt and pay proceeds as dividends, but doing so increases its future funding costs and may prevent financing of otherwise profitable investment opportunities. It is important to note that the model itself contributes to the literature by simultaneously characterizing firm's investments, debt issuances, and default decision in a very tractable manner, with many results available in a closed-form.

Think about the negative correlation between leverage and profitability. In the model, firms that eventually become the most profitable are those that ex-ante expect to find the best investment opportunities. Such firms, however, also face the greatest opportunity costs of issuing debt, and so they issue the smallest (if any) amount of debt that is not dedicated to finance investments. Furthermore, every time a firm invests, debtholders are willing to lend money at lower rates if firm's investment is more profitable. Therefore, firms with less-profitable investment opportunities issue debt with greater face value at every round of investment. Importantly, a firm with profitable investments keeps taking its investments even when its leverage is high, at times when a similar firm with less profitable investments does not, and so eventually has more debt. This, however, does not happen until after many rounds of investments, and an econometrician examining 100 years of investments and debt issuance of a high-profitable and a low-profitable firms will see 95 years, in which the leverage of the lower-profitable firm is higher, and 5 years, in which it is lower, and will conclude that leverage and profitability are negatively correlated. Figure 1 illustrates this logic: after many rounds of investments, distribution of profitability of firms with ex-ante good investment opportunities is shifted to the right, while the distribution of their leverage is shifted to the left.

Ex-ante difference in the quality of firms' investment opportunities also explains why cash flow is correlated with the investment amount even after controlling for the market-to-book ratio. Which firms have the greatest cash flows? Those that invested the most in the past, and whose investments were the most profitable. Therefore, the ex-ante expected quality of firm's investment opportunities naturally creates a spurious correlation over time between the cash flow and the investment amount. This correlation does not disappear after controlling for firm's market-to-book ratio, as the market-to-book ratio depends non-linearly on both the quality of firm's investment opportunities, and on firm's leverage (as leverage determines how many of those opportunities will actually be financed).

The model has a very good quantitative fit to the data. While firms in the model target fairly high leverage value, at any point in time there are firms that already invested a lot, and firms that are still waiting for the arrival of their investments. This way, the average value of market leverage produced by the model is very close to empirically observed (27% in the data vs. 25.2% in the model), and, thus, the model explains the underleverage puzzle. Moreover, the whole distribution of market leverage in the model closely resembles distribution in the data, and matches its median (22.2% in the data vs. 22.9% in the model), standard deviation (17.1% and 16.0%), and *all* quantiles between 5% and 95%. The average book leverage is also matched well, with its mean being 36.6% in the data and 37.3% in the model. Furthermore, the model produces a very good fir for the *whole* distribution of market-to-book ratios (as Figure 8 in the main part of the paper shows), with mean and median being 2.31 and 1.78 in the model and 2.42 and 1.85 in the data.

1.1 Literature review

One of the significant assumptions in the paper is that firms can only finance their investment opportunities with debt. While this assumption is certainly important, especially for the quantitative part of the paper, results qualitatively would not change even if firms in the model were allowed to issue equity. Nevertheless, there is some strong evidence suggesting that indeed firms issue equity infrequently, and that cost to issue equity also grow with leverage.

Empirical evidence on equity issuance by distressed firms is scant, but also mixed. Below is a review of papers that consider this question. The overall conclusion that follows is that most firms issue equity when their leverage is low; there are instances of equity issuances by financially distressed firms, but the costs are high, and such firms use equity financing because they cannot raise debt.

The first group of papers argues that most firms issue equity when their performance is good. For instance, Senber and Senber (1995) report a complete absence of equity issuance by distressed firms. Fama and French (2005) show that equity issuances are frequent, but most firms issue equity when their leverage is low. Similarly, Mikkelson and Partch (1986) and Eckbo, Masulis, and Norli (2007) find that equity issuances for cash are rare - both in absolute level and relative to public debt issuances. Some other studies provide indirect evidence that firms in distress do not issue equity. Korajczyk, Lucas, and McDonal (1990) find that firm's leverage does not increase significantly two years before an SEO; should firms issue equity to make required debt payments when internally generated cash flow is insufficient, one would observe an increase in leverage prior to an SEO. DeAngelo, DeAngelo and Stulz (2008) find that the average leverage of a firm before an SEO is only 27%. Denis and McKeon (2012) show that firms, whose leverage is above the target, tend to cover financial deficit by issuing new debt and increasing leverage further.

Other authors argue in contrary that a sizable number of distressed firms issue equity, but they sell new shares at a large discount, and do so because debt financing is unavailable. Park (2017) finds that public equity offerings decrease for firms in distress, but private placements increase. Walker and Wu (2017) find that a third of all SEOs are conducted by financially distressed firms. Both of these papers, however, use the distress measure from Campbell, Hilsher, and Szilagyi (2008), which is only partially related to firm's leverage. Indeed, the average leverage in the subsample of distressed firms in Walker and Wu is 32%, which implies that these firms are in distress for reasons other than their indebtedness, and they likely have very limited access to debt financing. This conclusion is further reinforced by Lim and Schwert (2017) who study all private placement of equity (PIPEs) by U.S. firms. They find that most firms issuing PIPEs are small distressed firms without access to debt markets: the median leverage of firms issuing PIPEs is only 7.2%, and 93% of all firms do not have credit rating. When such firms issue PIPEs, they offer shares to the market at a large discount.

Appendix D provides further empirical analysis of the correlation between the frequency of equity issuance and firm's leverage based on Thomson Reuters data. Results show that the amount of equity issuance decreases with firm's leverage, and the discount at which newly issued shares are offered to investors increases with leverage; this conclusion holds for all firms and also for the subsample of firms that have access to public debt markets. Similar conclusion follows from Figure **??** which is based on Compustat data. [1] The question why costs to issue equity grow for firms in distress is beyond the scope of this paper, but Appendix B provides a simple two-period model that shows that presence of leverage amplifies information asymmetry. Such explanation is consistent with empirical findings of Hertzel and Smith (1993) and Lim and Schwer (2017) who argue that distressed firms are characterized by severe information asymmetry.

The model derived in this paper assumes that firms always have access to debt capital markets.[2] Therefore, the assumption that such firms do not issue equity to pay required debt payments in distress is consistent with empirical evidence discussed above.

[1] The paper does not combine the two databases because they use different definitions of equity issuance. In particular, Thomson Reuters mostly considers SEOs, while Compustat partially considers private placements as well. Assuming that Compustat data is internally consistent, Figure **??** shows the relative scale of debt and equity issuance as firms leverage increases, which would not necessarily be consistent if Compustat and Thomson Reuters data for equity issuance is pooled. On the other hand, Thomson Reuters has data for the discount/premium paid for newly issued shares, which is not available in Compustat

[2] The empirical sample of firms that the paper quantitatively explains also consists of firms with access to debt capital markets (firms with S&P long-term credit rating).

2 Model

2.1 The economy

This paper models an infinite-horizon economy in continuous time. Markets are complete, and there is a riskless asset that pays a constant rate of interest r per unit of time[3]. In what follows, $\mathbb{P}$ denotes the physical probability measure, and $\mathbb{Q}$ denotes the risk-neutral probability measure in this economy.

A firm in the economy is characterized by the amount of capital K it has, and capital productivity X_t. Firm's production technology has a constant return to scale, and instantaneous profits equal $y_t = X_t K$ per unit of time. For simplicity, capital does not depreciate, and X_t evolves over time as a Geometric Brownian motion:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t^Q \tag{1}$$

where μ is the risk-neutral drift, σ is volatility of capital productivity, and W_t^Q is a Brownian motion under $\mathbb{Q}$.

Government taxes firm's profits at a constant corporate rate τ, and firm's after-tax profits are $(1-\tau)X_t K$ per unit of time. it is assumed that firms cannot save cash, and all profits have to be paid as dividends to equityholders immediately.

Capital is traded on the outside market, and its price depends on its productivity level: price of a unit is $(1-\tau)XH$, where H is a constant. To gain intuition, consider the value of a firm that operates one unit of capital with current productivity X_t that takes no actions:

$$V = \mathbb{E}\left[\int_t^{\infty} e^{-r(s-t)}(1-\tau)X_s ds\right] = \frac{(1-\tau)X_t}{r-\mu} \tag{2}$$

If firms in the economy indeed were not taking actions, and capital markets were competitive, equation (2) would give the exact price of capital (from no arbitrage condition). However, as discussed later in the text, firms can invest and issue debt to exploit benefits of tax-shield, and therefore price of capital may deviate from what equation (2) suggests. Nevertheless, equation (2)

[3]Here and below "per unit of time" means that investors earn approximately $r\Delta t$ within a short interval of time Δt

shows why price of capital should grow with its productivity level[4].

2.2 Investments

Firms buy new capital when they find investment opportunities. Investment opportunities arrive at a rate λ per unit of time; informally, probability to find an investment opportunity within a short period of time Δt equals $\lambda \Delta t$. An investment opportunity allows a firm to buy pre-specified amount of capital K_{new} with low-productivity X_{low} and install this capital within the firm. However, once the capital is installed, it productivity grows up to $X_{high} > X_{low}$. This is what makes investment opportunities profitable for firms: they pay for low-productive capital, but install it as high-productive capital. Importantly, investment opportunities do not change the amount or productivity of firm's existing capital: by taking an investment opportunity, the firm gets a new capital stock with its own productivity. A simple way to think about investment opportunities is that it allows a firm to open a new plant, which works independently of firm's other plants. This would imply, however, that a pair (K, X) does not fully characterize a firm, as it shows firm's total amount of capital and productivity, while the firm may have several capital units after a number of investments. The following assumption guarantees that capital and productivity can be aggregated:

Assumption. *Consider a firm at time t that has K units of capital with productivity X_t that finds an investment opportunity, which allows the firm to buy K_{new} units of capital with productivity X_{low} and install them within the firm with productivity X_{high}. Then:*

1. *Size of the investment opportunity is proportional to the amount of capital the firm already has:*

$$K_{new} = \delta K$$

2. *Productivity of capital that the firm buys is the same as the productivity of capital that the firm already has:*

$$X_{low} = X_t$$

[4] Notice that price of capital is also proportional to the level $(1-\tau)$: one way to look at this is to assume that after firms purchase capital, it is immediately depreciated for accounting purposes. In other words, firms pay HX for a unit of capital, but the government returns them τHX back, so the effective price that firms pay is only $(1-\tau)HX$. Return from the government may take a form of smaller other taxes that firms pay.

3. *Productivity of capital once it is installed is proportional to the productivity of capital the firm buys:*

$$X_{high} = (1+\gamma)X_t$$

4. *After new capital is installed, its productivity will evolve according to the same equation (1) as the productivity of firm's old capital; in other words, there are no additional idiosyncratic shocks to the productivity of new capital*

Assumption (4) guarantees that firm's different capital units can be aggregated into one: there is only one process X_t that characterizes productivity of all capital units, and production technology is constant return to scale. Appendix 1 formally proves that capital can be aggregated.

Note how firm's profits change when the firm takes an investment opportunity:

$$\begin{gathered} X_t K \to X_t K + \delta K(1+\gamma)X_t = \big(1+\delta(1+\gamma)\big)X_t K \\ y_t \to \big(1+\delta(1+\gamma)\big)y_t \end{gathered} \tag{3}$$

Note also that the price that the firm pays to take an investment opportunity $(1-\tau)HX_t\delta K = (1-\tau)\delta H y_t$ - is proportional to firm's profits. Therefore, y_t alone is a sufficient state variable to describe the firm. Evolution of y_t over time then can be computed using Ito's lemma:

$$\frac{dy_t}{y_t} = \mu dt + \sigma dW_t + I\{\text{firm invests}\}(1+\delta(1+\gamma))dN_t \tag{4}$$

where dN_t is a Poisson process with intensity λ, and $I\{\text{firm invests}\}$ is the indicator function that shows whether the firm invests when it gets an investment opportunity.

By construction, investment opportunities are profitable for firms: firms buy cheap low-productive capital, but install it as high-productive capital. Nevertheless, firms that have to finance investments by issuing debt may optimally choose not take investment opportunities. This is because firms that issued a lot of debt in the past may only be able to raise new debt at very high rates, which would not justify the investment. Before the discussion of this case, however, consider the benchmark example, in which equityholders have "deep pockets", meaning that firms can always issue equity to pay for investments.

Benchmark example with equity financing

Let $v(y)$ denote the equity value of a firm whose current profits equal y, and guess that the firm always invests when it finds an investment opportunity. $v(y)$ should then satisfy the following HJB equation:

$$\begin{aligned} rv(y) = (1-\tau)y + \mu y v'(y) + \frac{\sigma^2 y^2}{2} v''(y) + \\ + \lambda\Big(v\big((1+\delta(1+\gamma))y\big) - v(y) - (1-\tau)\delta H y\Big) \end{aligned} \tag{5}$$

The last term of equation (5) is the probability that the firm finds an investment opportunity (λ) multiplying the change in the value of equity after the investment is taken. Notice that because in this example investments are financed by issuing equity, there is a negative outflow of $(1-\tau)\delta H y$ every time the firm invests.

Guess that equation (5) has a linear solution $v(y) = H_0 y$. Then:

$$rH_0 y = (1-\tau)y + \mu H_0 y + \lambda\Big(H_0(1+\delta(1+\gamma))y - H_0 y - (1-\tau)\delta H y\Big) \tag{6}$$

$$H_0 = \frac{(1-\tau)(1-\lambda\delta H)}{r-\mu-\lambda\delta(1+\gamma)} \tag{7}$$

Now consider the case when capital markets are competitive, and price of a unit of capital exactly equals the value of a firm that operates this unit of capital, implying that $H_0 = H$. Substitute $H_0 = H$ into equation (7), and solve for $v(y) = H_0 y$:

$$v(y) = \frac{(1-\tau)y}{r-\mu-\lambda\delta\gamma} \tag{8}$$

This very simple equation is an analogue of the Gordon growth formula, which shows that the value of a firm equals to the value of its immediate dividends $(1-\tau)y$ divided by the difference between the discounting rate r and the expected growth rate $(\mu+\lambda\delta\gamma)$. The growth rate in this case is composed of the unconditional growth μ, and the growth coming from investments $\lambda\delta\gamma$. While y increases by $\delta(1+\gamma)$ every time the firm invests, the firm pays for δ part of it, and only $\delta\gamma$ is the additional growth.

2.3 Debt and debtholders

It is assumed that debt is the only source of external financing available to firms. While this assumption may seem extreme, it has been shown empirically that debt financing dominates equity financing.

Firms in the model issue debt for three purposes:

1. Interest expenses on debt are tax-deductible, and by issuing debt firms can exploit benefits of tax-shield.

2. Firms use debt to buy capital to finance investment opportunities.

3. When firm's cash flow is lower that required interest expenses on previously issued debt, it has to issue more debt to cover the shortfall.

Note that firms whose cash flow is not sufficient to pay interest expenses face immediate default, but, as the third bullet says, they may avoid it by issuing more debt. However, firms cannot increase the amount of outstanding debt infinitely as at some point debtholders would prefer to default the firm and take its assets rather than roll over its debt and hope that firm's financial situation recovers[5] .

Debt takes the form of a perpetuity that pays a constant coupon payment c per unit of time as long as there is no default. Interest expenses are tax-deductible, and, therefore, firm's instantaneous profits equal $(1-\tau)(y-c)dt$. For simplicity, it is assumed that when $(y-c)$ is negative, the firm pays negative taxes, which means it receives money from the government.

Debt markets are competetive, and price of debt equals the present value of future payments that debtholders expect to receive from the firm. Let $D(y,c)$ be the value of all firm's outstanding debt. The fact that debt can be aggregated requires that either all of firm's debt is held by one creditor, or that different creditors have equal seniority, which is assumed to be correct. When a firm issues more debt, it increases future coupon payments, and the amount of proceeds that the firm receives equals $\big(D(y,c_{new})-D(y,c)\big)$ - change in the debt value. This is a departure from the traditional assumption that debt issuances incur non-trivial transaction costs that prevent firms from continuously adjusting their leverage (i.e. Goldstein, Ju, and Leland (2001), Strebulaev

[5]Default happens endogenously and is discussed later. However, Belyakov (2018) provides a more detailed analysis and its implications for observed patterns of capital structure

(2007), Chen (2010), Bhamra, Kuehn, and Strebulaev (2009, 2010))[6]. Instead, the paper assumes that firms are allowed to issue new debt in any quantities and as often as they want. However, even without the assumption of costly debt issuances firms in the model do not issue debt continuously: most debt is issued to finance investment opportunities, which only arrive infrequently.

Consider a firm with current cash flow y and debt that is characterized by coupon payments c that finds an investment opportunity and decides to take it. To purchase necessary capital, the firm has to pay $(1-\tau)H\delta y$, and this amount needs to be raised by issuing more debt. The new coupon payment c_{new} that the firm has to promise to debtholders is then implicitly defined through the following equation:

$$D\big((1+\delta(1+\gamma))y, c_{new}\big) - D(y,c) = (1-\tau)H\delta y \tag{9}$$

The LHS of equation (9) shows how the value of firm's debt changes when it takes the investment opportunity: its y increases by the factor $(1+\delta(1+\gamma))$ as in equation (3), and c increases to c_{new}. As debt markets are competitive, change in debt value is the amount of proceeds that the firm receives, and this amount should equal to the price that the firm pays to buy necessary capital; therefore, RHS of equation (9).

For now consider the case when firms do not issue debt to exploit benefits of tax-shield (this feature is added later). Then firms with $y > c$ only issue debt to finance investment opportunities. The value of debt $D(y,c)$ should satisfy the following HJB equation:

$$\begin{aligned} rD(y,c) = c + \mu y D'_y + \frac{\sigma^2 y^2}{2} D''_{yy} + \\ \lambda I\big\{\text{firm invests}\big\}\Big(D\big((1+\delta(1+\gamma))y, c_{new})\big) - D(y,c) - (1-\tau)H\delta y\Big) \end{aligned} \tag{10}$$

where the last term indicates the probability that a firm finds an investment opportunity (λ) multiplied by whether the firm takes the investment opportunity once it finds it ($I\big\{\text{firm invests}\big\}$), multiplied by how the value of debt changes - both y and c increase, but debtholders provide funds for the firm to buy capital. Notice, however, that equation (9) shows that this last term of

[6] The assumption of small but non-trivial costs of debt issuance proved to be a powerful tool in explaining many stylized facts - from infrequent debt issuances, to underleverage puzzle and negative relationship between leverage and profitability (i.e. Goldstein, Ju, and Leland (2001) or Strebulaev (2007)). This paper assumes that debt can be issued at no costs, but it provides a different explanation for the above-mentioned phenomena

equation (10) equals zero. It should not be surprising: because debt capital markets are competitive, debtholders provide financing at rates which make them indifferent between whether the firm takes the investment opportunity or not. Effectively, equityholders capture the whole surplus arising from profitable investments. Therefore, equation (10) can be simplified:

$$rD(y,c) = c + \mu y D'_y + \frac{\sigma^2 y^2}{2} D''_{yy} \tag{11}$$

The fact that debt value does not increase when firms take profitable investment opportunities may sound unrealistic at first, as it seemingly implies that firm's profitability does not affect debt pricing. It is not the case: as clear from equation (9), higher value of γ unambiguously increases the value of debt, and therefore, c_{new} is lower to equalize the LHS and RHS. It means that firms with more profitable investment opportunities can issue debt at lower rates.

Notice, however, that equation (11) holds on the region $y > c$, where firm's cash flow is sufficient to pay required interest expenses. On the region $y < c$ the firm does not have enough cash to pay required interest expenses, and so it needs to issue more debt to cover the gap to avoid default. Let dD denote the additional amount of debt that needs to be issued when $y < c$, and dc the change in coupon payments. The following formula then links together dc and dD:

$$\Big(c - \big(y + (c-y)\tau\big)\Big)dt = (c-y)(1-\tau)dt = dD = dc\frac{\partial D}{\partial c} \tag{12}$$

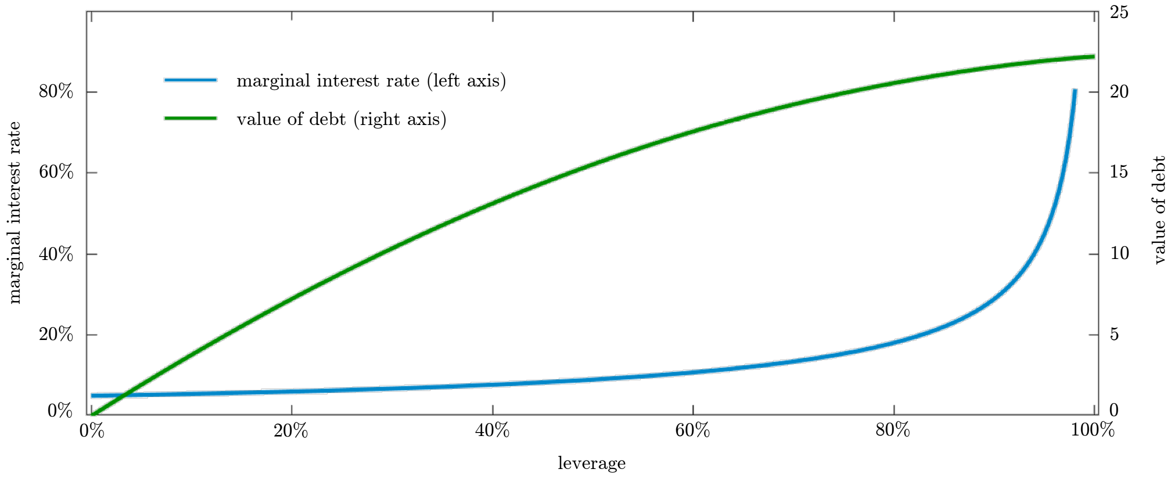
The very LHS of equation (12) is the difference between required coupon payments c and the amount of money the firm has on hands - its profits y and tax-return from the government $\tau(c-y)$. This difference is the amount that the firm must but cannot pay to debtholders; the firm, therefore, raises this amount by issuing new debt, and coupon payments increase accordingly[7]. From equation (12), the dynamic of dc on the region $y < c$ is then:

$$dc = \frac{(1-\tau)(c-y)}{\frac{\partial D}{\partial c}} dt \tag{13}$$

HJB equation for the value of debt on the region $y < c$ should account for the fact that both y and

[7]Notice that new debt can only be issued for the case when $\frac{\partial D}{\partial c} > 0$

Figure 2
Debt value and marginal interest rate for an additional unit of debt



c change[8]:

$$rD(y,c) = (y + \tau(c-y)) + \frac{(1-\tau)(c-y)}{\frac{\partial D}{\partial c}} D'_c + \mu y D'_y + \frac{\sigma^2 y^2}{2} D''_y y \tag{14}$$

which simplifies to the following:

$$rD(y,c) = c + \mu y D'_y + \frac{\sigma^2 y^2}{2} D''_y y \tag{15}$$

which is again the same as equation (11). Equation (15) implies that pricing of debt is not affected by the fact that the firm with $y < c$ cannot pay its interest expenses and has to issue more debt. This happens because an increase in c means greater future payments to debtholders, but also that probability of default is higher; because debt markets are competitive, an increase in c adjusts so that the two effects compensate each other, and the value of debt stays unchanged.

Equations (11) and (15)[9] have a closed form solution of the following form:

$$D(y,c) = \frac{c}{r} + Bc^{1-\beta} y^{\beta} + B_2 c^{1-\beta_2} y^{\beta_2} \tag{16}$$

where B and B_2 are constants to be determined from boundary conditions, and $\beta < 0$ and $\beta_2 > 0$

[8] Of course equation (14) should also account for the fact that the firm may find and take an investment opportunity, but due to the same argument as in equation (10), value of debt does not change in this case

[9] which are the same

are the roots of the quadratic growth equation:

$$r = \Big(\mu - \frac{\sigma^2}{2}\Big)\beta + \frac{\sigma^2}{2}\beta^2 \tag{17}$$

Consider the economic interpretation of the terms of equation (16): the first term $\frac{c}{r}$ denotes the value of the risk-free debt that always pays c. As β is negative, the second term is large when y is low, but it converges to zero when y increases[10]. This term captures the effect of default (and associated losses): when firm's cash flow is high, firm's default probability is low, and so firm's debt is almost risk-free; however, if firm's cash flow is low, the default probability gets bigger, and market value of debt adjusts downward. The last term, in contrast, converges to infinity as y increases. Value of risky debt can never exceed the value of the riskless debt, which implies that $B_2 = 0$.

To uniquely determine $D(y, c)$, one more boundary condition needs to be imposed, which is the value of debt when the firm defaults. Absolute priority rules apply, and equityholders receive nothing; firm's assets are transferred to debtholders, and they sell them at the market price. A fraction α of assets, however, is lost in the process, and so debtholders only receive $(1-\alpha)$ of assets value. Assume that a firm in default has K units of assets with productivity X. The value of this firm's debt then equals:

$$D(y, c) = \frac{c}{r} + By^{\beta}c^{1-\beta} = (1-\tau)(1-\alpha)XHK = (1-\tau)(1-\alpha)Hy \tag{18}$$

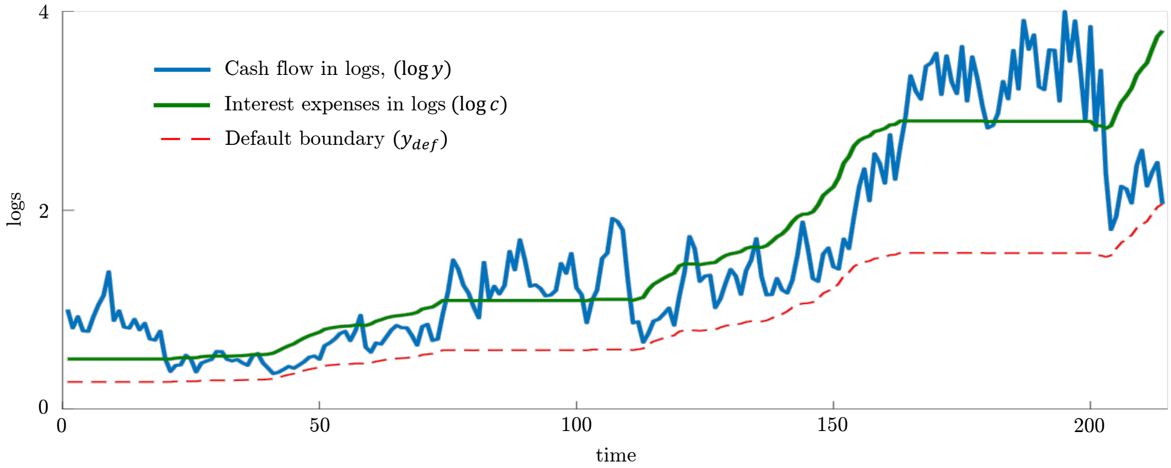
It only remains to show when firms in the model default.

2.3.1 Default

Notice that dividends to equityholders are always non-negative: positive on the interval $y > c$ and zero on the interval $y \leq c$. Therefore, equityholders never wish to voluntarily default the firm. However, firms that have $y < c$ cannot meet interest payments and, therefore, continuously issue debt to avoid default. Debt issuances in distress cannot last infinitely, though. As equation (13) implies, more debt can be issued only if $\frac{\partial D}{\partial c} > 0$. Therefore, default is determined by the following

[10] more formally, as the ratio of $\frac{y}{c}$ converges to infinity

Figure 3
Path of a typical firm in the model



This graph provides an illustration of how firm's cash flow develops over time; notice that this graph assumes that the firm does not invest. The firm takes no action as long as its cash flow (blue curve) stays above its coupon payment (green curve). When firm's cash flow is lower than its coupon payment but above the default boundary y_{def} (bottom red curve), the firm slowly raises new debt to cover the shortfall between the cash flow and required coupon payments. The firm defaults when cash flow level reaches the default boundary. Values on the graph are in logs.

condition:

$$\frac{\partial D}{\partial c} = 0 \tag{19}$$

Notice that because debt in the model is a function of two variables (y and c), default boundary is a curve $y_{def}(c)$ rather than a single number. Together, equations (18) and (19) allow to solve for B and firm's default threshold:

$$y_{def}(c) = \frac{-\beta}{1-\beta} \frac{1}{(1-\tau)(1-\alpha)rH} c \tag{20}$$

$$B = -\frac{1}{r(1-\beta)} \left(\frac{y_{def}(c)}{c} \right)^{-\beta} \tag{21}$$

Note that $y_{def}(c)$ is linear in c, and, therefore, the resulting B is indeed a constant. Moreover, the fact that $\beta < 0$ implies that $B < 0$, which means that value of debt gets smaller as y decreases and probability of default gets higher.

2.4 Equity and equityholders

The closed-form expression for the market value of debt allows to simplify equation (9), which shows how firm's coupon payments increase when it takes an investment opportunity and issues debt to finance it:

$$\frac{c_{new}}{r} + B\big((1+\delta(1+\gamma))y\big)^{\beta} c_{new}^{1-\beta} = \frac{c}{r} + By^{\beta}c^{1-\beta} + (1-\tau)H\delta y \tag{22}$$

Let $v(y,c)$ denote the value of firm's equity. If $y > c$ the firm produces enough money to pay required interest expenses, and so it only issues debt infrequently to finance investment opportunities. Therefore, HJB equation for $v(y,c)$ takes the following form on the interval $y \geq c$:

$$\begin{aligned} rv(y,c) = (y-c)(1-\tau) + \mu y v'_y(y,c) + \frac{\sigma^2 y^2}{2} v''_{yy}(y,c) + \\ \lambda I\big\{\text{Firm invests}\big\}\Big(v\big((1+\delta(1+\gamma))y, c_{new}\big) - v(y,c)\Big) \end{aligned} \tag{23}$$

where the last term shows how the value of equity changes when the firm finds an investment opportunity, and c_{new} is implicitly defined in equation (22). On the interval $y < c$ the firm does not produce enough cash to pay required interest expenses, and has to continuously issue debt to finance the shortfall, and so the HJB equation should account for that. Note that dc is defined in equation (13), and the firm does not pay dividends:

$$\begin{aligned} rv(y,c) = & \frac{(1-\tau)(c-y)}{\frac{1}{r} + (1-\beta)By^{\beta}c^{-\beta}} v'_c(y,c) + \mu y v'_y(y,c) + \frac{\sigma^2 y^2}{2} v''_{yy}(y,c) + \\ & \lambda I\big\{\text{Firm invests}\big\}\Big(v\big((1+\delta(1+\gamma))y, c_{new}\big) - v(y,c)\Big) \end{aligned} \tag{24}$$

Generally speaking, equations (23) and (24) describe a second order PDE of a function of two variables, and so may be hard to solve, even numerically. However, the model satisfies the scaling feature, meaning that function $v(y,c)$ should be homogeneous of degree one in y and c. Intuitively, a firm with $(2y, 2c)$ is simply a greater replica of a firm with (y,c), and, therefore, its equity value should be twice as high. Appendix 2 formally proves the homogeneity property.

Define a new variable $z = \frac{c}{y}$ - firm's inverse coverage ratio[11]. First note how firm's z changes

[11] Notice that the inverse coverage ratio is a more natural parameter in the model than the coverage ratio $\frac{y}{c}$. By construction y can never be zero, while there will be firms without any debt and, therefore, with $c = 0$, and so the

when it takes an investment opportunity. For this, divide both sides of equation (22) by y:

$$(1+\delta(1+\gamma))\Big(\frac{z_{new}}{r}+Bz_{new}^{1-\beta}\Big)=\frac{z}{r}+Bz^{1-\beta}+(1-\tau)H\delta \tag{25}$$

where $z_{new}=\frac{c_{new}}{(1+\delta(1+\gamma))y}$ - firm's new inverse coverage ratio after an investment is taken. Importantly, equation (25) shows that firm's inverse coverage ratio z_{new} after an investment is taken is only a function of the prior inverse coverage ratio z and not y and c separately.

Because $v(y,c)$ is homogeneous of degree one in y and c, there is a function $f(z)$ such that $v(y,c)=yf\big(\frac{c}{y}\big)=yf(z)$. Note that

$$v'_y(y,c)=\big(yf(z)\big)'_y=f(z)-zf'(z) \tag{26}$$

$$v''_{yy}(y,c)=\big(f(z)-zf'(z)\big)'_y=-f'(z)\frac{z}{y}+f'(z)\frac{z}{y}+\frac{z^2}{y}f''(z)=\frac{z^2}{y}f''(z) \tag{27}$$

Furthermore, note that equation (13) that determines dc on the interval $y<c$ can be partially rewritten in terms of z:

$$dc=\frac{(c-y)(1-\tau)}{\frac{1}{r}+(1-\beta)By^{\beta}c^{-\beta}}dt=y\frac{(z-1)(1-\tau)}{\frac{1}{r}+(1-\beta)Bz^{-\beta}}dt \tag{28}$$

Conditions $y>c$ and $y\leq c$ can be rewritten as $z<1$ and $z\geq 1$. Plug equations (26), (27), and (28) into equations (22) and (23) and divide both sides by y. Then:

if $z\leq 1$

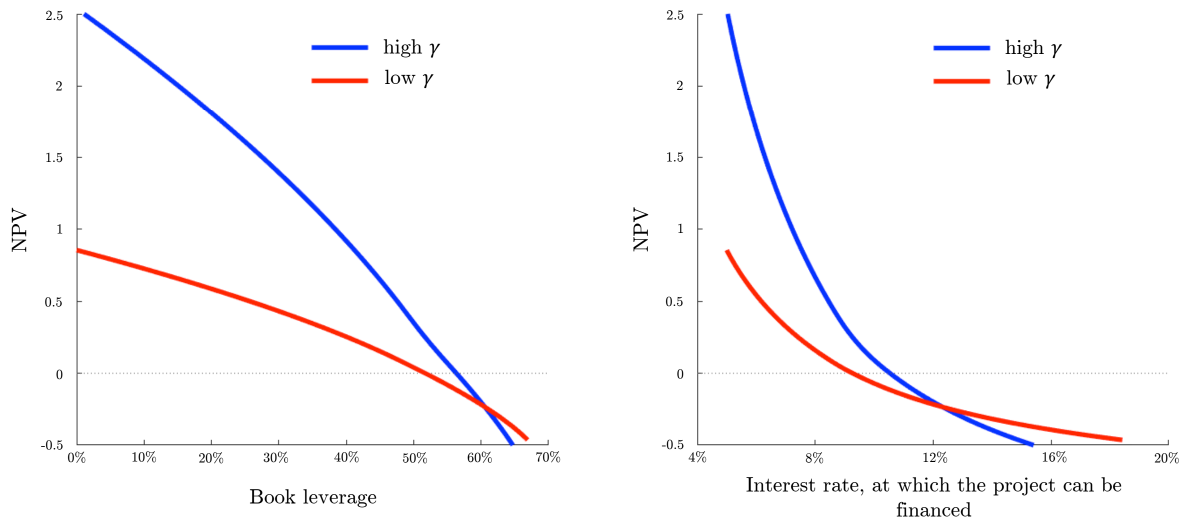
$$\begin{aligned}(r-\mu)f(z)=(1-z)(1-\tau)-\mu zf'(z)+\frac{\sigma^2z^2}{2}f''(z)+\\ \lambda I\big\{\text{Firm invests}\big\}\Big((1+\delta(1+\gamma))f(z_{new})-f(z)\Big)\end{aligned} \tag{29}$$

if $z>1$

$$\begin{aligned}(r-\mu)f(z)=\left(\frac{(z-1)(1-\tau)}{\frac{1}{r}+(1-\beta)Bz^{-\beta}}-\mu z\right)f'(z)+\frac{\sigma^2z^2}{2}f''(z)+\\ \lambda I\big\{\text{Firm invests}\big\}\Big((1+\delta(1+\gamma))f(z_{new})-f(z)\Big)\end{aligned} \tag{30}$$

coverage ratio for these firms would be undetermined. Furthermore, if $\frac{y}{c}$ was used as the state variable for the model, equation (22) would imply that $v(y,c)=cf\big(\frac{y}{c}\big)$, which would again be undetermined for firms that have no debt

Figure 4
NPV of an investment opportunity for a firm



NPV of a project in the model depends on two parameters: the quality of the project (firm's γ) and firm's leverage. Keeping firm's leverage constant, a higher γ implies a greater-NPV project, which on the Figure correspond to the blue curve being above the red curve. As leverage increases, the firm can only raise new debt at a high rate (because bankruptcy risks are high). Therefore, project NPV decreases with leverage, as Panel A shows. Panel B shows a different interpretation of the same message, specifically putting the rate to raise new debt on the horizontal axis. A firm will take a project as long as its NPV is positive, which implies that high γ firms will be investing more than low γ firms.

A firm will take arriving investment opportunity only if doing so increases its equity value. Therefore, the following condition determines when a firm with an investment opportunity is indifferent between investing and not:

$$v\big((1+\delta(1+\gamma))y, c_{new}\big) = v(y,c) \tag{31}$$

If LHS of (45) is greater than the RHS, firm's value increases if invests, and so the firm with an investment opportunity certainly takes it. On the other hand, if LHS of (45) is lower than the RHS, firm's value decreases if it invests, and so the firm will forgo the investment opportunity.

Note that equation (45) can be rewritten in terms of z:

$$(1+\delta(1+\gamma))f(z_{new}) = f(z_{inv}) \tag{32}$$

where z_{inv} denotes the level such that the firm takes arriving investment opportunities if and only if $z \leq z_{inv}$.

It only remains to specify boundary conditions. Equity value should be zero at default, as firm's assets are liquidated, and all proceeds go to debtholders. Therefore,

$$f(z_{def}) = 0 \tag{33}$$

where value of z_{def} follows from equation (20)

$$z_{def} = \frac{1-\beta}{-\beta}(1-\tau)(1-\alpha)rH \tag{34}$$

The second boundary condition is more complicated, and is discussed in the Appendix B

2.4.1 Tax-shield

The discussion so far has been focused on a firm that does not issue debt to exploit benefits of tax-shield. To add this important feature, consider a firm that is characterized by a pair of (y, c) that decided to issue more debt and increase its level of c to c_{res}. In doing so, the firm gets the amount of proceeds $D(y, c_{res}) - D(y, c)$, but its equity value changes to $v(y, c_{res})$. The firm, therefore, should choose c_{res} to maximize the value of the sum of proceeds and equity:

$$\max_{c_{res}} \Big(v(y, c_{res}) + D(y, c_{res}) - D(y, c) \Big) \tag{35}$$

Take first order conditions with respect to c_{res} and divide both sides by y:

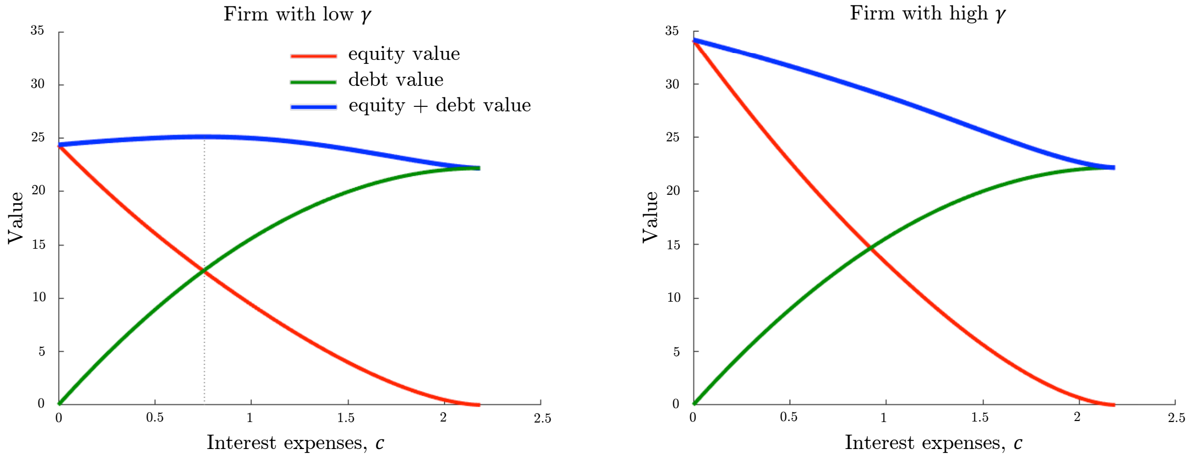
$$f'(z_{res}) + \frac{D'_c(y, c_{res})}{y} = 0 \tag{36}$$

At last, expand $D(y, c_{res})$ using the expression (16):

$$f'(z_{res}) + \frac{1}{r} + (1-\beta)Bz_{res}^{-\beta} = 0 \tag{37}$$

Note that solution z_{res} to equation (35) does not depend on firm's current z, which means firm's policy regarding debt issuances for tax-shield purposes is independent of its current state. Firms with $z < z_{res}$ will issue enough debt so that $z = z_{res}$, and firms with $z > z_{res}$ will be waiting before

Figure 5
Value of equity and debt, and the decision to issue debt to exploit benefits of tax-shield



These two graphs show the value of equity and debt for a firm with current productivity $X = 1$ and different interest expenses c. At any point in time, a firm is allowed to issue new debt and increase the level of interest expenses; a firm would only do that if it increases the value of the blue curve, which is the sum of the red and the green curves. Notice that a firm whose γ is high optimally chooses not to increase its debt level. However, a firm with a low value of γ does issue debt. This happens because of the trade-off between the time-value of money and inability to finance future investments with debt: a firm can issue debt immediately and pay dividends, or can delay the debt issuance decision until an investment opportunity arrives and finance it later.

their y increases sufficiently.

Two important points should be mentioned about the level z_{res}. First of all, it is possible that there is no value z that satisfies equation (37): such firms never issue debt to exploit benefits of tax-shield. To understand why this could happen, recall that firms also issue debt to finance investment opportunities. If firm's investment opportunities are very profitable or if they arrive frequently, the firm may optimally choose not to issue any debt for tax-shield purposes, and instead wait until it finds an investment opportunity. Second, it may happen that $z_{inv} < z_{res}$. This would imply that firm's investment opportunities are not profitable enough for the firm to wait until they arrive, and the firm would rather issue debt immediately and exploit tax-shield benefits right away instead of paying high taxes to the government in waiting for investment opportunities.

3 Model solution

3.1 Simulations

To analyze cross-sectional implications of the model, this paper uses simulations approach. A virtual economy that has $N = 40000$ firms is simulated over $T = 2400$ periods with every period being one month; the next subsection explains why these numbers are chosen. In every period of the simulation, each firm observes the realization of its idiosyncratic shock and whether it gets an investment opportunity. If the firm gets an investment opportunity, it optimally decides whether to take or forgo it. Firms are also free to issue more debt to better exploit benefits of tax-shield in any period. Firms whose cash flow is below the level of their interest expenses issue debt to cover the shortfall. A firm defaults if its interest expenses in a given period exceed its cash flow, and the firm fails to issue more debt to cover the shortfall.

Firms in the simulation are different by 1) the timing of the arrival of their investment opportunities, 2) the realization of their idiosyncratic shocks, and 3) the profitability of their investment opportunities γ. By assuming that firms in a cross-section differ by the profitability of their investment opportunities, I create an environment, in which firms are ex-post heterogeneous in their profitability, and this heterogeneity is achieved through the endogenous choice to invest. All other parameters are identical between firms.

At $t = 0$, firms start with one unit of capital $K = 1$ with productivity $X = 1$ and their own profitability of future investment opportunities γ. As the economy evolves, some firms naturally default, and these firms are replaces by newly-born firms that inherit old firm's productivity X, the number of units of capital minus bankruptcy costs $(1 - \alpha)K$, and the profitability of investment opportunities γ. There is a concern that this way of replacing defaulted firms with new firms may affect the relationship between leverage and other variables. For instance, suppose that the true relationship between leverage and profitability is positive, and firms with the greatest leverage are also the most profitable. Resetting firm's leverage to zero upon default will then mechanically induce the negative correlation between profitability and leverage because now most profitable firms will have no debt.[12] To alleviate the mechanical effects, the analysis excludes firms that were in a simulation for less than $t = 60$ periods (5 years). This approach does not seem restrictive as the target empirical distribution of firms consists of firms with S&P long-term credit ratings, which are usually old and mature.

The economy is simulated over $T = 2400$ periods with every period being one month. As it takes time for the economy to achieve its steady-state, I only conduct the analysis based on the

[12]Note that another standard approach when defaulted firms are replaced with firms that have $K = 1$ and $X = 1$ will create the opposite problem when positive correlation between leverage and profitability is mechanically induced. This would happen because cross-sectional X on average grows

Table 1
Benchmark parameter values used to solve the model

		This paper	S&W (2012)
Panel A: Economy-wide parameters			
risk-free rate	r	5%	5%
effective corporate tax rate	τ	20%	20%
risk premium	rp	5%	5%
price of capital*	H	30.8	
Panel B: Firm-specific parameters			
growth rate of the capital productivity	μ	1.5%	2%
volatility of the cash flow process	σ	25%	25%
bankruptcy costs	α	10%	10%
frequency of investment opportunities arrival	λ	1	
size of investment opportunities	δ	10%	
profitability of investment opportunities**	γ	$U[2\% - 6\%]$	

This table shows parameter values used to solve the model. S&W (2012) denotes numbers used in Strebulaev and Whited (2012), where authors review the literature on dynamic capital structure and simulate a number of models similar to the one used in this paper. *Price of capital is calibrated to match the equity value of a firm with no leverage and $\gamma = 0$ (so that market-to-book ratio for this firm equals one at time zero). **Firms in the simulation differ by the profitability of their investment opportunities γ, and γ is distributed uniformly between 2% and 6%

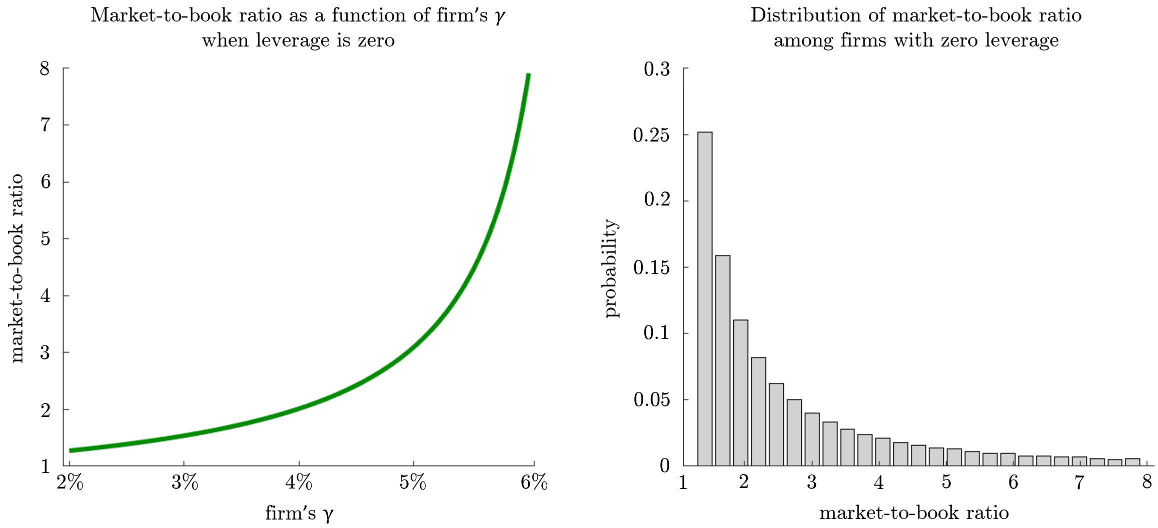
last $t = 48$ periods (4 years) of observations.

3.2 Parameters choice

Table 2 shows benchmark values of parameters that are used to solve the model. Most of the values are taken from Strebulaev and Whited (2012) who provide a review of literature on dynamic capital structure and choose neutral parameter values to simulate a number of models similar to the one discussed in this paper. Importantly, the literature does not have a consensus on the value of most of these parameters or their distribution among firms. In addressing this issue, where applicable, I discuss estimates of different authors or provide support based on the data from Compustat.

Specifically mentioned, estimates of bankruptcy costs α, which in this paper is assumed to be 10%, vary from very low to very high. For example, Gruber and Warner (1977) finds that direct bankruptcy costs are about 1% of the assets value, and Andrade and Kaplan (1998) report the value of about 20%. Some authors used a structural estimation approach to infer the bankruptcy costs from firms' observed decisions. In particular, Davydenko, Strebulaev and Zhao (2012) find that default costs are in the range of 10% and 30%, Hennessy and Whited (2007) report values between 8.4% and 15.1%, and Glover (2016) finds the value of about 45%. Glover's estimates are

Figure 6



This graph shows how the distribution of γ translates into the distribution of market-to-book ratios for firms with no leverage.

well-above estimates of other authors, but as argued by Reindl, Stoughton, and Zechner (2017), it is because Glover assumes that all firms follow optimal leverage policy, while it is not necessarily the case in the data. The authors estimate a similar model without imposing optimal capital structure and using firms stock prices instead, and find substantially lower values of bankruptcy costs (20%).

Cash flow volatility parameter σ also does not have a precise estimate in the literature. Faulkender and Petersen (2005) report that the average implied asset volatility of firms that have access to public debt is 19%, and Schaefer and Strebulaev (2008) find 23% (also among firms that issue bonds); Reindl, Stoughton, and Zechner (2017) take a structural estimation approach with Leland-type environment and find asset volatility between 25% and 42%.

The effective corporate tax-rate τ that this paper uses implicitly aggregates the effect of corporate and personal taxes on dividends and interest payments; the resulting value $\tau = 20\%$ is based on the estimates of Graham (2000). In a model, similar to mine, Chen (2010) considers different taxes explicitly, and the resulting effective corporate tax rate in his model is around 18%.

Cash flow growth rate μ is 2% in Strebulaev and Whited (2012), but they assume that the process for cash flow is purely exogenous. In this paper, firm's cash flow becomes higher because there is the unconditional growth rate μ, but also because firms invest. Therefore, the paper assumes a smaller unconditional growth rate value of 1.5%.

While the model is solved under the risk neutral probability measure $\mathbb{Q}$, actual shock realizations happen under the physical probability measure $\mathbb{P}$. Therefore, in the simulation procedures discussed below, risk-premium $rp = 5\%$ is added to the risk-neutral growth rate μ.

There are three parameters unique to this model (at least for the class of Leland-type models): size of firm's investment opportunities δ, frequency of investments arrival λ, and profitability of these investment opportunities γ. This paper assumes that investment opportunities arrive on average once a year ($\lambda = 1$), and size is 10% of firm's existing capital. Results of the model are not very sensitive to variations in δ.

Instead of assuming a single value of γ for all firms, this paper assumes that there is heterogeneity in γ among firms. Clearly, there is no way to measure γ for firms in the data directly. This parameter in the model, however, shows how good firm's investment opportunities are, and so should be directly linked to the distribution of market-to-book ratios. With this in mind, the paper assumes that γ is uniformly distributed on the interval $[2\%, 6\%]$. This region was chosen to target the distribution of market-to-book ratios in the data for *firms with zero leverage*; firms with zero leverage are chosen because in the model such firms correspond to newly born firms. Figure 6 shows how this distribution of γ's translates into the distribution of market-to-book ratios for firms with zero leverage. The mean and median values in this distribution is 2.59 and 2.01 vs. 2.89 and 2.50 in the data for firms with zero leverage. Even though distribution of γ's is chosen to target the distribtuion of market-to-book ratios for firms with no leverage, the following subsection shows that the simulated distribution of market-to-book ratios in the *full* cross-section (i.e. for firms with all values of leverage) is also very similar to the data.

To have γ uniformly distributed between 2% and 6% for simulation purposes, I first choose 1000 numbers equally spaced over the interval $[2\%, 6\%]$. For each value in the interval, I create 40 firms that have this value as their γ. This results in $N = 40000$ firms, and I simulate these firms over $T = 2400$ periods, with the length of each period being one month.

Price of capital is a free parameter in the model, and it is normalized so that market-to-book ratio equals one for a firm with no leverage whose $\gamma = 0$.

Table 2
Leverage-predicting regressions in the model

Panel A: LHS is market leverage (value of debt divided by the sum of the values of debt and equity)							
	I	II	III	IV	V	VI	VII
profitability	-0.003^*			-0.001^*	-0.001^*	-0.160^*	0.001
capital		-0.023^*		0.006^*	0.003^*	0.278^*	0.006^*
market-to-book			-0.068^*	-0.071^*	-0.272^*	-0.188^*	-0.525^*
γ FE	No	No	No	No	Yes	No	No
Firm FE	No	No	No	No	No	Yes	No

	Panel B: LHS is book leverage			Panel C: LHS is inverse coverage ratio		
	VIII	IX	X	XI	XII	XIII
profitability	-0.020^*	-0.001^*	-0.204^*	-0.050^*	-0.002^*	-0.425^*
capital	0.015^*	0.005^*	0.375	0.380^*	0.009^*	0.761
market-to-book	-0.026^*	-0.231^*	-0.099^*	-0.070^*	-0.641^*	-0.380^*
γ FE	No	Yes	No	No	Yes	No
Firm FE	No	No	Yes	No	No	Yes

This table shows results of the following regression $LHS = \beta_0 + \beta_1 \log(X) + \beta_2 \log(K) + \beta_3 + mb + \varepsilon$ based on the data from a simulated cross-section of firms; see the text for details of the simulation. Specifications V, VI, IX, X, XII, and XIII also control for fixed affects. LHS is: Panel A: firm's market leverage (the ratio of the market value of firm's debt to the sum of market values of debt and equity); Panel B: firm's book leverage (the ratio of the market value of firm's debt to the value of its assets); Panel C: firm's inverse coverage ratio (z in the model, the ratio of firm's interest expenses c to its cash flow y). Specification VII is only based on a sub-sample of firms all of which have $\gamma = 4\%$ (median value in the distribution). *denote variables significant at 5% level.

3.3 Leverage and profitability

Table 2 shows results of the leverage predicting regression in the model. Specifications I, II, III, and IV use a simple model when leverage is regressed on profitability ($\log(X)$), capital ($\log(K)$), market-to-book ratio ($v(y,c)/Ay$) in univariate regressions, and then in one multivariate regression. All three coefficients are significant, and their signs coincide with the signs found in similar cross-sectional regressions based on Compustat data.

What is the mechanism behind the negative correlation between leverage and profitability? This result is mainly driven by the effect of heterogeneity in γ's among firms. Firms with ex-post high profitability are firms that have high γ - their investment opportunities are better, and so they end up with a higher value of X after every round of investment. Firms with higher γ, on average, also have a smaller leverage. This effect is mainly driven by two factors. First, a firm with a good investment opportunity (a high-γ firm), when it raises debt to pay for an investment opportunity, is able to obtain a smaller interest rate from debtholders. Therefore, every time a high-γ firm invests, its total debt value increases by a smaller value than for a comparable low-γ firm. The second effect is purely pricing: the way leverage is measured in the model, it has firm's equity value in denominator. Naturally, equity value of firms with high γ is much greater than the equity value of low-γ firms, and so leverage is mechanically lower for high-γ firms than for low-γ firms with the same level of debt. There is also a third factor: firms with higher γ have more incentives to delay debt issuance until investment opportunities arrive, while firms with low γ may choose to issue debt to purely exploit benefits of tax-shield even when there are no investment opportunities. This third factor, however, does not play a role in the current calibration of the model: all firms optimally choose to never issue debt if they do not have investment opportunities.

Between the other two factors (cheaper debt and higher pricing of equity for firms with high γ) - which one has a more significant effect on the relationship between leverage and profitability? It appears that both factors have the same effect. To show that the effect of cheaper debt plays an important role, Panels B and C of Table 2 replaces leverage on the LHS of the regression by two alternative measures of firm's indebtedness that are not affected by the pricing of equity: book value of leverage and inverse coverage ratio. Book value of leverage puts the price of capital in denominator, and price of capital is the same for all companies independent of their γ. Inverse coverage is the ratio of firm's interest expenses and cash flow, which are not affected by firm's future investment opportunities. As evident from the Table 2, the negative relationship between leverage and profitability remains in these regression.

One can argue that since the negative correlation between leverage and profitability in the model is purely driven by firm's γ, the relationship should revert back to positive once the regression is controlled for γ-fixed effects (or firm fixed-effects, since γ is fixed for every firm). Such result would be problematic for the model: in the data, the relationship between leverage and profitability remains negative even if the regression controls for firm fixed-effects. Specifications V and VI in Table 2, however, show that the model addresses this concern and produces negative relationship between leverage and profitability even after accounting for fixed-effects. To understand why the coefficient for profitability remains negative in the regressions that control for fixed-effects, remember that fixed-effects only allow for different slopes in a regression, while the relationship between the variables in the model is highly-non linear. In particular, firms with different γ's follow different investment rules, and this interaction affects regression coefficients.

At last, to prove that the difference in γ is indeed the factor that drives the negative relationship between leverage and profitability in the model, I additionally run a regression on a sub-sample of

Table 3
Cross-sectional regressions in the model

Panel B: Investment-predicting regressions								
	I	II	III	IV	V	VI	VII	VIII
cash flow	0.003*	0.000*	0.045*	0.002	0.871*	0.083*	1.690*	0.141
market-to-book	0.018*	0.016*	0.016*	0.018*	0.091*	0.581*	0.559*	0.446*
γ FE	No	Yes	No	No	No	Yes	No	No
Firm FE	No	No	Yes	No	No	No	Yes	No

This table shows results of regressions based on the data from a simulated cross-section of firms; see the text for details of the simulation. LHS variable for specifications I - IV is a dummy variable which equals one for firm-year observations with non-zero investments, and zero otherwise. LHS variable for specifications V - VIII is a log of the investment amount for firm-year observations with positive investments, and zero otherwise; results are robust if a different negative number is used instead of zero for firm-year observations with no investments. Specification I is a probit regression, and all other specifications are linear regressions. Specifications IV and VIII are based on the data from a sub-sample of firms for which $\gamma = 4$ (median value of γ in the population). * denote variables significant at 5% level.

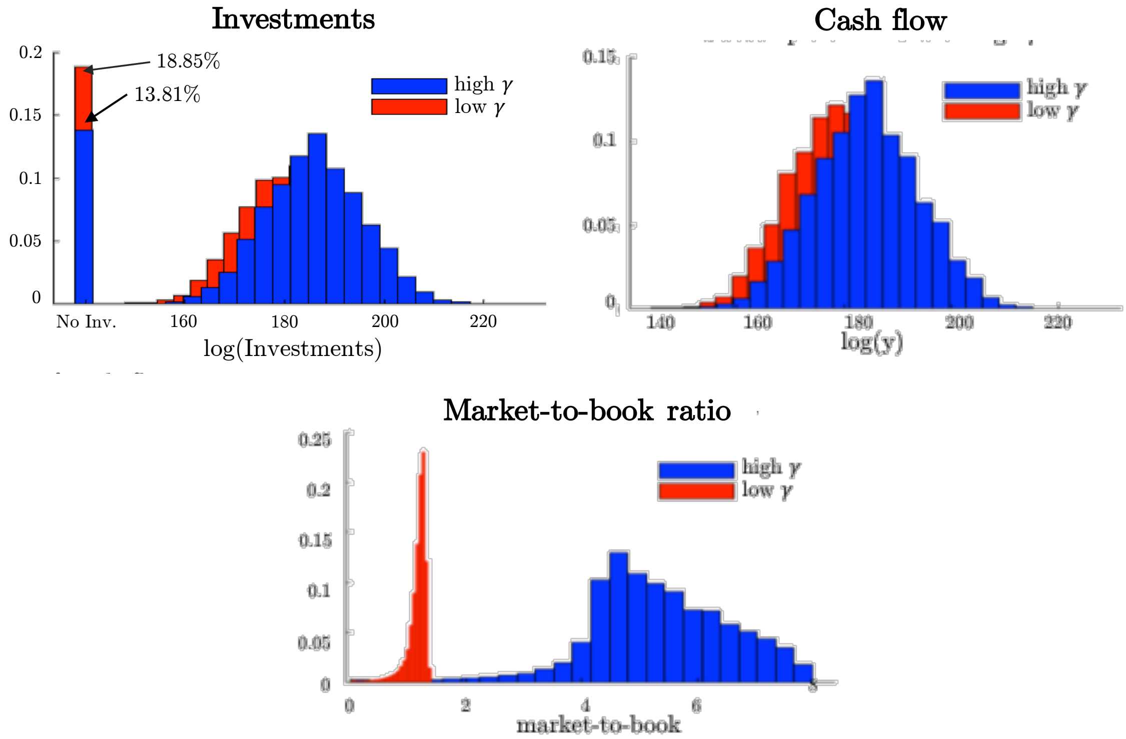
firms, all of which have the same γ. More specifically, Table 2 specification VII shows results for a regression of leverage on the three factors only based on observations for firms, for which $\gamma = 4\%$ (the average value of γ in the cross-section). The coefficient on profitability flips its sign to positive. This shows that heterogeneity in $\gamma's$ is indeed the driving factor for the negative correlation.

3.4 Investments and cash flow

The next stylized fact that the model attempts to explain is why cash flow has a significant effect in a regression of investments on cash flow and market-to-book ratio. First, the model manages to replicate this stylized fact. Table 3 shows results of a regression of firm's investments on market-to-book ratio and cash flow, and the same regressions when the LHS is the probability that a firm invests in a given period. It appers that inbth cases the cash flow has significant explanatory power.

To understand the mechanism of this result, recall that in the model investment opportunities arrive randomly; the decision to invest, however, is endogenous, and depends on two factors: 1) firm's current leverage, and 2) the profitability of the investment (firm's γ). While firm's market-to-book ratio is well correlated with firm's γ, these two are not the same things. Therefore, even when one controls for firms market-to-book ratio, the effect of the γ is not fully captured. On the other hand, firms with ex-post high cash flows are firms with high γ - their cash flow is higher precisely because all their projects were good. In this sense, γ drives spurious correlation between firm's investment amount and the cash flow, and this correlation makes cash flow significant in the investment regression, even after one controls for firm's market-to-book ratio. Figure 7 illustrates

Figure 7
Distributions in a simulated cross-section for firms with different values of γ



This figure shows results of a model simulation. Firms in the model are different in the quality of investment opportunities they expect, with some firms consistently getting better investment opportunities than others. As evident from the graph, firms with better investment opportunities on average have higher cash flows, higher market-to-book ratios, and invest more.

this result by showing the distribution of investments, cash flows, and market-to-book ratios in two sub-samples of firms.

Since the explanation for the significance of the cash flow in the investment regression comes from heterogeneity in γ, and γ is fixed for each firm at the origin, one can claim that the cash flow significance should disappear once firm-fixed effects are controlled for. Nevertheless, the model shows that the cash flow remains significant even when fixed-effects are added to the regression. This is shown in specifications II, III, XI, and XII of the Table 2. The explanation why the significance does not disappear is similar to the explanation in the previous section: firm fixed-effects only allow for different average levels of investments for firms with low γ. In the model, however, γ is not the only factor that determines whether and how much a firm invests; the decision to invest depends both on firm's γ and on its leverage, and the relationship between the variables is highly n0n-linear. Therefore, both variables - market-to-book ratio and cash flow - are significant

Table 4
Moments of leverage distribution in the data and in the steady-state cross-section in the model

Panel A: Moments of leverage distribution

	Data	Model
mean	26.3%	25.2%
median	22.9%	22.2%
standard deviation	17.1%	16.0%
quantiles		
5%	5.9%	5.8%
10%	8.33%	8.2%
25%	13.9%	13.6%
75%	34.7%	33.3%
90%	48.3%	45.9%
95%	59.47%	55.3%

Panel B: Moments of market-to-book distribution

	Data	Data, winsorized	Data, censored	Model
mean	2.54	2.42	2.25	2.31
median	1.85	1.85	1.85	1.78
min	-12.36	0	0	0
max	24.93	8	8	8

This table reports summary statistics on leverage market-to-book distributions in the model and in the data. For the data part, only firms that have access to the public debt markets are considered, and the access is proxied by whether a firm has S&P long-term credit rating in a particular year. See the text in Section 3.1 for the details of simulation.

even in the regressions with fixed-effects.

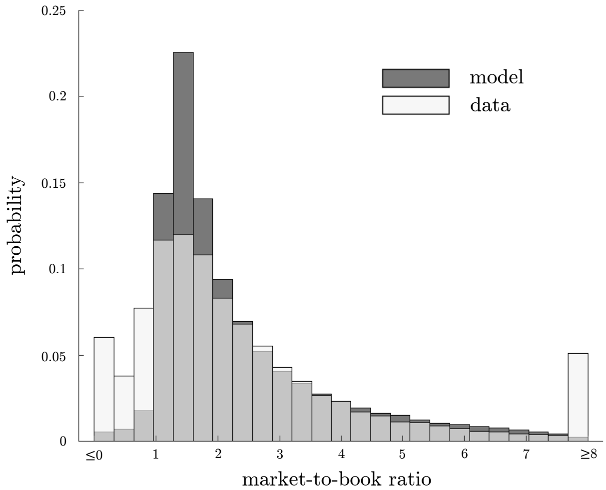
However, cash flow is no longer a significant factor in the regression once the regression is restricted to a sub-sample of firms that all have the same value of γ. Specifications IV and VIII of the Table 3 proves this by running the regression on a sub-sample of firms with $\gamma = 4\%$ (the mean value of γ in the cross-section). The cash flow significance disappears, which indeed confirms that γ drives the relationship in the model.

3.5 Quantitative results

To study quantitative implications of the model, I first collect the data for the empirical moments that are related to the distribution of leverage and market-to-book ratios. I focus on these moments because the papers gives predictions about these moments, and they are not directly assumed in the parametrization section. The sample of firms comes from Compustat for years 1981-2017.[13] Firms in the financial sector (6000s SICs) and the public sector (9000s SICs) are excluded from the

[13] 1981 is the first year that has data on S&P long-term credit ratings

Figure 8
Distribution of market-to-book ratio in the data and in the model



This Figure shows the distribution of market-to-book ratios for firms in the data (white bars) and firms in a simulated cross-section (black bars). See the text Section 3.1 for details of the simulation. Distribution in the data is based on market-to-book ratios for firms that have access to S&P long-term credit ratings in years 1978-2017.

analysis; observations with the book value of assets that is less than $1 million are also excluded.

As Faulkender and Petersen (2005) find, firm's capital structure depends a lot on whether the firm has access to public bond markets. The assumption that firms can issue debt easily is crucial in this model, and for this reason, the paper only consider firms that have S&P long-term credit rating, which is used as a proxy for whether the firm can issue public debt. Data on the S&P long-term ratings is available on monthly basis, but financial data is annual. To match the datasets, it is assumed that a firm has S&P long-term rating in a given year if it has S&P long-term rating in at least one month of that year. Data on S&P long-term ratings is available between years 1981 and 2017, and there is, on average, 1500 observations in each year. However, years 1981-1984 have only five observations combined, and year 2017 has only 146 observations.

As the Table 4 shows, the model manages to fully reproduces leverage distribution in the data. The model matches its mean (26.3% in the data vs. 25.2% in the model), median (22.9% vs. 22.2%), standard deviation (17.1% vs. 16.0%) and every quantile between 5% and 95%. In addition to market leverage, the model also matches the mean and median values of book leverage (37.3% and 37.1% in the model and 36.6% and 34.2% in the data, though the rest of the distribution is not matched as well as the distribution of market leverage. This is in part because the definition of book leverage that this paper uses and the definition that is used in the data are different: in the data, book leverage is measured as book value of debt divided by book value of assets. In the model, I measure book value of debt as the market value of debt divided by the value of assets. The concept of the face value of debt is not very meaningful in this model, and therefore there is no way to measure the book value of leverage in the model the same way as it is measured in the data.

The model also reproduces the distribution of the market-to-book ratios in the data well, as illustrated by the Figure 8. The model can't account for firms with extremely high values of market-to-book ratios (above 8) or negative values, but it matches the distribution in the middle. The mean and median values of the market-to-book ratio produced by the model (2.31 and 1.78) are very close to those in the data (2.42 and 1.85).

References

[1] Andrade, Gregor, and Steven N. Kaplan. "How costly is financial (not economic) distress? Evidence from highly leveraged transactions that became distressed." The Journal of Finance 53.5 (1998): 1443-1493.

[2] Bhamra, Harjoat S., Lars-Alexander Kuehn, and Ilya A. Strebulaev. "The aggregate dynamics of capital structure and macroeconomic risk." The Review of Financial Studies 23.12 (2010): 4187-4241.

[3] Bhamra, Harjoat S., Lars-Alexander Kuehn, and Ilya A. Strebulaev. "The levered equity risk premium and credit spreads: A unified framework." The Review of Financial Studies 23.2 (2009): 645-703.

[4] Chen, Hui. "Macroeconomic conditions and the puzzles of credit spreads and capital structure." The Journal of Finance 65.6 (2010): 2171-2212.

[5] Davydenko, Sergei A., Ilya A. Strebulaev, and Xiaofei Zhao. "A market-based study of the cost of default." The Review of Financial Studies 25.10 (2012): 2959-2999.

[6] Denis, David J., and Stephen B. McKeon. "Debt financing and financial flexibility evidence from proactive leverage increases." The Review of Financial Studies 25.6 (2012): 1897-1929.

[7] Gruber, Martin J., and Jerold B. Warner. "Bankruptcy costs: Some evidence." The journal of Finance 32.2 (1977): 337-347.

[8] Fama, Eugene F., and Kenneth R. French. "Financing decisions: who issues stock?." Journal of financial economics 76.3 (2005): 549-582

[9] Faulkender, Michael, and Mitchell A. Petersen. "Does the source of capital affect capital structure?." The Review of Financial Studies 19.1 (2005): 45-79.

[10] Gilson, Stuart C. "Transactions costs and capital structure choice: Evidence from financially distressed firms." The Journal of Finance 52.1 (1997): 161-196.

[11] Glover, Brent. "The expected cost of default." Journal of Financial Economics 119.2 (2016): 284-299.

[12] Goldstein, Robert, Nengjiu Ju, and Hayne Leland. "An EBITbased model of dynamic capital structure." The Journal of Business 74.4 (2001): 483-512.

[13] Graham, John R. "How big are the tax benefits of debt?." The Journal of Finance 55.5 (2000): 1901-1941.

[14] Gruber, Martin J., and Jerold B. Warner. "Bankruptcy costs: Some evidence." The Journal of Finance 32.2 (1977): 337-347.

[15] Hennessy, Christopher A., and Toni M. Whited. "How costly is external financing? Evidence from a structural estimation." The Journal of Finance 62.4 (2007): 1705-1745.

[16] Huang, Jing-Zhi, and Ming Huang. "How much of the corporate-treasury yield spread is due to credit risk?." The Review of Asset Pricing Studies 2.2 (2012): 153-202.

[17] Leland, Hayne E. "Corporate debt value, bond covenants, and optimal capital structure." The journal of finance 49.4 (1994): 1213-1252.

[18] Myers, Stewart C., and Nicholas S. Majluf. "Corporate financing and investment decisions when firms have information that investors do not have." Journal of financial economics 13.2 (1984): 187-221.

[19] Miller, Merton H. "Debt and taxes." the Journal of Finance 32.2 (1977): 261-275.

[20] Moody's special comment "Measuring Corporate Default Rates", 2006. Authors: Hamilton, T. David, and Cantor, Richard

[21] Moody's special comment "Moody's Ultimate Recovery Database", 2007. Author: Emery, Kenneth

[22] Reindl, Johann, Neal Stoughton, and Josef Zechner. "Market implied costs of bankruptcy." (2017).

[23] Smith, Clifford W. "Alternative methods for raising capital: Rights versus underwritten offerings." Journal of financial economics 5.3 (1977): 273-307

[24] Schaefer, Stephen M., and Ilya A. Strebulaev. "Structural models of credit risk are useful: Evidence from hedge ratios on corporate bonds." Journal of Financial Economics 90.1 (2008): 1-19.

[25] Strebulaev, Ilya A., and Toni M. Whited. "Dynamic models and structural estimation in corporate finance." Foundations and Trends in Finance 6.12 (2012): 1-163.

[26] Strebulaev, Ilya A. "Do tests of capital structure theory mean what they say?." The Journal of Finance 62.4 (2007): 1747-1787.

4 Appendix

4.1 Appendix A

This Appendix shows that assumptions 1-4 allows to aggregate firm's capital units into one. To see that, consider two firms. The first firm has K units of capital with productivity $X(1+\delta(1+\gamma))$. The second has two stocks of capital: K units of capital with productivity X and δK units of capital with productivity $(1+\gamma)X$. Denote y_t^1 total profits of the firs firm, and y_t^2 total profits of the second firm, and notice that the following equality holds for all $s \geq t$

$$y_s^1 = KX(1+\delta(1+\gamma)) = KX + \delta K(1+\gamma)X = y_s^2 \tag{38}$$

Therefore, at any point in time, profits of the first firm is the same as profits of the second firm. When there is an investment opportunity, the first firm will have the following stocks of capital:

$$\Big\{\big\{K, X(1+\delta(1+\gamma))\big\}, \big\{\delta K, X(1+\gamma)(1+\delta(1+\gamma))\big\}\Big\}$$

The second firm will have four stocks of capital:

$$\Big\{\big\{K, X\big\}, \big\{\delta K, X(1+\gamma)\big\}, \big\{\delta K, X(1+\gamma))\big\}, \big\{\delta^2 K, X(1+\gamma)^2)\big\}\Big\}$$

However, notice that immediate after-investment profits are the same for both firms:

$$XK\Big((1+\delta(1+\gamma)) + \delta(1+\gamma)(1+\delta(1+\gamma))\Big) = XK\Big(1 + (\delta(1+\gamma)) + (\delta(1+\gamma)) + \delta^2(1+\gamma)^2)\Big)$$

Therefore, at any future time s, combined profits of the first and second firms are the same, which means they should have the same price. This means that capital indeed can be aggregated.

4.2 Appendix B

Let $v(y,c)$ be the equity value of a firm. Llet $z = \frac{c}{y}$, and guess that $v(y,c) = yf(z)$. Sufficient conditions to confirm the guess are:

1. Equations (23) and (24) can be rewritten in terms of z
2. Boundary conditions should take the form $v(y,c) = yf(z)$
3. Boundaries on which boundary conditions are determined should only depend on z

To verify the first condition, note that

$$v'_y(y,c) = \big(yf(z)\big)'_y = f(z) - zf'(z) \tag{39}$$

$$v'_{yy}(y,c) = \big(f(z) - zf'(z)\big)'_y = -f'(z)\frac{z}{y} + f'(z)\frac{z}{y} + \frac{z^2}{y}f''(z) = \frac{z^2}{y}f''(z) \tag{40}$$

Furthermore, notice that dc can be partially rewritten in terms of z on the interval $y < c$:

$$dc = \frac{(c-y)(1-\tau)}{\frac{1}{r} + (1-\beta)By^{\beta}c^{-\beta}} = y\frac{(z-1)(1-\tau)}{\frac{1}{r} + (1-\beta)Bz^{-\beta}} \tag{41}$$

Plug equations (26), (27), and (28) into equations (22) and (23) and divide both sides by y. Then if $z \leq 1$

$$\begin{aligned}(r-\mu)f(z) = (1-z)(1-\tau) - \mu zf'(z) + \frac{\sigma^2 z^2}{2}f''(z) + \\ \lambda I\big\{\text{Firm invests}\big\}\Big((1+\delta(1+\gamma))f(z_{new}) - f(z)\Big)\end{aligned} \tag{42}$$

and if $z > 1$

$$\begin{aligned}(r-\mu)f(z) = \frac{(z-1)(1-\tau)}{\frac{1}{r} + (1-\beta)Bz^{-\beta}} - \mu zf'(z) + \frac{\sigma^2 z^2}{2}f''(z) + \\ \lambda I\big\{\text{Firm invests}\big\}\Big((1+\delta(1+\gamma))f(z_{new}) - f(z)\Big)\end{aligned} \tag{43}$$

where $z_{new} = \frac{c_{new}}{(1+\delta(1+\gamma))y}$ - firm's new inverse coverage ratio after an investment is taken. Note that in order for equations (29) and (2.4) to be only a function of z, it needs to be shows that z_{new} is only a function of z. To see that, divide both sides of equation (23) by y:

$$(1+\delta(1+\gamma))\Big(\frac{z_{new}}{r} + Bz_{new}^{1-\beta}\Big) = \frac{z}{r} + Bz^{1-\beta} + (1-\tau)H\delta \tag{44}$$

which verifies condition 1.

The next step is to prove that boundary conditions for $v(y,c)$ take the form of $yf(z)$. One of the boundary conditions is the value of equity at default. Note that default boundary is indeed a function of z only:

$$z_{def} = \frac{c_{def}}{y_{def}} = (1-\tau)(1-\alpha)rH\frac{1-\beta}{-\beta} \tag{45}$$

where the last equality is equation (20). Furthermore, value of equity at default should equal to zero, and so is independent of both y and c. The second boundary condition is more complicated and is discussed in the appendix, but also

4.3 Appendix C

Example developed in this Appendix shows that presence of leverage amplifies the problem of information asymmetry, and make equity issuance costs grow with leverage. In the example, for the same uncertainty structure, a firm with zero leverage optimally chooses to issue equity, while costs of equity issuance are preventive for a levered firm. The example is based on the seminal Myers and Majluf (1984) paper.

There is a firm that has assets in place, and an investment opportunity. Firm's quality is not known, but what is known is that it can either be good or bad with probabilities $p = (1-p) = 0.5$. Good firm's asset value is 190, and bad firm's asset value is 110. Required investment for the project is 100 for both types of firms, but the return is higher for the good firm: 120 vs. 110. The project has to be financed now or never, and the firm can only do it by issuing equity. For simplicity, investors are risk-neutral, and there is no discounting between periods, in which the investment takes place and return is realized. Figure 9 summarizes the example.

Figure 9

	Good $p = 0.5$	**Bad** $1 - p = 0.5$
Value of assets in place	190	110
Project required investment	100	100
Project return (gross)	120	100

Assume there is an equilibrium, in which both firms issue equity and finance the project. The value that outsiders assign to the firm is then the following:

$$V^{outs.} = \frac{1}{2}(190 + 120 + 110 + 110) = 265$$

Because the firm has to issue 100 of equity, the share that outside investors will require in return is $\frac{100}{265}$. In order for this to be an equilibrium, investors' beliefs should coincide with the actual behavior of firms. Therefore, both firms should be willing to take the investment opportunity. The good firm knows its type, and it will invest if the value that is left to current shareholders is higher with the investment:

$$V^G = \left(1 - \frac{100}{265}\right)(190 + 120) = 193 > 190$$

where 190 is the value of the good firm if the investment is not made, and no equity is issued. Equilibrium, in which both firms finance the project indeed exists.

Now consider a small modification of this example: assume that the firm, whose type is still unknown, issued debt in the past. Face value of debt is 100, and it has to be repaid next period. Figure **??** summarizes the modified example. Notice that firm's type cannot be revealed by the amount of debt it has, and no firm defaults next period, independent of whether the project is taken or not.

Assume again that there is an equilibrium, in which both firms issue equity and finance the project. The value that outsiders assign to the firm is then the following:

$$V^{outs.} = \frac{1}{2}(190 + 120 + 110 + 110) - 100 = 165$$

Naturally, firm's equity value is now smaller, as debtholders also have a claim on firm's assets.

In order for such equilibrium to exist, the good firm should be willing to pool with the bad firm. The good firm still knows its type, and in choosing whether to invest or not, it compares the value to its current equityholders with and without the investment:

$$V^G = \left(1 - \frac{100}{165}\right)(90 + 120) = 82.7 < 90$$

Therefore, the good firm chooses not to invest, which means there is no equilibrium in which both firms issue equity. The only difference between the two examples is that the firm is levered in the second example. This indeed shows that firm's equity issuance costs increase with leverage.