# Clustering Network Vertices in Sparse Contextual Multilayer Networks 

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#### Abstract

We consider the problem of learning the latent community structure in a Multi-Layer Contextual Block Model introduced by [27], where the average degree for each of the observed networks is of constant order and establish a sharp detection threshold for the community structure, above which detection is possible asymptotically, while below the threshold no procedure can perform better than random guessing. We further establish that the detection threshold coincides with the threshold for weak recovery of the common community structure using multiple correlated networks and co-variate matrices. Finally, we provide a quasi-polynomial time algorithm to estimate the latent communities in the recovery regime. Our results improve upon the results of [27], which considered the diverging degree regime and recovers the results of [26] in the special case of a single network structure.


## 1 Introduction

The problem of clustering the nodes of a network into interpretable communities, also known as the the community detection problem, has received wide attention in statistics, machine learning and probability and has found applications in diverse fields like the study of sociological interactions in [16], gene expressions in [11], and recommendation systems in [25], among others. A number of generative models have been suggested to understand the performance of different network clustering algorithms, the most popular being the Stochastic Block Model (SBM) from [20]. The SBM's are quite accurate in modeling several real world networks and exhibits rich phase transition phenomena (see, [13, 28, 30, 31]). The SBM's are used to study the interplay between the statistical and computational barriers of community detection (see, [2, 9]). We refer the reader to [1] for a detailed survey.

A recent but interesting direction in the community detection literature is the study of Contextual Block Models introduced by [15]. In this setup, the data comprise of a network of $n$ nodes with a $n \times p$ dimensional Gaussian co-variate matrix. In [15], a simple generative model was proposed to study the problem of community detection with such data and the information theoretic threshold for weak recovery was derived in the diverging average degree regime. Later, [26] studied the problem assuming constant average degree and established the sharp phase transition threshold above which a non-trivial estimate of the community assignment is possible while below the transistion threshold no such estimate exists. They also provided a quasi-polynomial time algorithm for the weak recovery of the community labels from the observed data.
[27] extended Contextual Block Models to introduce Contextual Multi-Layer Networks, where in addition to the $n \times p$ dimensional Gaussian co-variate matrix one observes multiple networks on $n$ subjects
each conveying some information about the underlying latent community structure. Specifically, the data conveys information about the community structure among $n$ nodes through different noise channels, a.k.a the Gaussian co-variate matrix and multiple SBM's. For example, one could potentially observe network data from the same set of students interacting in different social media platforms, or genetic data from two different experiments on same set of subjects. Another example of this scenario was described in [12] where the nodes represent proteins, edges in one network represent physical interactions between nodes and those in another network represent co-memberships in protein complexes. Data of this type have been studied in $[23,6,18,17,5,8,32,19,33,10,24]$.

In [27], a simple generative model is introduced to study the information theoretic thresholds of weak community recovery in the Contextual Multi-Layer Networks setup. The authors derive the limit of the mutual information between the latent communities and the observed data using an AMP algorithm. However, their work considers the diverging average degree regime, and the method described there breaks down when one considers the regime with non-diverging average degrees for each network which are more common in real applications where small world networks tend to be the predominant type. Moreover, prior works in this direction do not provide an algorithm implementable in polynomial time, to weakly recover the latent community structure from multiple correlated networks and a Gaussian co-variate matrix. This leaves a significant gap in the literature which we try to fill in this work.

Our major contributions in this paper are as follows:

1. We show that the phase boundary derived under the assumption of diverging average degrees for all the observed networks continues to hold if the average degrees of all networks are greater than 1.
2. We establish that the detection threshold i.e., the effective SNR threshold above which identification of non-trivial community structure in the observed networks is possible, overlaps with the weak recovery threshold i.e., the effective SNR threshold above one can provide a clustering of the nodes which is better than random guessing. To show the positive part of the result above the threshold, we provide a quasi-polynomial time algorithm to achieve weak recovery of the community structure. For the negative part, we use the popular second moment method. In the contiguity regime, we further derive the limit of the likelihood ratio in terms of appropriate cycle statistics.
3. Finally, we give an approximate Belief Propagation Algorithm to compute the MAP estimator of $\sigma$.

Our work recovers the results of [26] when $m=1$.

## 2 Contextual Multi-Layer Network

### 2.1 Setup and Assumptions

We consider $n$ subjects who are classified into two disjoint clusters indexed by $\{-1,+1\}$. Formally, the cluster assignment among $n$ subjects is denoted by $\boldsymbol{\sigma}=\left(\sigma_{i}\right)_{i=1}^{n} \in\{-1,+1\}^{n}$, where $\sigma_{i}$ denotes the cluster assignment of subject $i$. We use the notation $[n]$ to denote the set $\{1,2, \cdots, n\}$.

Let us assume that $\boldsymbol{\sigma}$ is sampled uniformly from $\{-1,+1\}^{n}$, or in other words, the subjects are randomly assigned to the clusters. We observe $m$ un-directed networks on these $n$ subjects (represented by random graphs on $n$ vertices), identified by $\left\{\boldsymbol{G}_{i}: i \in[m]\right\}$. Conditional on $\boldsymbol{\sigma},\left\{\boldsymbol{G}_{1}, \cdots, \boldsymbol{G}_{m}\right\}$ are mutually independent and in each $\boldsymbol{G}_{k}$, edges are generated independently with probability

$$
\mathbb{P}\left[(i, j) \in E\left(\boldsymbol{G}_{k}\right) \mid \boldsymbol{\sigma}\right]=\left\{\begin{array}{ll}
\frac{a_{k}}{n} & \text { if } \sigma_{i}=\sigma_{j} \\
\frac{b_{k}}{n} & \text { if } \sigma_{i} \neq \sigma_{j}
\end{array} \text { with } a_{k}>b_{k}, \text { for all } 1 \leqslant k \leqslant m .\right.
$$

where $E\left(\boldsymbol{G}_{k}\right)$ is the edge set of the random graph $\boldsymbol{G}_{k}$. For $1 \leqslant k \leqslant n$ we define $\boldsymbol{A}_{k}:=\left(A_{i j}^{(k)}\right) \in \mathbb{R}^{n \times n}$ to be the adjacency matrix of the network $\boldsymbol{G}_{k}$ and define $d_{k}:=\frac{a_{k}+b_{k}}{2}$ to be the average degree parameter.

Further consider the parametrization,

$$
a_{k}=d_{k}+\lambda_{k} \sqrt{d_{k}}, \text { and } b_{k}=d_{k}-\lambda_{k} \sqrt{d_{k}}
$$

where $\lambda_{k} \geqslant 0$ is the signal to noise ratio for the $k$-th network. Henceforth, we work under the sparse regime where $d_{k}>1$ is non-diverging for each graph $\boldsymbol{G}_{k}$. Furthermore, we also observe a $n \times p$ data matrix $\boldsymbol{B}=\left[B_{1}, B_{2}, \cdots, B_{n}\right]^{T} \in \mathbb{R}^{n \times p}$, where the $k^{t h}$ row represents a $p$-dimensional covariate vector obtained from the $k^{t h}$ individual. These covariates are also correlated with the underlying cluster assignment of the subject. Specifically, we assume the following structure on the covariate information,

$$
\begin{equation*}
B_{i}=\sqrt{\frac{\mu}{n}} \sigma_{i} \boldsymbol{u}+\boldsymbol{R}_{i}, i \in[n] \tag{1}
\end{equation*}
$$

where $\boldsymbol{u} \sim \mathrm{N}\left(\mathbf{0}, \boldsymbol{I}_{p \times p}\right)$ is a latent gaussian vector, and $\left\{\boldsymbol{R}_{i}\right\}_{i=1}^{n}$ are IID from $\mathrm{N}\left(\mathbf{0}, \boldsymbol{I}_{p \times p}\right)$. It can be easily seen that by definition, $\boldsymbol{B}$ is independent of $\left\{\boldsymbol{G}_{k}: k \in[m]\right\}$ given the cluster assignment vector $\boldsymbol{\sigma}$. Here, $\mu$ represents the signal to noise ratio of the co-variates. For $m=1$, the above model recovers the contextual stochastic block model of [26]. We assume $\mu \geqslant 0$, and work under the high dimensional proportional regime, where $\frac{n}{p} \rightarrow \gamma \in(0, \infty)$.

### 2.2 Detection Threshold

Let us consider the following hypothesis testing problem,

$$
\begin{equation*}
\boldsymbol{H}_{0}:\left(\lambda_{1}, \cdots, \lambda_{m}, \mu\right)=(0, \cdots, 0,0) \quad \text { vs } \quad \boldsymbol{H}_{1}:\left(\lambda_{1}, \cdots, \lambda_{m}, \mu\right) \neq(0, \cdots, 0,0) \tag{2}
\end{equation*}
$$

Let $\mathbb{P}_{\mathbf{0}, 0}$ refer to the joint distribution of the $m$ networks and the co-variate matrix when $\lambda_{k}=0$ for all $k$ and $\mu=0$. Similarly, let $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ refer to the same joint distribution with non-trivial signal to noise parameters.

Let us observe that the null hypothesis refers to the setup when there is no community structure in any of the observed networks or the data matrix. It is imperative, that for very small values of $\lambda_{i}$ 's and $\mu$, the null will be indistinguishable from the alternative. In such a situation, when the values of $\lambda_{i}$ 's and $\mu$ are all very small, we cannot hope to get any estimator of the community structure which performs better than random guessing. However, with the availability of multiple networks we can afford to have some of the $\lambda_{k}$ 's to be small, as long as the combined effect of all the $\lambda_{k}$ 's and $\mu$ is substantial. We formalize this intuition in the following theorem.

Theorem 2.1. If $\lambda_{1}^{2}+\cdots+\lambda_{m}^{2}+\mu^{2} / \gamma<1$, then $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ is contiguous to $\mathbb{P}_{\mathbf{0}, 0}$. On the other hand, if $\lambda_{1}^{2}+\cdots+\lambda_{m}^{2}+\mu^{2} / \gamma>1$, the distributions are asymptotically mutually singular.

Let us recall that the signal to noise ratio is $\lambda_{k}^{2}$ if we only observe the $k$-th network and is $\mu^{2} / \gamma$ if we only observe the co-variate matrix. So, observing multiple networks and the co-variate matrix significantly boosts the signal to noise ratio compared to the setup when one observes either of the networks or the co-variate matrix. The effective signal to noise ratio of the entire model comprising all the $m$ networks and the data matrix is $\lambda_{1}^{2}+\cdots+\lambda_{m}^{2}+\mu^{2} / \gamma$. Theorem 2.1 implies that if the effective signal to noise ratio is less than 1 , then we cannot have a consistent test between the null and the alternative. We observe that the signal to noise ratio from each of the individual data sources are boosted additively by the combination of different data sources. So, we can now test for the non-triviality of the community structure from the combined information even when it is impossible to test from any of the component information alone.

When the effective signal to noise ratio is larger than 1 we construct a sequence of consistent tests for the hypotheses (2) which exibits the mutual singularity of $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ and $\mathbb{P}_{\mathbf{0}, 0}$. In the following paragraph, we construct a class of cycle statistics on an appropriately defined Factor Graph, $G_{F}=\left(V_{F}, E_{F}\right)$. If $\lambda_{1}^{2}+\cdots+\lambda_{m}^{2}+\mu^{2} / \gamma>1$, this statistic has different distribution under $H_{0}$ and $H_{1}$, with the mean of the statistic significantly elevated under the alternative which provides the above mentioned sequence of consistent tests.


Figure 1: Example of Factor Graph where $V_{0}$ is the vertex set comprising of $n$ variable nodes, $V_{1, k}, 1 \leqslant$ $k \leqslant m$ are the factor nodes corresponding to networks $\boldsymbol{G}_{k}, 1 \leqslant k \leqslant m$ and $V_{2}$ are the factor nodes corresponding to the $p$ covariates, where corresponding edge weights are indicated to the left of such edge.

Cycle Statistics and Detecting Test. Let us consider a factor graph $G_{F}=\left(V_{F}, E_{F}\right)$. Here, the vertex set comprises of $n$ variable nodes, collectively denoted by $V_{0}$, representing the $n$ subjects under study, and $m+1$ layers of factor nodes. For each network $\boldsymbol{G}_{k}$, we have a layer of factor nodes representing each of $A_{i j}^{(k)}$, for all $1 \leqslant i<j \leqslant n$, collectively denoted by $V_{k 1}$. Additionally, we have $p$ factor nodes for the $p$ covariates, denoted by $V_{2}$. Therefore, $V_{F}=V_{0} \cup V_{11} \cup \cdots \cup V_{m, 1} \cup V_{2}$. For all $1 \leqslant i<j \leqslant n$ and $k \in[m]$, the factor node corresponding to $A_{i j}^{(k)}$ is connected to the variable nodes corresponding to $i$ and $j$ where the edge weights are $A_{i j}^{(k)}$. Let the edges between factor nodes of layer $k$ and the variable nodes be denoted by $E_{1_{k}}$. Moreover, the subgraph made of the $p$ factor nodes corresponding to the $p$ covariates and the $n$ variable nodes is a complete bi-partite graph. Let these edges be denoted by $E_{2}$ and each edge weight for edge $e=(i, j)$, where $i \in V_{0}$ and $j \in V_{2}$, be $B_{i j}$. Thus the edge set $E_{F}$ is given by $E_{F}=E_{1_{1}} \cup E_{1_{2}} \cup \cdots \cup E_{1_{m}} \cup E_{2}$. An illustration of this factor graph is given in Figure 1.

Now, let us consider cycles on this factor graph. We observe that all the edges from $E_{1_{k}}$ or $E_{2}$ occur in pairs. Such pairs are referred as wedges. If it is made of two edges of type $E_{1_{k}}$, for $1 \leqslant k \leqslant m$, then we call it a $\boldsymbol{A}_{k}$ type wedge and if it is made of two edges of type $E_{2}$, then we call it a $\boldsymbol{B}$ type wedge. We denote a cycle with $k_{r}$ type $\boldsymbol{A}_{r}$ wedges for $1 \leqslant r \leqslant m$ and $\ell$ type $\boldsymbol{B}$ wedges be denoted by $\omega_{k_{1}, \cdots, k_{m}, \ell}$. Let the subgraph induced by this cycle be denoted as $\left(V_{\omega}, E_{\omega}\right)$. Let the edges of $\omega_{k_{1}, \cdots, k_{m}, \ell}$ which are of type $E_{1_{k}}$ be denoted by $E_{\omega, 1_{k}}$ and those which are of type $E_{2}$ be denoted by $E_{\omega, 2}$. Now, let us consider the following statistic.

$$
Y_{k_{1}, k_{2}, \cdots, k_{m}, \ell}:=\frac{1}{n^{\ell}} \sum_{\omega_{k_{1}}, \cdots, k_{m}, \ell} \prod_{e_{1} \in E_{\omega, 1_{1}}} \cdots \prod_{e_{m} \in E_{\omega, 1_{m}}} \prod_{e_{\ell} \in E_{\omega, 2}} A_{e_{1}}^{(1)} \cdots A_{e_{m}}^{(m)} B_{e_{\ell}}
$$

This statistic counts the number of cycles in the observed networks with $k_{r}$ type $\boldsymbol{A}_{r}$ wedge for $1 \leqslant r \leqslant m$ and $\ell$ type $\boldsymbol{B}$ wedges; weighted by the product of the edge weights of type $E_{2}$ edges and scaled by $n^{\ell}$. The following theorem characterizes the distribution of $Y_{k_{1}, k_{2}, \cdots, k_{m}, \ell}$ under the null and the alternative generative models from (2).

Theorem 2.2. Let $k=k_{1}+\cdots+k_{m}+\ell$. Then as $n \rightarrow \infty$,

1. Under $H_{0}$,

$$
Y_{k_{1}, k_{2}, \cdots, k_{m}, 0} \xrightarrow{d} \operatorname{Poi}\left(\frac{1}{2 k} \frac{k!}{k_{1}!k_{2}!\cdots k_{m}!}\left\{d_{1}^{k_{1}} d_{2}^{k_{2}} \cdots d_{m}^{k_{m}}\right\}\right)
$$

and,

$$
\frac{Y_{k_{1}, k_{2}, \cdots, k_{m}, \ell}}{\sqrt{\frac{1}{2 k \gamma^{\ell}} \frac{k!}{\ell!k_{1}!k_{2}!\cdots k_{m}!} \prod_{j=1}^{m} d_{j}^{k_{j}}}} \stackrel{d}{\rightarrow} \mathrm{~N}(0,1)
$$

2. Under $H_{1}$,

$$
Y_{k_{1}, k_{2}, \cdots, k_{m}, 0} \xrightarrow{d} \operatorname{Poi}\left(\frac{1}{2 k} \frac{k!}{\prod_{j=1}^{m} k_{j}!}\left\{\prod_{j=1}^{m} d_{j}^{k_{j}}+\prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\right\}\right)
$$

and,

$$
\frac{Y_{k_{1}, k_{2}, \cdots, k_{m}, \ell}-\frac{1}{2 k} \frac{k!}{\ell!\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\left(\frac{\mu}{\gamma}\right)^{\ell}}{\sqrt{\frac{1}{2 k \gamma^{\ell}} \frac{k!}{\ell!k_{1}!k_{2}!\cdots k_{m}!} \prod_{j=1}^{m} d_{j}^{k_{j}}}} \stackrel{d}{\rightarrow} \mathrm{~N}(0,1) .
$$

Further $\left\{Y_{\left.k_{1}, \cdots, k_{m}, \ell\right\}}\right.$ are asymptotically independent and the asymptotic distribution continues to hold for $k=O\left(\log ^{1 / 4} n\right)$.

To prove the theorem in the case when $\ell=0$ we rely on the Method of Moments and the techniques described in [31] and for $\ell \neq 0$ we rely on the method of moments and Wick's Formula. We defer the elaborate proofs to the supplemental section.

Contiguity Regime and the Asymptotic Expansion of the Likelihood Ratio. In the contiguity regime, i.e., when $\sum_{i=1}^{m} \lambda_{i}^{2}+\mu^{2} / \gamma<1$, one can find an asymptotic expansion of the log Likelihood ratio as a weighted sum of the graph statistics $\left\{Y_{\left.k_{1}, k_{2}, \cdots, k_{m}, \ell\right\}}\right.$ for different $k_{1}, \cdots, k_{l}$ and $m$. Since these statistics are asymptotically independent by Theorem 2.2 ; this provides an analysis of variance style decomposition of the randomness in the log likelihood ratio. To obtain the expansion, we use the small subgraphs conditioning techniques described by [22] and later extended by [3, 4, 26].

Let us define $\delta_{k_{1}, \cdots, k_{m}}, \lambda_{k_{1}, \cdots, k_{m}}, \mu_{k_{1}, \cdots, k_{m}, \ell}$ and $\sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}$ for $k_{1}, \cdots, k_{m}, \ell \in \mathbb{N} \cup\{0\}$ as follows,

$$
\delta_{k_{1}, \cdots, k_{m}}:=\prod_{j=1}^{m}\left(\frac{a_{i}-b_{i}}{a_{i}+b_{i}}\right)^{k_{j}}, \quad \mu_{k_{1}, \cdots, k_{m}, \ell}:=\frac{1}{2 k} \frac{k!}{\ell!\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\left(\frac{\mu}{\gamma}\right)^{\ell}
$$

and

$$
\lambda_{k_{1}, \cdots, k_{m}}:=\frac{1}{2 k} \frac{k!}{\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m} d_{j}^{k_{j}}, \quad \sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}:=\frac{1}{2 k \gamma^{\ell}} \frac{k!}{\ell!\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m} d_{j}^{k_{j}}
$$

Let $\nu_{\left(k_{1}, \cdots, k_{m}\right)}$ be a sequence of independent $\operatorname{Poisson}\left(\lambda_{k_{1}, \cdots, k_{m}}\right)$ random variables, and $Z_{\left(k_{1}, \cdots, k_{m}, \ell\right)}$ be another sequence of independent $N\left(0, \sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}\right)$ random variables. Then the following theorem describes the asymptotic expansion of the log likelihood ratio.

Theorem 2.3. Consider the sequence of distributions $\mathbb{P}_{n}$, representing the sequence of distributions under $\mathrm{H}_{0}$, and $\mathbb{Q}_{n}$, representing the sequence of distributions under $\mathrm{H}_{1}$. If $\sum_{i=1}^{m} \lambda_{i}^{2}+\mu^{2} / \gamma<1$, then under $\mathbb{P}_{n}$,

$$
\begin{aligned}
& \log \frac{d \mathbb{Q}_{n}}{d \mathbb{P}_{n}} \xrightarrow{d} \sum_{K=1}^{\infty}\left\{\sum_{k_{1}+\cdots+k_{m}=K} \nu_{\left(k_{1}, \cdots, k_{m}\right)} \log \left(1+\delta_{k_{1}, \cdots, k_{m}}\right)-\prod_{j=1}^{m} \lambda_{k_{1}, \cdots, k_{m}} \delta_{k_{1}, \cdots, k_{m}}\right. \\
&\left.+\sum_{\substack{k_{1}+\cdots+k_{m}+\ell=K \\
\ell \neq 0}} \frac{2 \mu_{k_{1}, \cdots, k_{m}, \ell} Z_{\left(k_{1}, \cdots, k_{m}, \ell\right)}-\mu_{k_{1}, \cdots, k_{m}, \ell}^{2}}{2 \sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}}\right\} .
\end{aligned}
$$

Further, there exists a constant $K(\varepsilon, \delta)>0$, such that for all sequence $\left\{n_{k}\right\}$, there exists a further subsequence $\left\{n_{k_{\ell}}\right\}$ such that,

$$
\begin{aligned}
\limsup _{\ell \rightarrow \infty} \mathbb{P}_{n_{k_{\ell}}}\left[\left\lvert\, \log \frac{d \mathbb{Q}_{n}}{d \mathbb{P}_{n}}-\sum_{r=1}^{K}\{ \right.\right. & \sum_{k_{1}+\cdots+k_{m}=r} Y_{n_{k_{\ell}}, k_{1}, \cdots, k_{m}, 0} \log \left(1+\delta_{k_{1}, \cdots, k_{m}}\right)-\prod_{j=1}^{m} \lambda_{k_{1}, \cdots, k_{m}} \delta_{k_{1}, \cdots, k_{m}} \\
& \left.\left.+\sum_{\substack{k_{1}+\cdots+k_{m}+\ell=r \\
\ell \neq 0}} \frac{2 \mu_{k_{1}, \cdots, k_{m}, \ell} Y_{n_{k_{\ell}}, k_{1}, \cdots, k_{m}, \ell}-\mu_{k_{1}, \cdots, k_{m}, \ell}^{2}}{2 \sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}}\right\} \mid \geqslant \varepsilon\right] \leqslant \delta .
\end{aligned}
$$

This theorem decomposes the randomness in the asymptotic log likelihood ratio in terms of the randomness coming from simpler cycle statistics. This serves as a first step towards designing computationally efficient tests achieving same optimal power as the likelihood ratio test for testing the hypotheses defined in (2). We defer the proof of the theorem to the supplement.

### 2.3 Weak Recovery

Next, we turn to problem of weak recovery of the cluster assignment vector $\boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{n}\right)$.
Definition 2.1 ([26, 27]). An estimator $\hat{\boldsymbol{\sigma}}:=\hat{\boldsymbol{\sigma}}\left(\boldsymbol{A}_{1}, \ldots, \boldsymbol{A}_{m}, \boldsymbol{B}\right)$ of $\boldsymbol{\sigma}$ achieves weak recovery under $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ if

$$
\liminf _{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\boldsymbol{\lambda}, \mu}[|\langle\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}\rangle|]>0
$$

Weak recovery is said to be possible under $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ if there exists an estimator $\hat{\sigma}$ that achieves weak recovery under $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$.

In [27], it is shown that the weak recovery threshold coincides with the detection threshold mentioned in Theorem 2.1 for contextual multilayer networks with diverging average degree parameters $d_{k}$ 's. The result below generalizes this to the constant average degree setup of the current paper.
Theorem 2.4. When $\sum_{k=1}^{m} \lambda_{k}^{2}+\mu^{2} / \gamma<1$, for any estimator $\hat{\boldsymbol{\sigma}}$, we have

$$
\frac{1}{n} \mathbb{E}[|\langle\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}\rangle|] \rightarrow 0
$$

i.e., weak recovery is not possible. On the other hand, when $\sum_{k=1}^{m} \lambda_{k}^{2}+\mu^{2} / \gamma>1$, weak recovery is possible.

Under the contiguity regime $\sum_{k=1}^{m} \lambda_{k}^{2}+\mu^{2} / \gamma<1$, the impossibility of weak recovery follows by analyzing the posterior distribution of components $\boldsymbol{\sigma}_{u}$ of $\boldsymbol{\sigma}$ given the data $\boldsymbol{A}_{1}, \ldots, \boldsymbol{A}_{m}, \boldsymbol{B}$, and a disjoint component of $\boldsymbol{\sigma}_{S}$ (where $u \notin S \subset[n]$ ), as in [3, 26].

When $\sum_{k=1}^{m} \lambda_{k}^{2}+\mu^{2} / \gamma>1$, we describe in the following, a quasi-polynomial time algorithm using selfavoiding walks on the factor graph to compute an estimator that achieves weak recovery complementing the result in Theorem 2.4.

Weak recovery via self avoiding walks. Let

$$
\hat{A}_{i_{1}, i_{2}}^{(k)}=\frac{2 n}{a_{k}-b_{k}}\left(A_{i_{1}, i_{2}}^{(k)}-\frac{a_{k}+b_{k}}{2 n}\right)
$$

for each $k=1, \ldots, m$, where $i_{1}, i_{2} \in[n]$, and similarly define

$$
\hat{B}_{i_{1}, i_{2}}^{j}=\frac{n}{\mu} \hat{B}_{i_{1}, j} \hat{B}_{i_{2}, j}=\frac{n}{\mu}\left(\sqrt{\frac{\mu}{n}} \sigma_{i_{1}} u_{j}+Z_{i, j}\right)\left(\sqrt{\frac{\mu}{n}} \sigma_{i_{2}} u_{j}+Z_{i, j}\right) .
$$

The expectation and variance of the above terms are

$$
\begin{array}{ll}
\mathbb{E}\left[\hat{A}_{i_{1}, i_{2}}^{(k)} \mid \sigma\right]=\sigma_{i_{1}} \sigma_{i_{2}}, & \operatorname{Var}\left(\hat{A}_{i_{1}, i_{2}}^{(k)}\right)=\frac{n}{\lambda_{k}^{2}}, \\
\mathbb{E}\left[\hat{B}_{i_{1}, i_{2}}^{j} \mid \sigma\right]=\sigma_{i_{1}} \sigma_{i_{2}}, & \operatorname{Var}\left(\hat{B}_{i_{1}, i_{2}}^{j}\right)=\frac{n p}{\mu^{2} / \gamma} .
\end{array}
$$

We associate the weight $A_{i_{1}, i_{2}}^{(k)}$ to the type $\boldsymbol{A}_{k}$ wedge between variable nodes $i_{1}$ and $i_{2}$, and the weight $\hat{B}_{i_{1}, i_{2}}^{j}$ to the type $\boldsymbol{B}$ wedge between variable nodes $i_{1}$ and $i_{2}$ including the $j^{\text {th }} \boldsymbol{B}$-Type factor node. We call a path $\alpha$ starting at the variable node $i_{1}$ and ending at the variable node $i_{2}$ to be self-avoiding if it visits no factor (type $\boldsymbol{B}$ ) node twice, and does not have two type $\boldsymbol{A}$ wedges between the same pair of variable nodes. The total weight corresponding to $\alpha$ is defined as $p_{\alpha}:=\prod_{e \in \alpha} w(e)$, where $e$ are wedges in the path $\alpha$.

Consider a self-avoiding walk $\alpha$ with $k_{i}$ type $\boldsymbol{A}_{i}$ wedges and $\ell$ type $\boldsymbol{B}$ wedges. Then, we have

$$
\mathbb{E}\left[p_{\alpha} \mid \boldsymbol{\sigma}\right]=\sigma_{i_{1}} \sigma_{i_{2}}, \quad \operatorname{Var}\left(p_{\alpha}\right)=\prod_{j=1}^{m}\left(\frac{n}{\lambda_{j}^{2}}\right)^{k_{j}}\left(\frac{n p}{\mu^{2} / \gamma}\right)^{\ell}(1+o(1)) .
$$

Our estimator of the matrix $\Sigma=\boldsymbol{\sigma} \boldsymbol{\sigma}^{\top}$ is then given by

$$
\hat{\Sigma}_{i_{1} i_{2}}:=\frac{1}{\left|\mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)\right|} \sum_{\alpha \in \mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)} p_{\alpha},
$$

where $\mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)$ is the set of all self avoiding walks between factor nodes $i_{1}$ and $i_{2}$ on the factor graph that have $k_{j}$ type $\boldsymbol{A}_{j}$ edges for $1 \leqslant j \leqslant m$, and $\ell$ type $\boldsymbol{B}$ wedges and $\left|\boldsymbol{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, l\right)\right|$ denotes the cardinality of this set. Then the matrix $\widehat{\Sigma}$ satisfies the following reverse Cauchy Schwartz type inequality.
Lemma 2.1. If $\sum_{k=1}^{m} \lambda_{k}^{2}+\mu^{2} / \gamma \geqslant(1+\varepsilon)$, then there exists a constant $\delta=\delta\left(\lambda_{1}, \cdots, \lambda_{m}, \mu, \gamma, \varepsilon\right)>0$, such that,

$$
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\left\langle\hat{\Sigma}, \boldsymbol{\sigma} \boldsymbol{\sigma}^{\top}\right\rangle\right] \geqslant \delta \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\|\hat{\Sigma}\|_{F}^{2}\right]^{1 / 2}{ }_{n} .
$$

Now, let us consider the matrix $\hat{\Psi}$, the solution of the following convex program.

$$
\begin{aligned}
& \operatorname{minimize}_{\Psi}\|\Psi\|_{F} \\
& \text { s.t. } \operatorname{diag}(\Psi)=1, \\
& \quad\langle\hat{\Sigma}, \Psi\rangle \geqslant \delta n\|\widehat{\Sigma}\|_{F}, \quad \text { and } \quad \Psi \geq 0 .
\end{aligned}
$$

We obtain $\hat{\Psi}$ by solving the convex program involving $\delta$ from Lemma 2.1 and $\hat{\Sigma}$. The estimator for $\boldsymbol{\sigma}$, denoted by $\hat{\boldsymbol{\sigma}}$, is given by co-ordinate wise signs of $\boldsymbol{Z}$, a $n$ dimensional Gaussian vector with mean $\mathbf{0}$ and variance $\Psi$. Then by 2.1, Markov Inequality and Lemma 3.5 of [21] we deduce that if $k=O(\log n)$ with high probability $\hat{\sigma}$ weakly recovers $\boldsymbol{\sigma}$.

Now, let us observe that
where $\widetilde{G}$ is the adjacency matrix of the entire factor graph $G_{F}, i_{0}, \cdots, i_{2 k}$ 's are variable nodes and $f_{i_{0} i_{1}}, \cdots, f_{i_{2 k-1} i_{2 k}}$ 's are factor nodes. Thus computing this is equivalent to computing the matrix power $\widetilde{G}^{2 k}$ plus a checking in each step to ensure the resulting path is self avoiding. This is polynomial time in the number of total nodes, which in turn is of order $n^{k+\ell}$ in the proportional asymptotic regime. One can similarly compute $\sum_{\alpha \in \mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, l\right)} p_{\alpha}$ by considering appropriately weighted adjacency graphs.

Hence, for $k$ and $\ell$ fixed, the matrix $\widehat{\Sigma}$ can be computed in polynomial time. Further, solving the convex program to find $\widehat{\Psi}$ is also a polynomial time exercise and hence we can compute $\widehat{\boldsymbol{\sigma}}$ in $O\left(n^{\log n}\right)$ time. Hence this is a quasi-polynomial time algorithm. As remarked in [21], one can probably improve this to polynomial time by considering non-backtracking walks and color coding, but this is beyond the scope of this article.

## 3 Belief Propagation and MAP Estimation

Let us observe that the quasi-polynomial time algorithm described in the previous section is hard to implement in practice. A commonly used approach in these situations is to consider the maximum a posteriori (MAP) estimate of $\boldsymbol{\sigma}$ given the networks $\boldsymbol{G}_{1}, \cdots, \boldsymbol{G}_{m}$ and $\boldsymbol{B}$. However, this requires marginalization over $\boldsymbol{\sigma} \in\{ \pm 1\}^{n}$ and $\boldsymbol{u} \in \mathbb{R}^{p}$, which is intractable in practice. Hence, we require some type of approximate methods like variational inference or mean field approximation. For approximately tree like graphs, the best local algorithm is Belief Propagation (see, [14]). In the extremely sparse regime that we are considering, each of our networks is approximately tree like, hence we consider the following linearized version of Belief Propagation algorithm. We consider the iterates $T_{i}^{t}, S_{i, \ell}^{t}$ for all $i \in[n]$ and $\ell \in[m] ; S_{i \rightarrow j ; \ell}^{t}$ for all edges $(i, j)$ in $\boldsymbol{G}_{\ell}$; node messages $\eta_{k}^{t}$ for $k \in[n]$; and $m_{q}^{t}$ and $\tau_{q}^{t}$ for $q \in[p]$. They are updated in the following way.

$$
\begin{aligned}
S_{i \rightarrow j ; \ell}^{t+1} & =\sum_{k \in \partial_{\ell} i \backslash\{j\}} f\left(\eta_{k \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{k \in[n]} f\left(\eta_{k}^{t} ; \rho_{n, \ell}\right) \\
S_{i ; \ell}^{t+1} & =\sum_{k \in \partial_{\ell} i} f\left(\eta_{k \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{k \in[n]} f\left(\eta_{k}^{t} ; \rho_{n, \ell}\right) \\
\eta_{k \rightarrow i ; \ell}^{t+1} & =T_{k}^{t}+\sum_{r \neq \ell} S_{k ; r}^{t+1}+S_{k \rightarrow i ; \ell}^{t+1} \\
\eta_{k}^{t+1} & =T_{k}^{t}+\sum_{r=1}^{m} S_{k ; r}^{t+1} \\
\tau_{q}^{t+1} & =\left(1+\mu-\frac{\mu}{p \gamma} \sum_{j \in[n]} B_{q j}^{2} \operatorname{sech}^{2}\left(\eta_{j}^{t}\right)\right)^{-1} \\
m_{q}^{t+1} & =\frac{\sqrt{\mu / \gamma}}{\tau_{q}^{t+1}} \sum_{j \in[n]} \frac{B_{q j}}{\sqrt{p}} \tanh \left(\eta_{j}^{t}\right)-\frac{\mu}{\gamma \tau_{q}^{t+1}}\left[\sum_{j \in[n]} \frac{B_{q j}^{2}}{p} \operatorname{sech}^{2}\left(\eta_{j}^{t}\right)\right] m_{q}^{t-1} \\
T_{k}^{t+1} & =\sqrt{\frac{\mu}{\gamma}} \sum_{r \in[p]} \frac{B_{r k}}{\sqrt{p}} m_{r}^{t+1}-\frac{\mu}{p \gamma}\left(\sum_{r \in[p]} \frac{B_{r k}^{2}}{\tau_{r}^{t+1}}\right) \tanh \left(\eta_{k}^{t}\right) .
\end{aligned}
$$

Here,

$$
\begin{gathered}
f(z ; \rho)=\frac{1}{2} \log \frac{\cosh (z+\rho)}{\cosh (z-\rho)} \\
\rho_{\ell}=\tanh ^{-1}\left(\lambda_{\ell} / \sqrt{d_{\ell}}\right) \text { and } \rho_{n ; \ell}=\tanh ^{-1}\left(\lambda_{\ell} \sqrt{d_{\ell}} /\left(n-d_{\ell}\right)\right)
\end{gathered}
$$

Further, for all $i \in[n], \partial_{\ell} i$ is the neighborhood of $i$ in the $\ell$-th network. This linearized BP is obtained by linearizing the belief propagation update equations around a certain 'zero information' fixed point as in [15]. We consider the following estimate of $\boldsymbol{\sigma}$ denoted by $\hat{\boldsymbol{\sigma}}$ which closely approximates the MAP estimate. We run the algorithm for $T_{\max }$ many iterations and estimate the $i$-th node label by,

$$
\widehat{\sigma}_{i}=\operatorname{sgn}\left(\eta_{i}^{T_{\max }}\right)
$$

In Section 4, we empirically study the performance of this estimator in terms of average overlap with the true vertex labels.


Figure 2: Plots of power of the BP based test and of average overlap of $\hat{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ versus the threshold $\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2}+\mu^{2} / \gamma$.

Remark 3.1. Using the techniques used in [15], this BP algorithm can be further approximated to obtain an approximate linear message passing algorithm as in (13)-(15) of [15]. It is interesting in itself to study the spectrum of the version of non-backtracking walk on the multilayer network obtained by setting $\mu=0$ in this approximated update equations. However, that is beyond the scope of our current work and we relegate it to future work.

## 4 Experiments

First, we consider the testing problem (2). Under $\boldsymbol{H}_{0}$, the norm of $\boldsymbol{\eta}^{T_{\max }}$ is zero. However, as $\lambda_{i}$ 's and $\mu$ increase we expect increase in $\left\|\boldsymbol{\eta}^{T_{\max }}\right\|$. Hence, the test rejecting the null when $\left\|\boldsymbol{\eta}^{T_{\max }}\right\|>\left\|\boldsymbol{\eta}^{0}\right\|$, should be able to distinguish the null from the alternative. Motivated by this intuition, we devise the following experiment.

1. We take $n=300, p=450$ and $T_{\max }=50$. We take $m=3$ and consider $\lambda_{i}=r_{i} \times t_{k}, i \in\{1,2,3\}$ and $\mu=r_{4} \times \gamma \times t_{k}$, where $r_{1}+\cdots+r_{4}=1$, for a sequence $\left(t_{k}\right)_{k=1}^{10}$, chosen uniformly from $(0.5,4)$.
2. We sample $\boldsymbol{G}_{1}, \boldsymbol{G}_{2}, \boldsymbol{G}_{3}$ and $\boldsymbol{B}$ for each of $P_{\boldsymbol{\lambda}, \mu}$ and run the BP algorithm for $T_{\max }$ many iterations with the initialization $S_{i \rightarrow j ; \ell}^{0}, S_{i \rightarrow j ; \ell}^{-1}, S_{i ; \ell}^{0}, S_{i ; \ell}^{-1}, m_{q}^{0}, m_{q}^{-1}, T_{k}^{0}, T_{k}^{-1} \sim \mathrm{~N}(0,0.01)$ and $\tau_{q}^{0}, \tau_{q}^{-1} \sim$ Unif(0.9, 1.3).
3. We approximate the power of the test that rejects when $\left\|\boldsymbol{\eta}^{T_{\max }}\right\|>\left\|\boldsymbol{\eta}^{0}\right\|$ by Monte Carlo method and plot it in Figure 2 as a function of the detection threshold.
4. We also plot the average overlap of $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ as a function of the detection threshold.

This plot shows that, above the threshold the test based on our BP iterates achieves full power and below it the power is less than 1 but increasing. The power function is concave and increasing in the threshold. The overlap is also increasing function of the threshold. The fact that we have reasonable overlap above the threshold implies that using $\hat{\boldsymbol{\sigma}}$ gives an estimator of $\boldsymbol{\sigma}$ which performs quite well in terms of weak recovery.

The experiment was conducted using R in a Windows desktop running on Intel Core i7-10700 @ 2.9 GHz having 8 cores and 16 logical processors.

## 5 Conclusion and Limitations

In this paper, we have found the sharp detection threshold for the underlying community structure in a sparse multilayer contextual stochastic block model. Further, we have demonstrated that the detection threshold coincides with the weak recovery threshold and have provided a quasi polynomial algorithm to achieve weak recovery above the threshold. Finally, we have given an approximate version of Belief Propagation Algorithm to compute the MAP estimate of $\boldsymbol{\sigma}$ and shown that this estimator is also quite good in weakly recovering the true $\sigma$.

The techniques described in this work are limited to the balanced two community networks and are not easily extendable for multiple communities. Various interesting scenarios occur in multi-community setup, for example, each layer of network may be informative about a single community, and all the communities are identifiable only when all the layers are observed [29]. The detection threshold in such problems are interesting directions for future research.

Also it is interesting to analyse the situation when the conditional independence assumption between the layers and the covariates are relaxed. However extending our methods to such setup is not straightforward and beyond the scope of our current work.

Finally, we want to mention that clustering individuals into communities based on their interaction networks, particularly in the context of social media may lead to societal issues, like the filter bubble effect. It is important to consider these aspects while designing efficient algorithms for community detection.

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## A Proof of Theorem 2.1

## A. 1 Proof of the information theoretic lower bound

In this section we show that $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ is contiguous to $\mathbb{P}_{\mathbf{0}, 0}$ when $\frac{\mu^{2}}{\gamma}+\sum_{i=1}^{m} \lambda_{i}^{2}<1$. Consider $\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}$ to be the same distribution conditioned on $\boldsymbol{\sigma}$ and $\boldsymbol{u}$. Given $\delta>0$ define,

$$
\begin{equation*}
\mathscr{S}=\left\{\boldsymbol{u}:\|\boldsymbol{u}\|_{2} \leqslant(1+\delta) \sqrt{p}\right\} \tag{3}
\end{equation*}
$$

Observe that the likelihood ratio between $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ and $\mathbb{P}_{0,0}$ can be written as,

$$
L:=\frac{\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)}=\frac{\mathbb{E}_{\boldsymbol{\sigma}, \boldsymbol{u}}\left[\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\mathbb{A}_{m}, \boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u}\right)\right]}{\mathbb{P}_{\mathbf{0}, 0}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)}
$$

where $\mathbb{A}_{m}=\left\{\boldsymbol{A}_{i}: 1 \leqslant i \leqslant m\right\}$. Now consider the truncated likelihood ratio

$$
L_{t}:=\frac{\mathbb{E}_{\boldsymbol{\sigma}, \boldsymbol{u}}\left[\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\mathbb{A}_{m}, \boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u}\right) \mathbf{1}\{\boldsymbol{u} \in \mathscr{S}\}\right]}{\mathbb{P}_{\mathbf{0}, 0}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)}
$$

Lemma A.1. If there exists $C>0$ such that, $\mathbb{E}_{\boldsymbol{H}_{0}} L_{t}^{2} \leqslant C$, then $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ is contiguous to $\mathbb{P}_{\mathbf{0}, 0}$.
Proof. Let $\left\{E_{n}: n \geqslant 1\right\}$ be any sequence of events such that $\mathbb{P}_{\mathbf{0}, 0}\left(E_{n}\right) \rightarrow 0$, as $n \rightarrow \infty$. Observe that,

$$
\begin{equation*}
\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(E_{n}\right)=\mathbb{E}_{\mathbf{0}, 0}\left[L \mathbf{1}\left\{E_{n}\right\}\right]=\mathbb{E}_{\mathbf{0}, 0}\left[L_{t} \mathbf{1}\left\{E_{n}\right\}\right]+\mathbb{E}_{\mathbf{0}, 0}\left[\left(L-L_{t}\right) \mathbf{1}\left\{E_{n}\right\}\right] . \tag{4}
\end{equation*}
$$

By definition of $L_{t}$ it then follows:

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left[\left(L-L_{t}\right) \mathbf{1}\left\{E_{n}\right\}\right] \leqslant \mathbb{E}_{\mathbf{0}, 0}\left[L-L_{t}\right] \leqslant \mathbb{E}_{\boldsymbol{\sigma}, \boldsymbol{u}}[\mathbf{1}\{\boldsymbol{u} \notin \mathscr{S}\}] \xrightarrow{n \rightarrow \infty} 0 \tag{5}
\end{equation*}
$$

Using Cauchy-Schwarz Inequality along with the bound on $\mathbb{E}_{\boldsymbol{H}_{0}} L_{t}^{2}$ we have,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left[L_{t} \mathbf{1}\left\{E_{n}\right\}\right] \leqslant \sqrt{\mathbb{E}_{\mathbf{0}, 0}\left[L_{t}^{2}\right] \mathbb{P}_{\mathbf{0}, 0}\left(E_{n}\right)} \xrightarrow{n \rightarrow \infty} 0 . \tag{6}
\end{equation*}
$$

Combining (4),(5) and (6) completes the proof.
Proposition A.1. Under $\boldsymbol{H}_{0}, \mathbb{E} L_{t}^{2} \leqslant C_{0}<\infty$ for some universal constant $C_{0}>0$.
Proof. Observe that, by Fubini's Theorem,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0} L_{t}^{2}=\mathbb{E}_{(\boldsymbol{\sigma}, \boldsymbol{u}),(\boldsymbol{\tau}, \boldsymbol{v})}\left[\mathbb{E}_{\mathbf{0}, 0}\left[\frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\mathbb{A}_{m}, \boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}, \boldsymbol{B}\right)} \frac{\left.\mathbb{A}_{m}, \boldsymbol{B} \mid \boldsymbol{\tau}, \boldsymbol{v}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)} \mathbf{1}\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{S}\}\right]\right] \tag{7}
\end{equation*}
$$

where $\boldsymbol{\tau}$ and $\boldsymbol{v}$ are i.i.d copies of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ respectively. Recall that, by construction, given $\{\boldsymbol{\sigma}, \boldsymbol{u}\}, \boldsymbol{A}_{i}{ }^{\prime}$ s for $1 \leqslant i \leqslant m$ are independent and $\boldsymbol{A}_{i}$ 's are mutually independent and independent of $\boldsymbol{B}$. Then,

$$
\frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\mathbb{A}_{m}, \boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)}=\left\{\prod_{i=1}^{m} \frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\boldsymbol{A}_{i} \mid \boldsymbol{\sigma}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\boldsymbol{A}_{i}\right)}\right\} \frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}(\boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u})}{\mathbb{P}_{\mathbf{0}, 0}(\boldsymbol{B})}
$$

Define,

$$
W_{i j}^{(k)}:=W_{i j}^{(k)}\left(\boldsymbol{A}_{k}, \boldsymbol{\sigma}\right)=\left\{\begin{array}{l}
\frac{2 a_{k}}{a_{k}+b_{k}} \text { if } \sigma_{i}=\sigma_{j}, A_{i j}^{(k)}=1 \\
\frac{2 b_{k}}{a_{k}+b_{k}} \text { if } \sigma_{i} \neq \sigma_{j}, A_{i j}^{(k)}=1 \\
\frac{n-a_{k}}{n-\frac{a_{k}+b_{k}}{2}} \text { if } \sigma_{i}=\sigma_{j}, A_{i j}^{(k)}=0 \\
\frac{n-b_{k}}{n-\frac{a_{k}+b_{k}}{2}} \text { if } \sigma_{i} \neq \sigma_{j}, A_{i j}^{(k)}=0
\end{array}\right.
$$

and hence by definition,

$$
\frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\boldsymbol{A}_{k} \mid \boldsymbol{\sigma}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\boldsymbol{A}_{k}\right)}=\prod_{i<j} W_{i j}^{(k)}
$$

By definition of $\boldsymbol{B}$ from (1),

$$
\frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}(\boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u})}{\mathbb{P}_{\mathbf{0}, 0}(\boldsymbol{B})}=\exp \left(\sqrt{\frac{\mu}{n}} \sum_{i=1}^{n} \sigma_{i} \boldsymbol{R}_{i}^{T} \boldsymbol{u}-\frac{\mu}{2}\|\boldsymbol{u}\|_{2}^{2}\right)
$$

For all $1 \leqslant k \leqslant m$ consider $V_{i j}^{(k)}=V_{i j}^{(k)}\left(\boldsymbol{A}_{k}, \boldsymbol{\tau}\right), 1 \leqslant i<j \leqslant n$ to be defined similarly to $W_{i j}, 1 \leqslant i<$ $j \leqslant n$. Let $V_{i j}^{(k)}$ 's and $W_{i j}$ 's be independent. By Lemma 5.4 of [31],

$$
\begin{equation*}
\mathbb{E}\left[\prod_{i<j} W_{i j}^{(k)} V_{i j}^{(k)} \mid \boldsymbol{\sigma}, \boldsymbol{\tau}\right]=(1+o(1)) \exp \left(-\frac{\lambda_{k}^{2}}{2}-\frac{\lambda_{k}^{4}}{4}\right) \exp \left(\frac{\rho^{2} \lambda_{k}^{2}}{2}\left(d_{k}+n\right)\right) \tag{8}
\end{equation*}
$$

where $\rho:=\rho(\boldsymbol{\sigma}, \boldsymbol{\tau})=\frac{1}{n}\langle\boldsymbol{\sigma}, \boldsymbol{\tau}\rangle$. Using MGF of multivariate Gaussian distribution,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left[\left.\exp \left(\sqrt{\frac{\mu}{n}} \sum_{i=1}^{n} \boldsymbol{R}_{i}^{T}\left(\sigma_{i} \boldsymbol{u}+\tau_{i} \boldsymbol{v}\right)-\frac{\mu}{2}\left(\|\boldsymbol{u}\|_{2}^{2}+\|\boldsymbol{v}\|_{2}^{2}\right)\right) \right\rvert\, \boldsymbol{u}, \boldsymbol{v} ; \boldsymbol{\sigma}, \boldsymbol{\tau}\right]=\exp (\mu\langle\boldsymbol{u}, \boldsymbol{v}\rangle \rho) \tag{9}
\end{equation*}
$$

Observe that under $\boldsymbol{H}_{0}, \mathbb{A}_{m}$ and $\boldsymbol{B}$ are independent, hence

$$
\begin{aligned}
\mathbb{E}_{\mathbf{0}, 0} & {\left[\left.\frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\mathbb{A}_{m}, \boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)} \frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\mathbb{A}_{m}, \boldsymbol{B} \mid \boldsymbol{\tau}, \boldsymbol{v}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\mathbb{A}_{m}, \boldsymbol{B}\right)} \mathbf{1}\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{S}\} \right\rvert\, \boldsymbol{u}, \boldsymbol{v} ; \boldsymbol{\sigma}, \boldsymbol{\tau}\right] } \\
\quad= & \mathbf{1}\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{S}\} \mathbb{E}_{\mathbf{0}, 0}\left[\left.\prod_{k=1}^{m} \frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\boldsymbol{A}_{k} \mid \boldsymbol{\sigma}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\boldsymbol{A}_{k}\right)} \frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}\left(\boldsymbol{A}_{k} \mid \boldsymbol{\tau}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\boldsymbol{A} \boldsymbol{A}_{k}\right)} \right\rvert\, \boldsymbol{u}, \boldsymbol{v} ; \boldsymbol{\sigma}, \boldsymbol{\tau}\right] \\
& \mathbb{E}_{\mathbf{0}, 0}\left[\left.\frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}(\boldsymbol{B} \mid \boldsymbol{\sigma}, \boldsymbol{u})}{\mathbb{P}_{\mathbf{0}, 0}(\boldsymbol{B})} \frac{\widetilde{\mathbb{P}}_{\boldsymbol{\lambda}, \mu}(\boldsymbol{B} \mid \boldsymbol{\tau}, \boldsymbol{v})}{\mathbb{P}_{\mathbf{0}, 0}(\boldsymbol{B})} \right\rvert\, \boldsymbol{u}, \boldsymbol{v} ; \boldsymbol{\sigma}, \boldsymbol{\tau}\right] \\
\quad= & \mathbf{1}\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{S}\} \prod_{k=1}^{m} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i<j} W_{i j}^{(k)} V_{i j}^{(k)} \mid \boldsymbol{\sigma}, \boldsymbol{\tau}\right] \\
& \mathbb{E}_{\mathbf{0}, 0}\left[\left.\exp \left(\sqrt{\frac{\mu}{n}} \sum_{i=1}^{n} \boldsymbol{R}_{i}^{T}\left(\sigma_{i} \boldsymbol{u}+\tau_{i} \boldsymbol{v}\right)-\frac{\mu}{2}\left(\|\boldsymbol{u}\|_{2}^{2}+\|\boldsymbol{v}\|_{2}^{2}\right)\right) \right\rvert\, \boldsymbol{u}, \boldsymbol{v} ; \boldsymbol{\sigma}, \boldsymbol{\tau}\right] \\
\quad & (1+o(1)) \mathbf{1}\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{S}\} \exp \left(-\sum_{k=1}^{m} \frac{\lambda_{k}^{2}}{2}-\sum_{k=1}^{m} \frac{\lambda_{k}^{4}}{4}+\sum_{k=1}^{m} \frac{\lambda_{k}^{2} d_{k}^{2}}{2}\right) \\
& \exp \left(n\left(\frac{\rho^{2} \sum_{k=1}^{m} \lambda_{k}^{2}}{2}+\frac{\mu}{\gamma} \rho \frac{\langle\boldsymbol{u}, \boldsymbol{v}\rangle}{p}\right)\right)
\end{aligned}
$$

where the last inequality follows by (9) and (8) and noting that $\rho \leqslant 1$. Plugging the above bound into (7), we obtain,

$$
\begin{align*}
\mathbb{E}_{\mathbf{0}, 0} L_{t}^{2} \leqslant & (1+o(1)) \exp \left(-\sum_{k=1}^{m} \frac{\lambda_{k}^{2}}{2}-\sum_{k=1}^{m} \frac{\lambda_{k}^{4}}{4}+\sum_{k=1}^{m} \frac{\lambda_{k}^{2} d_{k}^{2}}{2}\right) \\
& \mathbb{E}_{(\boldsymbol{\sigma}, \boldsymbol{u}),(\boldsymbol{\tau}, \boldsymbol{v})}\left[\exp \left(n\left(\frac{\rho^{2} \sum_{k=1}^{m} \lambda_{k}^{2}}{2}+\frac{\mu}{\gamma} \rho \frac{\langle\boldsymbol{u}, \boldsymbol{v}\rangle}{p}\right)\right) \mathbf{1}\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{S}\}\right] \tag{10}
\end{align*}
$$

Recall that $\frac{\mu^{2}}{\gamma}+\sum_{i=1}^{m} \lambda_{i}^{2}<1$, then we can choose $\delta>0$, defined in (3), small enough such that $\frac{\mu^{2}}{\gamma}(1+\delta)^{2}+\sum_{i=1}^{m} \lambda_{i}^{2}<1$. Choosing such $\delta>0$ and following the proof of [26, Theorem 1] we have,

$$
\begin{equation*}
\mathbb{E}_{(\boldsymbol{\sigma}, \boldsymbol{u}),(\boldsymbol{\tau}, \boldsymbol{v})}\left[\exp \left(n\left(\frac{\rho^{2} \sum_{k=1}^{m} \lambda_{k}^{2}}{2}+\frac{\mu}{\gamma} \rho \frac{\langle\boldsymbol{u}, \boldsymbol{v}\rangle}{p}\right)\right) \mathbf{1}\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{S}\}\right]<C_{1} \tag{11}
\end{equation*}
$$

for some constant $C_{1}>0$. The proof if completed by substituting the bound from (11) into (10).
Finally observe that applying Lemma A. 1 along with Proposition A. 1 shows that $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ is contiguous to $\mathbb{P}_{\mathbf{0}, 0}$.

## A. 2 Proof of the information theoretic upper bound

In this section, for $\frac{\mu^{2}}{\gamma}+\sum_{i=1}^{m} \lambda_{i}^{2}>1$, we devise a consistent test using the cycle statistics $Y_{n, k_{1}, \cdots, k_{m}, \ell}$. This implies the asymptotic singularity of $\mathbb{P}_{\boldsymbol{\lambda}, \mu}$ and $\mathbb{P}_{\mathbf{0}, 0}$. Recall, $k=\sum_{j=1}^{m} k_{j}+\ell$. By Theorem 2.2, under $\boldsymbol{H}_{0}$, for $k=O\left(\log ^{1 / 4} n\right)$

$$
\frac{Y_{n, k_{1}, \cdots, k_{m}, \ell}}{\sigma_{k_{1}, \cdots, k_{m}, \ell}} \xrightarrow{d} \mathrm{~N}(0,1)
$$

where,

$$
\sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}=\frac{1}{2 k \gamma^{\ell}} \frac{k!}{\ell!k_{1}!k_{2}!\cdots k_{m}!} \prod_{j=1}^{m} d_{j}^{k_{j}}
$$

and under $\boldsymbol{H}_{1}$,

$$
\frac{Y_{n, k_{1}, \cdots, k_{m}, \ell}}{\sigma_{k_{1}, \cdots, k_{m}, \ell}}-\widetilde{\mu}_{k_{1}, \cdots, k_{m}, \ell} \xrightarrow{d} \mathrm{~N}(0,1)
$$

where,

$$
\tilde{\mu}_{k_{1}, \cdots, k_{m}, \ell}=\sqrt{\frac{1}{2 k} \frac{k!}{\ell!\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m}\left(\lambda_{j}^{2}\right)^{k_{j}}\left(\frac{\mu^{2}}{\gamma}\right)^{\ell}}
$$

Now choose,

$$
k_{j}=\left\lfloor\frac{\lambda_{j}^{2} k}{\mu^{2} / \gamma+\sum_{j=1}^{m} \lambda_{j}^{2}}\right\rfloor \text { for all } 1 \leqslant j \leqslant m, \text { and } \ell=k-\sum_{j=1}^{m} k_{j} .
$$

Then using Stirling's approximation,

$$
\begin{aligned}
\widetilde{\mu}_{k_{1}, \cdots, k_{m}, \ell}^{2} & \approx \frac{C_{1}}{2 k} \sqrt{\frac{k}{\ell \prod_{j=1}^{m} k_{j}}}\left(\frac{k}{\ell}\right)^{\ell} \prod_{j=1}^{m}\left(\frac{k}{k_{j}}\right)^{k_{j}} \prod_{j=1}^{m}\left(\lambda_{j}^{2}\right)^{k_{j}}\left(\frac{\mu^{2}}{\gamma}\right)^{\ell} \\
& \geqslant C_{2} \sqrt{\frac{1}{k \ell \prod_{j=1}^{m} k_{j}}}\left(\frac{\mu^{2}}{\gamma}+\sum_{j=1}^{m} \lambda_{j}^{2}\right)^{k} \\
& \geqslant C_{2} \frac{1}{k^{\frac{m+1}{2}}}\left(\frac{\mu^{2}}{\gamma}+\sum_{j=1}^{m} \lambda_{j}^{2}\right)^{k} \xrightarrow{k \rightarrow \infty} \infty
\end{aligned}
$$

where $C_{1}, C_{2}>0$ are universal constants. Thus for $k$ slowly growing in $n$ such that, $k=O\left(\log ^{1 / 4} n\right)$ we get a sequence of consistent tests, completing the proof.

## B Proof of Theorem 2.2

Fix $r>0$. Consider $\left\{\left(k_{j_{1}}, \cdots, k_{j_{m}}, \ell_{j}\right): 1 \leqslant j \leqslant r\right\}$ such that $k_{j}=\sum_{p=1}^{m} k_{j_{p}}+\ell_{j}$ for all $1 \leqslant j \leqslant r$ and $m_{1}, \cdots, m_{r} \geqslant 1$. Without loss of generality suppose there exists $r_{1} \leqslant r$ such that $\ell_{p}=0,1 \leqslant p \leqslant r_{1}$ and $\ell_{p}>0$ for $p>r_{1}$. Further suppose that $k_{1}<\cdots<k_{r_{1}}$ and $k_{r_{1}+1}<\cdots<k_{r}$. We shall show that,

$$
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right] \rightarrow \prod_{j=1}^{r_{1}} \mathbb{E}\left[\nu_{\left(k_{j_{1}}, \cdots, k_{j_{m}}\right)}^{m_{j}}\right] \prod_{j=r_{1}+1}^{r} \mathbb{E}\left[Z_{\left(k_{1}, \cdots, k_{m}, \ell\right)}^{m_{j}}\right]
$$

where $\nu_{\left(k_{j_{1}}, \cdots, k_{j_{m}}\right)}^{m_{j}}$ are $\operatorname{Poisson}\left(\lambda_{k_{j_{1}}, \cdots, k_{j_{m}}}\right)$ and $Z_{\left(k_{1}, \cdots, k_{m}, \ell\right)}$ are $N\left(0, \sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}\right)$. Similarly, we shall also show,

$$
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{j=1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right] \rightarrow \prod_{j=1}^{r_{1}} \mathbb{E}\left[\nu_{\left(k_{j_{1}}, \cdots, k_{j_{m}}\right)}^{m_{j}}\right] \prod_{j=r_{1}+1}^{r} \mathbb{E}\left[\widetilde{Z}_{\left(k_{1}, \cdots, k_{m}, \ell\right)}^{m_{j}}\right]
$$

where $\widetilde{Z}_{\left(k_{1}, \cdots, k_{m}, \ell\right)}$ are $N\left(\mu_{k_{1}, \cdots, k_{m}, \ell}, \sigma_{k_{1}, \cdots, k_{m}, \ell}^{2}\right)$. To show that we need the following lemma.
Lemma B.1. As $n \rightarrow \infty$ we have the following,

$$
\begin{aligned}
& \text { 1. } \mid \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right]-\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{r_{1}} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, 0}^{m_{j}}\right] \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=r_{1}+1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right] \rightarrow 0, \\
& \text { 2. } \mid \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{j=1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right]-\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{j=1}^{r_{1}} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, 0}^{m_{j}}\right] \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{j=r_{1}+1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right] \rightarrow 0 .
\end{aligned}
$$

The proof of Lemma B. 1 is omitted here and given in section B.3. By the decoupling shown in Lemma B.1, under both $\boldsymbol{H}_{0}$ and $\boldsymbol{H}_{1}$, it is enough to analyze the terms with $\ell=0$ and $\ell>0$ separately.

## B. 1 Proof of Theorem 2.2(1)

Let us first consider the $\ell=0$ case. To show Poisson convergence it is enough to show,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left[\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}\right] \xrightarrow{n \rightarrow \infty}\left(\frac{1}{2 k} \frac{k!}{\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m} d_{j}^{k_{j}}\right)^{M}, \text { for all } M \geqslant 1 \tag{12}
\end{equation*}
$$

where $(T)_{[M]}=T(T-1) \cdots(T-M+1)$ for any random variable $T$. Let $Y_{C}$ be the indicator that $C$ is a cycle in the factor graph having $k_{j}$ many type $\boldsymbol{A}_{j}$ wedges. Consider,

$$
\mathscr{C}_{k_{1}, \cdots, k_{m}, 0}=\left\{C: C \text { is a cycle in the factor graph having } k_{j} \text { many type } \boldsymbol{A}_{j} \text { wedges }\right\}
$$

Then it is easy to see that $Y_{n, k_{1}, \cdots, k_{m}, 0}$ can be rewritten as,

$$
\begin{equation*}
Y_{n, k_{1}, \cdots, k_{m}, 0}=\sum_{C \in \mathscr{C}_{k_{1}, \cdots, k_{m}, 0}} Y_{C} \tag{13}
\end{equation*}
$$

By a simple counting argument,

$$
\begin{equation*}
\left|\mathscr{C}_{k_{1}, \cdots, k_{m}, 0}\right|=\binom{n}{k} \frac{(k-1)!}{2} \frac{k!}{\prod_{j=1}^{m} k_{j}!} \tag{14}
\end{equation*}
$$

For any $C \in \mathscr{C}_{k_{1}, \cdots, k_{m}, 0}$ by definition, $\mathbb{E}_{\mathbf{0}, 0} Y_{C}=\prod_{j=1}^{m}\left(d_{j} / n\right)^{k_{j}}$ and hence,

$$
\mathbb{E}_{\mathbf{0}, 0} Y_{n, k_{1}, \cdots, k_{m}, 0}=\binom{n}{k} \frac{(k-1)!}{2} \frac{k!}{\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m}\left(\frac{d_{j}}{n}\right)^{k_{j}}
$$

It is well known that,

$$
\frac{n(n-1) \cdots(n-k+1)}{n^{k}} \rightarrow 1 \text { whenever } k=o(\sqrt{n})
$$

Taking $n \rightarrow \infty$ we get,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0} Y_{n, k_{1}, \cdots, k_{m}, 0}=(1+o(1)) \frac{1}{2 k} \frac{k!}{\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m} d_{j}^{k_{j}} \text { whenever } k=o(\sqrt{n}) \tag{15}
\end{equation*}
$$

Then (15) shows (12) for $M=1$. Define,

$$
\mu_{\mathrm{Poi}, \boldsymbol{H}_{0}}:=\frac{1}{2 k} \frac{k!}{\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m} d_{j}^{k_{j}} .
$$

We now show that

$$
\mathbb{E}_{\mathbf{0}, 0}\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]} \rightarrow \mu_{\mathrm{Poi}, \boldsymbol{H}_{0}}^{M}
$$

By definition, one can see that $\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}$ counts the number of $M$ tuples of cycles $\left(C_{1}, C_{2}, \cdots, C_{M}\right)$, where all $C_{i} \in \mathscr{C}_{k_{1}, \cdots, k_{m}, 0}$ are distinct for all $1 \leqslant i \leqslant M$. First, suppose ( $C_{1}, C_{2}, \cdots, C_{M}$ ) are all vertex disjoint and hence $Y_{C_{i}}$ are independent for $1 \leqslant i \leqslant M$, then,

$$
\begin{equation*}
\mathbb{P}_{\mathbf{0}, 0}\left(C_{i} \in G_{F}, 1 \leqslant i \leqslant M\right)=\mathbb{E}_{\mathbf{0}, 0} \prod_{i=1}^{M} Y_{C_{i}}=\prod_{i=1}^{M} \mathbb{E} Y_{C_{i}}=\left(\prod_{j=1}^{m}\left(d_{j} / n\right)^{k_{j}}\right)^{M}=n^{-k M}\left(\prod_{j=1}^{m} d_{j}^{k_{j}}\right)^{M} \tag{16}
\end{equation*}
$$

where $G_{F}$ is the factor graph. Observe that number of ways to choose such vertex disjoint cycles is given by,

$$
\begin{equation*}
\binom{n}{k M} \prod_{i=0}^{M-1}\left[\binom{k(M-i)}{k} \frac{(k-1)!}{2} \frac{k!}{\prod_{j=1}^{m} k_{j}!}\right]=\frac{n!}{(n-k M)!} \frac{1}{(2 k)^{M}}\left(\frac{k!}{\prod_{j=1}^{m} k_{j}!}\right)^{M} \tag{17}
\end{equation*}
$$

Then as long as $k=o(\sqrt{n})$, the contribution of vertex disjoint cycles in $\mathbb{E}_{\mathbf{0}, 0}\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}$ as $n \rightarrow \infty$ is given by,

$$
\left(\frac{1}{2 k} \frac{k!}{\prod_{j=1}^{m} k_{j}!} \prod_{j=1}^{m} d_{j}^{k_{j}}\right)^{M}=\mu_{\mathrm{Poi}, \boldsymbol{H}_{0}}^{M}
$$

Finally it remains to show that the contribution of $M$ tuples of cycles $\left(C_{1}, \cdots, C_{M}\right)$ such that at least one pair is not vertex disjoint is asymptotically negligible. Consider $\mathscr{C}_{k_{1}, \cdots, k_{m}, 0}^{(M, 2)}$ to be the collection of such $M$ tuples. Then the contribution of $\mathscr{C}_{k_{1}, \cdots, k_{m}, 0}^{(M, 2)}$ in $\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}$ is given by,

$$
\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}^{(2)}:=\sum_{\left(C_{1}, \cdots, C_{M}\right) \in \mathscr{C}_{k_{1}, \cdots, k_{m}, 0}^{(M, 2)}} \prod_{i=1}^{M} Y_{C_{i}}
$$

Now, observe that $\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}^{(2)}$ is stochastically dominated by the same random variable for a ErdősRényi random graph with connection probabilities $\max _{j=1}^{m} d_{j}$. Then, by [7] Chapter 4,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}^{(2)}=o(1) \text { whenever } d=O\left(\log ^{1 / 4} n\right) \tag{18}
\end{equation*}
$$

which completes the proof for (12).

Next, consider $0<\ell<k=\sum_{j=1}^{m} k_{j}+\ell$. Observe that under $\boldsymbol{H}_{0}, \boldsymbol{A}_{s}^{\prime}$ s are independent of $\boldsymbol{B}$. Hence,

$$
\mathbb{E}_{\mathbf{0}, 0}\left[Y_{n, k_{1}, \cdots, k_{m}, \ell}\right]=0
$$

Moving onto the variance calculations we observe that,

$$
\begin{align*}
& \mathbb{E}_{\mathbf{0}, 0}\left[Y_{n, k_{1}, \cdots, k_{m}, \ell}^{2}\right] \\
& \quad=\frac{1}{n^{2 \ell}} \sum_{\omega_{1: 2}} \mathbb{E}_{\mathbf{0}, 0}\left[\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{1}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega_{1}, 2}} B_{e_{\ell}}\right)\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{2}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega_{2}, 2}} B_{e_{\ell}}\right)\right] \tag{19}
\end{align*}
$$

where $\omega_{1: 2}$ is a collection of cycles $\omega_{1}, \omega_{2}$ having $k_{r}$ type $E_{1_{r}}$ wedges for $1 \leqslant r \leqslant m$ and $\ell$ type $E_{2}$ wedges. We decompose (19) as follows,

$$
\mathbb{E}_{\mathbf{0}, 0}\left[Y_{n, k_{1}, \cdots, k_{m}, \ell}^{2}\right]=T_{2}+T_{2}
$$

where

$$
T_{1}=\frac{1}{n^{2 \ell}} \sum_{\omega} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega, 2}} B_{e_{\ell}}^{2}\right]
$$

and

$$
\begin{equation*}
T_{2}=\frac{1}{n^{2 \ell}} \sum_{\omega_{1} \neq \omega_{2}} \mathbb{E}_{\mathbf{0}, 0}\left[\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{1}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega_{1}, 2}} B_{e_{\ell}}\right)\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{2}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega_{2}, 2}} B_{e_{\ell}}\right)\right] \tag{20}
\end{equation*}
$$

Now fix a cycle $\omega$, then by definition of B-type edges,

$$
\begin{aligned}
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega, 2}} B_{e_{\ell}}^{2}\right] & =\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega, 1_{j}}} A_{e_{j}}^{(j)}\right]=\prod_{j=1}^{m} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{e_{j} \in E_{\omega, 1_{j}}} A_{e_{j}}^{(j)}\right] \\
& =\prod_{j=1}^{m}\left(\frac{d_{j}}{n}\right)^{k_{j}}
\end{aligned}
$$

implying that,

$$
T_{1}=\frac{1}{n^{2 \ell}} \sum_{\omega} \prod_{j=1}^{m}\left(\frac{d_{j}}{n}\right)^{k_{j}}
$$

To complete the expression of $T_{1}$ we need to compute number of cycles $\omega$ having $k_{r}$ many type $E_{1_{r}}$ wedges for $1 \leqslant r \leqslant m$ and $\ell$ many type $E_{2}$ wedges. Recall $k=\ell+\sum_{j=1}^{m} k_{j}$ then the number of such cycles can be easily computed to be,

$$
\begin{equation*}
\frac{1}{2}\binom{n}{k}(k-1)!\binom{k}{\ell}\binom{k-\ell}{k_{1}}\binom{k-\ell-k_{1}}{k_{2}} \cdots p^{\ell} \tag{21}
\end{equation*}
$$

Then,

$$
\begin{align*}
T_{1} & =\frac{1}{n^{2 \ell}} \prod_{j=1}^{m}\left(\frac{d_{j}}{n}\right)^{k_{j}} \frac{1}{2}\binom{n}{k}(k-1)!\binom{k}{\ell}\binom{k-\ell}{k_{1}}\binom{k-\ell-k_{1}}{k_{2}} \cdots p^{\ell}  \tag{22}\\
& =\frac{1}{2 k} \prod_{j=1}^{m} d_{j}^{k_{j}} \frac{1}{\gamma^{\ell}} \frac{k!}{\ell!\prod_{j=1}^{m} k_{j}!}(1+o(1)) .
\end{align*}
$$

as long as $k=o(\sqrt{n})$. Now recalling the definition of $T_{2}$ from (20) it is easy to see that the expectation would be 0 unless $\omega_{1} \neq \omega_{2}$ have exactly the same $B$ wedges. Using independence under $\boldsymbol{H}_{0}$ along with the above observation,

$$
\begin{aligned}
T_{2} & =\frac{1}{n^{2 \ell}} \sum_{\substack{\omega_{1} \neq \omega_{2} \\
E_{\omega_{1}, 2}=E_{\omega_{2}, 2}}} \mathbb{E}_{\mathbf{0}, 0}\left[\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{1}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega_{1}, 2}} B_{e_{\ell}}\right)\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{2}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{\ell} \in E_{\omega_{2}, 2}} B_{e_{\ell}}\right)\right] \\
& =\frac{1}{n^{2 \ell}} \sum_{\substack{\omega_{1} \neq \omega_{2} \\
E_{\omega_{1}, 2}=E_{\omega_{2}, 2}}} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{1}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{j} \in E_{\omega_{2}, 1_{j}}} A_{e_{j}}^{(j)}\right] .
\end{aligned}
$$

Fix $\omega_{1} \neq \omega_{2}$ such that $E_{\omega_{1}, 2}=E_{\omega_{2}, 2}$ then suppose $\omega_{1}$ and $\omega_{2}$ share $a$ many $\mathbb{A}$ type wedges, that is,

$$
\left|\bigcup_{i=1}^{2} \bigcup_{j=1}^{m} E_{\omega_{i}, 1_{j}}\right|=2 a
$$

Since $\omega_{1} \neq \omega_{2}$, and they cannot differ in one edge, hence $0 \leqslant a \leqslant k-\ell-2$. Define $b=k-\ell-a$. Define,

$$
\mathscr{X}_{b}=\left\{\omega_{1} \neq \omega_{2}: E_{\omega_{1}, 2}=E_{\omega_{2}, 2} \text { and }\left|\bigcup_{i=1}^{2} \bigcup_{j=1}^{m} E_{\omega_{i}, 1_{j}}\right|=2(k-\ell-b)\right\} .
$$

Then,

$$
\begin{equation*}
T_{2}=\frac{1}{n^{2 \ell}} \sum_{b=2}^{k-\ell} \sum_{\mathscr{X}_{b}} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{1}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{j} \in E_{\omega_{2}, 1_{j}}} A_{e_{j}}^{(j)}\right] \tag{23}
\end{equation*}
$$

Fix $b$ and consider $\omega_{1}, \omega_{2} \in \mathscr{X}_{b}$. Suppose that there are $\delta_{r}$ many common wedges contributed by $\boldsymbol{A}_{r}$ for all $1 \leqslant r \leqslant m$. Then we must have,

$$
\sum_{j=1}^{m} \delta_{j}=k-\ell-b
$$

Recall that under $\boldsymbol{H}_{0}, \boldsymbol{A}_{r}, 1 \leqslant r \leqslant m$ are independent and hence for above choice of $\omega_{1}$ and $\omega_{2}$ we have,

$$
\begin{aligned}
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{1}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{j} \in E_{\omega_{2}, 1_{j}}} A_{e_{j}}^{(j)}\right] & =\prod_{j=1}^{m} \mathbb{E}\left[\prod_{e_{j} \in E_{\omega_{1}, 1_{j}}} A_{e_{j}}^{(j)} \prod_{e_{j} \in E_{\omega_{2}, 1_{j}}} A_{e_{j}}^{(j)}\right] \\
& =\prod_{j=1}^{m}\left(\frac{d^{(j)}}{n}\right)^{\delta_{j}+2\left(k_{j}-\delta_{j}\right)}=O\left(\frac{d}{n}\right)^{k-\ell+b}
\end{aligned}
$$

Hence by (23) we have,

$$
\begin{equation*}
T_{2}=O\left(\frac{1}{n^{2 \ell}} \sum_{b=2}^{k-\ell} \sum_{\mathscr{X}_{b}}\left(\frac{d}{n}\right)^{k-\ell+b}\right) . \tag{24}
\end{equation*}
$$

Now we need an upper bound on $\left|\mathscr{X}_{b}\right|$ for $2 \leqslant b \leqslant k-\ell$. Observe that similar to (21) we can choose the first cycle $\omega_{1}$ in $O\left(n^{k} p^{\ell}\right)$ many ways whenever $k=o(\sqrt{n})$. By definition of $\mathscr{X}_{b}$, the second cycle $\omega_{2}$ can be chosen in $O\left(n^{b-1}\right)$ ways. Hence,

$$
\left|\mathscr{X}_{b}\right|=O\left(n^{k+\ell+b-1}\right)
$$



Figure 3: An example of construction of quotient graph by identifying the $\mathbb{A}$ type blocks to be a single vertex (quotient operation vertices) in the newly constructed graph.

Recalling $k=o(\sqrt{\log n})$, then by (24) we conclude that $T_{2}=o(1)$. Finally combining with (22) we have,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left[Y_{n, k_{1}, \cdots, k_{m}, \ell}^{2}\right] \xrightarrow{n \rightarrow \infty} \frac{1}{2 k} \prod_{j=1}^{m} d_{j}^{k_{j}} \frac{1}{\gamma^{\ell}} \frac{k!}{\ell!\prod_{j=1}^{m} k_{j}!} \tag{25}
\end{equation*}
$$

Now to show asymptotic Gaussianity of $Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, \ell_{j}, r_{1}+1 \leqslant j \leqslant r \text { we show that the limits of the }}$ moments satisfy Wick's formula, that is we will show that for all $\zeta \in \mathbb{N}, T_{n, i} \in\left\{Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, \ell_{j}}: r_{1}+1 \leqslant\right.$ $j \leqslant r\}, i \in[\zeta]$,

$$
\mathbb{E}\left[\prod_{\nu=1}^{\zeta} T_{n, \nu}\right]= \begin{cases}\sum_{\eta} \prod_{i=1}^{\zeta / 2} \mathbb{E}\left[T_{n, \eta(i, 1)} T_{n, \eta(i, 2)}\right]+o(1) & \text { if } \zeta \text { is even }  \tag{26}\\ o(1) & \text { otherwise }\end{cases}
$$

where $\eta$ is a partition of [ $\zeta$ ] into $\frac{\zeta}{2}$ blocks of size two and for $j \in\{1,2\}, \eta(i, j)$ denotes the $j^{\text {th }}$ element of the $i^{t h}$ block of $\eta$.

Fix $\zeta \in \mathbb{N}$, and consider a choice $\in\left\{Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, \ell_{j}}: r_{1}+1 \leqslant j \leqslant r\right\}$ for $1 \leqslant i \leqslant \zeta$. For notational convenience in the following we will consider $T_{n, i}$ to have cycles with $k_{i_{s}}$ many $\boldsymbol{A}_{s}$ many wedges for $1 \leqslant s \leqslant m$ and $\ell_{i}$ many $\boldsymbol{B}$ type wedges. Then by definition,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{\nu=1}^{\zeta} T_{n, \nu}\right]=\frac{1}{n^{\sum_{i=1}^{\zeta} l_{i}}} \sum_{\omega_{1: \zeta}} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i=1}^{\zeta}\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{i}, 1_{j}}} A_{e_{j}}^{(j)}\right)\right] \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i=1}^{\zeta}\left(\prod_{e_{2} \in E_{\omega_{i}, 2}} B_{e_{2}}\right)\right] \tag{27}
\end{equation*}
$$

where $\omega_{1: \zeta}$ is a collection of cycles $\omega_{1}, \cdots, \omega_{\zeta}$ on the factor graph such that $\omega_{i}$ has $k_{i_{s}}$ many wedges coming from $\boldsymbol{A}_{s}$ for all $1 \leqslant s \leqslant m$ and $\ell_{i}$ many $\boldsymbol{B}$ type wedges with $x_{i}$ contiguous block of $\mathbb{A}$ type wedges and $\boldsymbol{B}$ type wedges where in a cycle $\omega$ we call a block of wedges to be $\mathbb{A}$ type wedges if there are no $\boldsymbol{B}$ type wedge in that block. Further consider $\Omega$ to be the collection of all such $\omega_{1: \zeta}$.

Given a cycle $\omega$ consider $\mathscr{G}(\omega)$ to be the graph corresponding to it. Suppose that $\omega$ has $x$ contiguous blocks of $\mathbb{A}$ type and $\boldsymbol{B}$ type wedges and for all $1 \leqslant j \leqslant x$ consider $\mathscr{\varphi}_{\mathbb{A}}(\omega, j)$ to be the $j^{\text {th }}$ block of $\mathbb{A}$ type wedges. Now construct a quotient graph $\mathscr{G}_{Q}(\omega)$ by identifying the block $\mathscr{G}_{\mathbb{A}}(\omega, j)$ to be a single vertex for all $1 \leqslant j \leqslant x$. We call the above construction a quotient operation. An example of quotient graph is given in Figure 3.

By construction, the $\boldsymbol{B}$ type vertices as well as edges remains unchanged in the quotient graph $\mathscr{G}_{Q}(\omega)$
and hence (27) becomes,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{\nu=1}^{\zeta} T_{n, \nu}\right]=\frac{1}{n^{\sum_{i=1}^{\zeta} l_{i}}} \sum_{\omega_{1: \zeta}} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i=1}^{\zeta}\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{i}, 1_{j}}} A_{e_{j}}^{(j)}\right)\right] \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i=1}^{t}\left(\prod_{e \in E\left(\mathscr{G}_{Q}\left(\omega_{i}\right)\right)} B_{e}\right)\right] \tag{28}
\end{equation*}
$$

where $E\left(\mathscr{G}_{Q}\left(\omega_{i}\right)\right)$ denotes the edges of the factor graph $\mathscr{G}_{Q}\left(\omega_{i}\right)$ constructed from $\mathscr{G}\left(\omega_{i}\right)$ for all $1 \leqslant i \leqslant \zeta$. Now consider the following equivalence relation among the vertices formed by the quotient operation. For two cycles $\omega_{i_{1}}$ and $\omega_{i_{2}}$ if the first and the last vertices of $\mathscr{G}_{\mathbb{A}}\left(\omega_{i_{1}}, j_{1}\right)$ and $\mathscr{G}_{\mathbb{A}}\left(\omega_{i_{2}}, j_{2}\right)$ are the same for some $1 \leqslant j_{1} \leqslant x_{i_{1}}$ and $1 \leqslant j_{2} \leqslant x_{i_{2}}$, then we consider the corresponding vertices of $\mathscr{G}_{Q}\left(\omega_{i_{1}}\right)$ and $\mathscr{G}_{Q}\left(\omega_{i_{2}}\right)$ to be the same. For each $\omega_{1: \zeta}$ consider $\mathscr{G}\left(\omega_{1: \zeta}, \mathbb{A}\right)$ to be the collection of $\mathbb{A}$ type wedges coming from $\omega_{1: \zeta}$ and $\mathscr{G}_{Q}\left(\omega_{1: \zeta}\right)$ to be the collection of quotient graph constructed from $\omega_{i}, 1 \leqslant i \leqslant \zeta$. Further consider,

$$
\mathbb{G}(\mathbb{A})=\left\{\mathscr{G}\left(\omega_{1: \zeta}, \mathbb{A}\right): \omega_{1: \zeta} \in \Omega\right\}
$$

and given $g \in \mathbb{G}(\mathbb{A})$ let $\omega_{1: \zeta}^{(g)}:=\left(\omega_{1}^{(g)}, \cdots, \omega_{\zeta}^{(g)}\right) \in \Omega$ be such that $\mathscr{G}\left(\omega_{1: \zeta}^{(g)}, \mathbb{A}\right)=g$. Observe that by construction $\omega_{g}$ is not unique. Then from (28) we have,

$$
\left.\left.\begin{array}{rl}
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{\nu=1}^{\zeta} T_{n, \nu}\right]=\frac{1}{n^{\sum_{i=1}^{\zeta} l_{i}}} & \left(\sum_{g \in \mathbb{G}(\mathbb{A})} \mathbb{E}_{\mathbf{0}, 0}\right.
\end{array}\right]\left[\prod_{i=1}^{\zeta}\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{i}^{(g)}, 1_{j}}} A_{e_{j}}^{(j)}\right)\right]\right), ~\left(\sum_{\mathscr{G}_{Q}\left(\omega_{1: \zeta)}\right.} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i=1}^{t}\left(\prod_{e \in E\left(\mathscr{G}_{Q}\left(\omega_{i}\right)\right)} B_{e}\right)\right]\right)
$$

Now consider,

$$
\mathbb{G}_{1}(\mathbb{A})=\left\{g \in \mathbb{G}(\mathbb{A}): \text { no overlap among } \mathbb{A} \text { type edges in } \omega_{i}^{(g)}, 1 \leqslant i \leqslant \zeta\right\} \text { and } \mathbb{G}_{2}(\mathbb{A})=\mathbb{G}(\mathbb{A}) \backslash \mathbb{G}_{1}(\mathbb{A})
$$

It is easy to observe that for $\omega_{1: \zeta} \in \mathbb{G}_{1}(\mathbb{A})$

$$
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i=1}^{\zeta}\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{i}, 1_{j}}} A_{e_{j}}^{(j)}\right)\right]=O\left(\frac{d}{n}\right)^{\sum_{i=1}^{\zeta} \sum_{j \leqslant x_{i}} \beta_{j}^{(i)}}
$$

where $\beta_{j}^{(i)}$ is the number of wedges in the $j^{\text {th }}$ block of $\omega_{i}$ and $d=\max _{j=1}^{m} d_{j}$. Further,

$$
\left|\mathbb{G}_{1}(\mathbb{A})\right|=O(n)^{\sum_{i=1}^{\zeta} \sum_{j \leqslant x_{i}}\left(\beta_{j}^{(i)}+1\right)}
$$

which follows since in a $\mathbb{A}$ type block having $\beta$ wedges, there must be $\beta+1$ variable nodes of which there are $O(n)$ many options (Note that for every wedge we have $m$ many options which is constant in $n$ ). Thus the total contribution coming from non-overlapping cycles is,

$$
\begin{equation*}
O\left(\frac{d}{n}\right)^{\sum_{i=1}^{\zeta} \sum_{j \leqslant x_{i}} \beta_{j}^{(i)}} O(n)^{\sum_{i=1}^{\zeta} \sum_{j \leqslant x_{i}}\left(\beta_{j}^{(i)}+1\right)} \tag{30}
\end{equation*}
$$

Notice that the dominant contribution comes from $\mathbb{G}_{1}(\mathbb{A})$, because if we consider cycles having $\xi$ many overlapping wedges then, we gain a term of $O\left(n^{\xi}\right)$ in computing the expectation, while we lose a term of $O\left(n^{\xi+1}\right)$ while counting above number of cycles as overlap in $\xi$ many wedges implies we must have overlap in $\xi+1$ many vertices. Hence,

$$
\sum_{g \in \mathbb{G}(\mathbb{A})} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{i=1}^{\zeta}\left(\prod_{j=1}^{m} \prod_{e_{j} \in E_{\omega_{i}^{(g)}, 1_{j}}} A_{e_{j}}^{(j)}\right)\right]=O\left(\frac{d}{n}\right)^{\sum_{i=1}^{\zeta} \sum_{j \leqslant x_{i}} \beta_{j}^{(i)}} O(n)^{\sum_{i=1}^{\zeta} \sum_{j \leqslant x_{i}}\left(\beta_{j}^{(i)}+1\right)}
$$

Next we investigate the term contributed by $\boldsymbol{B}$ type wedges. Define,

$$
\mathbb{G}_{W}(\boldsymbol{B})=\left\{\mathscr{G}_{Q}\left(\omega_{1: \zeta}\right): \forall e \in \bigcup_{i=1}^{\zeta} E\left(\mathscr{G}_{Q}\left(\omega_{i}\right)\right), \exists i_{1} \neq i_{2} \text { such that } e \in E\left(\mathscr{G}_{Q}\left(\omega_{i_{c}}\right)\right), c=1,2\right\} .
$$

Then it is easy to observe that only the contribution of $\mathbb{G}_{w}(\boldsymbol{B})$ is non-zero. Now, for every $\omega_{1: \zeta} \in \mathbb{G}_{W}(\boldsymbol{B})$ we consider a partition $\eta\left(\omega_{1: \zeta}\right)$ of [ $\zeta$ ] as follows, if $a$ and $b$ are in the same partition of $\eta\left(\omega_{1: \zeta}\right)$ then $\mathscr{C}_{Q}\left(\omega_{a}\right)$ and $\mathscr{G}_{Q}\left(\omega_{b}\right)$ share at least one edge. By considering the decomposition from (29) along with the bounds in (30) and the collection $\mathbb{G}(\boldsymbol{B})$, it is easy to see that (26) follows from the proof of Proposition 2 in [26]. The variance of the asymptotic Gaussian distribution was identified in (25). Finally we are left with verifying asymptotic independence. Observe that it is enough to show,

$$
\mathbb{E}_{\mathbf{0}, 0}\left[Y_{n, k_{i_{1}, \cdots, i_{m}, \ell_{i}}} Y_{n, k_{j_{1}, \cdots, j_{m}, \ell_{j}}}\right] \rightarrow 0
$$

whenever $r_{1}+1 \leqslant i \neq j \leqslant r$. Note that the above expectation is 0 is $\ell_{i} \neq \ell_{j}$. Then it suffices to consider $l_{i}=l_{j}$. Without loss of generality suppose $k_{i}<k_{j}$, then

$$
\mathbb{E}_{\mathbf{0}, 0}\left[Y_{n, k_{i_{1}, \cdots, i_{m}, \ell_{i}}} Y_{n, k_{j_{1}, \ldots, j_{m}, \ell_{j}}}\right]=\frac{1}{n^{2 \ell_{i}}} \sum_{\omega_{i}, \omega_{j}} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{s=1}^{m}\left(\prod_{e \in E_{\omega_{i}, 1_{s}}} A_{e}^{(s)}\right)\left(\prod_{e \in E_{\omega_{j}, 1_{s}}} A_{e}^{(s)}\right)\right]
$$

where for $s=i, j, \omega_{s}$ is a cycle having $k_{s_{a}}$ many type $\boldsymbol{A}_{s}$ wedges for all $1 \leqslant a \leqslant m$ and $l_{s}$ many type $\boldsymbol{B}$ wedges. Further the sum is over all those cycles $\omega_{i}, \omega_{j}$ who intersect on all the $l_{i}=l_{j}$ type $\boldsymbol{B}$ wedges. As before one can show that the dominant term is provided by cycles having no overlapping type $\mathbb{A}$ edges and hence,

$$
\mathbb{E}_{\mathbf{0}, 0}\left[Y_{n, k_{i_{1}, \ldots, i_{m}, \ell_{i}}} Y_{n, k_{j_{1}, \ldots, j_{m}, \ell_{j}}}\right]=n^{-2 \ell_{i}} O\left(\frac{d}{n}\right)^{k_{i}+k_{j}-2 \ell_{i}} O(n)^{k_{i}+k_{j}-\ell_{i}-1}=o(1)
$$

which holds whenever $k_{i}, k_{j}=o(\sqrt{\log n})$ and hence the proof is completed under $\boldsymbol{H}_{0}$.

## B. 2 Proof of Theorem 2.2(2)

Once again let us first consider the case when $\ell=0$. We shall show that as $n \rightarrow \infty$,

$$
\begin{equation*}
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]} \rightarrow\left(\frac{1}{2 k} \frac{k!}{\prod_{j=1}^{m} k_{j}!}\left\{\prod_{j=1}^{m} d_{j}^{k_{j}}+\prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\right\}\right)^{M} \tag{31}
\end{equation*}
$$

This, as was under $\boldsymbol{H}_{0}$, will imply the Poisson convergence. Recalling notations from (13) we have,

$$
Y_{n, k_{1}, \cdots, k_{m}, 0}=\sum_{C \in \mathscr{C}_{k_{1}}, \cdots, k_{m}, 0} Y_{C} .
$$

Let $\rho_{j}$ be the number of type $\boldsymbol{A}_{j}$ wedge such that the two variable nodes on its either sides have different community labels. Let us observe that, since the subgraph is a cycle, $\rho_{1}+\cdots+\rho_{m}$ is even. By the argument of Lemma 3.3 of [30],

$$
\begin{align*}
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[Y_{C}\right] & =n^{-k} 2^{-k} \sum_{\rho_{1}+\cdots+\rho_{m} \text { even }} \prod_{j=1}^{m}\binom{k_{j}}{\rho_{j}} a_{j}^{k_{j}-m_{j}} b_{j}^{m_{j}} \\
& =n^{-k} 2^{-k}\left(\prod_{j=1}^{m}\left(a_{j}+b_{j}\right)^{k_{j}}+\prod_{j=1}^{m}\left(a_{j}-b_{j}\right)^{k_{j}}\right) \\
& =n^{-k}\left(\prod_{j=1}^{m} d_{j}^{k_{j}}+\prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\right) \tag{32}
\end{align*}
$$

By (14) and (32) we get,

$$
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[Y_{n, k_{1}, \cdots, k_{m}, 0}\right]=(1+o(1)) \frac{1}{2 k} \frac{k!}{k_{1}!\cdots k_{m}!}\left(\prod_{j=1}^{m} d_{j}^{k_{j}}+\prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\right) \text { whenever } k=o(\sqrt{n})
$$

which proves (31) for $M=1$. Define,

$$
\mu_{\mathrm{Poi}, \boldsymbol{H}_{1}}=\frac{1}{2 k} \frac{k!}{k_{1}!\cdots k_{m}!}\left(\prod_{j=1}^{m} d_{j}^{k_{j}}+\prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\right)
$$

For $M>1$ we now show that,

$$
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]} \rightarrow \mu_{\mathrm{Poi}, \boldsymbol{H}_{\boldsymbol{1}}}^{M}
$$

Considering the $M$ tuple of vertex disjoint cycles $\left(C_{1}, \cdots, C_{M}\right)$ similar to (16) and using (32) we get,

$$
\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(C_{i} \in G_{F}, 1 \leqslant i \leqslant M\right)=n^{-k M}\left[\left(\prod_{j=1}^{m} d_{j}^{k_{j}}+\prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\right)\right]^{M} .
$$

Recalling the number of possible choices for such $M$ tuple of vertex disjoint cycles from (17) shows that contribution of collection of $M$ tuple of vertex disjoint cycles in $\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left(Y_{n, k_{1}, \cdots, k_{m}, 0}\right)_{[M]}$ as $n \rightarrow \infty$ is given by,

$$
\left(\frac{1}{2 k} \frac{k!}{k_{1}!\cdots k_{m}!}\left(\prod_{j=1}^{m} d_{j}^{k_{j}}+\prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\right)\right)^{M}=\mu_{\mathrm{Poi}, \boldsymbol{H}_{1}}^{M} \text { whenever } k=o(\sqrt{n})
$$

Considering the Erdős-Rényi random graph with edge probability $\max _{j=1}^{m}\left\{a_{i} / n, b_{i} / n\right\}$ and repeating the stochastic dominance argument from (18) shows that the contribution of $M$ tuple of vertex-overlapping cycles is asymptotically negligible completing the proof of (31).

Next, we consider the case when $\ell \neq 0$. First, we try to compute the mean of the cycle statistics. Let us observe that,

$$
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[B_{i_{1} j_{1}} B_{i_{2} j_{1}}\right]=\frac{\mu}{n} \sigma_{i_{1}} \sigma_{i_{2}}
$$

for any $\boldsymbol{B}$ wedge $\left(i_{1}, j_{1}, i_{2}\right)$. Now let us fix a cycle $\omega$ with $k_{1}$ type 1 wedge, $k_{2}$ type 2 wedge up to $k_{m}$ type $m$ wedge and $\ell$ type $\boldsymbol{B}$ wedges. For this cycle,
$\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{k=1}^{m} \prod_{e_{k} \in E_{\omega, 1_{k}}} \prod_{e_{\ell} \in E_{\omega, 2}} A_{e_{k}}^{(k)} B_{e_{\ell}} \mid \boldsymbol{\sigma}\right]=\left[\prod_{k=1}^{m} \prod_{\left(e_{k}^{+}, e_{k}^{-}\right) \in E_{\omega, 1}}\left(\frac{d_{k}+\lambda_{k} \sqrt{d_{k}} \sigma_{e_{k}^{+}} \sigma_{e_{k}^{-}}}{n} \prod_{\left(e_{\ell}^{+}, e_{\ell}^{-}\right) \in E_{\omega, 2}} \frac{\mu}{n} \sigma_{e_{\ell}^{+}} \sigma_{e_{\ell}^{-}}\right]\right.$.

Hence,

$$
\begin{aligned}
& \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[Y_{n, k_{1}, \cdots, k_{m}, \ell}\right]=\frac{1}{n^{\ell}} \sum_{\omega} \mathbb{E}_{\boldsymbol{\sigma}}\left[\prod_{k=1}^{m} \prod_{\left(e_{k}^{+}, e_{k}^{-}\right) \in E_{\omega, 1_{k}}}\left(\frac{d_{k}+\lambda_{k} \sqrt{d_{k}} \sigma_{e_{k}^{+}} \sigma_{e_{k}^{-}}}{n} \prod_{\left(e_{\ell}^{-}, e_{\ell}^{-}\right) \in E_{\omega, 2}} \frac{\mu}{n} \sigma_{e_{\ell}^{+}} \sigma_{e_{\ell}^{-}}\right]\right. \\
& =\frac{\mu^{\ell}}{n^{k+\ell}} \sum_{\omega} \mathbb{E}_{\boldsymbol{\sigma}}\left[\sum_{\substack{z_{e_{k}} \in\left\{d_{k}, \lambda_{k} \sqrt{d_{k}} \sigma_{e_{e}^{+}} \\
k \in\{1, \cdots, M\}\right.}} \prod_{k=1}^{M} \prod_{e_{k} \in E_{\omega, 1}} z_{e_{k}} \prod_{e_{e} \in E_{\omega, 2}} \sigma_{e_{\ell}^{+}} \sigma_{e_{\ell}^{-}}\right] \\
& \stackrel{(1)}{=} \frac{\mu^{\ell}}{n^{k+\ell}}\binom{n}{k} \frac{k!}{k_{1}!\cdots k_{m}!\ell!} \frac{(k-1)!}{2} p^{\ell} \prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}} \\
& =\left(\frac{\mu}{\gamma}\right)^{\ell} \frac{1}{2 k} \frac{k!}{k_{1}!\cdots k_{m}!\ell!} \prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}(1+o(1)),
\end{aligned}
$$

where equality (1) follows because the only contribution comes from the terms where all $z_{e_{k}}$ 's are $\lambda_{k} \sqrt{d_{k}} \sigma_{e_{k}^{+} e_{k}^{-}}$. Next, we observe that under $\boldsymbol{H}_{1}$, for all $\omega$ with $k_{j}$ type $j$ wedges for $k=1, \cdots, M$ and $\ell$ type $\boldsymbol{B}$ wedges and $(i, j) \in E_{\omega, 2}$,

$$
B_{i j}=X_{i j}+Z_{i j}, \quad \text { where } X_{i j}=\sqrt{\mu / n} \sigma_{i} u_{j} \text { and } Z_{i j} \text { are i.i.d Gaussians. }
$$

Thus we have,

$$
\begin{aligned}
Y_{n, k_{1}, \cdots, k_{m}, \ell} & =\frac{1}{n^{\ell}} \sum_{\omega} \prod_{k=1}^{M} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \prod_{e_{\ell} \in E_{\omega, 2}} B_{e_{\ell}} \\
& =T_{1}+T_{2}+T_{3}
\end{aligned}
$$

where

$$
\begin{aligned}
& T_{1}=\frac{1}{n^{\ell}} \sum_{\omega} \prod_{k=1}^{M} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \prod_{e_{\ell} \in E_{\omega, 2}} Z_{e_{\ell}}, \\
& T_{2}=\frac{1}{n^{\ell}} \sum_{\omega} \prod_{k=1}^{M} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \prod_{e_{\ell} \in E_{\omega, 2}} X_{e_{\ell}}
\end{aligned}
$$

and

$$
T_{3}=Y_{n, k_{1}, \cdots, k_{m}, \ell}-T_{1}-T_{2} .
$$

Let us observe that $T_{1}$ can be analyzed in the same way as in $\boldsymbol{H}_{0}$. Hence,

$$
T_{1} \xrightarrow{d} N\left(0, \frac{1}{2 k} \frac{k!}{k_{1}!\cdots k_{j}!\ell!} \prod_{j=1}^{m} d_{j}^{k_{j}}\right) .
$$

Next, we shall show that,

$$
T_{2} \xrightarrow{P} \frac{1}{2 k} \frac{k!}{k_{1}!\cdots k_{j}!\ell!} \prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}\left(\frac{\mu}{\gamma}\right)^{\ell}
$$

Finally, we shall show $T_{3} \xrightarrow{P} 0$. This will complete the proof of the theorem. Let us begin with $T_{3}$. We observe that $E_{\boldsymbol{\lambda}, \mu}\left[T_{3}\right]=0$. So it suffices to show that $\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[T_{3}^{2}\right]=o(1)$ as $n \rightarrow \infty$. Let us express $T_{3}=\sum_{\omega} V_{n, k_{1}, \cdots, k_{m}, \ell, \omega}$, where

$$
V_{n, k_{1}, \cdots, k_{m}, \ell, \omega}=\frac{1}{n^{\ell}} \prod_{k=1}^{m} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \sum_{E_{\omega, f} \subseteq E_{\omega, 2}} \prod_{e_{\ell} \in E_{\omega, f}} X_{e_{\ell}} \prod_{e_{\ell} \in E_{\omega, 2} \backslash E_{\omega, f}} Z_{e_{\ell}} .
$$

Now by argument similar to proof of Proposition 1, Calculations under $\boldsymbol{H}_{1}$ of [26] we can show $T_{3} \xrightarrow{P} 0$. Now we try to analyze $T_{2}$. Let us observe that, since,

$$
\begin{gathered}
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[Y_{n, k_{1}, \cdots, k_{m}, \ell}\right]=\left(\frac{\mu}{\gamma}\right)^{\ell} \frac{1}{2 k} \frac{k!}{k_{1}!\cdots k_{m}!\ell!} \prod_{j=1}^{m}\left(\lambda_{j} \sqrt{d_{j}}\right)^{k_{j}}(1+o(1)), \\
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[T_{1}\right]=o(1) \quad \text { and } \quad \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[T_{3}\right]=o(1)
\end{gathered}
$$

It is enough to show,

$$
\operatorname{Var}\left(T_{2}\right)=o(1)
$$

For a fixed set of type $B$ factor nodes $j_{1}, \cdots, j_{\ell}$, let us denote all cycles with $k_{j}$ type $j$ wedges and $\ell$ type B wedges involving the above mentioned factor nodes by $\mathscr{C}_{j_{1}: j_{\ell}}$. Let us observe that,

$$
T_{2}=\frac{1}{n^{\ell}}\left(\sum_{\omega \in \mathscr{C}_{j_{1}: j_{\ell}}} \prod_{k=1}^{m} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \prod_{e_{\ell} \in E_{\omega, 2}} \sigma_{e_{\ell}}^{+} \sigma_{e_{\ell}}^{-}\right)\left(\left(\frac{\mu}{n}\right)^{\ell} \sum_{j_{1}: j_{\ell}} \prod_{h=1}^{\ell} u_{j_{h}}^{2}\right) .
$$

As $u_{j_{h}}$ 's are independent, standard Gaussians; by the law of large numbers,

$$
\frac{1}{n^{\ell}} \sum_{j_{1}: j_{\ell}} \prod_{h=1}^{\ell} u_{j_{h}}^{2} \xrightarrow{P} 1
$$

So, it is enough to show,

$$
\operatorname{Var}_{\boldsymbol{\lambda}, \mu}\left(\frac{1}{n^{\ell}}\left(\sum_{\omega \in \mathscr{C}_{j_{1}: j_{\ell}}} \prod_{k=1}^{m} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \prod_{e_{\ell} \in E_{\omega, 2}} \sigma_{e_{\ell}}^{+} \sigma_{e_{\ell}}^{-}\right)\right)=o(1)
$$

or equivalently,

$$
\begin{aligned}
& \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\frac{1}{n^{2 \ell}}\left(\sum_{\omega \in \mathscr{C}_{j_{1}: j_{\ell}}} \prod_{k=1}^{m} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \prod_{e_{\ell} \in E_{\omega, 2}} \sigma_{e_{\ell}}^{+} \sigma_{e_{\ell}}^{-}\right)^{2}\right] \\
&=\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\frac{1}{n^{\ell}}\left(\sum_{\omega \in \mathscr{C}_{j_{1}: j_{\ell}}} \prod_{k=1}^{m} \prod_{e_{k} \in E_{\omega, 1_{k}}} A_{e_{k}}^{(k)} \prod_{e_{\ell} \in E_{\omega, 2}} \sigma_{e_{\ell}}^{+} \sigma_{e_{\ell}}^{-}\right)\right]^{2}+o(1)
\end{aligned}
$$

This follows by argument similar to the one used in the proof of Lemma B.1.

## B. 3 Proof of Lemma B. 1

Let us consider the following quantities for $\omega_{j}=\omega_{k_{j 1}, \cdots, k_{j m}, \ell_{j}}$ for $1 \leqslant j \leqslant r$.

$$
T_{1}:=\prod_{j=1}^{r_{1}}\left(\sum_{\omega_{j}} \prod_{e_{1} \in E_{\omega_{j}, 1_{1}}} \cdots \prod_{e_{m} \in E_{\omega_{j}, 1_{m}}} A_{e_{1}}^{(1)} \cdots A_{e_{m}}^{(m)}\right)^{m_{j}}
$$

and

$$
T_{2}:=\prod_{j=r_{1}+1}^{r}\left(\frac{1}{n^{\ell_{j}}} \sum_{\omega_{j}} \prod_{e_{1} \in E_{\omega_{j}, 1_{1}}} \cdots \prod_{e_{m} \in E_{\omega_{j}, 1_{m}}} \prod_{e_{\ell} \in E_{\omega_{j}, 2}} A_{e_{1}}^{(1)} \cdots A_{e_{m}}^{(m)} B_{e_{\ell}}\right)^{m_{j}}
$$

Expanding these terms we get,

$$
T_{1}:=\prod_{j=1}^{r_{1}}\left(\sum_{\omega_{j 1}, \cdots, \omega_{j m_{j}}} \prod_{q=1}^{m_{j}} \prod_{e_{1} \in E_{\omega_{j q}, 1_{1}}} \ldots \prod_{e_{m} \in E_{\omega_{j q}, 1_{m}}} A_{e_{1}}^{(1)} \cdots A_{e_{m}}^{(m)}\right)
$$

and

$$
T_{2}:=\prod_{j=r_{1}+1}^{r}\left(\frac{1}{n^{\ell_{j} m_{j}}} \sum_{\omega_{j 1}, \cdots, \omega_{j m_{j}}} \prod_{q=1}^{m_{j}} \prod_{e_{1} \in E_{\omega_{j q}, 1_{1}}} \ldots \prod_{e_{m} \in E_{\omega_{j q}, 1_{m}}} \prod_{e_{\ell} \in E_{\omega_{j q}, 2}} A_{e_{1}}^{(1)} \cdots A_{e_{m}}^{(m)} B_{e_{\ell}}\right)
$$

Under $\boldsymbol{H}_{0}$,
$\mathbb{E}_{0}\left[T_{1} T_{2}\right]=\sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \cdots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{k=1}^{m} \prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)} \prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}}\right]$

By model assumptions,

$$
\begin{aligned}
& \sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \cdots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}} \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{k=1}^{m} \prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1_{k}}}} A_{e_{t k}}^{(k)} \prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}}\right] \\
& =\sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \cdots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}}\left\{\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{k=1}^{m}\left(\prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)}\right)\left(\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)}\right)\right]\right. \\
& \left.\times \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}}\right]\right\} .
\end{aligned}
$$

Similar calculations also show that,

$$
\begin{aligned}
\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=1}^{r_{1}} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, 0}^{m_{j}}\right] \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{j=r_{1}+1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right] & =\sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \ldots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}} \\
& \times\left\{\mathbb{E}_{\mathbf{0}, 0}\left[\prod_{k=1}^{m} \prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)}\right]\right. \\
& \times \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)}\right] \\
& \left.\times \mathbb{E}_{\mathbf{0}, 0}\left[\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}}\right]\right\}
\end{aligned}
$$

Let us observe that these two terms seem to be different because of the presence of overlapping cycles. For example, if two cycles overlap on $m$-edges we gain a factor of $O\left(n^{m}\right)$ in the first term. But, then they also overlap in at least $m+1$ vertices and thus we lose $O\left(n^{m+1}\right)$ terms in choosing the cycles. Hence the dominant contribution comes from the non-overlapping cycles, implying part ( $i$ ) of the Lemma.

Under $\boldsymbol{H}_{1}$, we observe that conditioned on $\boldsymbol{\sigma}, \boldsymbol{A}_{k}$ for $k=1 \cdots, m$ and $\boldsymbol{B}$ are independent. Hence, to compute $\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{j=1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}}\right]$, we first condition on $\boldsymbol{\sigma}$. So we get,

$$
\begin{aligned}
& \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[T_{1} T_{2}\right] \\
= & \mathbb{E}_{\boldsymbol{\sigma}}\left[\sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \ldots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}} \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{k=1}^{m} \prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)} \prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}} \mid \boldsymbol{\sigma}\right]\right]
\end{aligned}
$$

Using model assumptions and calculations in previous part,

$$
\begin{aligned}
& \sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \cdots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}} \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{k=1}^{m} \prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)} \prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}} \mid \boldsymbol{\sigma}\right] \\
& =\sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \ldots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}}\left\{\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{k=1}^{m}\left(\prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1_{k}}}} A_{e_{t k}}^{(k)}\right)\left(\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1_{k}}}} A_{e_{t k}}^{(k)}\right) \mid \boldsymbol{\sigma}\right]\right. \\
& \left.\times \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}} \mid \boldsymbol{\sigma}\right]\right\} .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{\sigma}} & {\left[\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{j=1}^{r_{1}} Y_{\left.n, k_{j_{1}}, \cdots, k_{j_{m}, 0},{ }^{m_{j}} \mid \boldsymbol{\sigma}\right]}^{\mathbb{E}_{\boldsymbol{\lambda}, \mu}}\left[\prod_{j=r_{1}+1}^{r} Y_{n, k_{j_{1}}, \cdots, k_{j_{m}}, l_{j}}^{m_{j}} \mid \boldsymbol{\sigma}\right]\right]\right.} \\
= & \sum_{\omega_{11}, \cdots, \omega_{1 m_{1}}} \cdots \sum_{\omega_{r 1}, \cdots, \omega_{r m_{r}}} \frac{1}{n^{\sum_{j=r_{1}+1}^{r} \ell_{j} m_{j}}} \times \mathbb{E}_{\boldsymbol{\sigma}}\left\{\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{k=1}^{m} \prod_{t=1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)} \mid \boldsymbol{\sigma}\right]\right. \\
& \left.\times \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{t}} \prod_{e_{t k} \in E_{\omega_{t v, 1}}} A_{e_{t k}}^{(k)} \mid \boldsymbol{\sigma}\right] \times \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\prod_{t=r_{1}+1}^{r} \prod_{v=1}^{m_{\ell}} \prod_{e_{t \ell} \in E_{\omega_{t v}, 2}} B_{e_{t \ell}} \mid \boldsymbol{\sigma}\right]\right\}
\end{aligned}
$$

Again, as the dominant contributions come from the non-overlapping cycles by arguments of the previous part, the differences between the conditional expectations is $o(1)$. Hence, by Dominated Convergence Theorem, the second part of the lemma follows.

## C Proof of Theorem 2.3

Consider the random variables,

$$
\begin{aligned}
& W^{(M)}:=\prod_{K=1}^{M}\left\{\prod_{k_{1}+\cdots+k_{m}=K}\left(1+\delta_{k_{1}, \cdots, k_{m}}\right)^{\nu_{\left(k_{1}, \cdots, k_{m}\right)}} \exp \left(-\sum_{k_{1}+\cdots+k_{m}=K} \lambda_{k_{1}, \cdots, k_{m}} \delta_{k_{1}, \cdots, k_{m}}\right)\right. \\
& \left.\prod_{\substack{k_{1}+\cdots+k_{m}+\ell=K \\
\ell \neq 0}} \exp \left(\frac{2 \mu_{k_{1}, \cdots, k_{m}} Z_{\left(k_{1}, \cdots, k_{m}\right)}-\mu_{k_{1}, \cdots, k_{m}}^{2}}{2 \sigma_{k_{1}, \cdots, k_{m}}^{2}}\right)\right\},
\end{aligned}
$$

for $M>0$ and,

$$
\begin{align*}
& \left.\prod_{k_{1}+\cdots+k_{m}+\ell=K} \exp \left(\frac{2 \mu_{k_{1}, \cdots, k_{m}} Z_{\left(k_{1}, \cdots, k_{m}\right)}-\mu_{k_{1}, \cdots, k_{m}}^{2}}{2 \sigma_{k_{1}, \cdots, k_{m}}^{2}}\right)\right\} . \tag{33}
\end{align*}
$$

We prove it in two steps.

Step 1. Let us observe that,

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}_{n}}\left[\prod_{k_{1}+\cdots+k_{m}=K}\left(1+\delta_{k_{1}, \cdots, k_{m}}\right)^{\nu_{\left(k_{1}, \cdots, k_{m}\right)} \exp \left(-\sum_{k_{1}+\cdots+k_{m}=K} \lambda_{k_{1}, \cdots, k_{m}} \delta_{k_{1}, \cdots, k_{m}}\right), ~(1) .}\right. \\
& \left.\prod_{\substack{ \\
k_{1}+\cdots+k_{m}+\ell=K \\
\ell \neq 0}} \exp \left(\frac{2 \mu_{k_{1}, \cdots, k_{m}} Z_{\left(k_{1}, \cdots, k_{m}\right)}-\mu_{k_{1}, \cdots, k_{m}}^{2}}{2 \sigma_{k_{1}, \cdots, k_{m}}^{2}}\right)\right]=1 \quad \forall i \in \mathbb{N} .
\end{aligned}
$$

This implies $\left\{W^{(k)}\right\}$ is a martingale with respect to the natural filtration. Let us also observe that,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_{n}}\left[\left(W^{(k)}\right)^{2}\right]=\prod_{K=1}^{\infty}\left\{\prod_{k_{1}+\cdots+k_{m}=K} \exp \left(\lambda_{k_{1}, \cdots, k_{m}} \delta_{k_{1}, \cdots, k_{m}}^{2}+\lambda_{k_{1}, \cdots, k_{m}} \delta_{k_{1}, \cdots, k_{m}}\right)\right. \\
\left.\prod_{\substack{k_{1}+\cdots+k_{m}+\ell=K \\
\ell \neq 0}} \exp \left(\frac{\mu_{k_{1}, \cdots, k_{m}}^{2}}{\sigma_{k_{1}, \cdots, k_{m}}^{2}}\right)\right\}<\infty \quad \forall k .
\end{aligned}
$$

This implies $\left\{W^{(k)}\right\}$ is an $L_{2}$ bounded martingale. Hence there exists $W^{\infty}$ as defined in (33) in $L_{2}$ such that $W^{(k)} \xrightarrow{a . s / L_{2}} W^{(\infty)}$. Now let us observe that,

$$
\log \left(W^{(k)}\right)=Z_{k}+\widetilde{W}_{k},
$$

where $Z_{k}$ come from Gaussian distribution. Hence,

$$
\log \left(W^{(\infty)}\right)=Z+\widetilde{W}
$$

where $Z$ is Gaussian, implying $\mathbb{P}_{n}\left(W^{(\infty)}>0\right)=1$. That $W^{(\infty)} \in L_{2}$ immediately implies $\mathbb{P}_{n}\left(W^{(\infty)}<\right.$ $\infty)=1$. Hence, the term $W^{(\infty)}$ is well-defined.

Step 2. It is enough to show that

$$
T_{n}:=\frac{d \mathbb{Q}_{n}}{d \mathbb{P}_{n}} \xrightarrow{d} W^{(\infty)} .
$$

By the proof of Theorem 2.1, if $\lambda_{1}^{2}+\cdots+\lambda_{m}^{2}+\mu^{2} / \gamma \leqslant 1$,

$$
\limsup _{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{n}}\left[T_{n}^{2}\right]<\infty
$$

Hence the sequence $\left\{T_{n}\right\}$ is tight. By Prokhorov's Theorem, there exists a subsequence $\left\{n_{k}\right\}_{k=1}^{\infty}$ such that $T_{n_{k}}$ converges in distribution to $W\left(\left\{n_{k}\right\}\right)$. We shall show that the limit $W\left(\left\{n_{k}\right\}\right)$ does not depend on the subsequence and $W\left(\left\{n_{k}\right\}\right) \stackrel{d}{=} W^{(\infty)}$. For $\varepsilon>0$, let us choose $M$ large such that,

$$
\begin{align*}
& \left.\prod_{\substack{k_{1}+\cdots+k_{m}+\ell=K \\
\ell \neq 0}} \exp \left(\frac{2 \mu_{k_{1}, \cdots, k_{m}} Z_{\left(k_{1}, \cdots, k_{m}\right)}-\mu_{k_{1}, \cdots, k_{m}}^{2}}{2 \sigma_{k_{1}, \cdots, k_{m}}^{2}}\right)\right\}-  \tag{34}\\
& \prod_{K=1}^{\infty}\left\{\prod_{k_{1}+\cdots+k_{m}=K}\left(1+\delta_{k_{1}, \cdots, k_{m}}\right)^{\left.\nu_{\left(k_{1}, \cdots, k_{m}\right)} \exp \left(-\sum_{k_{1}+\cdots+k_{m}=K} \lambda_{k_{1}, \cdots, k_{m}} \delta_{k_{1}, \cdots, k_{m}}\right)\right) ~(-2)}\right. \\
& \left.\prod_{\substack{ \\
k_{1}+\cdots+k_{m}+\ell=K \\
\ell \neq 0}} \exp \left(\frac{2 \mu_{k_{1}, \cdots, k_{m}} Z_{\left(k_{1}, \cdots, k_{m}\right)}-\mu_{k_{1}, \cdots, k_{m}}^{2}}{2 \sigma_{k_{1}, \cdots, k_{m}}^{2}}\right)\right\} \mid<\varepsilon .
\end{align*}
$$

For this $M$, consider the joint distribution of $\left(T_{n_{k}},\left\{Y_{n_{k}, k_{1}, \cdots, k_{m}, \ell}\right\}\right.$ where $\left(k_{1}, \cdots, k_{m}, \ell\right) \in \mathscr{K}$ for $\mathscr{K}:=$ $\left\{\left(k_{1}, \cdots, k_{m}, \ell\right): k_{1}+\cdots+k_{m}+\ell \in\{1, \cdots, M\}\right\}$. By Theorem 2.2 , these random variables are tight with respect to $P_{n_{k}}$. So, there exists a further subsequence $\left\{n_{k_{\ell}}\right\}$ such that under $\mathbb{P}_{n_{k_{\ell}}}$,

$$
\left(T_{n_{k_{\ell}}},\left\{Y_{n_{k}, k_{1}, \cdots, k_{m}, \ell}\right\}_{\left(k_{1}, \cdots, k_{m}, \ell\right) \in \mathscr{K}}\right) \xrightarrow{d}\left(W\left(\left\{n_{k}\right\}\right), \boldsymbol{\nu}_{0, M}\right),
$$

where the set of random variables $\boldsymbol{\nu}_{0, M}$ is defined by Theorem 2.2. Since,

$$
\limsup _{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{n}}\left[T_{n}^{2}\right]<\infty
$$

the sequence $\left\{Y_{n_{k_{\ell}}}\right\}$ is uniformly integrable, implying,

$$
\mathbb{E}\left[W\left(\left\{n_{k}\right\}\right)\right]=\lim _{\ell \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{n_{k_{\ell}}}}\left[T_{n_{k_{\ell}}}\right]=1
$$

Now for any bounded positive continuous function $f$, by Fatou's Lemma,

$$
\begin{equation*}
\left.\left.\liminf _{\ell \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{n_{k_{\ell}}}}\left[f\left(\left\{Y_{n_{k}, k_{1}, \cdots, k_{m}, \ell}\right\}_{\left(k_{1}, \cdots, k_{m}, \ell\right) \in \mathscr{K}}\right)\right) T_{n_{k_{\ell}}}\right] \geqslant \mathbb{E}\left[f\left(\boldsymbol{\nu}_{M, 0}\right)\right) W\left(\left\{n_{k}\right\}\right)\right] \tag{35}
\end{equation*}
$$

For any constant $\xi$, by uniform integrability, $\xi=\xi \mathbb{E}\left[T_{n_{k_{\ell}}}\right] \rightarrow \xi \mathbb{E}\left[W\left(\left\{n_{k}\right\}\right)\right]=\xi$. So, (35) holds for any bounded continuous function. Replacing $f$ by $-f$, we get for any bounded continuous function,

$$
\left.\left.\lim _{\ell \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{n_{k_{\ell}}}}\left[f\left(\left\{Y_{n_{k}, k_{1}, \cdots, k_{m}, \ell}\right\}_{\left(k_{1}, \cdots, k_{m}, \ell\right) \in \mathscr{K}}\right)\right) T_{n_{k_{\ell}}}\right]=\mathbb{E}\left[f\left(\boldsymbol{\nu}_{0}, M\right)\right) W\left(\left\{n_{k}\right\}\right)\right]
$$

By Theorem 2.2,

$$
\begin{aligned}
\left.\int f\left(\left\{Y_{n_{k}, k_{1}, \cdots, k_{m}, \ell}\right\}_{\left(k_{1}, \cdots, k_{m}, \ell\right) \in \mathcal{K}}\right)\right) T_{n_{k_{\ell}}} d \mathbb{P}_{n_{k_{\ell}}} & \left.=\int f\left(\left\{Y_{n_{k}, k_{1}, \cdots, k_{m}, \ell}\right\}_{\left(k_{1}, \cdots, k_{m}, \ell\right) \in \mathcal{K}}\right)\right) d \mathbb{Q}_{n_{k_{\ell}}} \\
& \rightarrow \int f\left(\boldsymbol{\nu}_{M, 1}\right) d \mathbb{Q}
\end{aligned}
$$

where the set of random variables $\boldsymbol{\nu}_{M, 1}$ is defined by Theorem 2.2 and $Q$ is the measure induced by $\boldsymbol{\nu}_{M, 1}$. By definition of $W^{(M)}$,

$$
\int f\left(\boldsymbol{\nu}_{M, 1}\right) d \mathbb{Q}=\mathbb{E}\left[f\left(\boldsymbol{\nu}_{M, 0}\right) W^{(M)}\right]
$$

for any bounded continuous function $f$, and so $\int_{A} d Q=\mathbb{E}\left[1_{A} W^{(m)}\right]$ for any $A \in \sigma\left(\boldsymbol{\nu}_{M, 0}\right)$. This implies,

$$
W^{(m)}=\mathbb{E}\left[W\left(\left\{n_{k}\right\}\right) \mid \sigma\left(\boldsymbol{\nu}_{M, 0}\right)\right]
$$

From Fatou's Lemma and definition of $W\left(\left\{n_{k}\right\}\right)$,

$$
\mathbb{E}\left[W\left(\left\{n_{k}\right\}\right)^{2}\right] \leqslant \liminf _{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_{n}}\left[T_{n}^{2}\right]
$$

As a consequence, by (34),

$$
0 \leqslant \mathbb{E}\left|W\left(\left\{n_{k}\right\}\right)-W^{(M)}\right|^{2}=\mathbb{E}\left[W^{2}\left(\left\{n_{k}\right\}\right)\right]-\mathbb{E}\left[\left(W^{(M)}\right)^{2}\right]<\varepsilon
$$

So, $L_{2}\left(W\left(\left\{n_{k}\right\}\right), W^{(M)}\right)<\sqrt{\varepsilon}$. This implies $W^{(n)} \xrightarrow{L_{2} / d} W\left(\left\{n_{k}\right\}\right)$ as $n \rightarrow \infty$. But we have also shown $W^{(n)} \xrightarrow{L_{2} / d} W^{(\infty)}$. This implies $W\left(\left\{n_{k}\right\}\right) \stackrel{d}{=} W^{(\infty)}$, which completes the proof of the weak convergence statement.

The proof of the final statement follows exactly as the proof of equation (6) in [4].

## D Proof of Theorem 2.4

Weak recovery above the threshold follows via a proposed estimator defined using self avoiding walks in Section 2.3: it only remains to prove Lemma 2.1, which is done in the next section. In this section, we prove the impossibility of reconstruction under the threshold. We start with a proposition.

Proposition D.1. Let $\sum_{i} \lambda_{i}+\mu^{2} / \gamma<1$. Then, for any fixed $r$ with $1 \leqslant r \leqslant n$ and any $\left(\sigma_{1}, \ldots, \sigma_{r}\right),\left(\tau_{1}, \ldots, \tau_{r}\right) \in$ $\{ \pm 1\}^{r}$, we have

$$
\left\|\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\cdot \mid \sigma_{1: r}\right)-\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\cdot \mid \tau_{1: r}\right)\right\|_{\mathrm{TV}} \rightarrow 0
$$

Proof. This is an adaptation of Proposition 3 of [26] to our setup, and the proof proceeds along the same lines. Since $r$ is fixed, let $L_{\boldsymbol{\sigma}_{1: r}, n}:=\mathbb{E}_{\boldsymbol{\sigma}_{-r}, \boldsymbol{u}}\left[\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\sigma}, \boldsymbol{u}\right) / \mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)\right]$ and $L_{\boldsymbol{\tau}_{1: r}, n}$ (analogously defined) denote the conditional log likelihoods, and let $\mathscr{S}=\{\|\boldsymbol{u}\| \leqslant 2 \sqrt{p}\}$. Then, $\mathbb{P}(\mathscr{S}) \rightarrow 1$. We define the following truncations of the conditional $\log$ likelihood to $\mathscr{S}$ :

$$
\begin{aligned}
& \tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}:=\frac{\mathbb{E}_{\boldsymbol{\sigma}_{-r}, \boldsymbol{u}}\left[\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\sigma}, \boldsymbol{u}\right) \mathbf{1}_{[\boldsymbol{u} \in \mathscr{S}]}\right]}{\mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)} \\
& \tilde{L}_{\boldsymbol{\tau}_{1: r}, n}:=\frac{\mathbb{E}_{\boldsymbol{\tau}_{-r}, \boldsymbol{u}}\left[\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\sigma}, \boldsymbol{u}\right) \mathbf{1}_{[\boldsymbol{u} \in \mathscr{S}]}\right]}{\mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)}
\end{aligned}
$$

Denote by $Q_{\boldsymbol{\sigma}_{1: r}, n}$ and $Q_{\boldsymbol{\tau}_{1: r}, n}$ the probability measures induced by $\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}$ and $\tilde{L}_{\boldsymbol{\tau} 1: r, n}$, respectively, after normalization. Now, as long as, $\mathbb{E}_{0,0}\left[L_{\boldsymbol{\sigma}_{1: r}, n}^{2}\right], \mathbb{E}_{0,0}\left[L_{\boldsymbol{\tau}_{1: r}, n}^{2}\right]<\infty$, by Cauchy-Schwarz inequality, we have $\left\|\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\cdot \mid \sigma_{1: r}\right)-Q_{\sigma_{1: r}, n}(\cdot)\right\|_{\mathrm{TV}} \rightarrow 0$ and $\left\|\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\cdot \mid \tau_{1: r}\right)-Q_{\boldsymbol{\tau}_{1: r}, n}(\cdot)\right\|_{\mathrm{TV}} \rightarrow 0$. From the proof of Theorem 2.1, this holds under $\sum_{i} \lambda_{i}+\mu^{2} / \gamma<1$. Thus, it remains to show $\left\|Q_{\boldsymbol{\tau}_{1: r}, n}(\cdot)-Q_{\boldsymbol{\tau}_{1: r}, n}(\cdot)\right\|_{\mathrm{TV}} \rightarrow$ 0 , which amounts to showing $\frac{1}{\mathbb{P}(\mathscr{S})} \mathbb{E}_{0,0}\left[\left|\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}-\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}\right|\right] \rightarrow 0$. Again, $\mathbb{E}_{0,0}\left[\left|\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}-\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}\right|\right] \leqslant$ $\mathbb{E}_{0,0}\left[\left(\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}-\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}\right)^{2}\right]$, so that it is enough to show

$$
\frac{1}{\mathbb{P}(S)} \mathbb{E}_{0,0}\left[\left(\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}-\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}\right)^{2}\right] \rightarrow 0
$$

Expanding the square, we get

$$
\mathbb{E}_{0,0}\left[\left(\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}-\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}\right)^{2}\right]=\mathbb{E}_{0,0} \tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}^{2}+\mathbb{E}_{0,0} \tilde{L}_{\boldsymbol{\tau}_{1: r}, n}^{2}-2 \mathbb{E}_{0,0} \tilde{L}_{\boldsymbol{\sigma}_{1: r}, n} \tilde{L}_{\boldsymbol{\tau}_{1: r}, n}
$$

where

$$
\begin{aligned}
\tilde{L}_{\boldsymbol{\sigma}_{1: r}, n}^{2} & =\left(\mathbb{E}_{\boldsymbol{\sigma}_{-r}, \boldsymbol{u}}\left[\frac{d \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\sigma}, \boldsymbol{u}\right)}{d \mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)} \mathbb{1}\{\boldsymbol{u} \in \mathscr{S}\}\right]\right)^{2} \\
& =\mathbb{E}_{\boldsymbol{\sigma}_{-r}, \boldsymbol{\boldsymbol { \tau } _ { - r }}, \boldsymbol{u}, \boldsymbol{v}}\left[\frac{d \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\sigma}, \boldsymbol{u}\right)}{d \mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)} \frac{d \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\sigma}_{1: r}, \boldsymbol{\tau}_{-r}, \boldsymbol{v}\right)}{d \mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)} \mathbb{1}\{\boldsymbol{u}, v \in \mathscr{S}\}\right] .
\end{aligned}
$$

Using similar expansions for the other two terms, it is enough to show that

$$
\mathbb{E}_{0,0}\left[\frac{d \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\sigma}, \boldsymbol{u}\right)}{d \mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)} \frac{d \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \boldsymbol{\tau}, \boldsymbol{v}\right)}{d \mathbb{P}_{0,0}\left(A_{1}, \ldots, A_{m}, B\right)} \mathbb{1}\{\boldsymbol{u}, v \in \mathscr{S}\}\right]
$$

converges to a limit. This was shown in the proof of Theorem 2.1, which completes the proof.
We next translate the convergence of conditional data distributions in Proposition D. 1 to a convergence of posterior distributions of the cluster assignments $\sigma_{i}$.
Proposition D.2. Let $\sum_{i} \lambda_{i}+\mu^{2} / \gamma<1, S \subset[n],|S|=r$ where $r$ is finite and fixed, $u \in[n]$ any fixed index such that $u \notin S$. Then, as $n \rightarrow \infty$,

$$
\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\left\|\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\sigma_{u} \mid A_{1}, \ldots, A_{m}, B, \sigma_{S}\right)-\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}\right)\right\|_{\mathrm{TV}}\right] \rightarrow 0
$$

Proof.

$$
\begin{aligned}
& \mathbb{E}_{\boldsymbol{\lambda}, \mu} {\left[\left\|\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\sigma_{u} \mid A_{1}, \ldots, A_{m}, B, \sigma_{S}\right)-\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}\right)\right\|_{\mathrm{TV}}\right] } \\
&=\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}\right) \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\left\|\frac{\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\sigma_{u} \mid A_{1}, \ldots, A_{m}, B, \sigma_{S}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}\right)}-1\right\|_{\mathrm{TV}}\right] \\
&=\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}\right) \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\left\|\frac{\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(\sigma_{u}, A_{1}, \ldots, A_{m}, B, \sigma_{S}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}\right) \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B, \sigma_{S}\right)}-1\right\|_{\mathrm{TV}}\right] \\
&=\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}\right) \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\left\|\frac{\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}, \sigma_{S}\right) \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{u}, \sigma_{S}\right)}{\mathbb{P}_{\mathbf{0}, 0}\left(\sigma_{u}, \sigma_{S}\right) \mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{S}\right)}-1\right\|_{\mathrm{TV}}\right] \\
&=\frac{1}{2} \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\left.\left\|\frac{\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{u}, \sigma_{S}\right)}{\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{S}\right)}-1\right\|_{\mathrm{TV}} \right\rvert\, \sigma_{S}\right]\right] \\
&=\frac{1}{2} \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{S}\right) \times\right. \\
&\left.\mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\left\|\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{u}, \sigma_{S}\right)-\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{S}\right)\right\|_{\mathrm{TV}} \mid \sigma_{S}\right]\right] \\
&=\frac{1}{4} \mathbb{E}_{\boldsymbol{\lambda}, \mu}\left[\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{S}\right) \times\right. \\
&\left.\left\|\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{u}=+1, \sigma_{S}\right)-\mathbb{P}_{\boldsymbol{\lambda}, \mu}\left(A_{1}, \ldots, A_{m}, B \mid \sigma_{u}=-1, \sigma_{S}\right)\right\|_{\mathrm{TV}}\right]
\end{aligned}
$$

The term inside the expectation converges to 0 in probability by Proposition D.1, so that the outer expectation also converges to zero by the Dominated Convergence Theorem.

It remains to prove the impossibility of weak recovery using Propositions D. 1 and D.2. This follows using the technique of the proof of Theorem 2.3 of [3].

## E Proof of Lemma 2.1

It is enough to show, for a constant delta as specified in Lemma 2.1 and for each $i_{1}, i_{2} \in[n]$, that

$$
\mathbb{E}\left[\widehat{\Sigma}_{i_{1} i_{2}} \cdot \sigma_{i_{1}} \sigma_{i_{2}}\right] \geqslant \delta \sqrt{\mathbb{E}\left[\widehat{\Sigma}_{i_{1} i_{2}}^{2}\right]}
$$

For the left hand side, we have

$$
\begin{aligned}
\mathbb{E}\left[\widehat{\Sigma}_{i_{1} i_{2}} \cdot \sigma_{i_{1}} \sigma_{i_{2}}\right] & =\frac{1}{\left|\mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)\right|} \sum_{\alpha \in \mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)} \mathbb{E}\left[p_{\alpha} \sigma_{i_{1}} \sigma_{i_{2}}\right] \\
& =\frac{1}{\left|\mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)\right|} \sum_{\alpha \in \mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)} \mathbb{E}\left[\mathbb{E}\left[p_{\alpha} \mid \boldsymbol{\sigma}\right] \sigma_{i_{1}} \sigma_{i_{2}}\right] \\
& =1
\end{aligned}
$$

while, for the right hand side,

$$
\mathbb{E}\left[\widehat{\Sigma}_{i_{1} i_{2}}^{2}\right]=\frac{1}{\left|\mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)\right|^{2}} \sum_{\alpha, \beta \in \mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, l\right)} \mathbb{E}\left[p_{\alpha} p_{\beta}\right]
$$

For paths $\alpha, \beta \in \mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)$ with no wedges in common, we have

$$
\mathbb{E}\left[p_{\alpha} p_{\beta}\right]=\mathbb{E}\left[\mathbb{E}\left[p_{\alpha} p_{\beta} \mid \boldsymbol{\sigma}\right]\right]=1
$$

Next, we turn to controlling $\mathbb{E}\left[p_{\alpha} p_{\beta}\right]$ for paths $\alpha$ and $\beta$ with common wedges. Let $\alpha$ and $\beta$ coincide on $s_{i}$ type $\boldsymbol{A}_{i}$ wedges, for $i=1, \ldots, m$, and $t$ type $\boldsymbol{B}$ wedges. For these terms, we have

$$
\mathbb{E}\left[p_{\alpha} p_{\beta}\right]=O(1)\left[\left(\frac{n}{\lambda_{1}^{2}}\right)^{s_{1}} \ldots\left(\frac{n}{\lambda_{m}^{2}}\right)^{s_{m}}\left(\frac{n p}{\mu^{2} / \gamma}\right)^{t}\right] .
$$

Further, for fixed $s_{1}, \ldots, s_{m}$, the minimum possible number of factor and variable nodes common is between $\alpha$ and $\beta$ as described above is $t+\sum_{i} s_{i}$ variable nodes, $s_{i}$ type $\boldsymbol{A}_{i}$ factor nodes and $b$ type $\boldsymbol{B}$ factor nodes. This happens when two paths have the first $\sum_{i} s_{i}$ wedges common and of type $\mathbb{A}$, and the last $t$ wedges common and of type $\boldsymbol{B}$. Set $\boldsymbol{s}=\left(s_{1}, \ldots, s_{m}\right)$ and $\boldsymbol{k}=\left(k_{1}, \ldots, k_{m}\right)$. Number of path pairs having exactly this number of common nodes is then $n^{2(k-1)-\sum_{i} s_{i}-t} p^{2 \ell-t} g(\boldsymbol{s}, \boldsymbol{k})$, where $g(\boldsymbol{s}, \boldsymbol{k})$ is the number of valid choices of type $\mathbb{A}$ factor nodes for a pair of paths $\alpha, \beta$ as described above, that are determined except for this choice. In fact, it is not difficult to observe that

$$
g(\boldsymbol{s}, \boldsymbol{k}) \leqslant \frac{\left(\sum_{i=1}^{m} s_{i}\right)!}{\prod_{i=1}^{m}\left(s_{i}\right)!}\left(\frac{\left(\sum_{i=1}^{m}\left(k_{i}-s_{i}\right)\right)!}{\prod_{i=1}^{m}\left(k_{i}-s_{i}\right)!}\right)^{2}
$$

For terms with the minimum number of vertices common, the contribution to $\sum \mathbb{E}\left[p_{\alpha} p_{\beta}\right]$ is then

$$
O(1) n^{2(k-\ell-1)-\sum_{i=1}^{m} s_{i}-t} p^{2 \ell-t} \prod_{i=1}^{m}\left(\frac{n}{\lambda_{i}^{2}}\right)^{s_{i}}\left(\frac{n}{\mu}\right)^{t} g(\boldsymbol{s}, \boldsymbol{k})=O(1) n^{2(k-1)} p^{2 \ell} \prod_{i=1}^{m} \lambda_{i}^{-2 s_{i}}\left(\frac{\mu^{2}}{\gamma}\right)^{t} g(\boldsymbol{s}, \boldsymbol{k}),
$$

A similar calculation will show that the leading order term (in $n$ ) in the number of path pairs as described above is contributed by the pairs with minimum possible number of vertices common as the number of paths reduce when the paths intersect in more number of vertices. Then, summing over all possible number of intersecting wedges, we have, since $g(\boldsymbol{s}, \boldsymbol{k}) \leqslant 1$,

$$
\begin{aligned}
& \sum_{\alpha, \beta \in \mathcal{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)} \mathbb{E}\left[p_{\alpha} p_{\beta}\right] \\
\leqslant & O(1) n^{2(k-1)} p^{2 \ell} \sum_{i=1}^{m} \sum_{0 \leqslant s_{i} \leqslant k_{i}} \sum_{0 \leqslant t \leqslant \ell}\left(\prod_{i=1}^{m} \lambda_{i}^{-2 s_{i}}\left(\frac{\mu^{2}}{\gamma}\right)^{t} g(\boldsymbol{s}, \boldsymbol{k})\right) . \\
\leqslant & O(1) n^{2(k-1)} p^{2 \ell} \prod_{i=1}^{m} \lambda_{i}^{-2 k_{i}}\left(\frac{\mu^{2}}{\gamma}\right)^{-\ell} .
\end{aligned}
$$

Finally, to establish Lemma 2.1, it remains to show that

$$
\left|\mathscr{W}\left(i_{1}, i_{2}, k_{1}, \ldots, k_{m}, \ell\right)\right|^{2}=\frac{k!}{k_{1}!\ldots k_{m}!\ell!} n^{2(k-1)} p^{2 l} \geqslant n^{2(k-1)} p^{2 \ell} \prod_{i=1}^{m} \lambda_{i}^{-2 k_{i}}\left(\frac{\mu^{2}}{\gamma}\right)^{-\ell}
$$

for some choice of $\boldsymbol{k}$ and $\ell$. Choose

$$
\frac{k_{i}}{k}=\frac{\lambda_{i}^{2}}{\sum_{i=1}^{m} \lambda_{i}^{2}+\mu^{2} / \gamma}, \quad \frac{\ell}{k}=\frac{\mu^{2} / \gamma}{\sum_{i=1}^{m} \lambda_{i}^{2}+\mu^{2} / \gamma}
$$

Since $\sum_{i=1}^{m} \lambda_{i}^{2}+\mu^{2} / \gamma>1$, we have $\lambda_{i}^{2}>k_{i} / k$ and $\mu^{2} / \gamma>\ell / k$ Then,
$\frac{k!}{k_{1}!\ldots k_{m}!\ell!} \geqslant \exp \left(-2 l \log \frac{\ell}{k}-2 \sum_{i} \log \frac{k_{i}}{k}+o(1)\right) \gtrsim \exp \left(-\ell \log \frac{\mu^{2}}{\gamma}-k_{i} \log \lambda^{2}\right)=\prod_{i} \lambda^{-2 k_{i}}\left(\frac{\mu^{2}}{\gamma}\right)^{-\ell}$.
This completes the proof.


Figure 4: Combined Multi-Layered Factor graph for running the Belief Propagation Algorithm.

## F Proofs of results in Section 3

Let us compute the Belief Propagation Messages on the factor graph described by Figure 4. The messages from the variable nodes $i$ to $j$ along the edge $(i, j)$ in the graph $\boldsymbol{G}_{k}$, for $i, j \in[n], k \in[m]$, are given by,

$$
\begin{aligned}
\nu_{i \rightarrow j ; k}^{t+1}\left(v_{i}\right) \approx & \prod_{q \in[p]} \mathbb{E}_{q \rightarrow i}^{t}\left[\exp \left(\sqrt{\frac{\mu}{n}} B_{q i} u_{q} v_{i}\right)\right] \prod_{\ell \neq k, \ell=1}^{m}\left\{\prod_{c \in \partial_{\ell} i} \mathbb{E}_{c \rightarrow i ; \ell}^{t}\left[\frac{d_{\ell}+\lambda_{\ell} \sqrt{d_{\ell}} v_{i} v_{c}}{n}\right]\right. \\
& \left.\prod_{c \in\left(\partial_{\ell} i\right)^{c}} \mathbb{E}_{c \rightarrow i ; \ell}^{t}\left[1-\frac{d_{\ell}+\lambda_{\ell} \sqrt{d_{\ell}} v_{i} v_{c}}{n}\right]\right\} \\
& \left\{\prod_{c \in \partial_{\ell} k \backslash\{j\}} \mathbb{E}_{c \rightarrow i ; k}^{t}\left[\frac{d_{\ell}+\lambda_{k} \sqrt{d_{k}} v_{i} v_{c}}{n}\right] \prod_{c \in\left(\partial_{k} i\right)^{c} \backslash\{j\}} \mathbb{E}_{c \rightarrow i ; k}^{t}\left[1-\frac{d_{k}+\lambda_{k} \sqrt{d_{k}} v_{i} v_{c}}{n}\right]\right\}
\end{aligned}
$$

Next, the messages from the variable node $i$ to factor node $q$ for $i \in[n]$ and $q \in[p]$ are given by,

$$
\begin{aligned}
& \nu_{i \rightarrow q}^{t+1}\left(v_{i}\right) \approx \prod_{r \in[p] \backslash\{q\}} \mathbb{E}_{r \rightarrow i}^{t}\left[\exp \left(\sqrt{\frac{\mu}{n}} B_{r i} u_{r} v_{i}\right)\right] \prod_{\ell=1}^{m}\left\{\prod_{k \in \partial_{\ell} i} \mathbb{E}_{k \rightarrow i ; \ell}^{t}\left[\frac{d_{\ell}+\lambda_{\ell} \sqrt{d_{\ell}} v_{i} v_{k}}{n}\right]\right. \\
&\left.\prod_{k \in\left(\partial_{\ell} i\right)^{c}} \mathbb{E}_{k \rightarrow i, \ell}^{t}\left[1-\frac{d_{\ell}+\lambda_{\ell} \sqrt{d_{\ell}} v_{i} v_{k}}{n}\right]\right\} .
\end{aligned}
$$

Finally, the messages from factor nodes $q \in[p]$ to $i \in[n]$ are given by,

$$
\nu_{q \rightarrow i}^{t+1}\left(u_{q}\right) \approx \exp \left(-\frac{(1+\mu) u_{q}^{2}}{2}\right) \prod_{j \neq i} \mathbb{E}_{j \rightarrow q}^{t}\left[\exp \left(\sqrt{\frac{\mu}{n}} B_{q j} u_{q} v_{j}\right)\right]
$$

Now, to characterize the distributions $\nu_{i \rightarrow j ; k}^{t+1}$ 's and $\nu_{i \rightarrow q}^{t+1}$,s we define the following parameters characterizing the log odds ratio,

$$
\begin{equation*}
\eta_{i \rightarrow j ; k}^{t}:=\frac{1}{2} \log \frac{\nu_{i \rightarrow j ; k}^{t+1}(+1)}{\nu_{i \rightarrow j ; k}^{t+1}(-1)}, \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta_{i \rightarrow q}^{t}:=\frac{1}{2} \log \frac{\nu_{i \rightarrow q}^{t+1}(+1)}{\nu_{i \rightarrow}^{t+1}(-1)} \tag{37}
\end{equation*}
$$

For the messages $\nu_{q \rightarrow i}^{t}$, we use the Gaussian ansatz, i.e.,

$$
\nu_{q \rightarrow i}^{t}:=\mathrm{N}\left(m_{q \rightarrow i}^{t}, \tau_{q \rightarrow i}^{t}\right)
$$

Let us observe that,

$$
\begin{aligned}
\mathbb{E}_{q \rightarrow i}^{t}\left[\exp \left(\sqrt{\frac{\mu}{n}} B_{q i} u_{q} v_{i}\right)\right] & =\exp \left(\sqrt{\frac{\mu}{n}} B_{q i} v_{i} m_{q \rightarrow i}^{t}+\frac{\mu}{2 n} B_{q i}^{2} \tau_{q \rightarrow i}^{t}\right) \\
\mathbb{E}_{i \rightarrow j ; \ell}^{t}\left[\frac{d_{\ell}+\lambda_{\ell} \sqrt{d_{\ell}} v_{i} v_{j}}{n}\right] & =\frac{d_{\ell}}{n}\left(1+\frac{\lambda_{\ell} v_{j}}{\sqrt{d_{\ell}}} \tanh \left(\eta_{i \rightarrow j ; \ell}^{t}\right)\right) \\
\mathbb{E}_{i \rightarrow q}^{t}\left[\exp \left(\sqrt{\frac{\mu}{n}} B_{q i} u_{q} v_{i}\right)\right] & =\frac{\cosh \left(\eta_{i \rightarrow q}^{t}+\sqrt{\mu / n} B_{q i} u_{q}\right)}{\cosh \left(\eta_{i \rightarrow q}^{t}\right)}
\end{aligned}
$$

Using (36) and (37), we get,

$$
\begin{aligned}
\eta_{i \rightarrow j ; k}^{t+1}= & \sqrt{\frac{\mu}{\gamma p}} \sum_{q \in[p]} B_{q i} m_{q \rightarrow i}^{t}+\sum_{\ell=1 ; \ell \neq k}^{m}\left[\sum_{c \in \partial_{\ell} i} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{c \in\left(\partial_{\ell} i\right)^{c}} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{n, \ell}\right)\right] \\
& +\sum_{c \in \partial_{\ell} i \backslash\{j\}} f\left(\eta_{c \rightarrow i ; k}^{t} ; \rho_{\ell}\right)-\sum_{c \in\left(\partial_{\ell} i\right)^{c} \backslash\{j\}} f\left(\eta_{c \rightarrow i ; k}^{t} ; \rho_{n, \ell}\right), \\
\eta_{i \rightarrow q}^{t+1}= & \sqrt{\frac{\mu}{\gamma p}} \sum_{r \in[p] \backslash\{q\}} B_{r i} m_{r \rightarrow i}^{t}+\sum_{\ell=1}^{m}\left[\sum_{c \in \partial_{\ell} i} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{c \in\left(\partial_{\ell} i\right)^{c}} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{n, \ell}\right)\right]
\end{aligned}
$$

By Taylor Expansion,

$$
\begin{aligned}
\log \nu_{q \rightarrow i}^{t+1}\left(u_{q}\right)= & \text { const }-\frac{(1+\mu)}{2} u_{q}^{2}+\sum_{j \in[n] \backslash\{i\}} \log \cosh \left(\eta_{j \rightarrow q}^{t}+\sqrt{\frac{\mu}{n p}} B_{q j} u_{q}\right) \\
= & \text { const }-\frac{(1+\mu)}{2} u_{q}^{2}+\left(\sqrt{\frac{\mu}{n p}} \sum_{j \in[n] \backslash\{i\}} B_{q j} \tanh \left(\eta_{j \rightarrow q}^{t}\right)\right) u_{q} \\
& +\left(\frac{\mu^{2}}{2 n p} \sum_{j \in[n] \backslash\{i\}} B_{q j}^{2} \operatorname{sech}^{2}\left(\eta_{j \rightarrow q}^{t}\right)\right) u_{q}^{2}+O\left(n^{-1 / 2}\right) .
\end{aligned}
$$

This implies,

$$
\begin{aligned}
\tau_{q \rightarrow i}^{t+1} & =\left(1+\mu-\frac{\mu}{\gamma} \sum_{j \in[n] \backslash\{i\}} \frac{B_{q j}^{2}}{p} \operatorname{sech}^{2}\left(\eta_{j \rightarrow q}^{t}\right)\right)^{-1} \\
m_{q \rightarrow i}^{t+1} & =\tau_{q \rightarrow i}^{t+1} \sqrt{\frac{\mu}{\gamma}} \sum_{j \in[n] \backslash\{i\}} \frac{B_{q j}}{\sqrt{p}} \tanh \left(\eta_{j \rightarrow q}^{t}\right) .
\end{aligned}
$$

Now, we shall consider an approximation to simplify the iterates. Let us define,

$$
\begin{aligned}
T_{i}^{t} & =\sqrt{\frac{\mu}{\gamma p}} \sum_{q \in[p]} B_{q i} m_{q \rightarrow i}^{t} ; \quad T_{i \rightarrow q}^{t}=\sqrt{\frac{\mu}{\gamma p}} \sum_{r \in[p] \backslash\{q\}} B_{r i} m_{r \rightarrow i}^{t} \\
S_{i \rightarrow j ; \ell}^{t} & =\sum_{c \in \partial_{\ell} i \backslash\{j\}} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{c \in\left(\partial_{\ell} i\right)^{c} \backslash\{j\}} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{n, \ell}\right) \\
S_{i ; \ell}^{t} & =\sum_{c \in \partial_{\ell} i} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{c \in\left(\partial_{\ell} i\right)^{c}} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{n, \ell}\right)
\end{aligned}
$$

If $i$ and $j$ have no edge between them in $\boldsymbol{G}_{\ell}$, then,

$$
\begin{aligned}
\eta_{i \rightarrow j ; \ell}^{t} & =\eta_{i, \ell}^{t}-f\left(\eta_{j \rightarrow i ; \ell}^{t-1} ; \rho_{n, \ell}\right) \\
& =\eta_{i ; \ell}^{t}+o\left(\rho_{n, \ell}\right)=\eta_{i ; \ell}^{t}+O\left(n^{-1}\right)
\end{aligned}
$$

Using Taylor expansion,

$$
\begin{aligned}
f\left(\eta_{i \rightarrow j ; \ell}^{t} ; \rho_{n, l}\right) & =f\left(\eta_{i ; \ell}^{t} ; \rho_{n, l}\right)+O\left(\frac{\tanh \left(\rho_{n, \ell}\right)}{n}\right) \\
& =f\left(\eta_{i ; \ell}^{t} ; \rho_{n, l}\right)+O\left(n^{-2}\right)
\end{aligned}
$$

Hence we shall do the following approximation,

$$
S_{i \rightarrow j ; \ell}^{t+1} \approx \sum_{c \in \partial_{\ell} i \backslash\{j\}} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{c \in[n]} f\left(\eta_{c ; \ell}^{t} ; \rho_{n, \ell}\right)
$$

So the first layer of iterates look like the following.

$$
\begin{aligned}
S_{i \rightarrow j ; \ell}^{t+1} & =\sum_{c \in \partial_{\ell} i \backslash\{j\}} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{c \in[n]} f\left(\eta_{c ; \ell}^{t} ; \rho_{n, \ell}\right) \\
S_{i ; \ell}^{t+1} & =\sum_{c \in \partial_{\ell} i} f\left(\eta_{c \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{c \in[n]} f\left(\eta_{c ; \ell}^{t} ; \rho_{n, \ell}\right) \\
\eta_{i \rightarrow j ; k}^{t+1} & =T_{i}^{t}+\sum_{\ell=1 ; \ell \neq k}^{m} S_{i ; \ell}^{t+1}+S_{i \rightarrow j ; k}^{t+1} \\
\eta_{i ; k}^{t+1} & =T_{i}^{t}+\sum_{\ell=1}^{m} S_{i ; \ell}^{t+1} .
\end{aligned}
$$

Now we use the following ansatz,

$$
\begin{aligned}
\eta_{i \rightarrow q}^{t} & =\eta_{i}^{t}+\delta \eta_{i \rightarrow q}^{t} \\
T_{i \rightarrow q}^{t} & =T_{i}^{t}+\delta T_{i \rightarrow q}^{t} \\
m_{q \rightarrow i}^{t} & =m_{q}^{t}+\delta m_{q \rightarrow i}^{t} \\
\tau_{q \rightarrow i}^{t} & =\tau_{q}^{t}+\delta \tau_{q \rightarrow i}^{t}
\end{aligned}
$$

where $\delta T_{i \rightarrow q}^{t}, \delta \eta_{i \rightarrow q}^{t}, \delta m_{q \rightarrow i}^{t}, \delta \tau_{q \rightarrow i}^{t}$ are each $O\left(n^{-1 / 2}\right)$. Now, observing that,

$$
\begin{aligned}
\eta_{i \rightarrow q}^{t+1} & =T_{i \rightarrow q}^{t}+\sum_{\ell=1}^{m}\left\{\sum_{k \in \partial_{\ell} i} f\left(\eta_{k \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{k \in[n]} f\left(\eta_{k ; \ell}^{t} ; \rho_{n, \ell}\right)\right\} \\
& =T_{i}^{t}+\sum_{\ell=1}^{m} S_{i ; \ell}^{t+1}-\sqrt{\frac{\mu}{\gamma p}}\left(B_{q i} m_{q}^{t}+B_{q i} m_{q \rightarrow i}^{t} \delta\right)
\end{aligned}
$$

Since, the term $B_{q i} T_{q \rightarrow i}^{t} \delta / \sqrt{p}, B_{q i} m_{q \rightarrow i}^{t} \delta / \sqrt{p}=O\left(n^{-1}\right)$, we can ignore it. Hence, we have the following approximation,

$$
\begin{align*}
T_{i}^{t} & =\sqrt{\frac{\mu}{\gamma p}} \sum_{q \in[p]} B_{q i}\left(m_{q}^{t}+\delta m_{q \rightarrow i}^{t}\right)  \tag{38}\\
\eta_{i}^{t+1} & =T_{i}^{t}+\sum_{\ell=1}^{m}\left\{\sum_{k \in \partial_{\ell} i} f\left(\eta_{k \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{k \in[n]} f\left(\eta_{k ; \ell}^{t} ; \rho_{n, \ell}\right)\right\} \\
\delta \eta_{i \rightarrow q}^{t+1} & \approx-\sqrt{\frac{\mu}{\gamma p}} B_{q i} m_{q}^{t}
\end{align*}
$$

One can also show using similar techniques that it is valid to approximate $\delta T_{i \rightarrow q}^{t} \approx-\sqrt{\frac{\mu}{\gamma}} B_{q i} m_{q}^{t}$. Following the techniques described in (128)-(130) of [15], we get the following approximation,

$$
\begin{aligned}
\tau_{q}^{t+1} & =\left(1+\mu-\frac{\mu}{p \gamma} \sum_{j \in[n]} B_{q j}^{2} \operatorname{sech}^{2}\left(\eta_{j}^{t}\right)\right)^{-1} \\
\delta \tau_{q}^{t+1} & \approx 0
\end{aligned}
$$

Similarly, using (133)-(136) of [15], we get,

$$
\begin{align*}
m_{q}^{t+1} & =\frac{\sqrt{\mu / \gamma}}{\tau_{q}^{t+1}} \sum_{j \in[n]} \frac{B_{q j}}{\sqrt{p}} \tanh \left(\eta_{j}^{t}\right)-\frac{\mu}{\gamma \tau_{q}^{t+1}}\left[\sum_{j \in[n]} \frac{B_{q j}^{2}}{p} \operatorname{sech}^{2}\left(\eta_{j}^{t}\right)\right] m_{q}^{t-1}  \tag{39}\\
\delta m_{q}^{t+1} & \approx-\frac{\sqrt{\mu / \gamma}}{\tau_{q}^{t+1} \sqrt{p}} B_{q i} \tanh \left(\eta_{i}^{t}\right) .
\end{align*}
$$

Plugging in (39) in (38), we get the Belief Propagation updates,

$$
\begin{aligned}
S_{i \rightarrow j ; \ell}^{t+1} & =\sum_{k \in \partial_{\ell} i \backslash\{j\}} f\left(\eta_{k \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{k \in[n]} f\left(\eta_{k}^{t} ; \rho_{n, \ell}\right) \\
S_{i ; \ell}^{t+1} & =\sum_{k \in \partial_{\ell} i} f\left(\eta_{k \rightarrow i ; \ell}^{t} ; \rho_{\ell}\right)-\sum_{k \in[n]} f\left(\eta_{k}^{t} ; \rho_{n, \ell}\right) \\
\eta_{k \rightarrow i ; \ell}^{t+1} & =T_{k}^{t}+\sum_{r \neq \ell} S_{k ; r}^{t+1}+S_{k \rightarrow i ; \ell}^{t+1} \\
\eta_{k}^{t+1} & =T_{k}^{t}+\sum_{r=1}^{m} S_{k ; r}^{t+1} \\
\tau_{q}^{t+1} & =\left(1+\mu-\frac{\mu}{p \gamma} \sum_{j \in[n]} B_{q j}^{2} \operatorname{sech}^{2}\left(\eta_{j}^{t}\right)\right)^{-1} \\
m_{q}^{t+1} & =\frac{\sqrt{\mu / \gamma}}{\tau_{q}^{t+1}} \sum_{j \in[n]} \frac{B_{q j}}{\sqrt{p}} \tanh \left(\eta_{j}^{t}\right)-\frac{\mu}{\gamma \tau_{q}^{t+1}}\left[\sum_{j \in[n]} \frac{B_{q j}^{2}}{p} \operatorname{sech}^{2}\left(\eta_{j}^{t}\right)\right] m_{q}^{t-1} \\
T_{k}^{t+1} & =\sqrt{\frac{\mu}{\gamma}} \sum_{r \in[p]} \frac{B_{r k}}{\sqrt{p}} m_{r}^{t+1}-\frac{\mu}{p \gamma}\left(\sum_{r \in[p]} \frac{B_{r k}^{2}}{\tau_{r}^{t+1}}\right) \tanh \left(\eta_{k}^{t}\right) .
\end{aligned}
$$

