# Exact detection thresholds and minimax optimality of Chatterjee's correlation coefficient 

Arnab Auddy, ${ }^{\S, *}$ Nabarun Deb,,$^{\S, \dagger}$ and Sagnik Nandy ${ }^{\S, \neq}$<br>* Department of Biostatistics, University of Pennsylvania,<br>${ }^{\dagger}$ Booth School of Business, University of Chicago,<br>$\ddagger$ Department of Statistics, University of Pennsylvania


#### Abstract

: Recently, Chatterjee (2021) introduced a new rank-based correlation coefficient which can be used to measure the strength of dependence between two random variables. This coefficient has already attracted much attention as it converges to the Dette-SiburgStoimenov measure (see Dette et al. (2013)), which equals 0 if and only if the variables are independent and 1 if and only if one variable is a function of the other. Further, Chatterjee's coefficient is computable in (near) linear time, which makes it appropriate for large-scale applications. In this paper, we expand the theoretical understanding of Chatterjee's coefficient in two directions: (a) First we consider the problem of testing for independence using Chatterjee's correlation. We obtain its asymptotic distribution under any changing sequence of alternatives converging to the null hypothesis (of independence). We further obtain a general result that gives exact detection thresholds and limiting power for Chatterjee's test of independence under natural nonparametric alternatives converging to the null. As applications of this general result, we prove a $n^{-1 / 4}$ detection boundary for this test and compute explicitly the limiting local power on the detection boundary for popularly studied alternatives in the literature. (b) We then construct a test for non-trivial levels of dependence using Chatterjee's coefficient. In contrast to testing for independence, we prove that, in this case, Chatterjee's coefficient indeed yields a minimax optimal procedure with a $n^{-1 / 2}$ detection boundary. Our proof techniques rely on Stein's method of exchangeable pairs, a non-asymptotic projection result, and information theoretic lower bounds.


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## 1. Introduction

Suppose $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right) \stackrel{\text { i.i.d. }}{\sim} F_{X, Y}(\cdot, \cdot)$ for some bivariate distribution function $F_{X, Y}(\cdot, \cdot)$, with marginals $F_{X}(\cdot)$ and $F_{Y}(\cdot)$ for $X_{1}$ and $Y_{1}$, respectively. The problem of measuring and testing the extent of dependence between $X_{1}$ and $Y_{1}$ has attracted much attention for over a century (see e.g., $[11,21,43,56,60]$ ). A fundamental question in this regard is the classical independence testing problem

$$
\begin{equation*}
\mathrm{H}_{0}: F_{X, Y}(x, y)=F_{X}(x) F_{Y}(y) \forall x, y \in \mathbb{R} \quad \text { versus } \quad \mathrm{H}_{1}: \text { not } \mathrm{H}_{0} . \tag{1.1}
\end{equation*}
$$

Problem (1.1) has been studied extensively in the statistics literature along with a variety of applications (see $[6,10,11,24,25,30,32,37,38,43,53,60,68,18,20,23,35,57,66$, $67,69,34]$ ). Note however that (1.1) tests only whether $X$ and $Y$ are fully independent

[^0]or otherwise, and does not give any indication as to how strong the dependence between $X$ and $Y$ is. Therefore to better understand the dependence, an alternative approach is to use a measure of dependence, say $\rho\left(F_{X, Y}\right)$, which can be chosen as classical measures such as Pearson's linear correlation coefficient [56], Spearman's rank correlation coefficient [61] or more nonparametric measures such as $[21,30]$ to name a few, and consider the testing problem
\[

$$
\begin{equation*}
\mathrm{H}_{0}: \rho\left(F_{X, Y}\right)=\rho_{0} \quad \text { versus } \quad \mathrm{H}_{1}: \rho\left(F_{X, Y}\right)>\rho_{0}, \tag{1.2}
\end{equation*}
$$

\]

for general $\rho_{0}$. Depending on the choice of $\rho\left(F_{X, Y}\right)$, problem (1.2) gives a more interpretable understanding of the dependence between $X_{1}$ and $Y_{1}$. For example, if $\rho\left(F_{X, Y}\right)$ is chosen to be Pearson's correlation, then (1.2) helps understand how well $Y_{1}$ can be predicted from $X_{1}$ using a linear function; and if $\rho\left(F_{X, Y}\right)$ is chosen to be Spearman's correlation, then it helps understand how well $Y_{1}$ can be predicted from $X_{1}$ using a general monotone transform.

In view of problems (1.1) and (1.2), recently, Chatterjee [15] introduced a new nonparametric data rank-based correlation coefficient $\xi_{n}$ (see (1.3) below for definition). This coefficient in [15] possesses a combination of natural, but unique characteristics not exhibited by other measures. In particular, it converges to 0 if and only if $X_{1}$ and $Y_{1}$ are independent and to 1 if and only if $Y_{1}$ is a measurable function of $X_{1}$, as long as $X_{1}$ and $Y_{1}$ are non-degenerate. In fact, Chatterjee's coefficient converges to the Dette-Siburg-Stoimenov measure (see [21]; also see (1.4)) for general bivariate distributions. It also produces a consistent test against all fixed alternatives for problems (1.1) and (1.2), and is computable in (near) linear time (in terms of sample size), making it suitable for large-scale applications. Furthermore, through extensive simulations in [15], the author argued that $\xi_{n}$ converges to a population measure $\xi\left(F_{X, Y}\right)$ (introduced first in [21]; see (1.4) below) that captures how well $Y_{1}$ can be predicted using general measurable functions of $X_{1}$ (see Section 4 for more details). This gives the testing problem (1.2) using $\rho\left(F_{X, Y}\right)=\xi\left(F_{X, Y}\right)$, a natural and completely nonparametric interpretation. Consequently, it has attracted much attention in the past two years, in terms of both applications and theory (see $[4,12,13,19,28,40,58,63,5,36,50,48,49,17,29,70,59,9,1,39,33]$ ). The goal of this article is to expand the theoretical understanding of $\xi_{n}$ for the widely popular testing problems in (1.1) and (1.2) under general local alternatives (that is, when alternative converges to null as the sample size $n$ increases) and obtain exact detection thresholds. Before discussing our main results, let us first present Chatterjee's test statistic in [15].

We arrange the data as $\left(X_{(1)}, Y_{(1)}\right), \ldots,\left(X_{(n)}, Y_{(n)}\right)$ so that $X_{(1)} \leq \ldots \leq X_{(n)}$ and $Y_{(i)}$ is the $Y$ value concomitant to $X_{(i)}$. Let $R_{n, i}$ be the rank of $Y_{i}$, i.e.,

$$
R_{n, i}:=\sum_{j=1}^{n} \mathbb{1}\left(Y_{j} \leq Y_{i}\right) .
$$

Now consider the statistic given by

$$
\begin{equation*}
\xi_{n}:=1-\frac{3}{n^{2}-1} \sum_{i=1}^{n-1}\left|R_{n,(i+1)}-R_{n,(i)}\right| . \tag{1.3}
\end{equation*}
$$

Here $R_{n,(i)}$ is the rank of $Y_{(i)}$. Note that this statistic is well defined if there are no ties among $X_{i}$ 's. For a more general definition of $\xi_{n}$ that allows for ties, see [15, Page 2]. For technical convenience, we will work with the above definition of $\xi_{n}$ instead of the more general definition that takes potential ties into account. In particular, we will assume the existence of marginal densities of $X$ 's and $Y$ 's for the rest of the paper.

It has been established in Theorem 1.1 of [15] that $\xi_{n}$, as defined in (1.3), converges almost surely to the Dette-Siburg-Stoimenov measure (see [21])

$$
\begin{equation*}
\xi\left(f_{X, Y}\right):=6 \int \mathbb{E}[\mathbb{P}(Y \leq t \mid X)-\mathbb{P}(Y \leq t)]^{2} f_{Y}(t) d t \tag{1.4}
\end{equation*}
$$

where $f_{X, Y}(\cdot, \cdot)$ denotes the density corresponding to the distribution function $F_{X, Y}(\cdot, \cdot)$, with marginal densities $f_{X}(\cdot)$ and $f_{Y}(\cdot)$. In various bivariate copula based models, $\xi\left(f_{X, Y}\right)$ turns out to be a monotonic function of the natural dependence parameter (see examples 1.(a) - (d) in [21, Page 9 and Theorem 2]; also see Section 4), thereby making it natural and interpretable as a way to measure the strength of dependence between $X_{1}$ and $Y_{1}$.

### 1.1. Problem setup and summary of main contributions

We will use the standard framework of local power analysis taken from [41, 42] which is popularly used in independence testing procedures, see e.g., [7, 44]. Towards this direction, consider a triangular array of i.i.d. random variables $\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, n}, Y_{n, n}\right)$ from a bivariate density $f_{X, Y}^{(n)}(\cdot, \cdot)$ with marginals $f_{X}^{(n)}(\cdot), f_{Y}^{(n)}(\cdot)$. Note that the joint distribution is no longer fixed, but changes with the sample size. We analyze the testing problem

$$
\begin{equation*}
\mathrm{H}_{0, n}: \xi\left(f_{X, Y}^{(n)}\right)=\xi_{0} \quad \text { versus } \quad \mathrm{H}_{1, n}: \xi\left(f_{X, Y}^{(n)}\right)=\xi_{0}+c_{n}, \quad \xi_{0} \in[0,1) \tag{1.5}
\end{equation*}
$$

where $\xi(\cdot)$ is as defined in (1.4) and

$$
\begin{equation*}
c_{n} \downarrow 0 \quad \Longrightarrow \xi\left(f_{X, Y}^{(n)}\right) \downarrow \xi_{0} \quad \text { as } n \rightarrow \infty \tag{1.6}
\end{equation*}
$$

Clearly distinguishing between $\mathrm{H}_{0, n}$ and $\mathrm{H}_{1, n}$ becomes harder as $c_{n}$ converges to 0 faster. Also note that by [15, Theorem 1.1] and [21, Theorem 2], testing $\xi_{0}=0$ exactly corresponds to the test of independence as in (1.1). As $\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)$ converges almost surely to 0 under mild assumptions (see [15, Theorem 1.1]), a natural level $\alpha$ test function for (1.5) is given by

$$
\begin{equation*}
\phi_{n}:=\mathbb{1}\left(\xi_{n}-\xi_{0} \geq z_{n, \alpha}\right), \tag{1.7}
\end{equation*}
$$

where $z_{n, \alpha}$ is chosen appropriately so as to satisfy the level $\alpha$ condition. Define the power function of $\phi_{n}$ as

$$
\begin{equation*}
\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right):=\mathbb{P}\left(\phi_{n}=1\right) . \tag{1.8}
\end{equation*}
$$

Our goal is to investigate the following: "What is the fastest decaying $c_{n} \downarrow 0$ such that $\phi_{n}$ can distinguish between the null and the alternative, i.e., $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right) \rightarrow 1$ as $n \rightarrow \infty$ under $\mathrm{H}_{1, n}$ ? Further, is $\phi_{n}$ optimal for the testing problem (1.5)?"

In this paper, we provide an exact answer to this question. We find a dichotomy based on whether or not the limiting Dette-Siburg-Stoimenov measure $\xi_{0}$ is zero or positive. We make a two-fold contribution in both of these regimes. Note that the case where $\xi_{0}=0$ in (1.5) corresponds to the important problem of independence testing. On the other hand, when $\xi_{0}>0$, we refer to the null hypothesis in (1.5) as the problem of testing for degree of association between $X$ and $Y$.

Critical detection boundary of $\phi_{n}$ for independence testing $\left(\xi_{0}=0\right)$ : For this case, in Theorem 2.2, we show that the power of $\phi_{n}$ converges to $\alpha$, or 1 , or a number in ( $\alpha, 1$ ) depending on whether $\sqrt{n} c_{n}$ converges to 0 , or $\infty$, or some number in $(0, \infty)$, respectively. This indicates that the best choice of $c_{n}$ that ensure $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right) \rightarrow 1$ is $c_{n}=O\left(n^{-1 / 2}\right)$.

This is however a suboptimal threshold in terms of detecting dependence. To see this, consider the case where $\left(X_{1}, Y_{1}\right)$ is a bivariate normal distribution with correlation $\rho_{n}$. Theorem 2.2 implies that $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right)$ converges to $\alpha$, or 1 , or a number in $(\alpha, 1)$ depending on whether $n^{1 / 4} \rho_{n}$ converges to 0 , or $\infty$, or some number in $(0, \infty)$, respectively. This implies that $\phi_{n}$ has a detection boundary of $O\left(n^{-1 / 4}\right)$ in terms of $\rho_{n}$. However, it is well known from Le Cam's theory of local asymptotic normality that the optimal detection threshold for $\rho_{n}$ is of the order $n^{-1 / 2}$ and not $n^{-1 / 4}$. In Section 3, we give concrete examples of this detection boundary in some other local parametric alternatives, viz. simple mixtures, and noisy nonparametric regression.

Minimax optimality of $\phi_{n}$ for testing degree of association $\left(\xi_{0}>0\right)$ : For this case, in Theorem 4.1, part 1, we show that $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right)$ converges to 1 provided $\sqrt{n} c_{n} \rightarrow \infty$. In contrast to the $\xi_{0}=0$ case, we prove in Theorem 4.1, part 2, that $c_{n}=O\left(n^{-1 / 2}\right)$ is indeed the optimal threshold when $\xi_{0}>0$ in a local asymptotic minimax sense (see (4.4)). In other words, if $\sqrt{n} c_{n} \rightarrow(0, \infty)$, then no test can uniformly have power converging to 1 . Therefore $c_{n}=O\left(n^{-1 / 2}\right)$ is the correct detection boundary and this highlights the minimax optimality of Chatterjee's correlation coefficient for the testing problem (1.5) when $\xi_{0}>0$; see Section 4 for more details.

Additionally, as a technical device for the results above, we develop a central limit theorem for $\xi_{n}$.

Central limit theorem for shrinking alternatives: In Theorem 2.1, we show that for any sequence of alternatives specified by $\xi\left(f_{X, Y}^{(n)}\right) \rightarrow 0, \xi_{n}$ is asymptotically normal. Further, we characterize the limiting mean and the limiting variance explicitly. This is a wide generalization of the asymptotic normality results in $[12,15,58]$ which are stated under independence, or the fast shrinking alternatives $\xi\left(f_{X, Y}^{(n)}\right)=O\left(n^{-1 / 2}\right)$. Theorem 2.1 weakens this assumption only to $\xi\left(f_{X, Y}^{(n)}\right) \rightarrow 0$ and might be of independent interest. This CLT is obtained using Stein's method-based technique (see [14]). After the first draft of our paper, a number of other interesting limit theorems for $\xi_{n}$ or modified versions of $\xi_{n}$ have been established that highlight the interest in Chatterjee's correlation; see e.g. [36, 48, 49, 50, 1, 70] to name a few. In the following section, we will summarize the other relevant contributions to the problem considered here.

### 1.2. Comparison with existing literature

Prior to the first version of our paper, the theoretical analysis of $\xi_{n}$ had been carried out in three papers. In Chatterjee's paper [15], the asymptotic distribution of $\xi_{n}$ was derived under $\mathrm{H}_{0}$ as in (1.1) and its consistency against fixed (not changing with $n$ ) alternatives was proved. Two other papers [58] and [12] have analyzed $\xi_{n}$ under smooth contiguous alternatives (see [65, Chapter 6]). For example, under the mixture type alternatives in Section 3.1, their results show that $\phi_{n}$ is powerless along "contiguous" alternatives, i.e., in cases where the mixing probability shrinks to zero at a $n^{-1 / 2}$ rate. The proofs in $[12,58]$ use Le Cam's third lemma (see [65, Example 6.7]) which requires analyzing $\xi_{n}$ and the likelihood ratio jointly but only under the null, (that is, when $X_{n, 1}$ and $Y_{n, 1}$ are independent) followed by a change of measure formula that only holds under contiguous scales and not beyond. In contrast, the focus of our paper is characterizing the exact detection boundary of $\xi_{n}$, which as we shall see, happens to be in the non-contiguous regime. We therefore adopt a proof strategy using Stein's method-based technique of local dependence (see [14]) and
non-asymptotic projection results. While the focus of the paper is in the bivariate setting, it should be noted that multivariate versions of $\xi_{n}$ have been studied in the literature (see $[4,19,1])$ and asymptotic distributions under independence have been obtained in $[19,59,1]$.

After the first draft of our paper, a number of other results of interest have further solidified the understanding of $\xi_{n}$ or modified versions thereof. In [48], the authors modified $\xi_{n}$ by incorporating more "right nearest neighbors" in its definition. They then proved that in the bivariate Gaussian independence testing problem (see Section 3), as the number of right nearest neighbors grow, the detection boundary moves from $O\left(n^{-1 / 4}\right)$ to nearly $O\left(n^{-1 / 2}\right)$. On the other hand, in [49], the authors show that $\xi_{n}$ is asymptotically normal even when $\xi\left(f_{X, Y}^{(n)}\right)=\xi_{0}>0$. The limiting variance, in that case, is no longer universal and depends on the data distribution. The authors in [49] obtain a consistent, analytic estimator of this limiting variance. In the follow-up paper [50], the authors show that a natural bootstrap estimator of this limiting variance is not consistent under independence. On the other hand, [70] proved the asymptotic normality of a symmetrized version of $\xi_{n}$. We also refer the reader to the recent review paper [16] which provides a comprehensive overview of dependence/association measures that are based on $\xi_{n}$.

### 1.3. Organization

The rest of the paper is organized as follows. In Section 2 we describe our main results when $\xi_{0}=0$. In particular, Theorem 2.1 and Theorem 2.2 yield asymptotic limits of $\xi_{n}$ (centered and scaled) and asymptotic expressions for $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right)$ depending on how fast $c_{n} \downarrow 0$. Applications of these results to test for independence in popular local parametric models are provided in Section 3. In particular, Corollary 3.1 and Corollary 3.2 highlight the $n^{-1 / 4}$ detection boundary. In Section 4 (see Theorem 4.1), we show that the test $\phi_{n}$ constructed in (1.7) is minimax optimal for the testing problem (1.5) when $\xi_{0}>0$. The proofs of all main results are presented in the Appendix A. Finally, Appendix B contains the proofs of some additional results and technical lemmas.

## 2. Critical detection boundary of $\phi_{n}$ for independence testing

In this section, we first show that under a wide class of bivariate distributions satisfying (1.6), $\xi_{n}$, appropriately centered by its mean and scaled by its standard deviation, converges to a standard normal distribution. We provide a precise characterization of the limiting bias and standard deviation of $\xi_{n}$. We then use these findings to obtain the detection threshold and asymptotic power of the test $\phi_{n}$ as described in (1.7).

Recall the setting from the Introduction. We consider $\left(X_{n, k}, Y_{n, k}\right)_{1 \leq k \leq n}, n \in \mathbb{N}$, a triangular array of i.i.d. random variables drawn from a bivariate density $f_{X, Y}^{(n)}(\cdot, \cdot)$. All the probabilities and expectations taken in the sequel are with respect to the measure induced by $f_{X, Y}^{(n)}(\cdot, \cdot)$.

Notice that from (1.3), with the identity $\sum_{i \neq j}|i-j|=n\left(n^{2}-1\right) / 3$, one has

$$
\xi_{n}=\frac{3}{n^{2}-1}\left(\frac{1}{n} \sum_{i \neq j} \sum_{i}|i-j|-\sum_{i=1}^{n}\left|R_{n,(i+1)}-R_{n,(i)}\right|\right) .
$$

We will need the following definitions.

Definition 2.1 (Kantorovic-Wasserstein distance). Given any two probability measure $\mu$ and $\nu$ on the real line, the Kantorovic-Wasserstein distance between $\mu$ and $\nu$ is defined as

$$
\mathcal{W}(\mu, \nu):=\sup \left\{\left|\int h d \mu-\int h d \nu\right|: h(\cdot) \text { is Lipschitz, }\|h\|_{\text {Lip }} \leq 1\right\}
$$

In our applications, we will fix $\nu$ as the standard Gaussian distribution, which leads to the following natural notion of "distance to Gaussianity" based on Definition 2.1.

Definition 2.2 (Distance to Gaussianity). Let $\nu$ be the standard Gaussian law and $T$ be $a$ random variable with the law $\mu$. Then the distance between $T$ and the standard Gaussian is defined as

$$
\mathfrak{D}(T):=\mathcal{W}(\mu, \nu)
$$

where $\nu$ is the standard normal law.

We first state our assumptions.
Assumption (A1). There exist functions $L_{i}^{(n)}(\cdot): \mathbb{R} \rightarrow[0, \infty)$ for $i=1,2$ and numerical constants $\kappa_{1}>0, \eta \in(0,1]$, and $\theta \in(1, \infty]$ such that $\forall y, x_{1}, x_{2}$,

$$
\left|\mathbb{P}\left(Y_{n, 1} \geq y \mid X_{n, 1}=x_{1}\right)-\mathbb{P}\left(Y_{n, 1} \geq y \mid X_{n, 1}=x_{2}\right)\right| \leq\left(1+L_{1}^{(n)}\left(x_{1}, y\right)+L_{2}^{(n)}\left(x_{2}, y\right)\right)\left|x_{1}-x_{2}\right|^{\eta}
$$

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \int\left(L_{i}^{(n)}(x, y)\right)^{\theta} f_{X}^{(n)}(x) f_{Y}^{(n)}(y) d x d y \leq \kappa_{1} \tag{2.1}
\end{equation*}
$$

where $f_{X}^{(n)}(\cdot)$ and $f_{Y}^{(n)}(\cdot)$ are the marginal densities of $X_{n, 1}$ and $Y_{n, 1}$ under the joint density $f_{X, Y}^{(n)}(\cdot, \cdot)$.

Assumption (A2). There exist numerical constants $\gamma>1$ and $\kappa_{2}>0$ such that

$$
\limsup _{n \rightarrow \infty} \mathbb{E}\left|X_{n, 1}\right|^{\gamma} \leq \kappa_{2}
$$

Note that Assumption (A3) is weaker than the standard $\eta$ Hölder condition, in that the Hölder constants are allowed to depend on $y, x_{1}, x_{2}$ and also $n$. In this sense, it is weaker than the assumptions in [4] and related papers. Assumption (A4) is a standard moment assumption to control the tail of the distribution of $X_{n, i}$ 's. This tail behavior crucially affects the distance between $X_{n,(i)}$ and $X_{n,(i+1)}$, see e.g., [8, Section 2.2].

We are now in position to state our main result. In the following theorem (see Section A. 1 for a proof), we show that $\xi_{n}$, appropriately centered and scaled, has a limiting normal distribution in the asymptotic regime $\xi\left(f_{X, Y}^{(n)}\right) \rightarrow 0$.

Theorem 2.1. For any bivariate density $f_{X, Y}^{(n)}(x, y)$ satisfying Assumptions (A3) and (A4), there is a numerical constant $C\left(\eta, \theta, \gamma, \kappa_{1}, \kappa_{2}\right)>0$, i.e., depending only on the parameters $\eta, \theta, \gamma, \kappa_{1}, \kappa_{2}$ from Assumptions (A3) and (A4), such that:
(i) For all $n \geq 1$, we have

$$
\sqrt{n}\left|\mathbb{E}\left(\xi_{n}\right)-\xi\left(f_{X, Y}^{(n)}\right)\right| \leq C n^{-1 / 2}+C \sqrt{n} b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)
$$

(ii) Moreover,

$$
\mathfrak{D}\left(\frac{\sqrt{n}\left(\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right)}{\sqrt{2 / 5}}\right) \leq C n^{-1 / 2}+C\left(\xi\left(f_{X, Y}^{(n)}\right)+\sqrt{n \log n} b_{n}\right)^{1 / 2} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right),
$$

where $\mathfrak{D}$ is the Wasserstein distance to normality defined in Definition 2.2, and $b_{n}$ is defined as

$$
\begin{equation*}
b_{n}:=n^{-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)} . . . ~ . ~ . ~} \tag{2.3}
\end{equation*}
$$

In particular, part (ii) of Theorem 2.1 shows that if $\xi\left(f_{X, Y}^{(n)}\right) \rightarrow 0$ and $\frac{\theta-1}{\theta} \wedge \eta>\frac{\gamma+1}{2 \gamma}$, then

$$
\begin{equation*}
\sqrt{n}\left(\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right) \xrightarrow{w} \mathcal{N}(0,2 / 5) . \tag{2.4}
\end{equation*}
$$

Therefore, in the entire asymptotic regime $\xi\left(f_{X Y Y}^{(n)}\right) \rightarrow 0$, we see that $\xi_{n}$ has the same limiting variance, which matches the case where the $X$ 's and $Y$ 's are mutually independent (also see [15, Theorem 2.1]). A couple of remarks on the assumptions needed in Theorem 2.1 are now in order.

Remark 2.1. To understand the condition $\frac{\theta-1}{\theta} \wedge \eta>\frac{\gamma+1}{2 \gamma}$, let us first focus on a simple case. Assume that the conditional probability in (3.14) is uniformly Lipschitz in $y, x$. In that case, $\eta=1$ and $L_{1}^{(n)}(\cdot), L_{2}^{(n)}(\cdot)$ are both uniformly bounded. In view of (3.15), this implies $\theta=\infty$. Therefore $\frac{\theta-1}{\theta} \wedge \eta=1$. Recall from Assumption (A4) that $\gamma>1$ denotes the number of finite moments of $X_{n, 1}$. Therefore, $\frac{\gamma+1}{2 \gamma}<1$. As a result, the condition $\frac{\theta-1}{\theta} \wedge \eta>\frac{\gamma+1}{2 \gamma}$ holds in this case.
More generally speaking, the condition $\frac{\theta-1}{\theta} \wedge \eta>\frac{\gamma+1}{2 \gamma}$ can be rewritten as the combination of the following conditions:

$$
\frac{1}{\theta}+\frac{1}{2 \gamma}<1, \quad \frac{1}{\gamma}<2 \eta-1
$$

Therefore, if the Hölder exponent $\eta$ in (3.14) is greater than $\frac{1}{2}$, then the condition $\frac{\theta-1}{\theta} \wedge$ $\eta>\frac{\gamma+1}{2 \gamma}$ holds whenever $L_{1}^{(n)}\left(X_{n, 1}, Y_{n, 2}\right), L_{2}^{(n)}\left(X_{n, 1}, Y_{n, 2}\right)$, and $X_{n, 1}$ have sufficiently light tails.

Remark 2.2. In this paper, we have restricted ourselves to the case where the Hölder exponent $\eta \leq 1$ instead of expanding to higher-order Hölder regularity. This is because parts (i) and (ii) of Theorem 2.1 show that the order of the bias $\mathbb{E}\left(\xi_{n}\right)-\xi\left(f_{X, Y}^{(n)}\right)$ is o $(1 / \sqrt{n})$ which is already of a smaller order than its fluctuation $\left(\xi_{n}-\mathbb{E} \xi_{n}\right)=O_{p}(1 / \sqrt{n})$. As a result, imposing stronger regularity leads to no further gains here. The situation would be different for the multivariate version(s) of Chatterjee's correlation (see [4, 19]) where the bias would reflect a curse of dimensionality and be of a higher order than the fluctuations. While this is an interesting question, it is currently beyond the scope of this paper.

Note that Theorem 2.1 greatly generalizes the asymptotic normality theorems of $[12,15,58]$ that are only valid under the null hypothesis of independence, or for contiguous
parametric alternatives and cannot be used to analyze $\xi_{n}$ along non-contiguous and nonparametric alternatives. Therefore we believe Theorem 2.1 is of independent interest and hence we have presented it here as a separate result. Since Theorem 2.1 aims to provide asymptotic normality for any alternative with $\xi\left(f_{X, Y}^{(n)}\right) \rightarrow 0$, we can no longer use traditional instruments such as Le Cam's third lemma, which is the main tool in [12, 58]. Note that Theorem 2.1 holds for a large nonparametric class of distributions and comes with finite sample guarantees. In order to prove Theorem 2.1, we use Stein's method of normal approximation for locally dependent structures [14] and some explicit bias and variance computations. To elaborate briefly, we observe that $\xi_{n}$ can be rewritten as

$$
\begin{equation*}
\xi_{n}=\frac{3 n}{n^{2}-1}\left(\frac{1}{n} \sum_{i \neq j} \sum_{\mid}\left|\widehat{F}_{n}\left(Y_{n, i}\right)-\widehat{F}_{n}\left(Y_{n, j}\right)\right|-\sum_{i=1}^{n}\left|\widehat{F}_{n}\left(Y_{n,(i+1)}\right)-\widehat{F}_{n}\left(Y_{n,(i)}\right)\right|\right) \tag{2.5}
\end{equation*}
$$

where $\widehat{F}_{n}(\cdot)$ is the empirical cumulative distribution function (CDF) of $Y_{n, 1}, \ldots, Y_{n, n}$. Let $F_{Y}^{(n)}(\cdot)$ denote the population CDF of $Y_{n, 1}$. Recall that $X_{n,(1)} \leq \ldots \leq X_{n,(n)}$ and $Y_{n,(i)}$ is the $Y$ value concomitant to $X_{n,(i)}$. The main idea is to show that we can replace $\widehat{F}_{n}(\cdot)$ by $F_{Y}^{(n)}(\cdot)$ asymptotically. In other words, we show that $\xi_{n}$ is close (with quantitative error bounds) to $\xi_{n}^{*}$, where

$$
\begin{equation*}
\xi_{n}^{*}:=\frac{3 n}{n^{2}-1}\left(\frac{1}{n} \sum_{i \neq j} \sum_{j}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, j}\right)\right|-\sum_{i=1}^{n}\left|F_{Y}^{(n)}\left(Y_{n,(i+1)}\right)-F_{Y}^{(n)}\left(Y_{n,(i)}\right)\right|\right) \tag{2.6}
\end{equation*}
$$

Next, we quantify the proximity of the distribution of $\xi_{n}^{*}$ (appropriately centered and scaled) to a standard normal distribution, using [14, Theorem 3.4], to establish Theorem 2.1.

Now, we characterize the asymptotic behavior of $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right)$ defined in (1.8). First, suppose that $\sqrt{n} \xi\left(f_{X, Y}^{(n)}\right) \rightarrow \infty$. As $\sqrt{n}\left(\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right)=O_{p}(1)$, by (2.4), it is clear that

$$
\sqrt{n} \xi_{n}=\underbrace{\sqrt{n}\left(\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right)}_{O_{p}(1)}+\sqrt{n} \xi\left(f_{X, Y}^{(n)}\right) \xrightarrow{P} \infty
$$

Therefore, the last equality in (1.8) coupled with the above display implies that whenever $\sqrt{n} \xi\left(f_{X, Y}^{(n)}\right) \rightarrow \infty$, we have $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right) \rightarrow 1$. By using a similar sequence of arguments, we get the complete picture of the limiting behavior of $\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right)$, formalized in the theorem below.

Theorem 2.2. Suppose $\xi\left(f_{X, Y}^{(n)}\right) \rightarrow 0$ and Assumptions (A3), (A4) are satisfied with $\eta, \theta, \gamma$ such that $\frac{\theta-1}{\theta} \wedge \eta>\frac{\gamma+1}{2 \gamma}$. Let $z_{\alpha}$ be the upper $\alpha$ quantile of the standard normal distribution. Then the test $\phi_{n}$, defined in (1.7), with $z_{n, \alpha}=n^{-1 / 2} z_{\alpha} \sqrt{2 / 5}$ has a power function satisfying

$$
\beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right)=\mathbb{P}\left(\sqrt{n} \xi_{n} \geq z_{\alpha} \sqrt{2 / 5}\right)=1-\Phi\left(z_{\alpha}-\sqrt{n} \xi\left(f_{X, Y}^{(n)}\right) / \sqrt{2 / 5}\right)+o(1) .
$$

On the other hand, when $\xi\left(f_{X, Y}^{(n)}\right) \rightarrow c>0$, $\phi_{n}$ has asymptotic power equal to one. In
particular, we have the following explicit characterization of the asymptotic power of $\phi_{n}$ :

$$
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y}^{(n)}\right)= \begin{cases}\alpha & \text { if } \sqrt{n} \xi\left(f_{X, Y}^{(n)}\right) \rightarrow 0  \tag{2.7}\\ 1-\Phi\left(z_{\alpha}-c_{0} / \sqrt{2 / 5}\right) & \text { if } \sqrt{n} \xi\left(f_{X, Y}^{(n)}\right) \rightarrow c_{0} \in(0, \infty) \\ 1 & \text { if } \sqrt{n} \xi\left(f_{X, Y}^{(n)}\right) \rightarrow \infty\end{cases}
$$

Theorem 2.2 reduces the power calculation to calculating the population measure of association under the alternative. Once again we emphasize that these results do not depend on any specific form of the alternative distribution. In Section 3 we consider the applications of Theorem 2.2 to certain parametric classes of alternatives, previously considered in the literature. In doing so, we discover that for smooth parametric alternatives, the detection threshold is seen on a non-standard scale of $n^{-1 / 4}$. This is much larger than the optimal detection threshold of $n^{-1 / 2}$ in parametric problems, thereby leading to the suboptimality of $\xi_{n}$ for testing independence (also see [58, 12]). We provide a detailed account of this in the following section.

## 3. Applications

Throughout this section, we will use the test $\phi_{n}$ in (1.7) with $z_{n, \alpha}=n^{-1 / 2} z_{\alpha} \sqrt{2 / 5}$, where $z_{\alpha}$ is the upper $\alpha$ quantile of the standard normal distribution. To interpret the detection boundary from Theorem 2.2 in terms of its rate of decay with respect to $n$, it is crucial to note that $\xi\left(f_{X, Y}^{(n)}\right)$ (see ( $\left.\mathbf{P} 1\right)$ ) is the integrated squared distance between a conditional and a marginal distribution function. For example, consider the case where $f_{X, Y}^{(n)}(\cdot, \cdot)$ is the standard bivariate density Gaussian with correlation $\rho_{n}$. Then $Y \mid X=x \sim \mathcal{N}\left(\rho_{n} x, 1-\rho_{n}^{2}\right)$. Let $\phi(\cdot)$ and $\Phi(\cdot)$ be the standard normal density and distribution functions, respectively. By first-order Taylor approximations, we then get

$$
\xi\left(f_{X, Y}^{(n)}\right)=6 \int \mathbb{E}\left[\Phi\left(\frac{t-\rho_{n} X}{\sqrt{1-\rho_{n}^{2}}}\right)-\Phi(t)\right]^{2} \phi(t) d t=\rho_{n}^{2} \sqrt{3} / \pi+o\left(\rho_{n}^{2}\right)
$$

This implies $\xi\left(f_{X, Y}^{(n)}\right) / \rho_{n}^{2} \rightarrow \sqrt{3} / \pi$, that is, $\xi\left(f_{X, Y}^{(n)}\right)$ scales like $\rho_{n}^{2}$ instead of $\rho_{n}$. This is the result of $\xi\left(f_{X, Y}^{(n)}\right)$ being an integrated squared distance. Using this observation in (2.7), we get that

$$
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y, r_{n}}\right)= \begin{cases}\alpha & \text { if } n^{1 / 4} \rho_{n} \rightarrow 0  \tag{3.1}\\ 1-\Phi\left(z_{\alpha}-c_{1}\right) & \text { if } n^{1 / 4} \rho_{n} \rightarrow c_{0} \in(0, \infty) \\ 1 & \text { if } n^{1 / 4} \rho_{n} \rightarrow \infty\end{cases}
$$

This shows that a $n^{-1 / 4}$ detection boundary emerges naturally out of Theorem 2.2 in the bivariate Gaussian setting.

In this section we describe some applications of Theorem 2.2 in three popular local parametric models: mixture-type alternatives and noisy nonparametric regression. The detection thresholds and local powers for these two models are formally stated in Corollaries 3.1 and 3.2 , respectively. The analysis of local asymptotic power of various tests along parametric alternatives reveals specific features of popular parametric models that control the power of testing procedures. Consequently a lot of attention has been devoted to such analysis (see [22, 46, 47, 52, 55, 54]). Our general result as in Theorem 2.2 can be used directly
to get detection boundaries and local powers under a number of popular alternatives, both along "contiguous" (meaning $O\left(n^{-1 / 2}\right)$ perturbations around the null) and non-contiguous scales, all in one go. We consider two such local parametric models in Section 3:
(a) Mixture type alternatives used in [2, 22, 26, 27, 58, 62]; see Section 3.1,
(b) Noisy nonparametric regression used in [15]; see Section 3.2

### 3.1. Simple mixture model

Consider the bivariate density of ( $X, Y$ ) defined by

$$
\begin{equation*}
f_{X, Y, r}(x, y):=(1-r) f_{X}(x) f_{Y}(y)+r g_{X, Y}(x, y) \quad \forall x, y \in \mathbb{R} \tag{3.2}
\end{equation*}
$$

where $g_{X, Y}(\cdot, \cdot)$ is a bivariate density function that does not factor into the product of its marginals, $f_{X}(\cdot), f_{Y}(\cdot)$ are univariate densities, and $r \in[0,1]$. We also note that if $r=0$, then $X$ is independent of $Y$. Therefore it suffices to test if $r=0$ or otherwise. Suppose that $g_{X, Y}(\cdot, \cdot)$ has marginals $f_{X}(\cdot)$ and $f_{Y}(\cdot)$, i.e.,

$$
\begin{equation*}
\int_{x} g_{X, Y}(x, y) d x=f_{Y}(y), \quad \int_{y} g_{X, Y}(x, y) d y=f_{X}(x) . \tag{3.3}
\end{equation*}
$$

Note that (3.3) implies that the marginals do not change under the alternative. Consequently, we cannot use marginal-based tests (e.g., goodness-of-fit on marginal distributions) to distinguish between the null and alternative. Instead, we will require an independence test as demonstrated above. Furthermore, we also assume that there exist $\kappa_{1}, \kappa_{2}>0, \eta \in[0,1]$, and $\theta, \gamma>1$ such that

$$
\begin{gather*}
\int_{y}^{\infty}\left|g_{Y \mid X=x_{1}}(t)-g_{Y \mid X=x_{2}}(t)\right| d t \leq\left(1+L_{1}\left(x_{1}, y\right)+L_{2}\left(x_{2}, y\right)\right)\left|x_{1}-x_{2}\right|^{\eta}  \tag{3.4}\\
\limsup _{n \rightarrow \infty} \int\left(L_{i}(x, y)\right)^{\theta} f_{X}(x) f_{Y}(y) d x d y \leq \kappa_{1} \tag{3.5}
\end{gather*}
$$

and

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \int|x|^{\gamma} f_{X}(x) d x<\kappa_{2} \tag{3.6}
\end{equation*}
$$

Let us observe that (3.4) is a mild regularity assumption on the conditional distribution of $Y$ given $X$ when their joint distribution has density $g(\cdot, \cdot)$. Many common bivariate density functions, like bivariate normal with finite mean and variance, satisfy this assumption.

To perform local power analysis under model (3.2), we adopt the same framework as in [58, 26, 62]. Towards this direction, fix a sequence $\left\{r_{n}\right\}_{n \geq 1}$ with $r_{n} \in[0,1]$ for all $n \geq 1$ and consider the family of bivariate densities $f_{X, Y, r_{n}}(\cdot, \cdot)$ as in (3.2). It is easy to check that the condition (1.6), i.e., $\xi\left(f_{X, Y, r_{n}}\right) \rightarrow 0$ holds if $r_{n} \rightarrow 0$. In the same vein as in (1.5), we consider the following testing problem:

$$
\begin{equation*}
\mathrm{H}_{0}: r_{n}=0 \quad \text { vs. } \quad \mathrm{H}_{1, n}: r_{n}>0 . \tag{3.7}
\end{equation*}
$$

In view of (1.6) we focus on the shrinking alternative $r_{n} \rightarrow 0$. Our object of interest is the limiting power function, i.e.,

$$
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y, r_{n}}\right), \quad \text { where } \lim _{n \rightarrow \infty} r_{n}=0
$$

Crucially, Theorem 2.2 reduces the above problem to characterizing the asymptotic behavior of $\xi\left(f_{X, Y, r_{n}}\right)$. This is the subject of the following proposition.

Proposition 3.1. Let us consider $\xi\left(f_{X, Y, r_{n}}\right)$ defined in (1.4) for a sequence $\left\{r_{n}\right\}_{n \geq 1}$ such that $r_{n} \in[0,1]$ and $r_{n} \rightarrow 0$. Then we have

$$
\xi\left(f_{X, Y, r_{n}}\right)=r_{n}^{2} \xi\left(g_{X, Y}\right)
$$

From Proposition 3.1 it is evident that $\sqrt{n} \xi\left(f_{X, Y, r_{n}}\right) \rightarrow 0$ or $\infty$ accordingly as $n^{1 / 4} r_{n} \rightarrow 0$ or $\infty$. Further, $\xi\left(f_{X, Y, r_{n}}\right)$ also increases with $\xi\left(g_{X, Y}\right)$. Recall from (P1) that $\xi\left(g_{X, Y}\right)$ is a measure of association between $X, Y$ jointly sampled according to $g_{X, Y}(\cdot, \cdot)$. Therefore, Proposition 3.1 shows that the power of $\phi_{n}$ is governed by the strength of association between $X, Y$ when they are jointly sampled from $g_{X, Y}(\cdot, \cdot)$.

We now present the complete characterization of the asymptotic power of $\phi_{n}$ for the problem (3.7), which follows immediately from Proposition 3.1 coupled with Theorem 2.2.

Corollary 3.1. Consider the problem of testing $\mathrm{H}_{0}$ versus the sequence of alternatives $\mathrm{H}_{1, n}$ defined by (3.7), for a sequence $\left\{r_{n}\right\}_{n \geq 1}$ such that $r_{n} \in[0,1]$ and $r_{n} \rightarrow 0$. Suppose that (3.3) holds and Assumptions (3.4), (3.5) and (3.6) are satisfied with $\frac{\theta-1}{\theta} \wedge \eta>\frac{\gamma+1}{2 \gamma}$. Then the asymptotic power is given by

$$
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y, r_{n}}\right)= \begin{cases}\alpha & \text { if } n^{1 / 4} r_{n} \rightarrow 0  \tag{3.8}\\ 1-\Phi\left(z_{\alpha}-c_{0}^{2} \xi\left(g_{X, Y}\right) / \sqrt{2 / 5}\right) & \text { if } n^{1 / 4} r_{n} \rightarrow c_{0} \in(0, \infty) \\ 1 & \text { if } n^{1 / 4} r_{n} \rightarrow \infty\end{cases}
$$

Remark 3.1. In Corollary 3.1, the assumption (3.3) can be dropped. This will not change the detection threshold, but would alter the expression of the local power when $n^{1 / 4} r_{n} \rightarrow c_{0}$ to a more complicated and less interpretable expression. We refrain from presenting that version to facilitate easier understanding.

In Figures 1 and 2, we provide a numerical illustration of Corollary 3.1. We generate data using the model in Section 3.1 with $n=4000, f_{X} \equiv \mathcal{N}(0,1), f_{Y} \equiv \mathcal{N}(0,1), g_{X, Y} \equiv \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ where

$$
\boldsymbol{\Sigma}=\left(\begin{array}{cc}
1 & 0.95 \\
0.95 & 1
\end{array}\right)
$$

and the mixing probability $r_{n}=n^{-b}, b \in[0,0.5]$. The power of the test, when averaged over 10000 runs, is plotted in Figure 1 as $b$ varies in $[0,0.5]$. Here $\beta_{*}$ is a Monte Carlo estimate (with 10000 replications) of the limiting local power under model (3.2) with $r_{n}=n^{-0.25}$ and $\boldsymbol{\Sigma}$ as specified above; $\beta_{*} \approx 0.295$. Figure 1 clearly shows that the power decays sharply from 1 to 0 as $b$ varies between 0.2 and 0.3 . In fact, when $b$ is close to 0.25 , the empirical power is very close to the theoretical power $\beta^{*}$. A similar agreement between the empirical and the theoretical distributions is also observed in Figure 2 where we plot the histogram of $\sqrt{n}\left(\xi_{n}-r_{n}^{2} \xi\left(g_{X, Y}\right)\right) / \sqrt{2 / 5}$ under $r_{n}=n^{-0.25}$ and overlay it with the standard normal density curve, thus verifying (2.4) and Proposition 3.1.

### 3.2. Regression model

In our main motivating paper [15], the author considered the following model in numerical experiments:

$$
\begin{equation*}
Y=g(X)+\sigma Z, \tag{3.9}
\end{equation*}
$$



Figure 1: Power of $\phi_{n}$ for (3.7) at different values of $b$.


Figure 2: A histogram of $\sqrt{n}\left(\xi_{n}-r_{n}^{2} \xi\left(g_{X, Y}\right)\right) / \sqrt{2 / 5}$ for $r_{n}=n^{-0.25}$, with the standard normal density curve overlaid in red.
where $\sigma \geq 1, g: \mathbb{R} \rightarrow \mathbb{R}, \mathbb{E}\left[|g(X)|^{3}\right]<\infty, Z \sim N(0,1)$ and $X, Z$ are independent. We also assume that $X$ has a finite $\gamma$-th moment for some $\gamma>3$ and $g$ satisfies

$$
\left|g\left(x_{1}\right)-g\left(x_{2}\right)\right| \leq\left(1+\left|x_{1}\right|+\left|x_{2}\right|\right)\left|x_{2}-x_{1}\right| .
$$

This is the classical noisy nonparametric regression model. Note that, as $\sigma \rightarrow \infty$, the "noise" part of the model (3.9) dominates the "signal" part given by $g(X)$. Therefore, as $\sigma \rightarrow \infty$, the independence between the "noise" and the "signal" makes it harder and harder for the independence testing procedures to have a high power. This makes it interesting to study the performance of $\phi_{n}$ in the context of model (3.9) and to obtain detection thresholds in terms of $\sigma$.

Therefore, we consider the natural parametric model with $f_{X, Y, \sigma_{n}}(\cdot, \cdot)$ being the joint density of ( $X, Y$ ) drawn according to the model (3.9) with $\sigma \equiv \sigma_{n}$. It is easy to check that $\xi\left(f_{X, Y, \sigma_{n}}\right) \rightarrow 0$ as $\sigma_{n} \rightarrow \infty$. Consequently, in the same spirit as in the previous section, we are interested in the following limiting power function:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y, \sigma_{n}}\right), \quad \text { where } \lim _{n \rightarrow \infty} \sigma_{n}=\infty \tag{3.10}
\end{equation*}
$$

As in Section 3.1, we first state a proposition characterizing the asymptotics of $\xi\left(f_{X, Y, \sigma_{n}}\right)$ as $n \rightarrow \infty$.

Proposition 3.2. Consider the model (3.9) with $\sigma \equiv \sigma_{n}$. We then have the following identity:

$$
\xi\left(f_{X, Y, \sigma_{n}}\right)=\frac{\sqrt{3}}{\pi} \sigma_{n}^{-2} \operatorname{Var}(g(X))+o\left(\sigma_{n}^{-2}\right)
$$

Proposition 3.2 shows that $\sqrt{n} \xi\left(f_{X, Y, \sigma_{n}}\right) \rightarrow 0$ or $\infty$ accordingly as $n^{-1 / 4} \sigma_{n} \rightarrow \infty$ or 0 . Also $\xi\left(f_{X, Y, \sigma_{n}}\right)$ increases with $\operatorname{Var}(g(X))$. This is indeed very intuitive. Note that if $\operatorname{Var}(g(X))=0$ then $g(\cdot)$ is a constant function which means $X, Y$ are independent according to model (3.9). Also note that $\operatorname{Var}(\mathbb{E}(Y \mid X))=\operatorname{Var}(g(X))$ and $\mathbb{E} \operatorname{Var}(Y \mid X)=\sigma_{n}^{2}$, i.e., $\operatorname{Var}(g(X))$ measures the proportion of the total variance of $Y$ which is explained by $X$. Therefore, it is only natural that the larger the value of $\operatorname{Var}(g(X))$, the larger is the power of $\phi_{n}$.

We now present the complete answer to problem (3.10), which follows immediately from Proposition 3.2 coupled with Theorem 2.2.

Corollary 3.2. Consider the problem in (3.9). Then the asymptotic power is given by

$$
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y, \sigma_{n}}\right)= \begin{cases}\alpha & \text { if } n^{-1 / 4} \sigma_{n} \rightarrow \infty  \tag{3.11}\\ 1-\Phi\left(z_{\alpha}-c_{0}^{2} \sqrt{3} / \pi \operatorname{Var}(g(X)) / \sqrt{2 / 5}\right) & \text { if } n^{-1 / 4} \sigma_{n} \rightarrow c_{0}^{-1} \in(0, \infty) \\ 1 & \text { if } n^{-1 / 4} \sigma_{n} \rightarrow 0\end{cases}
$$

### 3.3. Rotation alternatives

As a third example, we now consider the pair of random variables $(X, Y)$ satisfying

$$
\binom{X}{Y}=\left(\begin{array}{cc}
1 & \Delta  \tag{3.12}\\
\Delta & 1
\end{array}\right)\binom{U}{V}
$$

where $U, V$ are independent, zero mean random variables with densities $f_{1}$ and $f_{2}$. We further assume that $f_{1}$ and $f_{2}$ are twice differentiable with the $i$-th derivative, $i \in\{0,1,2\}$ being denoted by $f_{1}^{(i)}$ and $f_{2}^{(i)}$ respectively. The 0 -th derivative is the function itself. Note that $X$ and $Y$, drawn according to model (3.12), are independent if and only if $\Delta=0$.

To perform a local power analysis, we adopt the same framework as in [31, 45, 58]. Consider $(X, Y) \sim f_{X, Y, \Delta_{n}}(\cdot, \cdot)$ as in (3.12) with $\Delta \equiv \Delta_{n}$. It is easy to check that $\xi\left(f_{X, Y, \Delta_{n}}\right) \rightarrow 0$ holds if $\Delta_{n} \rightarrow 0$. In the same vein of the earlier examples, we are interested in studying the following limiting power function

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y, \Delta_{n}}\right), \quad \text { where } \lim _{n \rightarrow \infty} \Delta_{n}=0 \tag{3.13}
\end{equation*}
$$

Before stating the main results of this section, we need some assumptions which are encapsulated below. Note that the first two assumptions are also required for the our main results in Section 2.

Assumption (A3). There exist functions $L_{i}^{(n)}(\cdot): \mathbb{R} \rightarrow[0, \infty)$ for $i=1,2$ and numerical constants $\kappa_{1}>0, \eta \in(0,1]$, and $\theta>1$ such that $\forall y, x_{1}, x_{2}$,

$$
\begin{gather*}
\left|\mathbb{P}\left(Y_{n, 1} \geq y \mid X_{n, 1}=x_{1}\right)-\mathbb{P}\left(Y_{n, 1} \geq y \mid X_{n, 1}=x_{2}\right)\right| \leq\left(1+L_{1}^{(n)}\left(x_{1}, y\right)+L_{2}^{(n)}\left(x_{2}, y\right)\right)\left|x_{1}-x_{2}\right|^{\eta} \\
\limsup _{n \rightarrow \infty} \int\left(L_{i}^{(n)}(x, y)\right)^{\theta} f_{X}^{(n)}(x) f_{Y}^{(n)}(y) d x d y \leq \kappa_{1}, \tag{3.14}
\end{gather*}
$$

where $f_{X}^{(n)}(\cdot)$ and $f_{Y}^{(n)}(\cdot)$ are the marginal densities of $X_{n, 1}$ and $Y_{n, 1}$ under the joint density $f_{X, Y}^{(n)}(\cdot, \cdot)$.

Assumption (A4). There exist numerical constants $\gamma>1$ and $\kappa_{2}>0$ such that

$$
\limsup _{n \rightarrow \infty} \mathbb{E}\left|X_{n, 1}\right|^{\gamma} \leq \kappa_{2} .
$$

Further, we make the following additional assumption for the rotation alternative.

## Assumption (A5).

(1) $\mathbb{E} U=\mathbb{E} V=0$ and $\mathbb{E} U^{2}=\mathbb{E} V^{2}=1$.
(2) Both $f_{1}$ and $f_{2}$ are continuous and twice differentiable. As $t \rightarrow \infty$,

$$
\left|t f_{1}^{\prime}(t)\right| \rightarrow 0, \quad \text { and } \quad\left|t f_{2}^{\prime}(t)\right| \rightarrow 0
$$

(3) There exist an $\epsilon>0$, and real-valued functions $R_{1, \epsilon}(\cdot, \cdot)$ and $R_{2, \epsilon}(\cdot, \cdot)$ such that

$$
\max _{k \in\{0,1,2\}} \sup _{|\Delta| \leq \epsilon}\left|\frac{\partial^{k}}{\partial \Delta^{k}} f_{i}\left(\frac{u-\Delta v}{1-\Delta^{2}}\right)\right| \leq R_{i, \epsilon}(u, v)
$$

for all $u, v$ and $i=1,2$. Further, for all $\ell \in\{0,1,2,3\}$,

$$
\mathbb{E}\left[\frac{R_{1, \epsilon}(U, V)}{f_{1}(U)}|U|^{\ell}\right]^{2}<\infty, \quad \mathbb{E}\left[\left(R_{2, \epsilon}(U, V)\right)|V|^{\ell}\right]^{2}<\infty .
$$

These assumptions are natural and hold in various commonly used models including the normal distribution, the $t$ distribution with sufficiently high degrees of freedom (10 or more) and various other distributions in the exponential family. Under these assumptions we have the following proposition which is proved in Section B.2.3.

Proposition 3.3. Consider the model defined by (3.12) with $\Delta \equiv \Delta_{n}$. Then, under the Assumption (A5), we have

$$
\begin{equation*}
\xi\left(f_{X, Y, \Delta_{n}}\right)=\Delta_{n}^{2} V_{0}+O\left(\Delta_{n}^{3}\right) \tag{3.16}
\end{equation*}
$$

where

$$
V_{0}:=6\left(\mathbb{E}_{V}\left[f_{2}^{2}(V)\right]+\mathbb{E}_{U}\left[\frac{f_{1}^{\prime}(U)}{f_{1}(U)}\right]^{2} \mathbb{E}_{V}\left[J^{2}(V)\right]-2 \mathbb{E}_{V}\left[J(V) f_{2}(V)\right]\right)
$$

and $J(t):=\int_{-\infty}^{t} y_{2} f_{2}\left(y_{2}\right) d y_{2}$.
The complete answer to problem (3.13) now follows by combining Proposition 3.3 with Theorem 2.2, and is presented below.

Corollary 3.3. Suppose Assumptions (A3), (A4), and (A5) hold with $\frac{\theta-1}{\theta} \wedge \eta>\frac{\gamma+1}{2 \gamma}$. Then the asymptotic power in (3.13) is given by

$$
\lim _{n \rightarrow \infty} \beta_{\phi_{n}}\left(f_{X, Y, \Delta_{n}}\right)= \begin{cases}\alpha & \text { if } n^{1 / 4} \Delta_{n} \rightarrow 0  \tag{3.17}\\ 1-\Phi\left(z_{\alpha}-c_{0}^{2} V_{0} / \sqrt{2 / 5}\right) & \text { if } n^{1 / 4} \Delta_{n} \rightarrow c_{0} \in(0, \infty) \\ 1 & \text { if } n^{1 / 4} \Delta_{n} \rightarrow \infty\end{cases}
$$

where $V_{0}$ is as defined in Proposition 3.3.

## 4. Minimax optimality of $\phi_{n}$ for testing degree of association

Recall our basic setting, that is, $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right) \stackrel{i . i . d .}{\sim} f_{X, Y}^{(n)}$. In the earlier sections, we focused on the case where $\xi\left(f_{X, Y}^{(n)}\right)$, the Dette-Siburg-Stoimenov measure, converges to 0 as $n \rightarrow \infty$. On the contrary, the focus of this section is on the other regime.

$$
\begin{equation*}
\xi\left(f_{X, Y}^{(n)}\right) \downarrow \xi_{0}>0 \quad \text { as } n \rightarrow \infty \tag{4.1}
\end{equation*}
$$

This regime is of particular importance because the value of $\xi\left(f_{X, Y}^{(n)}\right)$ provably encodes how noisy the functional relationship between $X$ and $Y$ is. In particular, in [21, Theorem 2] and [15, Theorem 1.1], the authors show that $\xi_{0}=1$ implies $Y$ is a noiseless function of $X$. Further, in [21, Theorems 1 and 2], the authors also show that $\xi\left(f_{X, Y}^{(n)}\right)$ can be used to define a natural notion of dependence ordering based on how well $Y$ can be predicted from $X$. Through some explicit computations, in [21, Section 4, Examples 1.(a) - (d)], the authors further prove that in multiple copula based dependence models, $\xi\left(f_{X, Y}^{(n)}\right)$ is a strictly monotonic function of the natural dependence parameters. Moving to [15], the author uses extensive numerical experiments to show that in a number of other examples, including variants of noisy nonparametric regression, [15] also shows through extensive simulations, that $\xi\left(f_{X, Y}^{(n)}\right)$ decreases monotonically with increasing noise levels. Overall, these
results show that through invertible transformations, one can directly convert $\xi\left(f_{X, Y}^{(n)}\right)$ to natural dependence parameters in a variety of models for ( $X_{1}, Y_{1}$ ). Therefore, drawing inference about $\xi\left(f_{X, Y}^{(n)}\right)$ immediately leads to interpretable inference about the nature of dependence in a large collection of models. We expand this collection of models further in the following proposition by showing that $\xi\left(f_{X, Y}^{(n)}\right)$ is a monotonic function of natural dependence parameters in all examples from Section 3.

Proposition 4.1 (Monotonicity of $\xi\left(f_{X, Y}^{(n)}\right)$ ). Recall the definition of $\xi\left(f_{X, Y}^{(n)}\right)$ from (1.4).
(1) Suppose

$$
\left(X_{1}, Y_{1}\right) \sim \mathcal{N}\left(\binom{\mu_{1}}{\mu_{2}},\left(\begin{array}{cc}
\sigma_{1}^{2} & \sigma_{1} \sigma_{2} \rho \\
\sigma_{1} \sigma_{2} \rho & \sigma_{2}^{2}
\end{array}\right)\right)
$$

for $\mu_{1}, \mu_{2} \in \mathbb{R}, \sigma_{1}, \sigma_{2}>0,|\rho| \leq 1$. Then $\xi\left(f_{X, Y}^{(n)}\right)$ is free of $\mu_{1}, \mu_{2}, \sigma_{1}, \sigma_{2}$ and a strictly increasing function of $|\rho|$.
(2) Suppose that $\left(X_{1}, Y_{1}\right)$ is distributed according to the mixture model in (3.2) with $g_{X, Y}(\cdot)$ further satisfying (3.3). Then $\xi\left(f_{X, Y}^{(n)}\right)$ is a strictly increasing function of $r$.
(3) Suppose $\left(X_{1}, Y_{1}\right)$ is distributed according to the regression model (3.9), for $\sigma>0$. Then $\xi\left(f_{X, Y}^{(n)}\right)$ is a strictly decreasing function of $\sigma$.

Motivated by the observations made above, we focus on the following natural hypothesis testing problem in this section:

$$
\begin{equation*}
\mathrm{H}_{0}: \xi\left(f_{X, Y}^{(n)}\right)=\xi_{0} \quad \text { versus } \quad \mathrm{H}_{1, n}:\left|\xi\left(f_{X, Y}^{(n)}\right)-\xi_{0}\right| \geq c_{n}, \tag{4.2}
\end{equation*}
$$

for some positive sequence $\left\{c_{n}\right\}_{n \geq 1}$ and $\xi_{0} \in(0,1)$. Clearly if $c_{n}=O(1)$, then from [15, Theorem 1.1], Chatterjee's correlation $\xi_{n}$ can be used to consistently separate $\mathrm{H}_{0}$ from $\mathrm{H}_{1, n}$. On the other hand, our focus here is when $c_{n}=o(1)$ which leads to (4.1). When $\xi_{0} \in(0,1)$, (4.2) can be viewed as a slightly general (two-sided) version of (1.5). The goal of this section is to address the following pair of subtle questions:

- What is the fastest decaying sequence $c_{n}$ such that Chatterjee's correlation coefficient $\xi_{n}$ can still separate $\mathrm{H}_{0}$ from $\mathrm{H}_{1, n}$ ?
- Conversely, what is the slowest decaying sequence $c_{n}$ such that no test can separate $\mathrm{H}_{0}$ from $\mathrm{H}_{1, n}$ in the local asymptotic minimax sense?
We will prove in this section that, for Chatterjee's correlation, the detection boundary occurs at $c_{n} \gg n^{-1 / 2}$. However, in sharp contrast to the $\xi_{0}=0$ case, $c_{n} \approx n^{-1 / 2}$ is indeed the minimax optimal threshold, in that no test can consistently separate the two hypotheses when $c_{n} \approx n^{-1 / 2}$. This indicates that Chatterjee's correlation based test is indeed minimax optimal for testing the degree of association when the two variables are not exactly independent. In the sequel, we will formalize these two notions.

We begin with some notation. Let $\Gamma:=(C, \eta, \theta, \gamma)$ where the aforementioned constants are taken from Assumptions (A3) and (A4). Consider the following family of distributions:
$\mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right):=\left\{f_{X, Y}^{(n)}(\cdot):\left|\xi\left(f_{X, Y}^{(n)}\right)-\xi_{0}\right| \geq c_{n}, f_{X, Y}^{(n)}\right.$ satisfies (A3), (A4) with constants $\left.\Gamma=(C, \eta, \theta, \gamma)\right\}$.
In other words, the family $\mathrm{H}_{1, n}^{\text {loc }}\left(c_{n} ; \Gamma\right)$ consists of the family of joint distributions which admit $\xi(\cdot)$ values at a $\geq c_{n}$ distance away from the null value $\xi_{0}$, and also satisfy Assumptions (A3) and (A4). Next, given a test function $\Phi_{n} \equiv \Phi_{n}\left(\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)\right)$, consider
its power function

$$
\begin{equation*}
\beta_{\Phi_{n}}\left(\mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)\right):=\inf _{f_{X, Y}^{(n)}(\cdot,) \in \mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)} \mathbb{P}\left(\Phi_{n} \text { rejects } \mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)\right) . \tag{4.4}
\end{equation*}
$$

In terms of the definitions in (4.3) and (4.4), the goal of a good test $\Phi_{n}$ would be to ensure that $\beta_{\Phi_{n}}\left(\mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)\right) \rightarrow 1$ as $n \rightarrow \infty$ for "small" values of $c_{n}$. Given a level parameter $\alpha \in(0,1)$, We define our candidate good test as follows:

$$
\begin{equation*}
\phi_{n}:=\mathbb{1}\left(\left|\xi_{n}-\xi_{0}\right| \geq \frac{1+5 \sqrt{\log (2 / \alpha)}}{\sqrt{n}}\right) \tag{4.5}
\end{equation*}
$$

The following theorem provides matching upper and lower bounds for $\beta_{(\cdot)}\left(\mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)\right)$.
Theorem 4.1 (Testing for degree of association).
(1) (Upper bound for $\xi_{n}$ ). Fix an arbitrary $\alpha \in(0,1)$ and consider the test $\phi_{n}$ from (4.5). Then it is an asymptotically level $\alpha$ under $\mathrm{H}_{0}$ and

$$
\beta_{\phi_{n}}\left(\mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)\right) \rightarrow 1
$$

provided $\sqrt{n} c_{n} \rightarrow \infty$.
(2) (Minimax lower bound). There exists some $\alpha \in(0,1)$ such that for any level $\alpha$ test $\Phi_{n}$ based on $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$, the following holds:

$$
\liminf _{n \rightarrow \infty} \beta_{\Phi_{n}}\left(\mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)\right)<1
$$

whenever $\liminf _{n \rightarrow \infty} \sqrt{n} c_{n}<\infty$.
The two parts of Theorem 4.1 suggest that Chatterjee's correlation based test is particularly suitable for testing the strength of association between two random variables and is able to detect small departures from a fixed non-zero value of $\xi\left(f_{X, Y}^{(n)}\right)$ (i.e., $\xi_{0}$ ) at the optimal rate. It is worth noting that the lower bound in Theorem 4.1 heavily relies on the fact that $0<\xi_{0}<1$.

Remark 4.1. In [49], the authors show that

$$
\frac{\sqrt{n}\left(\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right)}{\widehat{\sigma}_{n}} \stackrel{w}{\longrightarrow} \mathcal{N}(0,1)
$$

where $\widehat{\sigma}_{n}$ is a function of the data, and $\widehat{\sigma}_{n}=O_{p}(1)$. Therefore, a natural alternative to $\phi_{n}$ from (4.5) would be

$$
\phi_{n}=\mathbb{1}\left(\sqrt{n}\left|\xi_{n}-\xi_{0}\right| \geq z_{\alpha / 2}\right),
$$

where $z_{\alpha / 2}$ is the upper $\alpha / 2$ standard Gaussian quantile and $\xi\left(f_{X, Y}^{(n)}\right)=\xi_{0}$. This test was proposed in [49, Remark 1.4] and it has asymptotic size exactly equal to $\alpha$. On the fip side, from the definition of $\widehat{\sigma}_{n}$ (see [49, Theorem 1.1]), it seems that it has $O\left(n^{2}\right)$ time complexity which may be a greater computational burden depending on the application at hand.

Remark 4.2. In [48], the authors show that if $(X, Y)$ have a bivariate Gaussian distribution, then the detection boundary for $\xi_{n}$ when $\xi_{0}=0$ can be improved to near parametric
rates by incorporating more "right nearest neighbors". We would conjecture that the test in [48] achieves $O\left(n^{-1 / 2}\right)$ detection boundary when $\xi_{0} \in(0,1)$ and $(X, Y)$ are non-Gaussian. In view of Theorem 4.1, this would imply that the test in [48] attains (near) parametric efficiency whenever $\xi_{0} \in[0,1)$ (at the expense of greater computational complexity). A conclusive answer on this and an inspection of the relevant applications might be of independent interest.

## Appendix A: Proofs of main results

In this section, we will prove the main results in this paper. The proofs require a number of technical results, which are proved in Sections 2 and 3 of the Appendix B.

## A.1. Proofs from Section 2

## A.1.1. Proof of Theorem 2.1

Part (i). As mentioned in Section 2, our proof proceeds through studying an oracle version of $\xi_{n}$. Observe that $\xi_{n}$ can be rewritten as

$$
\begin{equation*}
\xi_{n}=\frac{3 n}{n^{2}-1}\left(\frac{1}{n} \sum_{i \neq j} \sum_{\mid}\left|\widehat{F}_{n}\left(Y_{n, i}\right)-\widehat{F}_{n}\left(Y_{n, j}\right)\right|-\sum_{i=1}^{n}\left|\widehat{F}_{n}\left(Y_{n,(i+1)}\right)-\widehat{F}_{n}\left(Y_{n,(i)}\right)\right|\right) \tag{A.1}
\end{equation*}
$$

where $\widehat{F}_{n}(\cdot)$ is the empirical cumulative distribution function (CDF) of $Y_{n, 1}, \ldots, Y_{n, n}$. Let $F_{Y}^{(n)}(\cdot)$ denote the population CDF of $Y_{n, 1}$. Recall that $X_{n,(1)} \leq \ldots \leq X_{n,(n)}$ and $Y_{n,(i)}$ is the $Y$ value concomitant to $X_{n,(i)}$. The main idea is to show that we can replace $\widehat{F}_{n}(\cdot)$ by $F_{Y}^{(n)}(\cdot)$ asymptotically. In other words, we show that $\xi_{n}$ is close (with quantitative error bounds) to $\xi_{n}^{*}$ where

$$
\begin{equation*}
\xi_{n}^{*}:=\frac{3 n}{n^{2}-1}\left(\frac{1}{n} \sum_{i \neq j} \sum_{j}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, j}\right)\right|-\sum_{i=1}^{n}\left|F_{Y}^{(n)}\left(Y_{n,(i+1)}\right)-F_{Y}^{(n)}\left(Y_{n,(i)}\right)\right|\right) . \tag{A.2}
\end{equation*}
$$

The following theorem characterizes the asymptotic variances and the distance between $\xi_{n}^{*}$ and $\xi_{n}$.

Theorem A.1. Suppose that Assumptions (A3) and (A4) hold. Then there is a constant $C>0$, such that for $n \geq 1$,

$$
\max \left\{n \operatorname{Var}\left(\xi_{n}-\xi_{n}^{*}\right),\left|n \operatorname{Var}\left(\xi_{n}\right)-\frac{2}{5}\right|\right\} \leq C n^{-1}+C \xi\left(f_{X, Y}^{(n)}\right)+C \sqrt{n \log n} b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)
$$

where

$$
\begin{equation*}
b_{n}:=n^{-\frac{\gamma}{\gamma+1}}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\frac{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}{} .} \tag{A.3}
\end{equation*}
$$

The next theorem characterizes the rate of convergence of $\frac{\sqrt{n}\left(\xi_{n}^{*}-\mathbb{E}\left(\xi_{n}^{*}\right)\right)}{\sigma_{n}^{*}}$ to normality.

Theorem A.2. There is a constant $C^{*}>0$ such that for all $n \geq 1$ we have

$$
\mathfrak{D}\left(\frac{\sqrt{n}\left(\xi_{n}^{*}-\mathbb{E}\left(\xi_{n}^{*}\right)\right)}{\sigma_{n}^{*}}\right) \leq \frac{C^{*}}{\sqrt{n}}\left(\frac{1}{\sigma_{n}^{* 2}}+\frac{1}{2 \sigma_{n}^{* 3}}\right)
$$

where $\sigma_{n}^{*}:=\sqrt{\operatorname{Var}\left(\xi_{n}^{*}\right)}$ and $\mathfrak{D}$ is the Wasserstein distance to normality defined in Definition 2.2 of the main paper.

We defer the proofs of these theorems to Section 2 of Appendix B, and proceed to use these results to prove Theorem 2.1.

For any $p \in(0,1)$ we can write

$$
\begin{equation*}
f_{Y \mid X}^{(n)}(y \mid x)=(1-p) f_{Y}^{(n)}(y)+p g(y \mid x), \text { where } g(y \mid x)=\frac{f_{Y \mid X}^{(n)}(y \mid x)-f_{Y}^{(n)}(y)}{p}+f_{Y}^{(n)}(y) \tag{A.4}
\end{equation*}
$$

Here $f_{Y}^{(n)}(y)=\int f_{X, Y}^{(n)}(x, y) d x$ is the marginal density of $Y_{n, 1}$. We have

$$
\begin{equation*}
G(y \mid x):=\int_{-\infty}^{y} g(t \mid x) d t=\frac{1}{p} F_{Y \mid X}^{(n)}(y \mid x)+\left(1-\frac{1}{p}\right) F_{Y}^{(n)}(y) ; \quad \int G(y \mid x) f_{X}^{(n)}(x) d x=F_{Y}^{(n)}(y) \tag{A.5}
\end{equation*}
$$

For each $i \in[n]$, let $N(i) \in\left\{j \in[n]: \operatorname{Rank}\left(X_{n, j}\right)=\operatorname{Rank}\left(X_{n, i}\right)+1\right\}$ be the unique index $j$ such that $X_{n, j}$ is immediately to the right of $X_{n, i}$ when $X_{n, i}$ 's are arranged in increasing order. If there are no such indices for some $i$, set the corresponding $N(i)=1$. To show part (i) of Theorem 2.1 we begin by observing that $\mathbb{E}\left(\sqrt{n} \xi_{n}\right)$ can be simplified as follows.

$$
\begin{align*}
& \mathbb{E}\left(\sqrt{n} \xi_{n}\right) \\
= & \sqrt{n} \mathbb{E}\left(1-\frac{3}{n^{2}-1} \sum_{i=1}^{n-1}\left|R_{i}-R_{N(i)}\right|\right) \\
= & \sqrt{n} \mathbb{E}\left[1-\frac{3}{n^{2}-1} \sum_{i=1}^{n-1}\left(R_{i}+R_{N(i)}-2 \min \left\{R_{i}, R_{N(i)}\right\}\right)\right] \\
= & \sqrt{n} \mathbb{E}\left(-2+\frac{6}{n^{2}-1} \sum_{i=1}^{n-1} \min \left\{R_{i}, R_{N(i)}\right\}\right)+O\left(n^{-1 / 2}\right) \\
= & -2 \sqrt{n}+\frac{6 n^{3 / 2}}{n^{2}-1}\left(1+\sum_{k \neq 1, N(1)} \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, 1}, Y_{n, N(1)}\right\}\right) \mid \mathbf{X}^{(n)}\right)\right)\right)+O\left(n^{-1 / 2}\right) \tag{A.6}
\end{align*}
$$

Hence, to characterize the asymptotic behavior of $\mathbb{E}\left(\sqrt{n} \xi_{n}\right)$, we focus on the asymptotics of the term $\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, 1}, Y_{n, N(1)}\right\}\right) \mid \mathbf{X}^{(n)}\right)\right)$.

Depending on which part of $f_{Y \mid X}^{(n)}$ contributes to the random variables $Y_{n, 1}, Y_{n, N(1)}, Y_{n, k}$, we can separate $\mathbb{E}\left(\sqrt{n} \xi_{n}\right)$, we focus on the asymptotics of the term $\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, 1}, Y_{n, N(1)}\right\}\right) \mid \mathbf{X}^{(n)}\right)\right)$ into the following terms.
$\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, 1}, Y_{n, N(1)}\right\}\right) \mid \mathbf{X}^{(n)}\right)\right)=:(1-p)^{3} T_{0}+(1-p)^{2} p T_{1}+(1-p) p^{2} T_{2}+p^{3} T_{3}$.

Here

$$
T_{0}:=\int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right) f_{Y}^{(n)}(y) f_{Y}^{(n)}\left(y_{1}\right) f_{Y}^{(n)}\left(y_{k}\right) d y d y_{1} d y_{k}
$$

while the other terms are defined as

$$
\begin{aligned}
T_{1}:= & \iint \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\iint \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) g\left(y_{k} \mid x_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\iint \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) f_{Y}\left(y_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
T_{2}:= & \iint \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) g\left(y_{k} \mid x_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\iint \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) g\left(y_{k} \mid x_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\iint \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
T_{3}:= & \iint \sum_{j \neq 1, k} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) g\left(y_{k} \mid x_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
= & \iint \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} .
\end{aligned}
$$

We can show that $T_{0}$ satisfies the following.

$$
\begin{align*}
T_{0} & =\int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right) f_{Y}^{(n)}(y) f_{Y}^{(n)}\left(y_{1}\right) f_{Y}^{(n)}\left(y_{k}\right) d y d y_{1} d y_{k} \\
& =\mathbb{P}\left(Y_{n, k}<\min \left\{Y_{n, 1}, Y_{n, 2}\right\} \mid Y_{n, 1}, Y_{n, 2}, Y_{n, k} \sim f_{Y}^{(n)}\right)=\frac{1}{3} \tag{A.8}
\end{align*}
$$

Let us define

$$
H(f, g):=\int \mathbb{E}\left(\int_{t}^{\infty} g\left(y \mid X_{n, 1}\right) d y\right)^{2} f_{Y}^{(n)}(t) d t=\int \mathbb{E}\left(1-G\left(t \mid X_{n, 1}\right)\right)^{2} f_{Y}^{(n)}(t) d t
$$

The second, third, and fourth terms can be controlled using the following lemma which has been proved in Section 3 of Appendix B:

## Lemma A.1.

$$
T_{1}=1+r_{n}, T_{2}=\frac{2}{3}+H(f, g)+r_{n}, T_{3}=H(f, g)+r_{n}
$$

where $\left|r_{n}\right| \lesssim b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)$.
We have from (A.7) that

$$
\mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, 1}, Y_{n, N(1)}\right\}\right) \mid \mathbf{X}^{(n)}\right)\right)
$$

Auddy, Deb, and Nandy/On Chatterjee's correlation coefficient

$$
\begin{align*}
& =(1-p)^{3} T_{0}+(1-p)^{2} p T_{1}+(1-p) p^{2} T_{2}+p^{3} T_{3} \\
& =(1-p)^{3} \cdot \frac{1}{3}+(1-p)^{2} p\left[1+r_{n}\right]+(1-p) p^{2}\left[\frac{2}{3}+H(f, g)+r_{n}\right]+p^{3} \cdot A_{3} \\
& =\frac{1}{3}+p(-1+1)+p^{2}\left(1-2+\frac{2}{3}+H(f, g)\right)+p^{3}\left(-\frac{1}{3}+1-\frac{2}{3}-H(f, g)+H(f, g)\right)+r_{n} \\
& =\frac{1}{3}+p^{2}\left(H(f, g)-\frac{1}{3}\right)+r_{n}, \tag{A.9}
\end{align*}
$$


Recall that $\xi\left(f_{X, Y}^{(n)}\right)$ is the population measure of association when $X \sim f_{X}^{(n)}, Y \sim f_{Y}^{(n)}$, and $Y \mid X \sim f_{Y \mid X}^{(n)}(y \mid x) \equiv(1-p) f_{Y}^{(n)}(y)+p g(y \mid x)$. Then

$$
\begin{aligned}
\xi\left(f_{X, Y}^{(n)}\right) / 6= & \int \mathbb{E}\left[\mathbb{P}\left(Y_{n, 1} \geq t \mid X_{n, 1}\right)-\mathbb{P}\left(Y_{n, 1} \geq t\right)\right]^{2} f_{Y}^{(n)}(t) d t \\
= & \int\left[(1-p)^{2} \operatorname{Var} \mathbb{P}\left(Y_{n, 1} \geq t\right)+p^{2} \operatorname{Var}\left(1-G\left(t \mid X_{n, 1}\right)\right)\right. \\
& \left.+2 p(1-p) \operatorname{Cov}\left(\mathbb{P}\left(Y_{n, 1} \geq t\right), 1-G\left(t \mid X_{n, 1}\right)\right)\right] f_{Y}^{(n)}(t) d t \\
= & p^{2} \int \operatorname{Var}\left(1-G\left(t \mid X_{n, 1}\right)\right) f_{Y}^{(n)}(t) d t .
\end{aligned}
$$

The last equality follows since $\mathbb{P}\left(Y_{n, 1} \geq t\right)$ does not depend on $X_{n, 1}$. By the definition, $H(f, g)=\int \mathbb{E}\left(\left(1-G\left(t \mid X_{n, 1}\right)\right)^{2}\right) f_{Y}^{(n)}(t) d t$. Next,
$\int\left(\mathbb{E}\left(1-G\left(t \mid X_{n, 1}\right)\right)\right)^{2} f_{Y}^{(n)}(t) d t=\int\left(1-F_{Y}^{(n)}(t)\right)^{2} f_{Y}^{(n)}(t) d t=\mathbb{E}\left(1-F_{Y}^{(n)}(Y)\right)^{2}=\int_{0}^{1} u^{2} d u=\frac{1}{3}$.
This means

$$
\begin{equation*}
6 p^{2}\left(H(f, g)-\frac{1}{3}\right)=6 p^{2} \int \operatorname{Var}\left(1-G\left(t \mid X_{n, 1}\right)\right) f_{Y}^{(n)}(t) d t=\xi\left(f_{X, Y}^{(n)}\right) \tag{A.10}
\end{equation*}
$$

Plugging (A.9) and (A.10) back into (A.6),

$$
\begin{aligned}
& \mathbb{E}\left(\sqrt{n} \xi_{n}\right) \\
= & -2 \sqrt{n}+\frac{6 n^{3 / 2}}{n^{2}-1}\left(1+\sum_{k \neq 1, N(1)} \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, 1}, Y_{n, N(1)}\right\}\right) \mid \mathbf{X}^{(n)}\right)\right)\right)+O\left(n^{-1 / 2}\right) \\
= & -2 \sqrt{n}+\frac{6 n^{5 / 2}}{n^{2}-1}\left[\frac{1}{3}+p^{2}\left(H(f, g)-\frac{1}{3}\right)+3 r_{n}\right]+O\left(\frac{1}{\sqrt{n}}\right) \\
= & 6 \sqrt{n} p^{2}\left(H(f, g)-\frac{1}{3}\right)+18 \sqrt{n} r_{n}+O\left(\frac{1}{\sqrt{n}}\right) \\
= & \sqrt{n} \xi\left(f_{X, Y}^{(n)}\right)+O\left(n^{-1 / 2}\right)+O\left(\sqrt{n} b_{n}\right) \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) .
\end{aligned}
$$

This finishes the proof of part (i) of Theorem 2.1.
$\operatorname{Part}$ (ii). Let $\mu_{n}^{*}$ and $\mu_{n}$ be the laws of $W_{n}^{*}=\left(\xi_{n}^{*}-\mathbb{E} \xi_{n}^{*}\right) \sqrt{\operatorname{Var}\left(\xi_{n}^{*}\right)}$ and $\sqrt{n}\left(\xi_{n}-\mathbb{E} \xi_{n}\right) / \sqrt{2 / 5}$. By Theorem A.1, we have the upper bound

$$
\mathcal{W}\left(\mu_{n}^{*}, \mu_{n}\right) \leq \mathcal{W}_{2}\left(\mu_{n}^{*}, \mu_{n}\right) \lesssim n^{-1 / 2}+\left(\xi\left(f_{X, Y}^{(n)}\right)+\sqrt{n \log n} b_{n}\right)^{1 / 2} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)
$$

where $\mathcal{W}_{2}$ is the Wasserstein-2 distance, and $\mathcal{W}$ is the Wasserstein- 1 distance defined in Definition 2.1.

Now if we define $\mu_{n}^{\prime}$ to be the law of $\sqrt{n}\left(\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right) / \sqrt{2 / 5}$, then by part (i) of the theorem

$$
\begin{aligned}
\mathcal{W}\left(\mu_{n}^{*}, \mu_{n}^{\prime}\right) & \leq \sqrt{n}\left|\mathbb{E} \xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right|+\mathcal{W}\left(\mu_{n}^{*}, \mu_{n}\right) \\
& \lesssim n^{-1 / 2}+\left(\xi\left(f_{X, Y}^{(n)}\right)+\sqrt{n \log n} b_{n}\right)^{1 / 2} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)
\end{aligned}
$$

Finally notice that for the standard normal law $\nu$,

$$
\begin{aligned}
\mathfrak{D}\left(\frac{\sqrt{n}\left(\xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right)}{\sqrt{2 / 5}}\right) & =\mathcal{W}\left(\mu_{n}^{\prime}, \nu\right) \leq \mathcal{W}\left(\mu_{n}^{\prime}, \mu_{n}^{*}\right)+\mathcal{W}\left(\mu_{n}^{*}, \nu\right) \\
& =\mathcal{W}\left(\mu_{n}^{*}, \mu_{n}^{\prime}\right)+\mathfrak{D}\left(W_{n}^{*}\right) \\
& \lesssim\left(\xi\left(f_{X, Y}^{(n)}\right)+\sqrt{n \log n} b_{n}\right)^{1 / 2} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)+n^{-1 / 2}
\end{aligned}
$$

where we use Theorem A. 2 in the last step.

## A.2. Proofs from Section 3

## A.2.1. Proof of Proposition 3.1

Let us observe that using equation (3.2), the marginal densities of $X$ and $Y$ under $f_{X, Y, r_{n}}(\cdot, \cdot)$ are $f_{X}(\cdot)$ and $f_{Y}(\cdot)$ respectively for any $r_{n}$. Let us define,

$$
\begin{equation*}
F_{Y \mid X ; r_{n}}(y \mid x):=\int_{-\infty}^{y} \frac{f_{X, Y, r_{n}}(t, x)}{f_{X}(x)} d t \tag{A.11}
\end{equation*}
$$

and observe that

$$
\begin{equation*}
\xi\left(f_{X, Y, r_{n}}\right)=6\left[\int_{-\infty}^{\infty} \mathbb{E}_{X}\left[F_{Y \mid X ; r_{n}}^{2}(t \mid X)\right] f_{Y}(t) d t-\frac{1}{3}\right] \tag{A.12}
\end{equation*}
$$

Let $F_{Y}(\cdot)$ be the CDF of $Y$. Next, note that (A.11) implies:

$$
F_{Y \mid X ; r_{n}}(t \mid x)=\left(1-r_{n}\right) F_{Y}(t)+r_{n} \int_{-\infty}^{t} \frac{g_{X, Y}(x, w)}{f_{X}(x)} d w
$$

Let $g_{X}(\cdot)$ be the marginal density of $X$ under $g_{X, Y}(\cdot, \cdot)$. By equation $(3.2), g_{X}(\cdot)=f_{X}(\cdot)$.
Since $\int_{-\infty}^{\infty} F_{Y}^{2}(t) f_{Y}(t) d t=\frac{1}{3}$, we have that

$$
\int_{-\infty}^{\infty} \mathbb{E}_{X}\left[F_{Y \mid X ; r_{n}}^{2}(t \mid X)\right] f_{Y}(t) d t=\left(1-r_{n}\right)^{2} \frac{1}{3}+2 r_{n}\left(1-r_{n}\right) \int_{-\infty}^{\infty} F_{Y}(t) \mathbb{E}_{X}\left[\int_{-\infty}^{t} \frac{g_{X, Y}(X, w)}{f_{X}(X)} d w\right] f_{Y}(t) d t
$$

$$
\begin{aligned}
& +r_{n}^{2} \int_{-\infty}^{\infty} \mathbb{E}_{X}\left[\int_{-\infty}^{t} \frac{g_{X, Y}(X, w)}{f_{X}(X)} d w\right]^{2} f_{Y}(t) d t \\
= & \frac{1}{3}-\frac{1}{3} r_{n}^{2}+r_{n}^{2} \int_{-\infty}^{\infty} \mathbb{E}_{X}\left[\int_{-\infty}^{t} \frac{g_{X, Y}(X, w)}{f_{X}(X)} d w\right]^{2} f_{Y}(t) d t . \\
= & \frac{1}{3}+\frac{r_{n}^{2}}{6}\left(-2+\int_{-\infty}^{\infty} \mathbb{E}_{X}\left[\int_{-\infty}^{t} \frac{g_{X, Y}(X, w)}{g_{X}(X)} d w\right]^{2} f_{Y}(t) d t\right) \\
= & \frac{1}{3}+\frac{r_{n}^{2} \xi\left(g_{X, Y}\right)}{6} .
\end{aligned}
$$

Plugging the above display in (A.12) completes the proof.

## A.2.2. Proof of Proposition 3.2

Let $\left(X^{\prime}, Y^{\prime}, Z^{\prime}\right) \stackrel{d}{=}(X, Y, Z)$ where $\left(X^{\prime}, Y^{\prime}, Z^{\prime}\right)$ is independent of $(X, Y, Z)$ and $(X, Y, Z)$ be drawn according the distribution defined by equation (3.9) with $\sigma \equiv \sigma_{n}$.

Let us observe that

$$
\begin{align*}
\xi\left(f_{X, Y, \sigma_{n}}\right) & =-2+6 \mathbb{E}_{Y^{\prime}} \mathbb{E}_{X}\left[\mathbb{P}\left(Y \leq Y^{\prime} \mid X, Y^{\prime}\right)\right]^{2} \\
& =-2+6 \mathbb{E}_{X^{\prime}, Z^{\prime}} \mathbb{E}_{X}\left[\mathbb{P}\left(Y \leq g\left(X^{\prime}\right)+\sigma_{n} Z^{\prime} \mid X, X^{\prime}, Z^{\prime}\right)\right]^{2} \tag{A.13}
\end{align*}
$$

Under equation (3.9), it is easy to check that

$$
\begin{equation*}
\mathbb{P}^{2}\left(Y \leq g\left(X^{\prime}\right)+\sigma_{n} Z^{\prime} \mid X, X^{\prime}, Z^{\prime}\right)=\Phi^{2}\left(Z^{\prime}+\frac{g\left(X^{\prime}\right)-g(X)}{\sigma_{n}}\right) \tag{A.14}
\end{equation*}
$$

where $\Phi(\cdot)$ is the CDF of $N(0,1)$. All the derivatives of $\Phi^{2}(\cdot)$ are uniformly bounded. Using this fact with a standard Taylor series expansion, we get that, for any $x, x^{\prime}, z^{\prime}$, we get:

$$
\begin{align*}
\Phi^{2}\left(z^{\prime}+\sigma_{n}^{-1}\left(g\left(x^{\prime}\right)-g(x)\right)\right)= & \Phi^{2}(z)+2 \sigma_{n}^{-1} \Phi(z) \phi(z)\left(g\left(x^{\prime}\right)-g(x)\right)+\frac{\sigma_{n}^{-2}}{2}\left[2 \phi^{2}(z)\left(g\left(x^{\prime}\right)-g(x)\right)^{2}\right. \\
& \left.+2 \Phi(z) \phi^{\prime}(z)\left(g\left(x^{\prime}\right)-g(x)\right)^{2}\right]+O\left(\sigma_{n}^{-3}\left|g\left(x^{\prime}\right)-g(x)\right|^{3}\right) \tag{A.15}
\end{align*}
$$

where $\phi(\cdot)$ is the density of the $N(0,1)$ distribution. By combining (A.13) and (A.14), we have,

$$
\begin{aligned}
\sqrt{n} \xi\left(f_{X, Y, \sigma_{n}}\right) & =-2+6 \sqrt{n} \mathbb{E}_{X^{\prime}, Z^{\prime}} \mathbb{E}_{X} \Phi^{2}\left(Z^{\prime}+\frac{g\left(X^{\prime}\right)-g(X)}{\sigma_{n}}\right) \\
& =6 \sqrt{n} \sigma_{n}^{-2} \mathbb{E}_{X, X^{\prime}}\left\{g(X)-g\left(X^{\prime}\right)\right\}^{2} \mathbb{E}_{Z^{\prime}}\left[\phi^{2}\left(Z^{\prime}\right)+\Phi\left(Z^{\prime}\right) \phi^{\prime}\left(Z^{\prime}\right)\right]+O\left(\sqrt{n} \sigma_{n}^{-3}\right) \\
& =6 \times 2 \sqrt{n} \sigma_{n}^{-2} \operatorname{Var}(g(X)) \times \frac{1}{4 \sqrt{3} \pi}+O\left(\sqrt{n} \sigma_{n}^{-3}\right) \\
& =(\sqrt{3 n} / \pi) \sigma_{n}^{-2} \operatorname{Var}(g(X))+O\left(\sqrt{n} \sigma_{n}^{-3}\right) .
\end{aligned}
$$

## A.2.3. Proof of Proposition 3.3

Let us begin by observing that the joint density of $(X, Y)$ is given by

$$
f_{X, Y, \Delta_{n}}(x, y)=\frac{1}{\left(1-\Delta_{n}^{2}\right)} f_{1}\left(\frac{x-\Delta_{n} y}{1-\Delta_{n}^{2}}\right) f_{2}\left(\frac{y-\Delta_{n} x}{1-\Delta_{n}^{2}}\right) .
$$

Let $\left(X^{\prime}, Y^{\prime}\right) \stackrel{d}{=}(X, Y)$ where $\left(X^{\prime}, Y^{\prime}\right)$ is independent of $(X, Y)$ and $(X, Y)$ is drawn according to (3.12) with $\Delta \equiv \Delta_{n}$. Notice that

$$
\begin{equation*}
\xi\left(f_{X, Y, \Delta_{n}}\right) / 6=\mathbb{E}_{Y^{\prime} \sim Y} \mathbb{E}_{X}\left[\mathbb{P}\left(Y \leq Y^{\prime} \mid X, Y^{\prime}\right)-P\left(Y \leq Y^{\prime} \mid Y^{\prime}\right)\right]^{2} \tag{A.16}
\end{equation*}
$$

Under the rotation model (3.12) for any $t$,

$$
\begin{equation*}
\mathbb{P}(Y \leq t \mid X)-\mathbb{P}(Y \leq t)=: \frac{\int_{-\infty}^{t} A\left(x, y, \Delta_{n}\right) d y}{\int_{-\infty}^{\infty} A\left(x, y, \Delta_{n}\right) d y}-\frac{1}{1-\Delta_{n}^{2}} \int_{-\infty}^{t} \int_{-\infty}^{\infty} A\left(x, y, \Delta_{n}\right) d x d y \tag{A.17}
\end{equation*}
$$

where $A\left(x, y, \Delta_{n}\right)=f_{1}\left(\frac{x-\Delta_{n} y}{1-\Delta_{n}^{2}}\right) f_{2}\left(\frac{y-\Delta_{n} x}{1-\Delta_{n}^{2}}\right)=: f_{1}\left(h_{1}\left(x, y, \Delta_{n}\right)\right) f_{2}\left(h_{2}\left(x, y, \Delta_{n}\right)\right)$. Without loss of generality, we can assume $\left|\Delta_{n}\right| \leq \epsilon$ where $\epsilon$ is as specified in Assumption (A3), part 3 from Section 3.3. Next, by Taylor expansion around 0 , we have
$f_{1}\left(\frac{x-\Delta_{n} y}{1-\Delta_{n}^{2}}\right)=f_{1}(x)-y f_{1}^{\prime}(x) \Delta_{n}+\left(\left.y^{2} \cdot \frac{\partial^{2}}{\partial \theta^{2}} f_{1}\left(h_{1}(x, y, \theta)\right)\right|_{\theta=\xi}+\left.2 x \frac{\partial}{\partial \theta} f_{1}\left(h_{1}(x, y, \theta)\right)\right|_{\theta=\xi}\right) \Delta_{n}^{2}$
$f_{2}\left(\frac{y-\Delta_{n} x}{1-\Delta_{n}^{2}}\right)=f_{2}(y)-x f_{2}^{\prime}(y) \Delta_{n}+\left(\left.x^{2} \cdot \frac{\partial^{2}}{\partial \theta^{2}} f_{2}\left(h_{2}(x, y, \theta)\right)\right|_{\theta=\xi^{\prime}}+\left.2 y \frac{\partial}{\partial \theta} f_{2}\left(h_{2}(x, y, \theta)\right)\right|_{\theta=\xi^{\prime}}\right) \Delta_{n}^{2}$
for some $\xi, \xi^{\prime} \in[-\epsilon, \epsilon]$. Multiplying the last two equations, we have

$$
A\left(x, y, \Delta_{n}\right)=: f_{1}(x) f_{2}(y)-\Delta_{n}\left(x f_{1}(x) f_{2}^{\prime}(y)+y f_{1}^{\prime}(x) f_{2}(y)\right)+\Delta_{n}^{2} g\left(x, y, \Delta_{n}\right)
$$

which implies (since $\left.\int x f_{1}(x) d x=\int y f_{2}(y) d y=0\right)$

$$
\begin{align*}
\int_{-\infty}^{t} A\left(x, y, \Delta_{n}\right) d y & =f_{1}(x) F_{2}(t)-\Delta_{n}\left[x f_{1}(x) f_{2}(t)+f_{1}^{\prime}(x) J(t)\right]+\Delta_{n}^{2} \int_{-\infty}^{t} g\left(x, y, \Delta_{n}\right) d y \\
\int_{-\infty}^{\infty} A\left(x, y, \Delta_{n}\right) d y & =f_{1}(x)+\Delta_{n}^{2} \int_{-\infty}^{\infty} g\left(x, y, \Delta_{n}\right) d y \\
\int_{-\infty}^{t} \int_{-\infty}^{\infty} A\left(x, y, \Delta_{n}\right) d x d y & =F_{2}(t)+\Delta_{n}^{2} \int_{-\infty}^{t} \int_{-\infty}^{\infty} g\left(x, y, \Delta_{n}\right) d x d y \tag{A.18}
\end{align*}
$$

We define $G\left(x, t, \Delta_{n}\right):=\int_{-\infty}^{t} g\left(x, y, \Delta_{n}\right) d y$. We also observe that the marginal density of $X$, say $f_{X, \Delta_{n}}(\cdot)$, satisfies the following:

$$
f_{X, \Delta_{n}}(x)=\frac{1}{1-\Delta_{n}^{2}} \int_{-\infty}^{\infty} A\left(x, y, \Delta_{n}\right) d y
$$

Then by (A.16), (A.17) and (A.18) we now have

$$
\xi\left(f_{X, Y}^{(n)}\right) / 6=\iint \frac{\left[-\Delta_{n}\left(x f_{1}(x) f_{2}(t)+f_{1}^{\prime}(x) J(t)\right)+\Delta_{n}^{2} g_{2}\left(x, t, \Delta_{n}\right)\right]^{2}}{f_{1}(x)+\Delta_{n}^{2} G\left(x, \infty, \Delta_{n}\right)} d x f_{Y, \Delta_{n}}(t) d t
$$

where $f_{Y, \Delta_{n}}(\cdot)$ is the marginal density of $Y$ and

$$
\begin{aligned}
g_{2}\left(x, t, \Delta_{n}\right):= & G\left(x, t, \Delta_{n}\right)-f_{1}(x) F_{2}(t) \int G\left(x, t, \Delta_{n}\right) d x-F_{2}(t) G\left(x, t, \Delta_{n}\right) \\
& +\Delta_{n}^{2} F_{2}(t) G\left(x, t, \Delta_{n}\right) \int G\left(x, t, \Delta_{n}\right) d x
\end{aligned}
$$

Next, let us observe that $G\left(x, t, \Delta_{n}\right)$ can be written as a sum of $\left(\frac{\partial^{k}}{\partial u^{k}} f_{1}(u)|u|^{l}\right)$ for $k \in$ $\{0,1,2\}$ and $\ell \in\{0,1,2,3\}$. It can be checked using Assumption (A3) from Section 3.3, that this implies $\int g_{2}\left(x, t, \Delta_{n}\right)^{2} / f_{1}(x) d x \lesssim 1$.

We now expand the square above (and use the Cauchy-Schwarz inequality for the cross terms) to obtain

$$
\xi\left(f_{X, Y}^{(n)}\right) / 6=\Delta_{n}^{2} \iint \frac{\left(x f_{1}(x) f_{2}(t)+f_{1}^{\prime}(x) J(t)\right)}{f_{1}(x)} d x f_{Y, \Delta_{n}}(t) d t+O\left(\Delta_{n}^{3}\right)
$$

Moreover, the marginal $f_{Y, \Delta_{n}}(y)$ under the alternative can be written, by a similar Taylor expansion, as

$$
f_{Y, \Delta_{n}}(y)=\frac{1}{1-\Delta_{n}^{2}} \int_{-\infty}^{\infty} A\left(x, y, \Delta_{n}\right) d x=f_{2}(y)+\Delta_{n}^{2} \int_{-\infty}^{\infty} g\left(x, y, \Delta_{n}\right) d x
$$

Consequently,

$$
\begin{align*}
& \xi\left(f_{X, Y}^{(n)}\right) / 6=\Delta_{n}^{2} \iint \frac{\left(x f_{1}(x) f_{2}(t)+f_{1}^{\prime}(x) J(t)\right)^{2}}{f_{1}(x)} d x f_{Y, \Delta_{n}}(t) d t+O\left(\Delta_{n}^{3}\right) \\
= & \Delta_{n}^{2} \iint \frac{x^{2} f_{1}(x)^{2} f_{2}(t)^{2}+\left(f_{1}^{\prime}(x)\right)^{2}(J(t))^{2}+2 x f_{1}(x) f_{2}(t) f_{1}^{\prime}(x) J(t)}{f_{1}(x)} d x f_{2}(t) d t+O\left(\Delta_{n}^{3}\right) . \tag{A.19}
\end{align*}
$$

It is not hard to see that

$$
\begin{aligned}
& \iint \frac{x^{2} f_{1}(x)^{2} f_{2}(t)^{2}}{f_{1}(x)} d x f_{2}(t) d t=\left(\mathbb{E} U^{2}\right) \mathbb{E}\left(f_{2}(V)^{2}\right)=\mathbb{E}\left(f_{2}(V)^{2}\right), \\
& \iint \frac{\left(f_{1}^{\prime}(x)\right)^{2} J^{2}(t)}{f_{1}(x)} d x f_{2}(t) d t=I\left(f_{1}\right) \mathbb{E}\left(J^{2}(V)\right),
\end{aligned}
$$

and finally

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{1}^{\prime}(x) f_{2}(t) J(t) d x f_{2}(t) d t=\int_{-\infty}^{\infty} J(t) f_{2}^{2}(t) d t \times \int_{-\infty}^{\infty} x f_{1}^{\prime}(x) d x=-\mathbb{E}\left(J(V) f_{2}(V)\right)
$$

Plugging these back into (A.19) finishes the proof.
Next, in Appendix B, we prove the theorems as stated in Section 4 above, and state some technical results required in the process. In Appendix C, we prove Theorems A1 and A2 from Appendix A. Finally in Appendix D, we present the proofs of the technical lemmas used to prove the theorems in the Appendix A as well as those in Sections B and C.

## Appendix B: Proofs from Section 4

## B.1. Proof of Proposition 4.1

Let $\left(X_{2}, Y_{2}\right)$ be generated independently of $\left(X_{1}, Y_{1}\right)$ and with the same distribution. Note that it suffices to show that

$$
\mathcal{J}:=\mathbb{E}_{\left(X_{1}, Y_{1}\right), Y_{2}}\left[\mathbb{P}\left(Y_{1} \leq Y_{2} \mid X_{1}, Y_{2}\right)\right]^{2}
$$

is a strictly increasing function of $|\rho|, r$ and $\sigma^{-1}$ in parts (1), (2) and (3) respectively.
Part (1). Clearly, by replacing $X_{1}, Y_{1}$, and $Y_{2}$ by $\left(X_{1}-\mu_{1}\right) / \sigma_{1},\left(Y_{1}-\mu_{2}\right) / \sigma_{2}$, and $\left(Y_{2}-\mu_{2}\right) / \sigma_{2}, \mathcal{J}$ does not change. Consequently we can assume without loss of generality $\mu_{1}=\mu_{2}=0$ and $\sigma_{1}=\sigma_{2}=1$. In the sequel, we will use $\Phi(\cdot)$ and $\phi(\cdot)$ to denote the probability distribution function and the probability density function of the standard normal distribution. With this in view, note that

$$
\begin{align*}
\mathcal{J} & =\int_{x} \int_{t} \Phi^{2}\left(\frac{t-\rho x}{\sqrt{1-\rho^{2}}}\right) \phi(t) \phi(x) d t d x \\
& =\sqrt{1-\rho^{2}} \int_{x} \int_{z} \Phi^{2}(z) \phi\left(\rho x+\sqrt{1-\rho^{2}} z\right) \phi(x) d z d x \quad\left[\text { Put } z=\frac{t-\rho x}{\sqrt{1-\rho^{2}}}\right] \\
& =\frac{\sqrt{1-\rho^{2}}}{2 \pi} \int_{x} \int_{z} \Phi^{2}(z) \exp \left(-\frac{1}{2}\left(\rho^{2} x^{2}+z^{2}\left(1-\rho^{2}\right)+2 z x \rho \sqrt{1-\rho^{2}}\right)\right) \exp \left(-\frac{x^{2}}{2}\right) d z d x \\
& =\frac{\sqrt{1-\rho^{2}}}{2 \pi} \int_{z} \Phi^{2}(z) \exp \left(-\frac{z^{2}}{2} \cdot \frac{1-\rho^{2}}{1+\rho^{2}}\right) \int_{x} \exp \left(-\frac{1+\rho^{2}}{2}\left(x+\frac{2 \rho \sqrt{1-\rho^{2}}}{\sqrt{1+\rho^{2}}}\right)^{2}\right) d x d z \\
& =\frac{1}{\sqrt{2 \pi} \rho_{0}} \int \Phi^{2}(z) \exp \left(-\frac{z^{2}}{2 \rho_{0}^{2}}\right) d z \\
& =\int_{z>0}\left(\Phi^{2}\left(\rho_{0} z\right)+\Phi^{2}\left(-\rho_{0} z\right)\right) \phi(z) d z . \tag{B.1}
\end{align*}
$$

By differentiating the above with respect to $\rho_{0}$, we get from (B.1) that:

$$
\frac{d}{d \rho_{0}} \mathcal{J}=\int_{z>0} z\left(2 \Phi\left(\rho_{0} z\right)-1\right) \phi\left(\rho_{0} z\right) \rho(z) d z>0
$$

which implies that $\mathcal{J}$ is a strictly increasing function of $\rho_{0}$ which in turn, is a strictly increasing function of $|\rho|$, thereby completing the proof.

Part (2). Note that $Y_{1}, Y_{2} \stackrel{i . i . d .}{\sim} f_{Y}(\cdot)$ and $X_{1} \sim f_{X}(\cdot)$ for all $r \in[0,1]$. Here, $F_{Y}(\cdot)$ and $f_{X}(\cdot)$ are probability densities, and we will write $F_{Y}(\cdot)$ and $F_{X}(\cdot)$ to denote the corresponding distribution functions. We will write $g_{Y \mid X}(\cdot)$ to denote the conditional density of $Y \mid X$
under $g_{X, Y}(\cdot)$. Therefore,

$$
\begin{aligned}
\mathcal{J} & =\int_{x} \int_{t}\left(\int_{-\infty}^{t}\left((1-r) f_{Y}(y)+r g_{Y \mid X=x}(y)\right) d y\right)^{2} f_{Y}(t) f_{X}(x) d t d x \\
& =\int_{t} F_{Y}^{2}(t) f_{Y}(t)+2 r \int_{t} \int_{-\infty}^{t}\left(\int_{x}\left(g_{Y \mid X=x}(y)-f_{Y}(y)\right) f_{X}(x) d x\right) f_{Y}(t) d y d t \\
& +r^{2} \int_{x} \int_{t}\left(\int_{-\infty}^{t}\left(g_{Y \mid X=x}(y)-f_{Y}(y)\right) d y\right)^{2} f_{Y}(t) f_{X}(x) d t d x \\
& =\frac{1}{3}+r^{2} \int_{x} \int_{t}\left(\int_{-\infty}^{t}\left(g_{Y \mid X=x}(y)-f_{Y}(y)\right) d y\right)^{2} f_{Y}(t) f_{X}(x) d t d x,
\end{aligned}
$$

which is clearly a strictly increasing function of $r$.
Part (3). We write $Y_{2}=g\left(X_{2}\right)+\sigma Z_{2}, Z_{2} \sim \mathcal{N}(0,1)$. Further, let $\widetilde{F}_{X}(\cdot)$ be the probability distribution function of the random variable $g\left(X_{2}\right)-g\left(X_{1}\right)$. Observe that $\widetilde{F}_{X}(\cdot)$ is symmetric around 0 , in the sense that $\widetilde{F}_{X}(t)=1-\widetilde{F}_{X}(-t)$ for all $t$. By simple computations, we then have:

$$
\begin{aligned}
\mathcal{J} & =\mathbb{E} \Phi^{2}\left(Z_{2}+\frac{g\left(X_{2}\right)-g\left(X_{1}\right)}{\sigma}\right) \\
& =\int_{z} \int_{x} \Phi^{2}\left(z+\frac{x}{\sigma}\right) \phi(z) d z d \widetilde{F}_{X}(x)
\end{aligned}
$$

Let us define $\sigma_{0}:=\sigma^{-1}$ and take derivative of $\mathcal{J}$ with respect to $\sigma_{0}$, to get:

$$
\begin{align*}
\frac{d}{d \sigma_{0}} \mathcal{J} & =2 \int_{z} \int_{x} \Phi\left(z+\sigma_{0} x\right) \phi\left(z+\sigma_{0} x\right) \phi(z) x d z d \widetilde{F}_{X}(x) \\
& =2 \int_{z} \int_{x>0} \Phi(z) \phi(z)\left(x \phi\left(z-\sigma_{0} x\right)-x \phi\left(z+\sigma_{0} x\right)\right) d z d \widetilde{F}_{X}(x) \\
& =2 \int_{z>0} \int_{x>0}(\Phi(z)-\Phi(-z)) \phi(z) x\left(\phi\left(z-\sigma_{0} x\right)-\phi\left(z+\sigma_{0} x\right)\right) d z d \widetilde{F}_{X}(x) \tag{B.2}
\end{align*}
$$

Note that for $z>0$, we have

$$
\Phi(z)>\Phi(-z)
$$

and for $x, z>0$, we have:

$$
\phi\left(z-\sigma_{0} x\right)-\phi\left(z+\sigma_{0} x\right)=\exp \left(-\frac{1}{2}\left(z^{2}+\sigma_{0}^{2} x^{2}\right)\right)\left(\exp \left(\sigma_{0} z x\right)-\exp \left(-\sigma_{0} z x\right)\right)>0
$$

Combining the two observations above, with (B.2), we get:

$$
\frac{d}{d \sigma_{0}} \mathcal{J}>0
$$

which implies $\mathcal{J}$ is a strictly increasing function of $\sigma_{0}$ and consequently a strictly decreasing function of $\sigma$.

## B.2. Proof of Theorem 4.1

Part (i). Consider $\left(X_{1}^{\prime}, Y_{1}^{\prime}\right), \ldots,\left(X_{n}^{\prime}, Y_{n}^{\prime}\right) \stackrel{i . i . d .}{\sim} f_{X, Y}^{(n)}$. Let $\xi_{n}^{i}$ be Chatterjee's correlation coefficient defined with $\left(X_{i}, Y_{i}\right)$ replaced by $\left(X_{i}^{\prime}, Y_{i}^{\prime}\right)$. By the same argument as in $[15$, Lemma 9.11], $\left|\xi_{n}-\xi_{n}^{i}\right| \leq 6 n^{-1}$. We apply the bounded differences inequality [51] to get:

$$
\mathbb{P}\left(\left|\xi_{n}-\mathbb{E} \xi_{n}\right| \geq t\right) \leq 2 \exp \left(-\frac{n t^{2}}{18}\right)
$$

Also by Theorem 2.1, there exists $N_{0}$ depending only on $\Gamma=(C, \eta, \theta, \gamma)$ from Assumptions (A1) and (A2) in the main paper, such that

$$
\sqrt{n}\left|\mathbb{E} \xi_{n}-\xi\left(f_{X, Y}^{(n)}\right)\right| \leq \frac{1}{2}
$$

for all $n \geq N_{0}$. Define $K:=1+5 \sqrt{\log (2 / \alpha)}$. Combining the two displays above, under $\mathrm{H}_{0}$, we get:

$$
\mathbb{P}_{\mathrm{H}_{0}}\left(\sqrt{n}\left|\xi_{n}-\xi_{0}\right| \geq K\right) \leq \mathbb{P}_{\mathrm{H}_{0}}\left(\sqrt{n}\left|\xi_{n}-\mathbb{E}_{\mathrm{H}_{0}} \xi_{n}\right| \geq 5 \sqrt{\log (2 / \alpha)}\right) \leq \alpha
$$

for all $n \geq N_{0}$.
Next note that, by the triangle inequality, for all $f_{X, Y}^{(n)} \in \mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)$, we have:

$$
\sqrt{n}\left|\xi_{n}-\xi_{0}\right| \geq \sqrt{n} c_{n}-\sqrt{n}\left|\xi_{n}-\mathbb{E}_{\mathrm{H}_{1, n}} \xi_{n}\right|-\sqrt{n}\left|\mathbb{E}_{\mathrm{H}_{1, n}} \xi_{n}-\xi_{0}\right| .
$$

As $\sqrt{n} c_{n} \rightarrow \infty$ and the other two terms are $O_{p}(1)$ and $O(1)$ from the preceding displays, we have

$$
\beta_{\phi_{n}}\left(\mathrm{H}_{1, n}^{\mathrm{loc}}\left(c_{n} ; \Gamma\right)\right) \rightarrow 1 \quad \text { as } \quad n \rightarrow \infty
$$

Part (ii). The proof of this result will use Le Cam's two-point method; see [64, Chapter 2]. Towards this direction, let $f_{X, Y}(\cdot, \cdot)$ be a joint distribution on $\mathbb{R}^{2}$, with marginals $f_{X}(\cdot)$ and $f_{Y}(\cdot)$ such that the following conditions hold:

- $\xi\left(f_{X, Y}\right)=\xi_{0}$.
- $f_{X, Y}(\cdot, \cdot)$ satisfies Assumptions (A1) and (A2) in the main paper with paramers $C, \eta, \theta, \gamma$ given in the problem statement.
- $f_{X, Y}(\cdot, \cdot)$ is compactly supported on $[-1,1]^{2}$ and is uniformly upper and lower bounded on $[-1,1]^{2}$.
This can be easily ensured by choosing $f_{X, Y}(\cdot, \cdot)$ to be a truncated bivariate Gaussian with appropriate parameters depending on $C, \eta, \theta, \gamma$. Fix $\left\{r_{n}\right\}_{n \geq 1}$ and define

$$
f_{X, Y}^{(n)}(\cdot, \cdot)=\left(1-r_{n}\right) f_{X, Y}(\cdot, \cdot)+r_{n} f_{X}(\cdot) f_{Y}(\cdot),
$$

where $r_{n} \in(0,1)$. Note that $f_{X, Y}^{(n)}$ also satisfies Assumptions (A1) and (A2) with the same parameters. It then suffices to show the following:
(1). For all $n$ large enough and some $c_{1}>0$, we have $\left|\xi\left(f_{X, Y}^{(n)}\right)-\xi_{0}\right| \geq c_{1} r_{n}$.
(2). For all $n$ large enough and some $c_{2}>0$, we have $\operatorname{KL}\left(\otimes_{n} f_{X, Y}^{(n)} \| \otimes_{n} f_{X, Y}\right) \leq c_{2} n r_{n}^{2}$. Here $K L(p \| q)$ denotes the Kullback-Leibler divergence between probability measures $p$ and $q$, and $\otimes_{n}$ denotes the $n$-fold product measure.

Proof of 1. Let $F_{Y \mid X}^{(n)}(\cdot \mid X)$ and $F_{Y \mid X}(\cdot \mid X)$ denote the conditional distribution functions of $Y \mid X$ under $f_{X, Y}^{(n)}$ and $f_{X, Y}$ respectively. Also, let $F_{Y}(\cdot)$ be the distribution function of $Y$. Then the following holds:

$$
\begin{align*}
& \xi\left(f_{X, Y}^{(n)}\right)  \tag{B.3}\\
= & 6 \int \mathbb{E}\left[F_{Y \mid X}^{(n)}(t \mid X)\right]^{2} f_{Y}(t) d t-2 \\
= & 6 \int \mathbb{E}\left[F_{Y \mid X}(t \mid X)+r_{n}\left(F_{Y}(t)-F_{Y \mid X}(t \mid X)\right)\right]^{2} f_{Y}(t) d t-2 \\
= & \xi_{0}+2 r_{n} \underbrace{\int\left(\mathbb{E}\left[F_{Y}(t)\right]^{2}-\mathbb{E}\left[F_{Y \mid X}(t \mid X)\right]^{2}\right) f_{Y}(t) d t}_{D}+r_{n}^{2} \underbrace{\int \mathbb{E}\left(F_{Y}(t)-F_{Y \mid X}(t \mid X)\right)^{2} f_{Y}(t) d t}_{E} . \tag{B.4}
\end{align*}
$$

Note that, by the conditional version of Jensen's inequality

$$
\mathbb{E}\left[F_{Y}(t)\right]^{2} \leq \mathbb{E}\left[F_{Y \mid X}(t \mid X)\right]^{2} \quad \text { for all } t \in \mathbb{R}
$$

As $\xi_{0}>0$, this implies $X$ and $Y$ are not independent (see [15, Theorem 1.1]) and consequently, the above display implies $D<0$. For the same reason $E>0$. By replacing $r_{n}$ by $c r_{n}$ for a small constant $c$ if necessary, we can ensure

$$
r_{n} \leq \frac{|D|}{E}
$$

Combining the above display with (B.3), we get

$$
\left|\xi\left(f_{X, Y}^{(n)}\right)-\xi_{0}\right| \geq 2 r_{n}|D|-r_{n}^{2} E \geq r_{n}|D|
$$

This proves 1 .
Proof of 2. As $\operatorname{KL}\left(\otimes_{n} p \| \otimes_{n} q\right)=n \operatorname{KL}(p \| q)$, it suffices to show that $\operatorname{KL}\left(f_{X, Y}^{(n)} \| f_{X, Y}\right) \leq r_{n}^{2}$. Towards this direction, note that

$$
\begin{aligned}
& \operatorname{KL}\left(f_{X, Y}^{(n)} \| f_{X, Y}\right) \\
&= \int\left[f_{X, Y}(x, y)+r_{n}\left(f_{X}(x) f_{Y}(y)-f_{X, Y}(x, y)\right)\right] \log \frac{f_{X, Y}(x, y)+r_{n}\left(f_{X}(x) f_{Y}(y)-f_{X, Y}(x, y)\right)}{f_{X, Y}(x, y)} d x d y \\
& \stackrel{(a)}{=} \int\left[f_{X, Y}(x, y)+r_{n}\left(f_{X}(x) f_{Y}(y)-f_{X, Y}(x, y)\right)\right]\left(r_{n} \frac{f_{X}(x) f_{Y}(y)-f_{X, Y}(x, y)}{f_{X, Y}(x, y)}\right)+\mathcal{O}\left(r_{n}^{2}\right) \\
& \leq c_{2} r_{n}^{2}
\end{aligned}
$$

where (a) follows from a Taylor Series expansion and $c_{2}$ depends on the parameters of $f_{X, Y}(\cdot, \cdot)$. This proves 2.

## Appendix C: Proofs of Theorems A. 1 and A. 2

## C.1. Proof of Theorem A. 1

We consider a triangular array $\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, n}, Y_{n, n}\right)$ from a bivariate density $f_{X, Y}^{(n)}(\cdot, \cdot)$. We shall show that

$$
\max \left\{\left|n \operatorname{Var}\left(\xi_{n}^{*}\right)-\frac{2}{5}\right|,\left|n \operatorname{Var}\left(\xi_{n}\right)-\frac{2}{5}\right|,\left|n \operatorname{Cov}\left(\xi_{n}^{*}, \xi_{n}\right)-\frac{2}{5}\right|\right\}
$$

$$
\begin{equation*}
\lesssim \frac{1}{n}+\xi\left(f_{X, Y}^{(n)}\right)+\sqrt{n \log n} b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) \tag{C.1}
\end{equation*}
$$

for $b_{n}:=n^{-\frac{\gamma}{\gamma+1}}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}$.
By the standard Glivenko-Cantelli Theorem, we know that $\widehat{F}_{n}(\cdot)$ and $F_{Y}^{(n)}(\cdot)$ are "close" almost surely in the $L^{\infty}$ norm. This motivates the definition of an oracle version of $\xi_{n}$ (see Section 2 in the main paper) as follows:

$$
\begin{equation*}
\xi_{n}^{*}:=\frac{3 n}{n^{2}-1}\left(\frac{1}{n} \sum_{i \neq j} \sum_{Y}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, j}\right)\right|-\sum_{i=1}^{n}\left|F_{Y}^{(n)}\left(Y_{n,(i+1)}\right)-F_{Y}^{(n)}\left(Y_{n,(i)}\right)\right|\right) \tag{C.2}
\end{equation*}
$$

Intuitively, of course, $\xi_{n}^{*}$ is mathematically more tractable than $\xi_{n}$ as it replaces the random function $\widehat{F}_{n}(\cdot)$ by the deterministic function $F_{Y}^{(n)}(\cdot)$.

Let $N(i) \in\left\{j \in[n]: \operatorname{Rank}\left(X_{n, j}\right)=\operatorname{Rank}\left(X_{n, i}\right)+1\right\}$ be the unique index $j$ such that $X_{n, j}$ is immediately to the right of $X_{n, i}$ when $X_{n, i}$ 's are arranged in increasing order. If there are no such indices for some $i$, set the corresponding $N(i)=1$. Let us define

$$
\begin{align*}
\xi_{n}^{\prime} & :=\frac{6}{n^{2}-1} \sum_{i=1}^{n-1} \min \left\{R_{i}, R_{N(i)}\right\} \\
\xi_{n}^{* \prime} & :=\frac{6 n}{n^{2}-1}\left(\sum_{i=1}^{n} \min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right\}-\frac{1}{n} \sum_{i \neq j} \sum_{i} \min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, j}\right)\right\}\right) \tag{C.3}
\end{align*}
$$

Here and in the rest of the supplement, we remove the subscript $n$ in $R_{n, i}$ and write $R_{i}$ for notational convenience.

Since $\left\{R_{i}\right\}$ and $\left\{R_{N(i)}\right\}$ form two permutations of $[n]$, we have that $\sum R_{i}=\sum R_{N(i)}=$ $n(n-1) / 2$. Then using the simple identity $|a-b|=a+b-2 \min \{a, b\}$, one can check that

$$
n \mathbb{E}\left(\xi_{n}^{*}-\xi_{n}^{* \prime}\right)^{2} \lesssim n^{-1} \quad \text { and } \quad n \mathbb{E}\left(\xi_{n}-\xi_{n}^{\prime}\right)^{2} \lesssim n^{-1}
$$

Therefore it suffices to prove (C.1) with $\xi_{n}^{\prime}$ and $\xi_{n}^{* \prime}$ instead of $\xi_{n}$ and $\xi_{n}^{*}$ respectively. Define $\mathbf{X}^{(n)}:=\left(X_{n, 1}, \ldots, X_{n, n}\right)$. Conditioning on $\mathbf{X}^{(n)}$, we have

$$
\begin{aligned}
\operatorname{Var}\left(\xi_{n}^{* \prime}\right)= & \mathbb{E}\left(\operatorname{Var}\left(\xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)\right)+\operatorname{Var}\left(\mathbb{E}\left(\xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)\right), \quad \operatorname{Var}\left(\xi_{n}^{\prime}\right)=\mathbb{E}\left(\operatorname{Var}\left(\xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right)\right)+\operatorname{Var}\left(\mathbb{E}\left(\xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right)\right) \\
& \operatorname{Cov}\left(\xi_{n}^{\prime}, \xi_{n}^{* \prime}\right)=\mathbb{E}\left(\operatorname{Cov}\left(\xi_{n}^{\prime}, \xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)\right)+\operatorname{Cov}\left(\mathbb{E}\left(\xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right), \mathbb{E}\left(\xi_{n}^{* \prime}\right) \mid \mathbf{X}^{(n)}\right)
\end{aligned}
$$

We now decompose each of the six terms on the right hand side, starting with the three expectation of covariance terms. To analyze the first term, that is, $\mathbb{E}\left(\operatorname{Var}\left(\xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)\right)$, let us introduce the following notations.

$$
\begin{aligned}
T_{1}^{*} & :=\frac{1}{n} \sum_{i} \operatorname{Var}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right\} \mid \mathbf{X}^{(n)}\right) \\
T_{2}^{*} & :=\frac{1}{n} \sum_{i} \operatorname{Cov}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right\}, \min \left\{F_{Y}^{(n)}\left(Y_{n, N(i)}\right), F_{Y}^{(n)}\left(Y_{n, N(N(i))}\right)\right\} \mid \mathbf{X}^{(n)}\right)
\end{aligned}
$$

$$
\begin{align*}
& T_{3}^{*}:=\frac{1}{n^{3}} \sum_{(i, j, l)} \sum_{\text {distinct }} \operatorname{Cov}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, j}\right)\right\}, \min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, l}\right)\right\} \mid \mathbf{X}^{(n)}\right) \\
& T_{4}^{*}:=\frac{1}{n^{2}} \sum_{(i, j, N(i))} \sum_{\text {distinct }} \operatorname{Cov}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, j}\right)\right\}, \min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right\} \mid \mathbf{X}^{(n)}\right) \\
& T_{5}^{*}:=\frac{1}{n^{2}} \sum_{(i, j, N(i))} \sum_{\text {distinct }} \operatorname{Cov}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, N(i)}\right), F_{Y}^{(n)}\left(Y_{n, j}\right)\right\}, \min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right\} \mid \mathbf{X}^{(n)}\right) \\
& T_{6}^{*}:=\frac{1}{n^{3}} \sum_{(i, j, k, l)} \sum_{\text {distinct }} \sum_{\operatorname{Cov}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, j}\right)\right\}, \min \left\{F_{Y}^{(n)}\left(Y_{n, k}\right), F_{Y}^{(n)}\left(Y_{n, l}\right)\right\} \mid \mathbf{X}^{(n)}\right), ~(n)} \\
& T_{7}^{*}:=\frac{1}{n^{2}} \sum_{(i, j, k, N(k))} \sum_{\text {distinct }} \sum_{\operatorname{Cov}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, j}\right)\right\}, \min \left\{F_{Y}^{(n)}\left(Y_{n, k}\right), F_{Y}^{(n)}\left(Y_{n, N(k)}\right)\right\} \mid \mathbf{X}^{(n)}\right), ~}^{n} \\
& T_{8}^{*}:=\frac{1}{n^{3}} \sum_{(i, j, N(i), N(j))} \sum_{\text {distinct }} \operatorname{Cov}\left(\min \left\{F_{Y}^{(n)}\left(Y_{n, i}\right), F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right\}, \min \left\{F_{Y}^{(n)}\left(Y_{n, j}\right),\right.\right. \\
& \left.\left.F_{Y}^{(n)}\left(Y_{n, N(j)}\right)\right\} \mid \mathbf{X}^{(n)}\right) \tag{C.4}
\end{align*}
$$

Let us define

$$
\begin{equation*}
T_{j, \text { ind }}^{*}:=\text { the value of } T_{j}^{*} \text { when } X, Y \text { are independent } \tag{C.5}
\end{equation*}
$$

and the corresponding deviation terms:

$$
\begin{equation*}
Q_{j}^{*}:=T_{j}^{*}-T_{j, \mathrm{ind}}^{*} . \tag{C.6}
\end{equation*}
$$

Through an explicit calculation of the $T_{j, \text { ind }}$ 's, it follows that

$$
\begin{aligned}
T_{1}^{*}=\frac{1}{18}+Q_{1}^{*}, & T_{2}^{*}=\frac{1}{45}+Q_{2}^{*}, \\
T_{3}^{*}=\frac{1}{45}+Q_{3}^{*}, & T_{4}^{*}=\frac{1}{45}+Q_{4}^{*}, \\
T_{5}^{*}=\frac{1}{45}+Q_{5}^{*}, & T_{6}^{*}=0, \\
T_{7}^{*}=0, & T_{8}^{*}=0,
\end{aligned}
$$

where the last three equations follow once again by the fact that conditional on $\mathbf{X}^{(n)}$ the random variables $Y_{n, 1}, \ldots, Y_{n, n}$ are independent. Now, we observe that

$$
\begin{equation*}
\frac{\left(n^{2}-1\right)^{2}}{36 n^{3}} \operatorname{Var}\left(\xi_{n}^{* /} \mid \mathbf{X}^{(n)}\right)=T_{1}^{*}+2 T_{2}^{*}+4 T_{3}^{*}-4 T_{4}^{*}-4 T_{5}^{*}+T_{6}^{*}-2 T_{7}^{*}+T_{8}^{*}+O\left(n^{-1}\right) \tag{C.7}
\end{equation*}
$$

Hence, to bound the absolute value of $\frac{\left(n^{2}-1\right)^{2}}{36 n^{3}} \operatorname{Var}\left(\xi_{n}^{* *} \mid \mathbf{X}^{(n)}\right)$, we need to bound the $Q_{i}^{*}$ s. To bound the $Q_{i}^{*}$ 's stochastically, we shall use the following lemma, which is proved in Appendix D.1.

Lemma C.1. $\max \left\{\mathbb{E}\left|Q_{1}^{*}\right|, \ldots, \mathbb{E}\left|Q_{5}^{*}\right|\right\} \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)$.
Using Lemma C.1, adding the equations in (C.4) and taking expectation over $\mathbf{X}^{(n)}$, we have

$$
\left|n \mathbb{E} \operatorname{Var}\left(\xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)-\frac{2}{5}\right|
$$

$$
\begin{align*}
& =\left|36 \mathbb{E}\left(T_{1}^{*}+2 T_{2}^{*}+4 T_{3}^{*}-4 T_{4}^{*}-4 T_{5}^{*}+T_{6}^{*}-2 T_{7}^{*}+T_{8}^{*}\right)-\frac{2}{5}\right|+O\left(n^{-1}\right) \\
& =36\left|\mathbb{E}\left(Q_{1}^{*}+2 Q_{2}^{*}+4 Q_{3}^{*}-4 Q_{4}^{*}-4 Q_{5}^{*}\right)\right|+O\left(n^{-1}\right) \\
& \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) \tag{C.8}
\end{align*}
$$

Moving on to $\mathbb{E}\left(\operatorname{Var}\left(\xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)\right)$ we notice that

$$
\min \left\{R_{i}, R_{N(i)}\right\}=\sum_{k=1}^{n} \mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, i}, Y_{n, N(i)}\right\}\right)=1+\sum_{k \neq i, N(i)} \mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, i}, Y_{n, N(i)}\right\}\right) .
$$

Let us now define the following notation.

$$
\begin{align*}
T_{1}:= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n-1} \operatorname{Var}\left(\min \left\{R_{i}, R_{N(i)} \mid \mathbf{X}^{(n)}\right\}\right) \\
= & \mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 3}, Y_{n, 4}\right\}, Y_{n, 2} \leq \min \left\{Y_{n, 3}, Y_{n, 4}\right\}\right) \\
& -\mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 3}, Y_{n, 4}\right\}\right) \mathbb{P}\left(Y_{n, 2} \leq \min \left\{Y_{n, 3}, Y_{n, 4}\right\}\right)+Q_{1} \\
= & \frac{1}{18}+Q_{1},  \tag{C.9}\\
T_{2}:= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n-1} \operatorname{Cov}\left(\min \left\{R_{i}, R_{N(i)}\right\}, \min \left\{R_{N(i)}, R_{N(N(i))}\right\} \mid \mathbf{X}^{(n)}\right) \\
= & \mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 3}, Y_{n, 4}\right\}, Y_{n, 2} \leq \min \left\{Y_{n, 4}, Y_{n, 5}\right\}\right) \\
& -\mathbb{P}\left(Y_{1} \leq \min \left\{Y_{3}, Y_{4}\right\}\right) \mathbb{P}\left(Y_{2} \leq \min \left\{Y_{4}, Y_{5}\right\}\right)+Q_{2} \\
= & \frac{1}{45}+Q_{2},  \tag{C.10}\\
T_{3}:= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i} \sum_{j \neq i} \sum_{l \neq i, j, j} \operatorname{Cov}\left(\mathbb{1}\left(Y_{n, l} \leq \min \left\{Y_{n, i}, Y_{n, N(i)}\right\}\right), \mathbb{1}\left(Y_{n, l} \leq \min \left\{Y_{n, j}, Y_{n, N(j)}\right\}\right) \mid \mathbf{X}^{(n)}\right) \\
= & \mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 2}, Y_{n, 3}\right\}, Y_{n, 1} \leq \min \left\{Y_{n, 4}, Y_{n, 5}\right\}\right) \\
& -\mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 2}, Y_{n, 3}\right\}\right) \mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 4}, Y_{n, 5}\right\}\right)+Q_{3} \\
= & \frac{4}{45}+Q_{3},  \tag{C.11}\\
T_{4}:= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i} \sum_{j \neq i} \sum_{l \neq j, N(j)} \operatorname{Cov}\left(\mathbb{1}\left(Y_{n, j} \leq \min \left\{Y_{n, i}, Y_{n, N(i)}\right\}\right), \mathbb{1}\left(Y_{n, l} \leq \min \left\{Y_{n, j}, Y_{n, N(j)}\right\}\right) \mid \mathbf{X}^{(n)}\right) \\
= & \mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 2}, Y_{n, 3}\right\}, Y_{n, 4} \leq \min \left\{Y_{n, 1}, Y_{n, 5}\right\}\right) \\
& -\mathbb{P}\left(Y_{n, 1} \leq \min \left\{Y_{n, 2}, Y_{n, 3}\right\}\right) \mathbb{P}\left(Y_{n, 4} \leq \min \left\{Y_{n, 1}, Y_{n, 5}\right\}\right)+Q_{4}, \\
= & -\frac{2}{45}+Q_{4} .  \tag{C.12}\\
T_{5}:= & \frac{n}{\left(n^{2}-1\right)^{2}} \\
\times & \sum_{i} \sum_{j} \sum_{k} \sum_{l} \quad \operatorname{Cov}\left(\mathbb{1}\left(Y_{n, k} \leq \min \left\{Y_{n, i}, Y_{n, N(i)}\right\}\right), \mathbb{1}\left(Y_{n, l} \leq \min \left\{Y_{n, j}, Y_{n, N(j)}\right\}\right) \mid \mathbf{X}^{(n)}\right) \\
& i, j, k, l, N(i), N(j) \operatorname{distinct}  \tag{C.13}\\
= & 0,
\end{align*} \quad(\mathrm{C} .13)
$$

where $Q_{j}$ 's are defined in a way similar to (C.5)-(C.6). More explicitly, let

$$
\begin{equation*}
T_{j, \text { ind }} \text { be the value of } T_{j} \text { when } X, Y \text { are independent. } \tag{C.14}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{j}:=T_{j}-T_{j, \text { ind }} . \tag{C.15}
\end{equation*}
$$

When conditioned on $\mathbf{X}^{(n)}$, the variables $N(1), \ldots, N(n)$ are measurable. Using this observation, some straightforward but tedious calculation then yields

$$
\frac{n}{36} \operatorname{Var}\left(\xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right)=T_{1}+2 T_{2}+T_{3}+4 T_{4}+T_{5}+O\left(n^{-1}\right)
$$

We have (C.13) because, conditioned on $\mathbf{X}^{(n)}$, the random variables $Y_{n, 1}, \ldots, Y_{n, n}$ are independent. To bound the $Q_{i}$ 's stochastically, we shall use the following lemma, which is proved in Appendix D.2.

Lemma C.2. $\max \left\{\mathbb{E}\left|Q_{1}\right|, \mathbb{E}\left|Q_{2}\right|, \mathbb{E}\left|Q_{3}\right|, \mathbb{E}\left|Q_{4}\right|\right\} \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)$.
Using Lemma C.2, adding equations (C.9)-(C.13) and taking expectation over $\mathbf{X}^{(n)}$ we have

$$
\begin{align*}
\left|n \mathbb{E} \operatorname{Var}\left(\xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right)-\frac{2}{5}\right| & =\left|36 \mathbb{E}\left(T_{1}+2 T_{2}+T_{3}+4 T_{4}+T_{5}\right)-\frac{2}{5}\right|+O\left(n^{-1}\right) \\
& =36\left|\mathbb{E}\left(Q_{1}+2 Q_{2}+Q_{3}+4 Q_{4}\right)\right|+O\left(n^{-1}\right) \\
& \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) . \tag{C.16}
\end{align*}
$$

Similarly, we can also prove

$$
\begin{equation*}
\left|n \mathbb{E}\left(\operatorname{Cov}\left(\xi_{n}^{\prime}, \xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)\right)-\frac{2}{5}\right| \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) . \tag{C.17}
\end{equation*}
$$

We now move on to the 'variance of expectation' terms. We begin with the following lemma which is proved in Appendix D.3.

Lemma C.3. Suppose $f_{X, Y}^{(n)}(\cdot, \cdot)$ satisfies Assumptions (A1) and (A2) in the main paper. Then,
$\max \left\{\operatorname{Var}\left(\mathbb{E}\left[\sqrt{n} \xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right]\right), \operatorname{Var}\left(\mathbb{E}\left[\sqrt{n} \xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right]\right)\right\} \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+\sqrt{n \log n} b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)$.
By the Cauchy-Schwarz inequality, the same upper bound holds for $\operatorname{Cov}\left(\mathbb{E}\left(\xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right), \mathbb{E}\left(\xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right)\right)$. Putting together equations (C.16), (C.8), (C.17), Lemma C. 3 and finally (C.3), the proof of (C.1) is complete.

## C.2. Proof of Theorem A.2

Recall that $\mathbf{X}^{(n)}=\left(X_{n, 1}, \ldots, X_{n, n}\right)$. Also, recall that $N(i)=\left\{j \in[n]: \operatorname{Rank}\left(X_{n, j}\right)=\right.$ $\left.\operatorname{Rank}\left(X_{n, i}\right)+1\right\}$ is the unique index $j$ such that $X_{n, j}$ is immediately to the right of $X_{n, i}$
when the $X_{n, i}$ 's are arranged in increasing order. If there are no such indices for some $i$, set the corresponding $N(i)=1$. We shall use the oracle statistic $\xi_{n}^{*}$ defined as

$$
\begin{align*}
\xi_{n}^{*} & =\frac{3 n}{n^{2}-1}\left(\frac{1}{n} \sum_{i \neq j} \sum_{Y}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, j}\right)\right|-\sum_{i=1}^{n}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right|\right) \\
& =\frac{3}{n(n-1)} \sum_{i \neq j} \sum_{Y}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, j}\right)\right|-\frac{3}{n} \sum_{i=1}^{n}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right|+O\left(n^{-1}\right) . \tag{C.18}
\end{align*}
$$

By the U-statistic projection theorem (see Theorem 12.3 of [65]),

$$
\begin{align*}
& \sqrt{n}\left[\frac{1}{\binom{n}{2}} \sum_{i<j}\left\{\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, j}\right)\right|-\mathbb{E}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, j}\right)\right|\right\}\right] \\
= & \frac{2}{\sqrt{n}} \sum_{i=1}^{n} h\left(Y_{n, i}\right)+O\left(n^{-1 / 2}\right) \tag{C.19}
\end{align*}
$$

where $h(y):=\mathbb{E}\left|F_{Y}^{(n)}(y)-U\right|-\mathbb{E}\left|U_{1}-U_{2}\right|$ with $U, U_{1}, U_{2}$ being i.i.d Uniform $(0,1)$ random variables. Combining equations (C.18) and (C.19) we have

$$
\begin{align*}
& \sqrt{n}\left(\xi_{n}^{*}-\mathbb{E} \xi_{n}^{*}\right) \\
= & \frac{3}{\sqrt{n}} \sum_{i=1}^{n}\left\{2 h\left(Y_{n, i}\right)-\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right|+\mathbb{E}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right|\right\}+O\left(n^{-1 / 2}\right) \\
= & S_{n}+O\left(n^{-1 / 2}\right) \tag{C.20}
\end{align*}
$$

where

$$
\begin{equation*}
S_{n}=\frac{3}{\sqrt{n}} \sum_{i=1}^{n}\left\{2 h\left(Y_{n, i}\right)-\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right|+\mathbb{E}\left|F_{Y}^{(n)}\left(Y_{n, i}\right)-F_{Y}^{(n)}\left(Y_{n, N(i)}\right)\right|\right\} . \tag{C.21}
\end{equation*}
$$

In the rest of the proof, we will study the asymptotic distribution of $S_{n}$. Let us define $W_{n}:=S_{n} / \sqrt{\operatorname{Var}\left(S_{n}\right)}$ and $W_{n}^{*}:=\sqrt{n}\left(\xi_{n}^{*}-\mathbb{E} \xi_{n}^{*}\right) / \sqrt{\operatorname{Var}\left(\xi_{n}^{*}\right)}$. Using the standard properties of the Wasserstein-1 distance (same as the Kantorovic-Wasserstein distance in Definition 1.1 of the main paper) and (C.20) we get

$$
\begin{equation*}
\mathfrak{D}\left(W_{n}^{*}\right) \leq \mathfrak{D}\left(W_{n}\right)+C n^{-1 / 2} . \tag{C.22}
\end{equation*}
$$

Now we proceed to bound $\mathfrak{D}\left(W_{n}\right)$. Let us define

$$
\mathcal{M}:=\left\{\left(X_{n, 1}, Y_{n, 1}\right), \cdots,\left(X_{n, n}, Y_{n, n}\right)\right\} \text { and } \mathcal{M}^{\prime}:=\left\{\left(X_{n, 1}^{\prime}, Y_{n, 1}^{\prime}\right), \cdots,\left(X_{n, 1}^{\prime}, Y_{n, n}^{\prime}\right)\right\}
$$

where $\mathcal{M}$ and $\mathcal{M}^{\prime}$ are independent and identically distributed. Let

$$
\begin{aligned}
\mathcal{M}^{i} & :=\left\{\left(X_{n, 1}, Y_{n, 1}\right), \cdots,\left(X_{n, i}^{\prime}, Y_{n, i}^{\prime}\right), \cdots,\left(X_{n, n}, Y_{n, n}\right)\right\} \\
\text { and } \quad \mathcal{M}^{i j} & :=\left\{\left(X_{n, 1}, Y_{n, 1}\right), \cdots,\left(X_{n, i}^{\prime}, Y_{n, i}^{\prime}\right), \cdots,\left(X_{n, j}^{\prime}, Y_{n, j}^{\prime}\right), \cdots,\left(X_{n, n}, Y_{n, n}\right)\right\} .
\end{aligned}
$$

Let us define $W_{\ell}:\left(\mathbb{R}^{2}\right)^{n} \rightarrow \mathbb{R}$ as,
$W_{\ell}\left(\left(x_{1}, y_{1}\right), \cdots,\left(x_{n}, y_{n}\right)\right):=3\left[2 h\left(y_{\ell}\right)-\left|F_{Y}^{(n)}\left(y_{\ell}\right)-F_{Y}^{(n)}\left(y_{N(\ell)}\right)\right|+\mathbb{E}\left|F_{Y}^{(n)}\left(y_{\ell}\right)-F_{Y}^{(n)}\left(y_{N(\ell)}\right)\right|\right]$.
By graphical rule on a measure space $\mathcal{X}^{n}$ we mean a mapping from $\mathcal{X}^{n}$ to the space of undirected graphs on $n$ vertices. For $\boldsymbol{x} \in \mathcal{X}^{n}$, the graphical rule at $\boldsymbol{x}$ will be denoted by $G(\boldsymbol{x})$. Such graphical rules will be called symmetric if for any permutation $\pi$ on $[n]$, the edges in $G\left(x_{\pi(1)}, \cdots, G_{\pi(n)}\right)$ will precisely be $\{(\pi(i), \pi(j)):(i, j) \in \mathcal{E}(G(x))\}$. This definition is in line with Section 2.3 of [14]. As defined in Section 2.3 of [14], we also consider the definition of a vector $\boldsymbol{x} \in \mathcal{X}^{n}$ being embedded in a vector $\boldsymbol{y} \in \mathcal{X}^{m}$ for $m>n$. For a function $f$ defined on $\mathcal{X}^{n}$, we shall call indices the $i$ and $j$ non interacting with respect to the triplet $\left(f, \boldsymbol{x}, \boldsymbol{x}^{\prime}\right)$ if

$$
f(\boldsymbol{x})-f\left(\boldsymbol{x}^{j}\right)=f\left(\boldsymbol{x}^{i}\right)-f\left(\boldsymbol{x}^{i j}\right),
$$

where

$$
x_{k}^{i}= \begin{cases}x_{k} & \text { if } k \neq i, \\ x_{i}^{\prime} & \text { if } k=i,\end{cases}
$$

and

$$
x_{k}^{i j}= \begin{cases}x_{k} & \text { if } k \in\{i, j\} \\ x_{k}^{\prime} & \text { if } k \in\{i, j\} .\end{cases}
$$

A graphical rule is called an interaction rule for a function $f$ if the edge $(i, j)$ is absent in all of $G(\boldsymbol{x}), G\left(\boldsymbol{x}^{i}\right), G\left(\boldsymbol{x}^{j}\right)$ and $G\left(\boldsymbol{x}^{i j}\right)$ implies the pair $(i, j)$ is non-interacting with respect to the triplet $\left(f, \boldsymbol{x}, \boldsymbol{x}^{\prime}\right)$.

Now let us consider the measure space $\mathcal{X}=\mathbb{R}^{2}$. For $\boldsymbol{x} \in \mathcal{X}^{n}$, let us define the following distance of $\{1, \cdots, n\}$.

$$
D_{\boldsymbol{x}}(i, j):= \begin{cases}n+20 & \text { if } x_{i}>x_{j}  \tag{C.24}\\ \#\left\{\ell: x_{i}<x_{\ell}<x_{j}\right\} & \text { if } x_{i} \leq x_{j} .\end{cases}
$$

Given a configuration $\boldsymbol{x} \in \mathcal{X}^{n}$, let us define a graph $\mathcal{G}(\boldsymbol{x})$ on $[n]$ as follows. For a pair $\{i, j\}$, there exist an edge between $i$ and $j$ if there exists an $\ell \in[n]$ such that,

$$
D_{\boldsymbol{x}}(\ell, i) \leq 2 \quad \text { and } \quad D_{\boldsymbol{x}}(\ell, j) \leq 2 .
$$

Note that this graphical rule is a symmetric rule. The next lemma shows that this is an interaction rule for the function $W_{\ell}$ for all $1 \leq \ell \leq n$, and has been proved in Appendix D.4.

Lemma C.4. Consider the graphical rule $\mathcal{G}(\boldsymbol{x})$ for $\boldsymbol{x} \in\left(\mathbb{R}^{2}\right)^{n}$, as defined above. For any pair of vertices $i, j$ if there exists no edge $\{i, j\}$ in $\mathcal{G}(\boldsymbol{x}), \mathcal{G}\left(\boldsymbol{x}^{i}\right), \mathcal{G}\left(\boldsymbol{x}^{j}\right)$ and $\mathcal{G}\left(\boldsymbol{x}^{i j}\right)$, then for all $1 \leq \ell \leq n$, we have,

$$
W_{\ell}(\boldsymbol{x})-W_{\ell}\left(\boldsymbol{x}^{i}\right)-W_{\ell}\left(\boldsymbol{x}^{j}\right)+W_{\ell}\left(\boldsymbol{x}^{i j}\right)=0,
$$

where $W_{\ell}$ is defined in (C.23).
By Lemma C. $4, \mathcal{G}$ is a symmetric graphical interaction rule. Let us define

$$
\Delta_{j}:=S_{n}(\mathcal{M})-S_{n}\left(\mathcal{M}^{j}\right),
$$

where $S_{n}$ is defined in (C.21), and $S_{n}(\mathcal{M})$ is $S_{n}$ computed with dataset $\mathcal{M}$. Let us define

$$
M:=\max _{j}\left|\Delta_{j}\right|
$$

Since $F_{Y}^{(n)}(y) \leq 1$ for all $y$, we have a constant $C>0$ such that for all $j \in[n]$,

$$
\begin{equation*}
\left|\Delta_{j}\right| \leq \frac{C}{\sqrt{n}} \quad \text { and } \quad M \leq \frac{C}{\sqrt{n}} . \tag{C.25}
\end{equation*}
$$

Let us now construct a new graph $\mathcal{G}^{\prime}(\mathcal{M})$ on vertices $\{1, \cdots, n+4\}$ as follows. There exists an edge between the vertices $i$ and $j$ if and only if there exists an $\ell \in[n+4]$ such that,

$$
D_{\mathcal{M}}(\ell, i) \leq 6 \quad \text { and } \quad D_{\mathcal{M}}(\ell, j) \leq 6
$$

Clearly all the edges of $\mathcal{G}(\mathcal{M})$ are in $\mathcal{G}^{\prime}(\mathcal{M})$, meaning that $\mathcal{G}(\mathcal{M})$ is embedded in $\mathcal{G}^{\prime}(\mathcal{M})$. Moreover, $\mathcal{G}^{\prime}(\mathcal{M})$ is a symmetric graphical rule. The degree of any vertex in $\mathcal{G}^{\prime}(\mathcal{M})$ is bounded by 14 as if there exists an $\ell \in[n]$ such that,

$$
D_{\mathcal{M}}(\ell, i) \leq 2 \quad \text { and } \quad D_{\mathcal{M}}(\ell, j) \leq 2
$$

then $i-7 \leq j \leq i+7$. Hence, almost surely

$$
\begin{equation*}
\delta:=1+\text { degree of vertex } 1 \text { in } \mathcal{G}^{\prime} \leq 15 \tag{C.26}
\end{equation*}
$$

Now, using Theorem 2.5 of [14], we have an absolute constant $C_{1}>0$ such that

$$
\begin{equation*}
\mathfrak{D}\left(W_{n}\right) \leq \frac{C_{1} n^{1 / 2}}{\sigma_{n}^{* 2}} \mathbb{E}\left(M^{8}\right)^{1 / 4} \mathbb{E}\left(\delta^{4}\right)^{1 / 4}+\frac{1}{2 \sigma_{n}^{* 3}} \sum_{j=1}^{n} \mathbb{E}\left|\Delta_{j}\right|^{3}, \tag{C.27}
\end{equation*}
$$

where,

$$
\sigma_{n}^{* 2}=\operatorname{var}\left(\frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} W_{\ell}(\mathcal{M})\right)
$$

and $W_{\ell}(\mathcal{M})$ is the function $W_{\ell}$ computed with the dataset $\mathcal{M}$. The proof of Theorem A. 2 is now completed by plugging in the bounds from (C.20), (C.25), and (C.26), into (C.22).

## Appendix D: Proofs of Lemmas

## D.1. Proof of Lemma C. 1

To reduce notation, hide the dependence on $n$ and write $f_{X, Y}(\cdot, \cdot), f_{X}(\cdot), f_{Y}(\cdot)$ and $F_{Y}(\cdot)$ to mean $f_{X, Y}^{(n)}(\cdot, \cdot), f_{X}^{(n)}(\cdot), f_{Y}^{(n)}$ and $F_{Y}^{(n)}(\cdot)$ respectively. We will also write $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$ to mean the random variables $\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, n}, Y_{n, n}\right) \sim f_{X, Y}^{(n)}$ from the triangular array.

We defer the steps for $T_{1}^{*}$ to the proof of Lemma C. 2 and start with the possibly more complicated term $T_{2}^{*}$. Observe that for any $i, j$,

$$
\begin{equation*}
\min \left\{F\left(Y_{i}\right), F\left(Y_{j}\right)\right\}=\int \mathbb{1}\left(y \leq \min \left\{Y_{i}, Y_{j}\right\}\right) f_{Y}(y) d y \tag{D.1}
\end{equation*}
$$

Using (D.1) in the definition of $T_{2}^{*}$ and writing $Q_{21}^{*}$ and $Q_{22}^{* *}$ for terms to be bounded at the end of the proof, we get:

$$
\begin{align*}
& \mathbb{E} T_{2}^{*}=\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{3} \mid X_{N(N(i))}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& -\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& \quad f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{3} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{4} \mid X_{N(N(i))}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \\
& =Q_{21}^{*}+\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& \quad f_{Y \mid X}\left(y_{2} \mid X_{i}\right) f_{Y \mid X}\left(y_{3} \mid X_{i}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& =\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& \quad f_{Y \mid X}\left(y_{2} \mid X_{i}\right) f_{Y \mid X}\left(y_{3} \mid X_{i}\right) f_{Y \mid X}\left(y_{4} \mid X_{i}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \\
& =\frac{1}{45}+Q_{22}^{*}+\mathbb{P}\left(Y_{4} \leq \min \left\{Y_{1}, Y_{2}\right\}, Y_{5} \leq \min \left\{Y_{2}, Y_{3}\right\}\right) \\
& \quad-\mathbb{P}\left(Y_{5} \leq \min \left\{Y_{1}, Y_{2}\right\}\right) \mathbb{P}\left(Y_{6} \leq \min \left\{Y_{3}, Y_{4}\right\}\right)
\end{align*}
$$

In terms of the notation used in the proof of Theorem 2.1 in the main paper, $Q_{2}^{*}=Q_{21}^{*}+Q_{22}^{*}$. We now move on to the error terms $Q_{21}^{*}$ and $Q_{22}^{*}$.

Bound for $Q_{21}^{*}$. Note that by our definition of $Q_{1}$, one has

$$
\begin{aligned}
& \left|Q_{21}^{*}\right| \\
& =\left\lvert\, \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right)\right. \\
& f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{3} \mid X_{N(N(i))}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& -\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{3} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{4} \mid X_{N(N(i))}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \\
& -\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& \quad f_{Y \mid X}\left(y_{2} \mid X_{i}\right) f_{Y \mid X}\left(y_{3} \mid X_{i}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& +\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& \quad f_{Y \mid X}\left(y_{2} \mid X_{i}\right) f_{Y \mid X}\left(y_{3} \mid X_{i}\right) f_{Y \mid X}\left(y_{4} \mid X_{i}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \mid \\
& \begin{array}{l}
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int\left|\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\} \mid X_{i}\right)-\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\} \mid X_{N(i)}\right)\right| f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5}
\end{array}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int\left|\mathbb{P}\left(Y \geq y_{5} \mid X_{i}\right)-\mathbb{P}\left(Y \geq y_{5} \mid X_{N(N(i))}\right)\right| f_{Y}\left(y_{5}\right) d y_{5} \\
& +\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int\left|\mathbb{P}\left(Y \geq y_{5} \mid X_{i}\right)-\mathbb{P}\left(Y \geq y_{5} \mid X_{N(i)}\right)\right| f_{Y}\left(y_{5}\right) d y_{5}+O\left(n^{-1}\right) \tag{D.3}
\end{align*}
$$

Let us now focus on the first term on the right hand side of (D.3). Towards this direction, by using Assumption (A1), (2.2) from the main paper, we further get:

$$
\begin{align*}
& \mathbb{E} \int\left|\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\} \mid X_{1}\right)-\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\} \mid X_{N(1)}\right)\right| f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5} \\
\leq & \mathbb{E}\left[\min \left\{\left(1+L_{1}\left(X_{1}, \max \left\{Y_{4}, Y_{5}\right\}\right)+L_{2}\left(X_{N(1)}, \max \left\{Y_{4}, Y_{5}\right\}\right)\right)\left|X_{1}-X_{N(1)}\right|^{\eta}, 1\right\}\right] \\
\lesssim & \mathbb{P}\left(\left|X_{1}-X_{N(1)}\right| \geq 1\right)+\left\{\mathbb{E}\left(1+\left(L_{1}\left(X_{1}, \max \left\{Y_{4}, Y_{5}\right\}\right)\right)^{\theta}+\left(L_{2}\left(X_{1}, \max \left\{Y_{4}, Y_{5}\right\}\right)\right)^{\theta}\right)\right\}^{\frac{1}{\theta}} \\
\{ & \left.\mathbb{E}\left|X_{1}-X_{N(1)}\right|^{\frac{\theta \eta}{\theta-1}} \mathbb{1}\left(\left|X_{1}-X_{N(1)}\right| \leq 1\right)\right\}^{\frac{\theta-1}{\theta}} \tag{D.4}
\end{align*}
$$

where the last two lines follow from Assumption (A1), (2.2) in the main paper and Hölder's inequality. Now we will bound each term on the right hand side of (D.4) separately. First note that,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \mathbb{E}\left(L_{1}^{(n)}\left(X_{1}, \max \left\{Y_{4}, Y_{5}\right\}\right)\right)^{2} \leq 2 \limsup _{n \rightarrow \infty} E\left(L_{1}^{(n)}\left(X_{1}, Y_{4}\right)\right)^{2} \lesssim 1 \tag{D.5}
\end{equation*}
$$

where the last line follows from Assumption (A2), (2.3) in the main paper. Next, on using [15, Lemma 9.4] and (D.5), it further follows that,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \mathbb{E}\left(L_{1}^{(n)}\left(X_{N(1)}, \max \left\{Y_{4}, Y_{5}\right\}\right)\right)^{2} \lesssim 1 \tag{D.6}
\end{equation*}
$$

Using Lemma D.4, we then get:

$$
\begin{align*}
\left\{\mathbb{E}\left|X_{1}-X_{N(1)}\right|^{\frac{\theta \eta}{\theta-1}} \mathbb{1}\left(\left|X_{1}-X_{N(1)}\right| \leq 1\right)\right\}^{\frac{\theta-1}{\theta}} & \leq\left\{\mathbb{E} \left\lvert\, X_{1}-X_{N(1)}^{\frac{\theta \eta}{\mid-1} \wedge 1} \mathbb{1}\left(\left|X_{1}-X_{N(1)}\right| \leq 1\right)\right.\right\}^{\frac{\theta-1}{\theta}} \\
& \lesssim\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)} \tag{D.7}
\end{align*}
$$

Plugging in the conclusions from (D.5), (D.6), and (D.7) into (D.4), we get that the first term on the right hand side of (D.3) satisfies,

$$
\begin{align*}
& \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int\left|\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\} \mid X_{i}\right)-\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\} \mid X_{N(i)}\right)\right| f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5} \\
& \leq n^{-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}:=b_{n} .} \tag{D.8}
\end{align*}
$$

By using a similar argument as above, we get the same bound for the other two terms on the right hand side of (D.3), which implies,

$$
\left|Q_{21}^{*}\right| \lesssim b_{n}
$$

Also, from (D.3), provided $X_{1}$ and $Y_{1}$ are independent, i.e., $\xi\left(f_{X, Y}^{(n)}\right)=0$, we also have $R^{(1)}=O\left(n^{-1}\right)$. Therefore,

$$
\begin{equation*}
\left|Q_{21}^{*}\right| \lesssim b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)+n^{-1} . \tag{D.9}
\end{equation*}
$$

Bound for $Q_{22}^{*}$. We have

$$
\begin{aligned}
& Q_{22}^{*}=\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& \quad f_{Y \mid X}\left(y_{2} \mid X_{i}\right) f_{Y \mid X}\left(y_{3} \mid X_{i}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& -\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) \\
& f_{Y \mid X}\left(y_{2} \mid X_{i}\right) f_{Y \mid X}\left(y_{3} \mid X_{i}\right) f_{Y \mid X}\left(y_{4} \mid X_{i}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \\
& -\mathbb{P}\left(Y_{4} \leq \min \left\{Y_{1}, Y_{2}\right\}, Y_{5} \leq \min \left\{Y_{2}, Y_{3}\right\}\right)+\mathbb{P}\left(Y_{5} \leq \min \left\{Y_{1}, Y_{2}\right\}\right) \mathbb{P}\left(Y_{6} \leq \min \left\{Y_{3}, Y_{4}\right\}\right)
\end{aligned}
$$

Then it can be checked that

$$
\begin{equation*}
\left|Q_{22}^{*}\right| \leq\left|S_{1}\right|+2\left|S_{2}\right|+\left|S_{3}\right|+2\left|S_{4}\right|+\left|S_{5}\right|+4\left|S_{6}\right|+2\left|S_{7}\right|+4\left|S_{8}\right|+4\left|S_{9}\right|+\left|S_{10}\right|, \tag{D.10}
\end{equation*}
$$

where

$$
\begin{aligned}
& S_{1}:=\mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{1}\right) \\
&\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right) f_{Y}\left(y_{3}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& S_{2}:=\mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
& f_{Y}\left(y_{2}\right) f_{Y}\left(y_{3}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& S_{3}:=\mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
& f_{Y}\left(y_{2}\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& S_{4}:=\mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{1}\right) \\
&\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& S_{5}:=\mathbb{E} \int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right)\left(f_{X, Y}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
&\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} \\
& S_{6}:=\mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
& f_{Y}\left(y_{2}\right) f_{Y}\left(y_{3}\right) f_{Y}\left(y_{4}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \\
& S_{7}:=\mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
&\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right) f_{Y}\left(y_{3}\right) f_{Y}\left(y_{4}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6}
\end{aligned}
$$

$$
\begin{aligned}
& S_{8}:=\mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
& f_{Y}\left(y_{2}\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right) f_{Y}\left(y_{4}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \\
& S_{9}:=\mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
&\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right) f_{Y}\left(y_{4}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5} d y_{6} \\
& S_{10}:=\mathbb{E} \int \mathbb{1}\left(y_{5} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{6} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{6}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right) \\
& \quad\left(f_{Y \mid X}\left(y_{2} \mid X_{i}\right)-f_{Y}\left(y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right)\left(f_{Y \mid X}\left(y_{4} \mid X_{1}\right)-f_{Y}\left(y_{4}\right)\right) d y_{1} d y_{2} \ldots d y_{6}
\end{aligned}
$$

Now we will bound each of the $S_{i}$ 's individually. We start with $S_{1}$. Observe that

$$
\begin{align*}
S_{1} & =\int \mathbb{1}\left(y_{4} \leq \min \left\{y_{1}, y_{2}\right\}\right) \mathbb{1}\left(y_{5} \leq \min \left\{y_{2}, y_{3}\right\}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) f_{Y}\left(y_{1}\right) \times \\
& \times\left(\int\left(f_{X, Y}\left(y_{2} \mid X_{1}=x\right)-f_{Y}\left(y_{2}\right)\right) f_{X}(x) d x\right) f_{Y}\left(y_{3}\right) d y_{1} d y_{2} d y_{3} d y_{4} d y_{5}=0 \tag{D.11}
\end{align*}
$$

A similar calculation shows that $S_{2}=0$. We now look at $S_{3}$. Note that,

$$
\begin{align*}
S_{3}= & \mathbb{E} \int_{y_{4}, y_{5}}\left(\int_{y_{2}} \mathbb{1}\left(y_{2} \geq \max \left\{y_{4}, y_{5}\right\}\right) f_{Y}\left(y_{2}\right) d y_{2}\right)\left(\int _ { y _ { 1 } } \mathbb { 1 } ( y _ { 1 } \geq y _ { 4 } ) \left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)\right.\right. \\
& \left.\left.-f_{Y}\left(y_{1}\right)\right) d y_{1}\right)\left(\int_{y_{3}} \mathbb{1}\left(y_{3} \geq y_{5}\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right) d y_{3}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5} \\
= & \mathbb{E} \int_{y_{4}, y_{5}}\left(\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\}\right)\right)\left(\mathbb{P}\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right)\left(\mathbb{P}\left(Y \geq y_{5} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{5}\right)\right) \\
\leq & f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5} \\
\leq & \left.\mathbb{E} \iint_{y_{4}}\left|\mathbb{P}\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right| f_{Y}\left(y_{4}\right) d y_{4}\right]^{2} \\
& \left.\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right)^{2} f_{Y}\left(y_{4}\right) d y_{4}=\frac{1}{6} \xi\left(f_{X, Y}^{(n)}\right), \tag{D.12}
\end{align*}
$$

where the last step uses the Cauchy-Schwarz inequality. In order to bound $S_{4}$, let us start with:

$$
\begin{aligned}
& S_{4}= \mathbb{E} \int_{y_{4}, y_{5}}\left(\int_{y_{2}} \mathbb{1}\left(y_{2} \geq \max \left\{y_{4}, y_{5}\right\}\right)\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right) d y_{2}\right)\left(\int_{y_{1}} \mathbb{1}\left(y_{1} \geq y_{4}\right) f_{Y}\left(y_{1}\right) d y_{1}\right) \\
&\left(\int_{y_{3}} \mathbb{1}\left(y_{3} \geq y_{5}\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{1}\right)-f_{Y}\left(y_{3}\right)\right) d y_{3}\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5} \\
&=\mathbb{E} \int_{y_{4}, y_{5}}\left(\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\} \mid X_{1}\right)-\mathbb{P}\left(Y \geq \max \left\{y_{4}, y_{5}\right\}\right)\right)\left(\mathbb{P}\left(Y \geq y_{5} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{5}\right)\right) \\
&=\mathbb{E} \int_{y_{4}} \int_{y_{5} \leq y_{4}}\left(\mathbb{P}\left(Y \geq y_{4}\right)\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5} \\
&\left.\left.f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d X_{1}\right)-\mathbb{P}\left(Y \geq y_{5}\right)\right)\left(\mathbb{P}\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right)\left(\mathbb{P}\left(Y \geq y_{4}\right)\right)
\end{aligned}
$$

$$
\begin{align*}
& +\mathbb{E} \int_{y_{4}} \int_{y_{5} \geq y_{4}}\left(\mathbb{P}\left(Y \geq y_{5} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{5}\right)\right)^{2}\left(\mathbb{P}\left(Y \geq y_{4}\right)\right) f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{5} d y_{4} \\
\leq & \mathbb{E}\left[\int_{y_{4}}\left|\mathbb{P}\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right| f_{Y}\left(y_{4}\right) d y_{4}\right]^{2}+\mathbb{E} \int\left(\mathbb{P}\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right)^{2} f_{Y}\left(y_{4}\right) d y_{4} \\
\leq & \frac{1}{6} \xi\left(f_{X, Y}^{(n)}\right)+\frac{1}{6} \xi\left(f_{X, Y}^{(n)}\right)=\frac{1}{3} \xi\left(f_{X, Y}^{(n)}\right) \tag{D.13}
\end{align*}
$$

For $S_{5}$, we observe that

$$
\begin{align*}
& S_{5}=\mathbb{E} \int_{y_{4}, y_{5}}\left(\mathbb{P}\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right)\left(\mathbb{P}\left(Y \geq y_{5} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{5}\right)\right) \\
& \mathbb{1}\left(y_{2} \geq \max \left\{y_{4}, y_{5}\right\}\right)\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right) d y_{2} f_{Y}\left(y_{4}\right) f_{Y}\left(y_{5}\right) d y_{4} d y_{5} \\
& \leq 2 \mathbb{E} \int\left(\mathbb{P}\left(Y \geq y_{4} \mid X_{1}\right)-\mathbb{P}\left(Y \geq y_{4}\right)\right)^{2} f_{Y}\left(y_{4}\right) d y_{4}=\frac{1}{3} \xi\left(f_{X, Y}^{(n)}\right) \tag{D.14}
\end{align*}
$$

A similar set of calculations can be used to show that $\max _{i \geq 6}\left|S_{i}\right| \lesssim \xi\left(f_{X, Y}^{(n)}\right)$. Combining the above observation with (D.11), (D.12), (D.13) and (D.14), we get $\max _{i \geq 1}\left|S_{i}\right| \lesssim \xi\left(f_{X, Y}^{(n)}\right)$. Applying this observation in (D.10), we further have:

$$
\left|Q_{22}^{*}\right| \lesssim \sum_{i \geq 1}\left|S_{i}\right| \lesssim \xi\left(f_{X, Y}^{(n)}\right)
$$

Combining the above display with (D.9) and (D.2) we finally obtain,

$$
\begin{equation*}
\left|\mathbb{E} T_{2}^{*}-\frac{1}{45}\right| \lesssim Q_{21}^{*}+Q_{22}^{*} \lesssim b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)+\xi\left(f_{X, Y}^{(n)}\right)+\frac{1}{n} \tag{D.15}
\end{equation*}
$$

A similar set of computations can be used in $\mathbb{E} T_{i}^{*}, i=1,3,4,5$. We skip the relevant algebraic details for brevity. We present the corresponding conclusions below:

$$
\begin{equation*}
\max \left\{\left|\mathbb{E} T_{1}^{*}-\frac{1}{18}\right|,\left|\mathbb{E} T_{3}^{*}-\frac{1}{45}\right|,\left|\mathbb{E} T_{4}^{*}-\frac{1}{45}\right|,\left|\mathbb{E} T_{5}^{*}-\frac{1}{45}\right|\right\} \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) \tag{D.16}
\end{equation*}
$$

By the definition of $Q_{i}^{*}$ for $i=1, \ldots, 5$ this completes the proof of Lemma C.1.

## D.2. Proof of Lemma C.2

The proof is similar to that of Lemma C.1. We decompose the first term $T_{1}$. Notice that $T_{1}^{*}$ from Lemma C. 1 is very similar and can be analysed similarly. To reduce notation, we will hide the dependence on $n$ and write $f_{X, Y}(\cdot, \cdot), f_{X}(\cdot), f_{Y}(\cdot)$ and $F_{Y}(\cdot)$ to mean $f_{X, Y}^{(n)}(\cdot, \cdot), f_{X}^{(n)}(\cdot), f_{Y}^{(n)}(\cdot)$ and $F_{Y}^{(n)}(\cdot)$ respectively. We will write $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$ to mean the random variables $\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, n}, Y_{n, n}\right) \sim f_{X, Y}^{(n)}$ from the triangular array. Also we will use $\mathbf{Y}$ and $\mathbf{X}$ for the set of random variables $\left(Y_{1}, \ldots, Y_{n}\right)$ and $\left(X_{1}, \ldots, X_{n}\right)$ respectively.

$$
\begin{aligned}
& \mathbb{E} T_{1} \\
= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \mathbb{E} \operatorname{Var}\left(\min \left\{R_{i}, R_{N(i)} \mid \mathbf{X}\right\}\right)
\end{aligned}
$$

$$
\begin{align*}
& =\mathbb{E} \int \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n-1}\left(\min \left\{R_{i}, R_{N(i)}\right\}\right)^{2} f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y}) d \mathbf{y}-\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \mathbb{E}\left[\int \min \left\{R_{i}, R_{N(i)}\right\} f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y}) d \mathbf{y}\right]^{2} \\
& =\mathbb{E} \int\left[\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n-1}\left(\min \left\{R_{i}, R_{N(i)}\right\}\right)^{2}\right]\left\{f_{\mathbf{Y}}(\mathbf{y})+f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y})-f_{\mathbf{Y}}(\mathbf{y})\right\} d \mathbf{y} \\
& \quad-\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \mathbb{E}\left[\int \min \left\{R_{i}, R_{N(i)}\right\}\left\{f_{\mathbf{Y}}(\mathbf{y})+f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y})-f_{\mathbf{Y}}(\mathbf{y})\right\} d \mathbf{y}\right]^{2} \\
& = \\
& =\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n-1} \mathbb{E}\left[\int\left(\min \left\{R_{i}, R_{N(i)}\right\}\right)^{2} f_{\mathbf{Y}}(\mathbf{y}) d \mathbf{y}-\left(\int \min \left\{R_{i}, R_{N(i)}\right\} f_{\mathbf{Y}}(\mathbf{y}) d \mathbf{y}\right)^{2}\right]+Q_{11}+Q_{12} \\
& =\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \sum_{k \neq i, N(i)} \mathbb{E V a r}\left(\mathbb{1}\left(Y_{k} \leq \min \left\{Y_{i}, Y_{N(i)}\right\}\right)\right) \\
& +\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \sum_{k_{1} \neq k_{2}} \sum_{k_{1}, k_{2} \neq i, N(i)} \mathbb{E} \operatorname{Cov}\left(\mathbb{1}\left(Y_{k_{1}} \leq \min \left\{Y_{i}, Y_{N(i)}\right\}\right), \mathbb{1}\left(Y_{k_{2}} \leq \min \left\{Y_{i}, Y_{N(i)}\right\}\right)\right)+Q_{11}+Q_{12} \\
& =\mathbb{P}\left(Y_{1} \leq \min \left\{Y_{3}, Y_{4}\right\}, Y_{2} \leq \min \left\{Y_{3}, Y_{4}\right\}\right)-\mathbb{P}\left(Y_{1} \leq \min \left\{Y_{3}, Y_{4}\right\}\right) \mathbb{P}\left(Y_{2} \leq \min \left\{Y_{3}, Y_{4}\right\}\right) \\
& +O\left(n^{-1}\right)+Q_{11}+Q_{12} \\
& =\frac{1}{6}-\frac{1}{3} \cdot \frac{1}{3}+Q_{11}+Q_{12}+O\left(n^{-1}\right)  \tag{D.17}\\
& =\frac{1}{18}+Q_{11}+Q_{12}+O\left(n^{-1}\right) .
\end{align*}
$$

In terms of the notation used in the proof of Theorem 2.1 in the main paper, $Q_{1}=$ $Q_{11}+Q_{12}$.
Bounding $Q_{11}$. Since $\min \left\{R_{i}, R_{N(i)}\right\}=1+\sum_{k \neq i, N(i)} \mathbb{1}\left(Y_{k} \leq Y_{i}, Y_{N(i)}\right)$, we have

$$
\begin{align*}
Q_{11}:= & \int\left[\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n-1}\left(\min \left\{R_{i}, R_{N(i)}\right\}\right)^{2}\right]\left(f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y})-f_{\mathbf{Y}}(\mathbf{y})\right) d \mathbf{y} \\
= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \sum_{k_{1} \neq k_{2}} \sum \int\left[\mathbb{1}\left(y_{k_{1}} \leq y_{i} \wedge y_{N(i)}\right) \mathbb{1}\left(y_{k_{2}} \leq y_{i} \wedge y_{N(i)}\right)\right]\left(f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y})-f_{\mathbf{Y}}(\mathbf{y})\right) d \mathbf{y} \\
& +O\left(n^{-1}\right) \\
= & \int \mathbb{1}\left(y_{1} \leq \min \left\{y_{3}, y_{4}\right\}\right) \mathbb{1}\left(y_{2} \leq \min \left\{y_{3}, y_{4}\right\}\right)\left(\prod_{i=1}^{4} f_{Y \mid X}\left(y_{i} \mid X_{i}\right)-\prod_{i=1}^{4} f_{Y}\left(y_{i}\right)\right) d y_{1} d y_{2} d y_{3} d y_{4} \\
& +O\left(n^{-1}\right) . \tag{D.18}
\end{align*}
$$

The difference of product pdfs can be written out as in the expansion of

$$
a^{4}-b^{4}=a^{3}(a-b)+a^{2} b(a-b)+a b^{2}(a-b)+b^{3}(a-b) .
$$

For example, term 1 (corresponding to $a^{3}(a-b)$ ) will be

$$
\mathbb{E} \int \mathbb{1}\left(y_{1}, y_{2} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right) f_{Y \mid X}\left(y_{3} \mid X_{3}\right)\left(f_{Y \mid X}\left(y_{4} \mid X_{N(3)}\right)-f_{Y}\left(y_{4}\right)\right) d \mathbf{y}
$$

$$
\begin{align*}
& =Q_{11}^{\prime}+\mathbb{E} \int \mathbb{1}\left(y_{1}, y_{2} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right) f_{Y \mid X}\left(y_{3} \mid X_{3}\right)\left(f_{Y \mid X}\left(y_{4} \mid X_{3}\right)-f_{Y}\left(y_{4}\right)\right) d \mathbf{y} \\
& =Q_{11}^{\prime}+\mathbb{E} \int \mathbb{1}\left(y_{1}, y_{2} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right) \\
& \quad\left(f_{Y \mid X}\left(y_{3} \mid X_{3}\right)-f_{Y}\left(y_{3}\right)\right)\left(f_{Y \mid X}\left(y_{4} \mid X_{3}\right)-f_{Y}\left(y_{4}\right)\right) d \mathbf{y} \\
& +\mathbb{E} \int \mathbb{1}\left(y_{1}, y_{2} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right) f_{Y}\left(y_{3}\right)\left(f_{Y \mid X}\left(y_{4} \mid X_{3}\right)-f_{Y}\left(y_{4}\right)\right) d \mathbf{y} \\
& =Q_{11}^{\prime}+\mathbb{E} \int \mathbb{1}\left(y_{1}, y_{2} \leq \min \left\{y_{3}, y_{4}\right\}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right) \\
& \quad\left(f_{Y \mid X}\left(y_{3} \mid X_{3}\right)-f_{Y}\left(y_{3}\right)\right)\left(f_{Y \mid X}\left(y_{4} \mid X_{3}\right)-f_{Y}\left(y_{4}\right)\right) d \mathbf{y}+0 \\
& \leq Q_{11}^{\prime}+\int\left(\int\left[\int_{y_{1} \vee y_{2}}^{\infty}\left(f_{Y \mid X}\left(y_{3} \mid X_{3}\right)-f\left(y_{3}\right)\right) d y_{3}\right]^{2} f\left(x_{3}\right) d x_{3}\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{2}\right) d y_{1} d y_{2} \\
& \leq Q_{11}^{\prime}+6 \xi\left(f_{X, Y}^{(n)}\right) . \tag{D.19}
\end{align*}
$$

The last expectation term on the third line above is zero by independence of $X_{i}$ since $\int f_{Y \mid X}\left(y \mid X_{3}\right) f_{X}\left(x_{3}\right) d x_{3}=f_{Y}(y)$. We also use Cauchy-Schwarz inequality in the second last line, and the definition of $\xi$ in the last line. The remainder $Q_{11}^{\prime}$ can be bounded as

$$
\begin{align*}
Q_{11}^{\prime}= & : \mathbb{E} \int \mathbb{1}\left(y_{1}, y_{2} \leq y_{3}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right) f_{Y \mid X}\left(y_{3} \mid X_{3}\right) \\
& \int_{\max \left(y_{1}, y_{2}\right)}^{\infty}\left(f_{Y \mid X_{3}}\left(y_{4}\right)-f_{Y \mid X_{N(3)}}\left(y_{4}\right)\right) d \mathbf{y} \\
= & \mathbb{E} \int \mathbb{1}\left(y_{1}, y_{2} \leq y_{3}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right) f_{Y \mid X}\left(y_{3} \mid X_{3}\right) \\
& \left(F_{X, Y}\left(y_{1} \vee y_{2} \mid X_{N(3)}\right)-F_{X, Y}\left(y_{1} \vee y_{2} \mid X_{3}\right)\right) d \mathbf{y} \\
\leq & \mathbb{E} \int f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{2}\right)\left(F_{X, Y}\left(y_{1} \vee y_{2} \mid X_{N(3)}\right)-F_{X, Y}\left(y_{1} \vee y_{2} \mid X_{3}\right)\right) d y_{1} d y_{2} \\
\leq & \mathbb{E}\left[\min \left\{\left(1+L_{1}^{(n)}\left(X_{3}, \max \left\{Y_{1}, Y_{2}\right\}\right)+L_{2}^{(n)}\left(X_{N(3)}, \max \left\{Y_{1}, Y_{2}\right\}\right)\right)\left|X_{3}-X_{N(3)}\right|^{\eta}, 1\right\}\right] \\
\lesssim & \mathbb{P}\left(\left|X_{3}-X_{N(3)}\right| \geq 1\right)+\left\{\mathbb{E}\left(1+\left(L_{1}^{(n)}\left(X_{3}, \max \left\{Y_{1}, Y_{2}\right\}\right)\right)^{\theta}+\left(L_{2}^{(n)}\left(X_{3}, \max \left\{Y_{1}, Y_{2}\right\}\right)\right)^{\theta}\right)\right\}^{\frac{1}{\theta}} \\
& \times\left\{\mathbb{E} \left\lvert\, X_{3}-X_{N(3)}{ }^{\frac{\theta}{\theta-1}} \mathbb{1}\left(\left|X_{3}-X_{N(3)}\right| \leq 1\right)\right.\right\}^{\frac{\theta-1}{\theta}}, \tag{D.20}
\end{align*}
$$

where the last two lines follow by Assumptions (A1) and (A2) from the main paper and Hölder's inequality. Plugging in the conclusions from (D.5), (D.6), and (D.7) into (D.20), we get that

$$
\begin{equation*}
Q_{11}^{\prime} \leq n^{-\frac{\gamma}{\gamma+1}}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}:=b_{n}, \tag{D.21}
\end{equation*}
$$

By using a similar argument as above, we get the same bound for the other terms in $Q_{11}$, which implies

$$
Q_{11} \leq \xi\left(f_{X, Y}^{(n)}\right)+b_{n}
$$

Also from (D.18), provided $X_{1}$ and $Y_{1}$ are independent, i.e., $\xi\left(f_{X, Y}^{(n)}\right)=0$, we also have $Q_{11}=O\left(n^{-1}\right)$. Therefore,

$$
\begin{equation*}
\mathbb{E} Q_{11} \lesssim b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)+\frac{1}{n} \tag{D.22}
\end{equation*}
$$

Bounding $Q_{12}$. Similarly, since $\min \left\{R_{i}, R_{N(i)}\right\} \leq n$, we have

$$
\begin{aligned}
& Q_{12}:= \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \mathbb{E}\left[\int \min \left\{R_{i}, R_{N(i)}\right\}\left(f_{\mathbf{Y}}(\mathbf{y})-f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y})\right) d \mathbf{y} \times\right. \\
&\left.\times \int \min \left\{R_{i}, R_{N(i)}\right\}\left(f_{\mathbf{Y}}(\mathbf{y})+f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y})\right) d \mathbf{y}\right] \\
& \leq 2 n^{-1} \mathbb{E}\left|\int \min \left\{R_{1}, R_{N(1)}\right\}\left(f_{\mathbf{Y} \mid \mathbf{X}}(\mathbf{y})-f_{\mathbf{Y}}(\mathbf{y})\right) d \mathbf{y}\right| \\
& \leq 2\left|\mathbb{E} \int \mathbb{1}\left(y_{3} \leq\left(y_{1}, y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{3}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{N(1)}\right)-\prod_{i=1}^{3} f_{Y}\left(y_{i}\right)\right) d y_{1} d y_{2} d y_{3}\right| \\
& \leq 2\left|\mathbb{E} \int \mathbb{1}\left(y_{3} \leq\left(y_{1}, y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{N(1)}\right)-f_{Y}\left(y_{1}\right) f_{Y}\left(y_{2}\right)\right) d y_{1} d y_{2} f_{Y}\left(y_{3}\right) d y_{3}\right| \\
& \leq 2\left|\mathbb{E} \int \mathbb{1}\left(y_{3} \leq\left(y_{1}, y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right) f_{Y \mid X}\left(y_{2} \mid X_{N(1)}\right)-f_{Y}\left(y_{1}\right) f_{Y}\left(y_{2}\right)\right) d y_{1} d y_{2} f_{Y}\left(y_{3}\right) d y_{3}\right| \\
&+2\left|\mathbb{E} \int \mathbb{1}\left(y_{3} \leq y_{1}\right) f_{Y \mid X}\left(y_{1} \mid X_{1}\right) d y_{1}\left(F_{X, Y}\left(y_{3} \mid X_{1}\right)-F_{X, Y}\left(y_{3} \mid X_{1}\right)\right) f_{Y}\left(y_{3}\right) d y_{3}\right| \\
& \leq 2\left|\mathbb{E} \int \mathbb{1}\left(y_{3} \leq\left(y_{1}, y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{1}\right)-f_{Y}\left(y_{1}\right)\right)\left(f_{Y \mid X}\left(y_{2} \mid X_{1}\right)-f_{Y}\left(y_{2}\right)\right) d y_{1} d y_{2} f_{Y}\left(y_{3}\right) d y_{3}\right| \\
&+2 \mathbb{E}\left[\min \left\{\left(1+L_{1}^{(n)}\left(X_{1}, Y_{2}\right)+L_{2}^{(n)}\left(X_{N(1)}, Y_{2}\right)\right)\left|X_{1}-X_{N(1)}\right|^{\eta}, 1\right\}\right] \\
& \leq \quad \mathbb{E}\left(\int \mathbb{1}\left(y_{3} \leq y_{1}\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{2}\right)-f_{Y}\left(y_{1}\right)\right) d y_{1}\right)^{2} f_{Y}\left(y_{3}\right) d y_{3} \\
& \lesssim \quad n^{-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)} \\
& \leq\left(f_{X, Y}^{(n)}\right)+n^{-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)} .
\end{aligned}
$$

In the above, we have used the fact that $\int f_{Y \mid X}(y \mid x) f_{X}(x) d x=f_{Y}(y)$ in the third and fifth inequalities. The last line uses Cauchy-Schwarz inequality for the first term and Assumption (A1) in the main paper, coupled with equations (D.5)-(D.7) for the second term.

On the other hand, notice that when $\xi\left(f_{X, Y}^{(n)}\right)=0, X_{1}$ and $Y_{1}$ are independent and $Q_{12}=0$ from definition. Thus

$$
\begin{equation*}
Q_{12} \lesssim \xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) \tag{D.23}
\end{equation*}
$$

Plugging in (D.22) and (D.23) into (D.17) yields

$$
\left|\mathbb{E} T_{1}-\frac{1}{18}\right| \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) .
$$

A similar set of computations can be used in $\mathbb{E} T_{i}, i=2,3,4$. We skip the relevant algebraic details for brevity. We present the corresponding conclusions below:

$$
\max \left\{\mathbb{E}\left|T_{1}-\frac{1}{18}\right|, \mathbb{E}\left|T_{2}-\frac{1}{45}\right|, \mathbb{E}\left|T_{3}-\frac{4}{45}\right|, \mathbb{E}\left|T_{4}+\frac{2}{45}\right|\right\} \lesssim n^{-1}+\xi\left(f_{X, Y}^{(n)}\right)+b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) .
$$

By the definitions of $Q_{1}, \ldots, Q_{5}$ this finishes the proof of Lemma C.2.

## D.3. Proof of Lemma C.3

As before, we hide the dependence on $n$ and write $f_{X, Y}, f_{X}, f_{Y}, F_{Y}$ to mean $f_{X, Y}^{(n)}, f_{X}^{(n)}$, $f_{Y}^{(n)}, F_{Y}^{(n)}$ respectively. We will write $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$ to mean the random variables $\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, n}, Y_{n, n}\right)$ for some fixed $n$ in the triangular array.

Bounding $\operatorname{Var}\left(\mathbb{E}\left[\sqrt{n} \xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right]\right)$.
We will start this proof with some definitions:

$$
\begin{aligned}
H_{1} & :=\frac{6 n}{n^{2}-1} \sum_{i=1}^{n} \int \min \left\{F_{Y}\left(y_{1}\right), F_{Y}\left(y_{2}\right)\right\} f_{Y \mid X}\left(y_{1} \mid X_{i}\right) f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) d y_{1} d y_{2}, \\
H_{1}^{\prime} & :=\frac{6 n}{n^{2}-1} \sum_{i=1}^{n} \int \min \left\{F_{Y}\left(y_{1}\right), F_{Y}\left(y_{2}\right)\right\} f_{Y \mid X}\left(y_{1} \mid X_{i}\right) f_{Y \mid X}\left(y_{2} \mid X_{i}\right) d y_{1} d y_{2}, \\
H_{2} & :=\frac{6}{n^{2}-1} \sum_{i \neq j} \sum_{j} \int \min \left\{F_{Y}\left(y_{1}\right), F_{Y}\left(y_{2}\right)\right\} f_{Y \mid X}\left(y_{1} \mid X_{i}\right) f_{Y \mid X}\left(y_{2} \mid X_{j}\right) d y_{1} d y_{2} .
\end{aligned}
$$

Based on the above notation, note that:

$$
\begin{equation*}
\operatorname{Var}\left(\mathbb{E}\left[\sqrt{n} \xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right]\right)=n \mathbb{E}\left(H_{1}-H_{2}-\mathbb{E} H_{1}+\mathbb{E} H_{2}\right)^{2} \tag{D.24}
\end{equation*}
$$

Using (D.24), it is easy to see that:

$$
\begin{equation*}
\operatorname{Var}\left(\mathbb{E}\left[\sqrt{n} \xi_{n}^{* \prime} \mid \mathbf{X}^{(n)}\right]\right) \lesssim n \mathbb{E}\left(H_{1}-H_{1}^{\prime}-\mathbb{E}\left(H_{1}^{\prime}-H_{2}\right)\right)^{2}+n \operatorname{Var}\left(H_{1}^{\prime}-H_{2}\right) \tag{D.25}
\end{equation*}
$$

We will now bound the two terms on the right hand side of (D.25) separately.
We start with the first term. Towards this direction, define $Z_{n}:=\sqrt{n}\left(H_{1}-H_{1}^{\prime}\right)$. The first term on the right hand side of (D.25) then equals $\operatorname{Var}\left(Z_{n}\right)$. Next observe that:

$$
\begin{align*}
\mathbb{E}\left|Z_{n}\right| & \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E} \int \mathbb{P}\left(Y \geq y \mid X_{i}\right)\left|\mathbb{P}\left(Y \geq y \mid X_{i}\right)-\mathbb{P}\left(Y \geq y \mid X_{N(i)}\right)\right| f_{Y}(y) d y \\
& \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E} \min \left\{\int\left(1+L_{1}^{(n)}\left(X_{i}, y\right)+L_{2}^{(n)}\left(X_{N(i)}, y\right)\right)\left|X_{i}-X_{N(i)}\right|^{\eta} f_{Y}^{(n)}(y) d y, 1\right\} \\
& \leq \sqrt{n} b_{n} \tag{D.26}
\end{align*}
$$

where $b_{n}$ is defined as in (C.1) and the last line follows using similar computations as in (D.7) and (D.20). Note that, if $X_{1}$ and $Y_{1}$ are independent, then $Z_{n}=0$, i.e.,

$$
\begin{equation*}
\xi\left(f_{X, Y}^{(n)}\right)=0 \quad \Longrightarrow \quad Z_{n}=0 \tag{D.27}
\end{equation*}
$$

Combining (D.27) with (D.26), we get:

$$
\begin{equation*}
\mathbb{E}\left|Z_{n}\right| \lesssim \sqrt{n} b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) \tag{D.28}
\end{equation*}
$$

The above gives us a bound on the first moment of $\left|Z_{n}\right|$. However, we are interested in the second moment. To make this transition, we will use Lemma D.5.

By Lemma D.5, there exists $K>0$, a positive constant, such that

$$
\begin{equation*}
\mathbb{P}\left(\left|Z_{n}-\mathbb{E} Z_{n}\right| \geq K \sqrt{\log n}\right) \lesssim \frac{1}{n^{6}} \tag{D.29}
\end{equation*}
$$

Choose such a $K>0$ satisfying (D.29) and consider the following decomposition:

$$
\begin{align*}
& \mathbb{E}\left(Z_{n}-\mathbb{E} Z_{n}\right)^{2} \\
= & \mathbb{E}\left[\left(Z_{n}-\mathbb{E} Z_{n}\right)^{2} \mathbb{1}\left(\left|Z_{n}-\mathbb{E} Z_{n}\right| \geq K \sqrt{\log n}\right)\right]+\mathbb{E}\left[\left(Z_{n}-\mathbb{E} Z_{n}\right)^{2} \mathbb{1}\left(\left|Z_{n}-\mathbb{E} Z_{n}\right| \leq K \sqrt{\log n}\right)\right] . \tag{D.30}
\end{align*}
$$

For the first term, we begin by observing that $\mathbb{E}\left(Z_{n}-\mathbb{E} Z_{n}\right)^{4} \lesssim 1$ by Lemma D. 5 . Consequently by the Cauchy-Schwarz inequality,

$$
\begin{align*}
& \mathbb{E}\left[\left(Z_{n}-\mathbb{E} Z_{n}\right)^{2} \mathbb{1}\left(\left|Z_{n}-\mathbb{E} Z_{n}\right| \geq K \sqrt{\log n}\right)\right] \\
\leq & \sqrt{\mathbb{E}\left(Z_{n}-\mathbb{E} Z_{n}\right)^{4}} \sqrt{\mathbb{P}\left(\left|Z_{n}-\mathbb{E} Z_{n}\right| \geq K \sqrt{\log n}\right)} \lesssim n^{-3} \tag{D.31}
\end{align*}
$$

where the last line follows from (D.29).
For the second term, note that

$$
\begin{equation*}
\mathbb{E}\left[\left(Z_{n}-\mathbb{E} Z_{n}\right)^{2} \mathbb{1}\left(\left|Z_{n}-\mathbb{E} Z_{n}\right| \leq K \sqrt{\log n}\right)\right] \leq K \sqrt{\log n} \mathbb{E}\left|Z_{n}\right| \lesssim b_{n} \sqrt{n \log n} \tag{D.32}
\end{equation*}
$$

where the last line follows from (D.26). Combining (D.32), (D.31) and (D.27), we then get:

$$
\begin{equation*}
n \mathbb{E}\left(H_{1}-H_{1}^{\prime}-E\left(H_{1}^{\prime}-H_{2}\right)\right)^{2} \lesssim\left(b_{n} \sqrt{n \log n}+n^{-3}\right) \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right), \tag{D.33}
\end{equation*}
$$

which provides a bound for the first term in (D.25).
For the second term in (D.25), we can use the same technique that we used to bound the $S_{i}$ 's in the proof of Lemma C.1, to get:

$$
\begin{equation*}
n \operatorname{Var}\left(H_{1}^{\prime}-H_{2}\right) \lesssim \xi\left(f_{X, Y}^{(n)}\right)+n^{-1} \tag{D.34}
\end{equation*}
$$

Finally, combining (D.33) and (D.34) completes the proof of Part 1.

Bounding $\operatorname{Var}\left(\mathbb{E}\left[\sqrt{n} \xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right]\right)$. Recall that

$$
\begin{equation*}
\min \left(R_{i}, R_{N(i)}\right)=1+\sum_{j \notin\{i, N(i)\}} \mathbb{1}\left(Y_{j} \leq \min \left(Y_{i}, Y_{N(i)}\right)\right) . \tag{D.35}
\end{equation*}
$$

We will decompose $f_{Y \mid X}(y \mid x)=f_{Y}(y)+f_{Y \mid X}(y \mid x)-f_{Y}(y)$. Then using (D.35) in the expression of $\xi_{n}^{\prime}$, we get:

$$
\begin{aligned}
& \mathbb{E}\left[\sqrt{n} \xi_{n}^{\prime} \mid \mathbf{X}^{(n)}\right] / 6 \\
= & \frac{\sqrt{n}}{n^{2}-1} \sum_{\substack{i=1 \\
j \neq i, N(i)}}^{n} \int \mathbb{1}\left(y_{3} \leq \min \left\{y_{1}, y_{2}\right\}\right) f_{Y \mid X}\left(y_{1} \mid X_{i}\right) f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{3} \mid X_{j}\right) d y_{1} d y_{2} d y_{3}+O\left(\frac{1}{n}\right) \\
= & \frac{\sqrt{n}}{n^{2}-1} \sum_{i=1}^{n} \sum_{j \neq\{i, N(i)\}}\left[\int \mathbb{1}\left(y_{3} \leq \min \left(y_{1}, y_{2}\right)\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{2}\right) f_{Y}\left(y_{3}\right) d y_{1} d y_{2} d y_{3}\right.
\end{aligned}
$$

$$
\begin{align*}
& +\int \mathbb{1}\left(y_{3} \leq \min \left(y_{1}, y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{i}\right)-f_{Y}\left(y_{1}\right)\right) f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{3} \mid X_{j}\right) d y_{1} d y_{2} d y_{3} \\
& +\int \mathbb{1}\left(y_{3} \leq \min \left(y_{1}, y_{2}\right)\right) f_{Y}\left(y_{1}\right)\left(f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right)-f_{Y}\left(y_{2}\right)\right) f_{Y \mid X}\left(y_{3} \mid X_{j}\right) d y_{1} d y_{2} d y_{3} \\
& \left.+\int \mathbb{1}\left(y_{3} \leq \min \left(y_{1}, y_{2}\right)\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{2}\right)\left(f_{Y \mid X}\left(y_{3} \mid X_{j}\right)-f_{Y}\left(y_{3}\right)\right) d y_{1} d y_{2} d y_{3}\right]+O\left(n^{-1}\right) . \tag{D.36}
\end{align*}
$$

Note that the first term within the sum is just $\mathbb{P}\left(Y_{3} \leq \min \left\{Y_{1}, Y_{2}\right\}\right)+O\left(n^{-1}\right)=\frac{1}{3}+O\left(n^{-1}\right)$.
Among the rest, let us define

$$
\begin{align*}
& T_{1} \\
&:= \frac{\sqrt{n}}{n^{2}-1} \sum_{i=1}^{n} \sum_{j \neq i, N(i)} \int \mathbb{1}\left(y_{3} \leq \min \left(y_{1}, y_{2}\right)\right)\left(f_{Y \mid X}\left(y_{1} \mid X_{i}\right)-f_{Y}\left(y_{1}\right)\right) \\
& f_{Y \mid X}\left(y_{2} \mid X_{N(i)}\right) f_{Y \mid X}\left(y_{3} \mid X_{j}\right) d y_{1} d y_{2} d y_{3} \\
&= \frac{\sqrt{n}}{n^{2}-1} \sum_{i=1}^{n} \sum_{j \neq i, N(i)} \int\left(F_{Y}(y)-F_{Y \mid X}\left(y \mid X_{i}\right)\right)\left(1-F_{Y \mid X}\left(y \mid X_{N(i)}\right)\right) f_{Y \mid X}\left(y \mid X_{j}\right) d y \\
&=: \frac{\sqrt{n}}{n^{2}-1} \sum_{i=1}^{n} \sum_{j \neq i, N(i)} \int\left(F_{Y}(y)-F_{Y \mid X}\left(y \mid X_{i}\right)\right)\left(1-F_{Y \mid X}\left(y \mid X_{i}\right)\right) f_{Y \mid X}\left(y \mid X_{j}\right) d y+T_{1}^{\prime}\left(\mathbf{X}^{(n)}\right) \\
&=: \frac{\sqrt{n}}{\left(n^{2}-1\right)} \sum_{i=1}^{n} \sum_{j \neq i, N(i)} \mathfrak{F}\left(X_{i}, X_{j}\right)+T_{1}^{\prime} \\
&=: T_{1}^{*}+T_{1}^{\prime} . \tag{D.37}
\end{align*}
$$

Note that $\mathfrak{F}\left(X_{i}, X_{j}\right) \leq 1$. Then

$$
\begin{align*}
& \operatorname{Var}\left(T_{1}^{*}\right) \\
= & \operatorname{Var}\left(\frac{\sqrt{n}}{n^{2}-1} \sum_{i=1}^{n} \sum_{j \neq i, N(i)} \int\left(F_{Y}(y)-F_{Y \mid X}\left(y \mid X_{i}\right)\right)\left(1-F_{Y \mid X}\left(y \mid X_{i}\right)\right) f_{Y \mid X}\left(y \mid X_{j}\right) d y\right) \\
= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i_{1}=1}^{n} \sum_{i_{2}=1}^{n} \sum_{j_{1} \neq i_{1}, N\left(i_{1}\right)} \sum_{j_{2} \neq i_{2}, N\left(i_{2}\right)} \operatorname{Cov}\left(\mathfrak{F}\left(X_{i_{1}}, X_{j_{1}}\right), \mathfrak{F}\left(X_{i_{2}}, X_{j_{2}}\right)\right) \\
= & \frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i_{1} \neq i_{2}} \sum_{j \neq i_{1}, N\left(i_{1}\right), i_{2}, N\left(i_{2}\right)} \operatorname{Cov}\left(\mathfrak{F}\left(X_{i_{1}}, X_{j}\right), \mathfrak{F}\left(X_{i_{2}}, X_{j}\right)\right) \\
& +\frac{n}{\left(n^{2}-1\right)^{2}} \sum_{i=1}^{n} \sum_{j_{1} \neq j_{2} \neq i, N(i)} \sum^{\operatorname{Cov}\left(\mathfrak{F}\left(X_{i}, X_{j_{1}}\right), \mathfrak{F}\left(X_{i}, X_{j_{2}}\right)\right)+O\left(n^{-2}\right)} \tag{D.38}
\end{align*}
$$

where the last step follows by the independence of $X_{1}, \ldots, X_{n}$. For any $1 \leq i_{1}, i_{2} \leq n$ and $j \neq i_{1}, N\left(i_{1}\right), i_{2}, N\left(i_{2}\right)$

$$
\begin{aligned}
& \mathbb{E}\left(\mathfrak{F}\left(X_{i_{1}}, X_{j}\right) \mathfrak{F}\left(X_{i_{2}}, X_{j}\right)\right) \\
= & \iint\left[\int\left(F_{Y}\left(y_{1}\right)-F_{Y \mid X}\left(y_{1} \mid X_{i_{1}}\right)\right)\left(1-F_{Y \mid X}\left(y_{1} \mid X_{i_{1}}\right)\right) f_{X}\left(x_{i_{1}}\right) d x_{i_{1}}\right] f_{Y \mid X}\left(y_{1} \mid X_{j}\right) d y_{1} \times
\end{aligned}
$$

$$
\begin{align*}
& \times \int\left[\int\left(F_{Y}\left(y_{2}\right)-F_{Y \mid X}\left(y_{2} \mid X_{i_{2}}\right)\right)\left(1-F_{Y \mid X}\left(y_{2} \mid X_{i_{2}}\right)\right) f_{X}\left(x_{i_{2}}\right) d x_{i_{2}}\right] f_{Y \mid X}\left(y_{2} \mid X_{j}\right) d y_{1} f_{X}\left(x_{j}\right) d x_{j} \\
\leq & \iint\left[\int\left(F_{Y}(y)-F_{Y \mid X}(y \mid X)\right)\left(1-F_{Y \mid X}(y \mid X)\right) f_{X}(x) d x\right]^{2} f_{Y \mid X}\left(y \mid X_{j}\right) d y f_{X}\left(x_{j}\right) d x_{j} \\
\leq & \int\left[\int\left(F_{Y}(y)-F_{Y \mid X}(y \mid X)\right)^{2}\left(1-F_{Y \mid X}(y \mid X)\right)^{2} f_{X}(x) d x\right] f_{Y}(y) d y \\
\leq & \xi\left(f_{X, Y}^{(n)}\right) . \tag{D.39}
\end{align*}
$$

Here we have used Cauchy-Schwarz inequality in the second and third inequalities. The exact same calculation also implies that

$$
\mathbb{E}\left(\mathfrak{F}\left(X_{i}, X_{j_{1}}\right) \mathfrak{F}\left(X_{i}, X_{j_{2}}\right)\right) \leq \xi\left(f_{X, Y}^{(n)}\right)
$$

By (D.38) and (D.39) we have

$$
\begin{equation*}
\operatorname{Var}\left(T_{1}^{*}\right) \lesssim \xi\left(f_{X, Y}^{(n)}\right)+n^{-2} \tag{D.40}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
& \mathbb{E}\left|T_{1}^{\prime}\right| \\
= & \frac{\sqrt{n}}{n^{2}-1} \mathbb{E} \sum_{i=1}^{n} \sum_{j \neq i, N(i)} \int\left|\left(F_{Y}(y)-F_{Y \mid X}\left(y \mid X_{i}\right)\right)\left(F_{Y \mid X}\left(y \mid X_{N(i)}\right)-F_{Y \mid X}\left(y \mid X_{i}\right)\right)\right| f_{Y \mid X}\left(y \mid X_{j}\right) d y \\
\lesssim & \frac{\sqrt{n}}{n+1} \sum_{i=1}^{n} \mathbb{E} \int\left|\mathbb{P}\left(Y \geq y \mid X_{i}\right)-\mathbb{P}\left(Y \geq y \mid X_{N(i)}\right)\right| f_{Y}(y) d y \lesssim \sqrt{n} b_{n} \tag{D.41}
\end{align*}
$$

following (D.26). Note that if $X_{1}$ and $Y_{1}$ are independent then $T_{1}^{\prime}\left(\mathbf{X}^{(n)}\right)=0$, i.e.,

$$
\begin{equation*}
\xi\left(f_{X, Y}^{(n)}\right)=0 \Longrightarrow T_{1}^{\prime}=0 \tag{D.42}
\end{equation*}
$$

which together with (D.41) implies

$$
\begin{equation*}
\mathbb{E}\left|T_{1}^{\prime}\right| \lesssim \sqrt{n} b_{n} \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) . \tag{D.43}
\end{equation*}
$$

As in part 1 of the lemma, we make a transition to the second moment via McDiarmid's inequality. Notice that just as in Lemma D.5, we have a constant $C>0$ such that for any $n \geq 1$ and $t \geq 0$

$$
\mathbb{P}\left(\left|T_{1}^{\prime}\left(\mathbf{X}^{(n)}\right)-\mathbb{E} T_{1}^{\prime}\left(\mathbf{X}^{(n)}\right)\right| \geq t\right) \leq 2 \exp \left(-C n t^{2}\right)
$$

In particular, there exists a constant $K>0$ such that

$$
\begin{equation*}
\mathbb{P}\left(\left|T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right| \geq K \sqrt{\log n}\right) \lesssim \frac{1}{n^{6}} \tag{D.44}
\end{equation*}
$$

One can then follow (D.30)-(D.32) to get

$$
\begin{aligned}
& \mathbb{E}\left(T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right)^{2} \\
= & \mathbb{E}\left[\left(T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right)^{2} \mathbb{1}\left(\left|T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right| \geq K \sqrt{\log n}\right)\right]+\mathbb{E}\left[\left(T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right)^{2} \mathbb{1}\left(\left|T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right| \leq K \sqrt{\log n}\right)\right] \\
\leq & \sqrt{\mathbb{E}\left(T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right)^{4}} \sqrt{\mathbb{P}\left(\left|T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right| \geq K \sqrt{\log n}\right)}+K \sqrt{\log n} \mathbb{E}\left|Z_{n}\right|
\end{aligned}
$$

$$
\begin{equation*}
\lesssim n^{-3}+b_{n} \sqrt{n \log n} \tag{D.45}
\end{equation*}
$$

which, together with (D.42) implies that

$$
\begin{equation*}
\mathbb{E}\left(T_{1}^{\prime}-\mathbb{E} T_{1}^{\prime}\right)^{2} \lesssim\left(n^{-3}+b_{n} \sqrt{n \log n}\right) \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) \tag{D.46}
\end{equation*}
$$

Combining (D.40), (D.46) and (D.37), we have

$$
\begin{equation*}
\operatorname{Var}\left(T_{1}\right) \leq \xi\left(f_{X, Y}^{(n)}\right)+\left(n^{-3}+b_{n} \sqrt{n \log n}\right) \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right)+n^{-2} \tag{D.47}
\end{equation*}
$$

Using a similar calculation, we have the same bound for the variance of the third and fourth terms on the right-hand side of (D.36). This finishes the proof of the second bound, and hence Lemma C. 3 follows.

## D.4. Proof of Lemma C.4

Let us consider $W_{\ell}$ defined in (C.23) and observe that $W_{\ell}(\boldsymbol{x})$ depends on $\left(x_{\ell}, y_{\ell}\right)$ and $\left(x_{N(\ell)}, y_{N(\ell)}\right)$. Consider $\boldsymbol{x}, \boldsymbol{x}^{\prime} \in\left(\mathbb{R}^{2}\right)^{n}$ and $\{i, j\} \in[n] \times[n]$, such that the edge $\{i, j\}$ does not exist in $\mathcal{G}(\boldsymbol{x}), \mathcal{G}\left(\boldsymbol{x}^{i}\right), \mathcal{G}\left(\boldsymbol{x}^{j}\right)$ and $\mathcal{G}\left(\boldsymbol{x}^{i j}\right)$. We shall show that for all $\ell \in[n]$,

$$
\begin{equation*}
W_{\ell}(\boldsymbol{x})-W_{\ell}\left(\boldsymbol{x}^{i}\right)-W_{\ell}\left(\boldsymbol{x}^{j}\right)+W_{\ell}\left(\boldsymbol{x}^{i j}\right)=0 \tag{D.48}
\end{equation*}
$$

By (C.24), we have

$$
\begin{equation*}
\left|D_{\boldsymbol{x}}(i, j)-D_{\boldsymbol{x}^{\prime}}(i, j)\right| \leq \#\left\{t: x_{t} \neq x_{t}^{\prime}\right\} \tag{D.49}
\end{equation*}
$$

Let us fix an $\ell \in[n]$ such that $D_{\boldsymbol{x}}(\ell, j) \leq 1$. That means there exists at most one $k$, such that, $x_{\ell}<x_{k}<x_{j}$. We shall show

$$
W_{\ell}(\boldsymbol{x})=W_{\ell}\left(\boldsymbol{x}^{i}\right) \quad \text { and } \quad W_{\ell}\left(\boldsymbol{x}^{j}\right)=W_{\ell}\left(\boldsymbol{x}^{i j}\right)
$$

As the edge $\{i, j\}$ is absent in $\mathcal{G}(\boldsymbol{x})$, we have

$$
\begin{equation*}
D_{\boldsymbol{x}}(\ell, i)>2 \tag{D.50}
\end{equation*}
$$

In particular, $i$ is different from $\ell$ and $j$. Again, as the edge $\{i, j\}$ is absent in $\mathcal{G}\left(\boldsymbol{x}^{i}\right)$, we further have,

$$
\begin{equation*}
D_{\boldsymbol{x}^{i}}(\ell, i)>2 \tag{D.51}
\end{equation*}
$$

This implies $N(\ell)$ does not change in $\boldsymbol{x}$ and $\boldsymbol{x}^{i}$. Hence,

$$
W_{\ell}(\boldsymbol{x})=W_{\ell}\left(\boldsymbol{x}^{i}\right)
$$

Next we show that if $D_{\boldsymbol{x}}(\ell, j) \leq 1$ then we have $D_{\boldsymbol{x}^{j}}(\ell, i) \geq 2$ and $D_{\boldsymbol{x}^{i j}}(\ell, i) \geq 2$. Suppose not, let $D_{\boldsymbol{x}^{j}}(\ell, i) \leq 1$. If $j=\ell$, then clearly this is false as the edge $\{i, j\}$ is absent in $\mathcal{G}\left(\boldsymbol{x}^{j}\right)$. If $j \neq \ell$, then by (D.49) and (D.50), we get,

$$
D_{\boldsymbol{x}^{j}}(\ell, i) \geq 2
$$

This is a contradiction. Further, if $D_{\boldsymbol{x}^{i j}}(\ell, i) \leq 1$ and $j=\ell$, then this is similarly false as the edge $\{i, j\}$ edge is absent in $\mathcal{G}\left(\boldsymbol{x}^{i j}\right)$. If $j \neq \ell$, then by (D.49) and (D.51), we get,

$$
D_{\boldsymbol{x}^{i j}}(\ell, i) \geq 2
$$

Again we get a contradiction. As $D_{\boldsymbol{x}^{j}}(\ell, i) \geq 2$ and $D_{\boldsymbol{x}^{i j}}(\ell, i) \geq 2, N(\ell)$ is same in $\boldsymbol{x}^{j}$ and $\boldsymbol{x}^{i j}$. Hence,

$$
W_{\ell}\left(\boldsymbol{x}^{j}\right)=W_{\ell}\left(\boldsymbol{x}^{i j}\right) .
$$

This implies if $D_{\boldsymbol{x}}(\ell, j) \leq 1$, then (D.48) holds. Similarly if $D_{\boldsymbol{x}^{i}}(\ell, j) \leq 1$ or $D_{\boldsymbol{x}^{j}}(\ell, j) \leq 1$ or $D_{\boldsymbol{x}^{i j}}(\ell, j) \leq 1$, then (D.48) holds. Now if $D_{\boldsymbol{x}}(\ell, j), D_{\boldsymbol{x}^{i}}(\ell, j), D_{\boldsymbol{x}^{j}}(\ell, j), D_{\boldsymbol{x}^{i j}}(\ell, j)>1$, then $N(\ell)$ does not change in $\boldsymbol{x}$ and $\boldsymbol{x}^{j}$ implying

$$
W_{\ell}(\boldsymbol{x})=W_{\ell}\left(\boldsymbol{x}^{j}\right)
$$

Also $N(\ell)$ does not change in $\boldsymbol{x}^{i}$ and $\boldsymbol{x}^{i j}$ implying

$$
W_{\ell}\left(\boldsymbol{x}^{i}\right)=W_{\ell}\left(\boldsymbol{x}^{i j}\right)
$$

This implies the lemma.

## D.5. Proof of Lemma A. 1

To reduce notation we write $f_{X, Y}(\cdot, \cdot), f_{X}(\cdot), f_{Y}(\cdot)$ and $F_{Y}(\cdot)$ to mean $f_{X, Y}^{(n)}(\cdot), f_{X}^{(n)}(\cdot)$, $f_{Y}^{(n)}(\cdot)$ and $F_{Y}^{(n)}(\cdot)$ respectively. We will write $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$ to mean the random variables $\left(X_{n, 1}, Y_{n, 1}\right), \ldots,\left(X_{n, n}, Y_{n, n}\right) \sim f_{X, Y}^{(n)}$ from the triangular array. We will also write $\mathbf{X}$ to mean $\left(X_{1}, \ldots, X_{n}\right)$. It is useful to note some properties of $g(y \mid x)$ which we will use throughout the proof.
a)

$$
\begin{equation*}
\int g(y \mid x) d y=1 \quad \text { and } \quad \int g(y \mid x) f_{X}(x) d x=f_{Y}(y) \tag{D.52}
\end{equation*}
$$

In this sense, $g(y \mid x)$ is like a conditional density, although it can take negative values.
b) Recalling the definition of $g$, we write

$$
\begin{equation*}
G(y \mid x)=\int_{-\infty}^{y} g(t \mid x) d t=\frac{1}{p} f_{X, Y}(y \mid x)+\left(1-\frac{1}{p}\right) F_{Y}(y) ; \quad \int G(y \mid x) f_{X}(x) d x=F_{Y}(y) \tag{D.53}
\end{equation*}
$$

c) For any $y, x_{1}, x_{2}$, Assumption (A1) in the main paper implies there exist $\eta \in(0,1]$, $\theta>1$ and $C>0$ such that

$$
\begin{gather*}
\left|G\left(y \mid x_{1}\right)-G\left(y \mid x_{2}\right)\right| \leq\left(1+L_{1}^{(n)}\left(x_{1}, y\right)+L_{2}^{(n)}\left(x_{2}, y\right)\right)\left|x_{1}-x_{2}\right|^{\eta} \\
\limsup _{n \rightarrow \infty} \int\left(L_{1}^{(n)}(x, y)\right)^{\theta} f_{X}(x) f_{Y}(y) d y \leq C \tag{D.54}
\end{gather*}
$$

We will prove the three parts of Lemma A. 1 as Lemmas D. 1 to D. 3 presented below.


## D.5.1. Proof of Lemma D. 1

$T_{1}$ involves two contributions from $f_{Y}(y)$ and one from $g(y \mid x)$. Thus

$$
\begin{align*}
T_{1} & =\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) g\left(y_{k} \mid x_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) f_{Y}\left(y_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} . \tag{D.55}
\end{align*}
$$

Now for the first term of $T_{1}$, we use Fubini's theorem to write

$$
\begin{aligned}
& \int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
&= \int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right)\left(g\left(y_{j} \mid x_{1}\right)+g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
&=f_{Y}\left(y_{k}\right) f_{Y}\left(y_{1}\right) d y_{1} d y_{k} d y_{j} \\
&= \int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right) \int g\left(y \mid x_{1}\right) f_{X}\left(x_{1}\right) d x_{1} f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) d y d y_{1} d y_{k}+E_{n} \\
&= \int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right) f_{Y}(y) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) d y d y_{1} d y_{k}+E_{n}=\frac{1}{3}+E_{n},
\end{aligned}
$$

where we use the fact that $\int g(y \mid x) f_{X}(x) d x=f_{Y}(y)$ from (D.52). The remainder term is

$$
\begin{align*}
E_{n}= & \int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \int\left(g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
= & \iint_{y_{k}}^{\infty} \int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \int_{y_{k}}^{\infty}\left(g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) f_{Y}\left(y_{k}\right) f_{Y}\left(y_{1}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
= & \iint_{y_{k}}^{\infty} \int \sum_{j \neq 1} \mathbb{1}(N(1)=j)\left[G_{Y \mid X_{1}=x_{1}}\left(y_{k}\right)-G_{Y \mid X_{j}=x_{j}}\left(y_{k}\right)\right] f_{Y}\left(y_{k}\right) f_{Y}\left(y_{1}\right) d y_{1} d y_{k} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
\leq & \iint_{y_{k}}^{\infty} \int\left|G_{Y \mid X_{1}}\left(y_{k}\right)-G_{Y \mid X_{N(1)}}\left(y_{k}\right)\right| f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} f_{Y}\left(y_{k}\right) f_{Y}\left(y_{1}\right) d y_{1} d y_{k} \\
\leq & \iint\left\{1 \wedge\left(1+L_{1}^{(n)}\left(x_{1}, y_{k}\right)+L_{2}^{(n)}\left(x_{N(1)}, y\right)\right)\left|x_{1}-x_{N(1)}\right|^{\eta}, 1\right\} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x}\left(1-F_{Y}\left(y_{k}\right)\right) f_{Y}\left(y_{k}\right) d y_{k} \\
\lesssim & \mathbb{P}\left(\left|X_{1}-X_{N(1)}\right| \geq 1\right) \\
& +\left\{\mathbb{E}\left(1+\left(L_{1}^{(n)}\left(X_{1}, Y_{k}\right)\right)^{\theta}+\left(L_{2}^{(n)}\left(X_{1}, Y_{k}\right)\right)^{\theta}\right)\right\}^{\frac{1}{\theta}}\left\{\mathbb{E}\left|X_{1}-X_{N(1)}\right|^{\frac{\theta \eta}{\theta-1}} \mathbb{1}\left(\left|X_{1}-X_{N(1)}\right| \leq 1\right)\right\}^{\frac{\theta-1}{\theta}} \\
\lesssim & n^{-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)},} \tag{D.56}
\end{align*}
$$

where we use (D.54) in the second last line, and Lemma D. 4 in the last line. By analogous calculation, the second and third terms in $T_{1}$ can be written as

$$
\begin{aligned}
& \int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right) f_{Y}(y) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) d y d y_{1} d y_{k}+O\left(\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}\right) \\
= & \frac{1}{3}+O\left(n^{\left.-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}\right) .}\right.
\end{aligned}
$$

Finally, when $\xi\left(f_{X, Y}^{(n)}\right)=0 X_{i}$ and $Y_{i}$ are independent, and hence $g(y \mid x)=f_{Y}(y)$ for all $x$. It can be checked that in this case $E_{n}=0$ and similarly the other error terms are all zero. This finishes the proof of Lemma D.1, i.e., part i) of Lemma A.1.

## Lemma D.2.

$$
\left|T_{2}-\left(\frac{2}{3}+H(f, g)\right)\right| \lesssim\left(n^{\left.-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}\right) \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) . . ~}\right.
$$

## D.5.2. Proof of Lemma D.2

The second term deals with the case where one of the three is from $f_{Y}(y)$ and the other two are from $g(y \mid x)$ :

$$
\begin{align*}
T_{2} & =\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) g\left(y_{k} \mid x_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) g\left(y_{k} \mid x_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& +\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} . \tag{D.57}
\end{align*}
$$

For the first term of $T_{2}$, by interchanging the integrals,

$$
\begin{align*}
& \int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) g\left(y_{k} \mid x_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
= & \int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) g\left(y_{k} \mid x_{k}\right) f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} \\
= & \int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right)\left[\int g\left(y_{1} \mid x_{1}\right) f_{X}\left(x_{1}\right) d x_{1} \int g\left(y_{k} \mid x_{k}\right) f_{X}\left(x_{k}\right) d x_{k}\right] f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j} \\
= & \int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j}=\frac{1}{3} \tag{D.58}
\end{align*}
$$

where the last line follows from (D.52). The second term in $T_{2}$ is

$$
\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) g\left(y_{k} \mid x_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x}
$$

$$
\begin{align*}
& =\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) f_{Y}\left(y_{1}\right) g\left(y_{k} \mid x_{k}\right) \\
& \qquad\left(g\left(y_{j} \mid x_{1}\right)+g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} d y_{1} d y_{k} d y_{j} \\
& =: \int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right)\left[\int g\left(y_{j} \mid x_{1}\right) f_{X}\left(x_{1}\right) d x_{1} \int g\left(y_{k} \mid x_{k}\right) f_{X}\left(x_{k}\right) d x_{k}\right] f_{Y}\left(y_{1}\right) d y_{1} d y_{k} d y_{j}+E_{n}^{\prime} \\
& =\int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right) f_{Y}\left(y_{1}\right) f_{Y}\left(y_{k}\right) f_{Y}\left(y_{j}\right) d y_{1} d y_{k} d y_{j}+E_{n}^{\prime}=\frac{1}{3}+E_{n}^{\prime} \tag{D.59}
\end{align*}
$$

where the last line follows from (D.52). Just as in equation (D.56), the remainder term

$$
\begin{align*}
& =E_{n}^{\prime} \sum_{j \neq 1, k} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right)\left(g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) g\left(y_{k} \mid x_{k}\right) d \mu(\mathbf{x}) f_{Y}\left(y_{1}\right) d y_{1} d y_{k} d y_{j} \\
& =\int\left[\int \sum_{j \neq 1, k} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right)\left(g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x}\right] \\
& \quad \times\left[\int g\left(y_{k} \mid x_{k}\right) f_{X}\left(x_{k}\right) d x_{k}\right] f_{Y}\left(y_{1}\right) d y_{1} d y_{k} d y_{j} \\
& =E_{n} \\
& \lesssim n^{-\frac{\gamma}{\gamma+1}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}}
\end{align*}
$$

We use the fact that $j \neq k$ in the second equality and Lemma D. 4 in the last step. With a remainder term $E_{n}^{\prime \prime}$ defined similarly as $E_{n}$ and $E_{n}^{\prime}$, we can write the third term as

$$
\begin{align*}
& \int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
= & : \int \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y\right\}\right)\left[\int g\left(y_{1} \mid x_{1}\right) g\left(y \mid x_{1}\right) f_{X}\left(x_{1}\right) d x_{1}\right] f_{Y}\left(y_{k}\right) d y_{1} d y_{k} d y+E_{n}^{\prime \prime} \\
= & \int \mathbb{E}\left(\int_{t}^{\infty} g(y \mid X) d y\right)^{2} f_{Y}(t) d t+E_{n}^{\prime \prime} \\
= & H(f, g)+E_{n}^{\prime \prime} . \tag{D.61}
\end{align*}
$$

Indeed

$$
\begin{aligned}
& E_{n}^{\prime \prime} \\
= & \int \sum_{j \neq 1, k} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<y_{1} \wedge y_{j}\right)\left(g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) g\left(y_{1} \mid x_{1}\right) f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} f_{Y}\left(y_{k}\right) d y_{1} d y_{k} d y_{j} \\
= & \int \sum_{j \neq 1, k} \mathbb{1}(N(1)=j) \int_{y_{k}}^{\infty}\left(g\left(y_{j} \mid x_{j}\right)-g\left(y_{j} \mid x_{1}\right)\right) d y_{j} \int_{y_{k}}^{\infty} g\left(y_{1} \mid x_{1}\right) d y_{1} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} f_{Y}\left(y_{k}\right) d y_{k} \\
= & \int\left[\int \sum_{j \neq 1, k} \mathbb{1}(N(1)=j)\left[G_{Y \mid X_{1}=x_{1}}\left(y_{k}\right)-G_{Y \mid X_{j}=x_{j}}\left(y_{k}\right)\right]\left(1-G_{Y \mid X_{1}=x_{1}}\left(y_{k}\right)\right) f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x}\right] f_{Y}\left(y_{k}\right) d y_{k}
\end{aligned}
$$

$$
\begin{align*}
= & \int\left[\int\left[G_{Y \mid X_{1}}\left(y_{k}\right)-G_{Y \mid X_{N(1)}}\left(y_{k}\right)\right]\left(1-G_{Y \mid X_{1}}\left(y_{k}\right)\right) f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x}\right] f_{Y}\left(y_{k}\right) d y_{k} \\
\lesssim & \mathbb{P}\left(\left|X_{1}-X_{N(1)}\right| \geq 1\right) \\
& +\left\{\mathbb{E}\left(1+\left(L_{1}^{(n)}\left(X_{1}, Y_{k}\right)\right)^{\theta}+\left(L_{2}^{(n)}\left(X_{1}, Y_{k}\right)\right)^{\theta}\right)\right\}^{\frac{1}{\theta}}\left\{\mathbb{E}\left|X_{1}-X_{N(1)}\right|^{\frac{\theta \eta}{\theta-1}} \mathbb{1}\left(\left|X_{1}-X_{N(1)}\right| \leq 1\right)\right\}^{\frac{\theta-1}{\theta}} \\
\lesssim & n^{-\frac{\gamma}{\gamma+1}}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left.\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)} \tag{D.62}
\end{align*}
$$

where the last step follows by mimicking the last line of (D.56). Adding (D.58)-(D.62) proves Lemma D. 2 when $\xi\left(f_{X, Y}^{(n)}\right)>0$.

Once again $\xi\left(f_{X, Y}^{(n)}\right)=0$ means $g(y \mid x)=f_{Y}(y)$ for all $x$, which implies that $H(f, g), E_{n}^{\prime}$ and $E_{n}^{\prime \prime}$ are equal to zero.

## Lemma D.3.

$$
\left|T_{3}-H(f, g)\right| \lesssim\left(n^{-\frac{\gamma}{\gamma+1}}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}\right) \mathbb{1}\left(\xi\left(f_{X, Y}^{(n)}\right)>0\right) .
$$

## D.5.3. Proof of Lemma D. 3

Finally in the third term, the contributions for all of $Y_{k} ; Y_{1} ; Y_{N(1)}$ are from $g(y \mid x)$ :

$$
\begin{aligned}
T_{3} & =\int \sum_{j \neq 1, k} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) g\left(y_{k} \mid x_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& =\int \sum_{j \neq 1} \mathbb{1}(N(1)=j) \mathbb{1}\left(y_{k}<\min \left\{y_{1}, y_{j}\right\}\right) g\left(y_{1} \mid x_{1}\right) f_{Y}\left(y_{k}\right) g\left(y_{j} \mid x_{j}\right) d y_{1} d y_{k} d y_{j} f_{\mathbf{X}}(\mathbf{x}) d \mathbf{x} \\
& =H(f, g)+E_{n}^{\prime \prime} \\
& =H(f, g)+O\left(n^{-\frac{\gamma}{\gamma+1}}(\log n)^{2}+\left(\frac{(\log n)^{2}}{n}\right)^{\left(\frac{\gamma(\theta-1)}{\theta(\gamma+1)} \wedge \frac{\eta \gamma}{\gamma+1}\right)}\right) .
\end{aligned}
$$

The first line is by definition of $T_{3}$. The second line follows by integrating over $x_{k}$ (notice that $j \neq k$ ). The third and fourth lines follow by comparing to (D.61) and (D.62) respectively. Once again $\xi\left(f_{X, Y}^{(n)}\right)=0$ means $g(y \mid x)=f_{Y}(y)$ for all $x$, which implies that $H(f, g), E_{n}^{\prime \prime}$ are equal to zero.

Lemmas D. 1 to D. 3 together prove Lemma A. 1 .

## D.6. Auxiliary lemmas

Lemma D.4. Under Assumption (A2) from the main paper, given any $\epsilon>0$,

$$
\mathbb{P}\left(\left|X_{1}-X_{N(1)}\right| \geq \epsilon\right) \lesssim n^{-\frac{\gamma}{\gamma+1}}+\frac{(\log n)^{2} n^{-\frac{\gamma}{\gamma+1}}}{\epsilon}
$$

Consequently, for any $p \leq 1$, uxil

$$
\mathbb{E}\left|X_{1}-X_{N(1)}\right|^{p} \mathbb{1}\left(\left|X_{1}-X_{N(1)}\right| \leq 1\right) \lesssim\left(\frac{(\log n)^{2}}{n}\right)^{p}
$$

Proof of Lemma D.4. The proof follows by retracing the steps of [4, Lemma 14.1] and a straightforward application of Lyapunov's inequality.

Lemma D.5. There exists a fixed positive constant $C>0$ such that the following holds for any $n \geq 1$ and $t \geq 0$ :

$$
\mathbb{P}\left(\left|Z_{n}-\mathbb{E} Z_{n}\right| \geq t\right) \leq 2 \exp \left(-C n t^{2}\right)
$$

Proof of Lemma D.5. We omit the proof of this lemma since it follows simply from McDiarmid's inequality (see [51]) as used in [15, Lemma 9.11].

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[^0]:    ${ }^{\S}$ Alphabetical order. All authors have equal contribution.

