Community Detection with Contextual Multilayer Networks

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January 13, 2023

Abstract

In this paper, we study community detection when we observe m sparse networks and a high dimensional covariate matrix, all encoding the same community structure among nsubjects. In the asymptotic regime where the number of features p and the number of subjects n grow proportionally, we derive an exact formula of asymptotic minimum mean square error (MMSE) for estimating the common community structure in the balanced two block case using an orchestrated approximate message passing algorithm. The formula implies the necessity of integrating information from multiple data sources. Consequently, it induces a sharp threshold of phase transition between the regime where detection (i.e., weak recovery) is possible and the regime where no procedure performs better than random guess. The asymptotic MMSE depends on the covariate signal-to-noise ratio in a more subtle way than the phase transition threshold. In the special case of m = 1, our asymptotic MMSE formula complements the pioneering work [17] which found the sharp threshold when m = 1. A practical variant of the theoretically justified algorithm with spectral initialization leads to an estimator whose empirical MSEs closely approximate theoretical predictions over simulated examples.

Keywords: clustering; contextual SBM; integrative data analysis; multilayer network; phase transition; stochastic block model; approximate message passing.

1 Introduction

Network data is a prevalent form of relational data. It appears in many different fields such as social science, economics, epidemiology, biological science, among others. Many networks come with inherent community structures. Nodes within the same community connect in different ways than nodes between different communities. The community labels of the nodes are usually unknown. It is of interest to uncover such latent community structures based on observed networks. This inference problem is usually called *community detection* which in essence is *clustering* of network nodes. The stochastic block model (SBM) [21] is a popular model for studying community detection. There has been a large literature on theoretical approaches, algorithmic, and application aspects of SBM's. We refer interested readers to several recent survey papers [1, 18, 25] for more detailed accounts of this large and growing literature.

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A more traditional clustering problem in statistics is clustering based on covariates, which is also a leading example of unsupervised learning. Standard techniques include, but are not limited to k-means clustering, hierarchical clustering, and the EM algorithm. The multivariate Gaussian mixture model has been a popular model for theoretical study on this front which has received renewed interest in recent years. The model is closely related to the spiked covariance model [24] which is widely adopted in Random Matrix Theory.

Ever-growing techniques for data acquisition have led us to a new paradigm where one could have multiple data sets as multiple sources of information about the community structure. For instance, for a set of n people, one could potentially have several social networks observed on them (Facebook, LinkedIn, etc.) together with a large collection of socioeconomic (and/or genomic, neuroimaging, etc.) covariates for each individual. This poses a new challenge: *How* can we best integrate information from these multiple sources to uncover the common underlying community structure?

When the network is at least part of the observation, there are two different scenarios. The first is where one observes multiple networks without any covariate. A practical example of this scenario was described in [14] where the nodes represent proteins, the edges in one network represent physical interactions between nodes and those in another network represent co-memberships in protein complexes. This scenario has been studied in the *multilayer network* literature. The arguably more interesting scenario is where one observes one or more networks together with a collection of covariates. In the pioneering work by Deshpande, et al. [17], the authors considered the case where the available data is an $n \times n$ adjacency matrix of an SBM and a high dimensional $p \times n$ Gaussian covariate matrix, both containing the same balanced two block community structure. Under this stylized yet informative model, they rigorously established a sharp information-theoretic threshold for detecting the community structure (i.e., to uncover the community structure better than random guessing) when p and n tend to infinity proportionally and the average degree of the network diverges with n. In addition, they proposed a heuristic algorithm which supports the information-theoretic threshold empirically. Subsequently, the sharp threshold was extended to the case where the average degree is bounded [27]. See also [11, 2] for investigations of spectral clustering and [38] for an SDP approach in similar models.

The present paper is motivated by two important questions that remain unanswered by [17].

- Multiple networks with or without covariates. How does the phase transition phenomenon found in [17] exhibit itself when one observes multiple networks with or without high-dimensional covariates? How does the threshold depend on individual signal-to-noise ratios in these multiple data sources?
- Precise characterization of the information-theoretic limit achieved by Bayes optimal estimator. Even in the special case considered by [17] where only one network is observed together with covariates, it is not clear what the information-theoretic limit of the performance by the best estimator is when the signal-to-noise ratio is above the phase transition threshold. The authors provided a spectral estimator that achieves nontrivial performance above the threshold. However, it is not Bayes optimal, and the exact form of the information-theoretic limit is unknown.

In this paper, we provide affirmative answers to both of the above questions. Without loss of generality, we propose to consider an observation model where one observes m independent adjacency matrices from m SBM's and a high-dimensional Gaussian data-set with p covariates,

all carrying the same latent community structure (balanced two block). Focusing on this model and assuming that the average degrees diverge, our contributions are the following.

- Sharp phase transition threshold. We establish sharp thresholds for phase transition between the regime where detecting the community structure is feasible and the regime where no procedure performs better than random guess.
- Exact formula for asymptotic minimum mean square error (MMSE). We give an exact formula for the asymptotic MMSE achieved by the Bayes optimal estimator of the community structure.

To facilitate the derivation of asymptotic MMSE, we also provide convergence analysis of an orchestrated approximate message passing (AMP) algorithm with multiple parallel and information-sharing orbits. This could be of independent interest.

Last but not least, our results continue to hold for the special case where one only observes m networks and there is no covariate. In this case, our model is reduced to a multilayer SBM. Results in [23, 33] provide sharp information-theoretic thresholds between the regimes of exact recovery (where the best procedure uncovers the community structure perfectly) and almost exact recovery (where the best procedure only makes mistakes on a vanishing proportion of nodes). In addition, [33, 37] derived the minimax rates of convergence that are sharp in exponents in the regime of almost exact recovery under the Hamming loss. In contrast, the present paper provides sharp thresholds for detection (a.k.a. weak recovery) and the exact asymptotic minimax risk under the squared error loss.

On the technical front, the main novelty of the present manuscript lies in designing an orchestrated AMP algorithm with multiple orbits that synchronizes the extraction of information about community structure from multiple data sources. In addition, we provide a rigorous proof of the almost sure convergence of the AMP average sequence. The idea underpinning the algorithm design is potentially applicable to other settings where the integration of information from multiple sources is needed.

The setting of the present paper can be reformulated as a multi-view spiked matrix model which has been studied in [7, 34]. The results of the aforementioned papers give the asymptotic MMSE of *joint* estimation of the covariates. More specifically, these papers have used the adaptive interpolation technique described in [6] to obtain the limit of per-vertex mutual information between data and both the community labels and the covariate means. Then they have used the I-MMSE identity to get the asymptotic MMSE for the joint estimation of the community labels and the covariate means. A different proof technique related to a similar model was described in [13], where the authors identified the limiting free energy as the viscosity solution to a certain Hamilton-Jacobi equation. In contrast to the foregoing papers, the present manuscript focuses on the optimal estimation of the community label vector *only*. The asymptotic joint estimation MMSE results in the foregoing papers do not lead to the asymptotic community label estimation MMSE we shall derive in this paper, since different priors (Rademacher vs. Gaussian) have been put on the community labels and the covariate means, respectively, while the connection between the joint and individual estimation MMSEs depend crucially on the choices of the priors. Due to the generality of the models considered, the asymptotic MMSE's in [7, 34] were expressed in complicated variational forms with matrix arguments. In contrast, under the specific setting considered in the present manuscript, we shall obtain an explicit 'single-letter' characterization of the asymptotic MMSE. While it remains possible to derive asymptotic per-vertex mutual information between data and community labels alone by using the adaptive interpolation technique and further obtain the asymptotic MMSE result in the current paper by differentiation, our proof technique is constructive and hence is entirely different from the approaches in [7, 34, 13]. While those investigations found the limit of the free energy and used it to find the limit of the per-vertex mutual information, we shall use an AMP algorithm to explicitly construct a Bayes optimal sequence of estimators and directly obtain the asymptotic MMSE as the limiting mean squared error of that sequence. The limit of the per-vertex mutual information will be obtained as a side result of our calculations.

Paper organization The rest of the paper is organized as follows. Section 2 introduces our observation models and presents key results on detection threshold and asymptotic MMSE along with introducing the new *orchestrated AMP setup*. It also lays out three major steps in the proof of main results, which are executed in Sections 3–5 in order, and formally summarized in Section 6. In Section 7 we collect the results on the asymptotics of the orchestrated AMP algorithm. In Section 8 we describe a practical algorithm for estimating the community labels using orchestrated AMP with appropriate spectral initialization and demonstrate its performance through simulations. Finally, we discuss the wider applicability of our techniques and discuss some potential future research directions in Section 9. The technical proofs are deferred to the appendices.

2 Detection Threshold and Asymptotic MMSE

2.1 Model

Suppose that n subjects are partitioned into two disjoint groups (labeled by ± 1) according to an n-dimensional vector $x^* \in {\pm 1}^n$. Throughout the paper we assume that the elements x_i^* 's are i.i.d. Rademacher random variables which take values ± 1 with equal probability $\frac{1}{2}$.

The observed data consists of two parts. The first part is a collection of m undirected networks on these n subjects denoted by their adjacency matrices, $\boldsymbol{G} = \{\boldsymbol{G}^{(i)} : i \in [m]\}$. Throughout the paper, for any positive integer k, we let $[k] = \{1, \ldots, k\}$. The second part is a $p \times n$ data matrix \boldsymbol{B} where the *i*-th column records the observed values of p covariates on the *i*-th subject. Conditional on an instance of \boldsymbol{x}^* , the adjacency matrix $\boldsymbol{G}^{(i)}$ has zero diagonal entries, and for all $k \neq l$, we assume

$$G_{kl}^{(i)} = G_{lk}^{(i)} \stackrel{ind}{\sim} \begin{cases} \operatorname{Bern}\left(\frac{a_n^{(i)}}{n}\right), & \text{if } x_k^* = x_l^*, \\ \operatorname{Bern}\left(\frac{b_n^{(i)}}{n}\right), & \text{if } x_k^* \neq x_l^*. \end{cases}$$
(2.1)

For any $\rho \in [0, 1]$, Bern (ρ) denotes a Bernoulli distribution with success probability ρ . Further, we assume that $a_n^{(i)} > b_n^{(i)}$ for all $n, i \ge 0$. In addition, the data matrix **B** is assumed to admit the representation

$$\boldsymbol{B} = \sqrt{\frac{\mu}{n}} \boldsymbol{v}^* (\boldsymbol{x}^*)^\top + \boldsymbol{R}, \qquad (2.2)$$

where \mathbf{R} is an $p \times n$ matrix consisting of i.i.d. standard Gaussian variates and $\mathbf{v}^* \sim N_p(\mathbf{0}, \mathbf{I}_p)$. Finally, we assume that conditional on \mathbf{x}^* , $\mathbf{G}^{(1)}, \ldots, \mathbf{G}^{(m)}$ and \mathbf{B} are mutually independent. In other words, conditional on \mathbf{x}^* , the first part of our data consists of m stochastic block models (a.k.a. a multi-layer stochastic block model with m layers [20]) with a common community structure \mathbf{x}^* . Given \mathbf{v}^* , the columns of the covariate matrix \mathbf{B} is also partitioned into two

groups by x^* , where those corresponding to $x_i^* = +1$ are i.i.d. realizations of a $N_p(\sqrt{\mu/n} v^*, I_p)$ distribution, and those with $\boldsymbol{x}_i^* = -1$ are i.i.d. random vectors following $N_p(-\sqrt{\mu/n} \boldsymbol{v}^*, \boldsymbol{I}_p)$. Our goal is to estimate \boldsymbol{x}^* upon observing $\{\boldsymbol{G}^{(1)}, \ldots, \boldsymbol{G}^{(m)}\}$ and \boldsymbol{B} . We focus on an asymp-

totic regime where as $n \to \infty$, m is fixed and

$$\lim_{n \to \infty} \frac{p}{n} = \frac{1}{c} \in (0, \infty).$$
(2.3)

For each $1 \leq i \leq m$, define

$$\bar{p}_n^{(i)} = \frac{a_n^{(i)} + b_n^{(i)}}{2n}, \quad \Delta_n^{(i)} = \frac{a_n^{(i)} - b_n^{(i)}}{2n}, \quad \lambda_n^{(i)} = \frac{n(a_n^{(i)} - b_n^{(i)})^2}{(a_n^{(i)} + b_n^{(i)})(2n - a_n^{(i)} - b_n^{(i)})}.$$

We further assume that $\mu \ge 0$ is a fixed constant while

$$\lim_{n \to \infty} n \overline{p}_n^{(i)} (1 - \overline{p}_n^{(i)}) = \infty, \quad \text{and}$$
(2.4)

$$\lambda_n^{(i)} = \lambda^{(i)} \in (0, \infty), \quad \text{for all } n, i.$$
(2.5)

For brevity, we let

$$\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)}) \text{ and } \boldsymbol{\lambda} = \sum_{i=1}^{m} \lambda^{(i)}$$
 (2.6)

in the rest of this paper. For convenience, we shall further assume that there are some constants $r^{(1)}, \ldots, r^{(m)} \in (0, 1)$ such that

$$\sum_{i=1}^{m} r^{(i)} = 1 \quad \text{and} \quad \lambda^{(i)} = r^{(i)}\lambda, \quad \text{for all } i.$$
(2.7)

2.2**Detection Threshold**

For every fixed n, λ and μ , we define

$$\operatorname{Overlap}_{n}(\boldsymbol{\lambda}, \mu) = \sup_{\hat{s}_{n}:\mathcal{G}_{n}\times\mathcal{B}_{n}\to\{\pm 1\}^{n}} \frac{1}{n} \mathbb{E} \left| \langle \boldsymbol{x}^{*}, \hat{s}_{n}(\boldsymbol{G}, \boldsymbol{B}) \rangle \right|.$$

Here \mathcal{G}_n is the set of all collections of m undirected networks on n vertices, \mathcal{B}_n is the set of all $p \times n$ real-valued matrices, and $\hat{s}_n(G, B)$ is a generic estimator of the community vector x^* based on observing G and B.

We say detection (a.k.a. weak recovery) of x^* is possible if

$$\liminf_{n \to \infty} \operatorname{Overlap}_n(\boldsymbol{\lambda}, \mu) > 0.$$

Otherwise, we perform no better than random guessing. Indeed, if we simply estimate by random guessing, then our estimator is essentially a vector \boldsymbol{x} with i.i.d. Rademacher entries that is independent of x^* , and we have $\frac{1}{n}\mathbb{E}|\langle x^*, x\rangle| \to 0$.

The following theorem characterizes the phase transition of detection under our model.

Theorem 2.1. Let the data be generated by (2.1) and (2.2). Suppose that as $n \to \infty$, (2.3), (2.4), (2.5) and (2.7) hold. Then we have

$$\limsup_{n \to \infty} \operatorname{Overlap}_{n}(\boldsymbol{\lambda}, \mu) = 0, \quad if \ \lambda + \frac{\mu^{2}}{c} \leq 1,$$

$$\liminf_{n \to \infty} \operatorname{Overlap}_{n}(\boldsymbol{\lambda}, \mu) > 0, \quad if \ \lambda + \frac{\mu^{2}}{c} > 1.$$
(2.8)

where λ is defined in (2.6).

Proof. The theorem follows from our later Theorem 2.2 and (2.10).

Remark 2.1. Note that $r^{(1)}, \ldots, r^{(m)} \in (0, 1)$ are allowed to take any values as long as (2.7) holds.

The foregoing theorem determines a sharp detection threshold in terms of the joint signalto-noise ratio (SNR) contained in the two different data sources, namely the m networks and the data matrix. Here λ can be understood as the joint SNR of the m networks. The phase transition described in (2.8) asserts that the joint SNR of the two parts has an additive form $\lambda + \mu^2/c$. In the special case of m = 1, Theorem 2.1 reconstructs the threshold found in [17]. When m = 0, it coincides with the famous Baik–Ben Arous–Peche phase transition for PCA [4, 5, 32]. When $\mu = 0$, we could simply discard the data matrix \boldsymbol{B} as it contains no information about \boldsymbol{x}^* and Theorem 2.1 leads to a new detection threshold for multi-layer stochastic block models.

2.3 Asymptotic MMSE

We now seek a more precise characterization of the optimal estimator of x^* based on observing G and B, where the optimality is measured through the mean square error.

Minimum mean square error To this end, define the (matrix) minimum mean square error for estimating the community labels from (G, B) as

$$\mathsf{MMSE}_{n}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{1}{n^{2}} \mathbb{E}\left[\left\| \boldsymbol{x}^{*}(\boldsymbol{x}^{*})^{\top} - \mathbb{E}\left[\boldsymbol{x}^{*}(\boldsymbol{x}^{*})^{\top} \right| \boldsymbol{G}, \boldsymbol{B} \right] \right\|_{F}^{2} \right].$$
(2.9)

Following the lines of the proof of Lemma 4.6 in [15], one has

$$1 - \mathsf{MMSE}_n(\boldsymbol{\lambda}, \mu) + O(n^{-1}) \leq \mathrm{Overlap}_n(\boldsymbol{\lambda}, \mu) \leq \sqrt{1 - \mathsf{MMSE}_n(\boldsymbol{\lambda}, \mu)} + O(n^{-1/2}).$$
(2.10)

In particular, $\operatorname{Overlap}_n(\lambda, \mu) \to 0$ if and only if $\mathsf{MMSE}_n(\lambda, \mu) \to 1$. Therefore, a more precise characterization of the overlap than that in Theorem 2.1 can be made if we could describe the exact asymptotic behavior of $\mathsf{MMSE}_n(\lambda, \mu)$.

A scalar Gaussian model As a useful device for describing the asymptotic behavior of $\mathsf{MMSE}_n(\lambda, \mu)$, we follow [29, 16, 15] to introduce the following scalar Gaussian model:

$$Y = Y(\eta) = \sqrt{\eta} X_0 + Z_0,$$
(2.11)

where $X_0 \sim \text{Rademacher and } Z_0 \sim N(0,1)$. In (2.11), every term is a scalar. We assume knowledge of η and the goal is to estimate X_0 based on the observed Y. For this model, we can



Figure 1: Plots of asymptotic MMSE. Left panel: Asymptotic MMSE as a function of λ for fixed μ and c combinations. Right panel: Asymptotic MMSE as a function of μ for fixed λ and c combinations.

compute the mutual information between X_0 and Y and the minimum mean square error for estimating X_0 respectively as

$$I(\eta) = \mathbb{E}\left[\frac{dp_{Y|X_0}(Y(\eta) \mid X_0)}{dp_Y(Y(\eta))}\right] = \eta - \mathbb{E}\left[\log\cosh(\eta + \sqrt{\eta}Z_0)\right],\tag{2.12}$$

$$\operatorname{mmse}(\eta) = \mathbb{E}\left[X_0 - \mathbb{E}(X_0 \mid Y(\eta))\right]^2 = 1 - \mathbb{E}\left[\tanh^2(\eta + \sqrt{\eta}Z_0)\right].$$
(2.13)

Representation of the asymptotic MMSE With the foregoing definitions in (2.11), (2.12) and (2.13), we define $z_* = z_*(\lambda, \mu)$ as the largest non-negative solution to

$$z = 1 - \text{mmse}\left(\lambda z + \frac{\mu^2}{c} \frac{z}{1 + \mu z}\right).$$
(2.14)

The following theorem gives a precise characterization of the limiting behavior of $\mathsf{MMSE}_n(\boldsymbol{\lambda}, \mu)$.

Theorem 2.2. Suppose that the conditions in Theorem 2.1 hold. Then we have

$$\lim_{n \to \infty} \mathsf{MMSE}_n(\boldsymbol{\lambda}, \mu) = 1 - z_*^2(\boldsymbol{\lambda}, \mu), \tag{2.15}$$

where λ is defined by (2.6). This implies

1. If $\lambda + \mu^2/c \leq 1$, $\lim_{n \to \infty} \mathsf{MMSE}_n(\lambda, \mu) = 1$, 2. If $\lambda + \mu^2/c > 1$, $\lim_{n \to \infty} \mathsf{MMSE}_n(\lambda, \mu) < 1$.

Remark 2.2. If $\lambda + \mu^2/c > 1$, then $z_*(\lambda, \mu)$, the largest non-negative solution to (2.14) is strictly greater than zero. Consequently, the limit of MMSE is strictly less than 1, or equivalently, we strictly perform better than random guessing.

Remark 2.3. Together with (2.10), phase transition of matrix MMSE for estimating x^* in Theorem 2.2 implies the phase transition of the overlap in Theorem 2.1.

Remark 2.4. By Theorems 2.1 and 2.2, the parameter λ affects the asymptotic behavior of both Overlap_n and MMSE_n only through λ . So from here on, we shall slightly abuse notation to also write $\text{Overlap}_n(\lambda,\mu)$ and $\text{MMSE}_n(\lambda,\mu)$, which can be viewed as fixing a set of $r^{(i)}$'s in (2.7) and hence treating Overlap_n and MMSE_n as functions of λ and μ only for this fixed set of $r^{(i)}$'s.

In Figure 1, we show how $\lim_{n\to\infty} \mathsf{MMSE}_n(\lambda,\mu)$ behaves as a function of λ for different values of μ and c, and as a function of μ for different values of λ and c. It is worth noting that even for the same values of λ and μ^2/c , asymptotic MMSE's could differ, as its dependence on the three parameters is more subtle than the phase transition threshold.

2.4 Outline of Proof

We now outline the three major steps in the proof of Theorem 2.2. We follow the same overall proof structure as in [15]. Compared with [17] and [15], our major novelty lies in the proposal of an orchestrated AMP algorithm which synchronizes updates about community structure from multiple data sources.

Gaussian approximation In the first step, we define a Gaussian observation model whose asymptotic per-vertex mutual information about x^* is the same as that in the original observation model in (2.1)–(2.2). To this end, let $\{Y^{(i)} : i \in [m]\}$ be a collection of symmetric Gaussian matrices defined as

$$\boldsymbol{Y}^{(i)} = \sqrt{\frac{\lambda^{(i)}}{n}} \boldsymbol{x}^* (\boldsymbol{x}^*)^\top + \boldsymbol{Z}^{(i)}, \quad i \in [m],$$
(2.16)

where $Z^{(i)}$'s are i.i.d. Gaussian Wigner matrices. In other words, each $Z^{(i)}$ is symmetric and

$$Z_{kl}^{(i)} = Z_{lk}^{(i)} \sim \begin{cases} N(0,1) & \text{if } k \neq l. \\ N(0,2) & \text{if } k = l. \end{cases}$$
(2.17)

We shall denote the collection of $\mathbf{Y}^{(i)}$'s by \mathbf{Y} , i.e.

$$\boldsymbol{Y} = \{ \boldsymbol{Y}^{(i)} : i \in [m] \}.$$

Using Lindeberg's interpolation argument, we show that the per-vertex mutual information between x^* and the Gaussian observation model $\{Y, B\}$ is asymptotically the same as that between x^* and the original observation $\{G, B\}$, in the sense that

$$\frac{1}{n}I(\boldsymbol{x}^*;\boldsymbol{G},\boldsymbol{B}) - \frac{1}{n}I(\boldsymbol{x}^*;\boldsymbol{Y},\boldsymbol{B}) \to 0 \quad \text{as} \quad n \to \infty.$$
(2.18)

As a final reduction in our first step, we shall show that

$$I(x^*; T, B) = I(x^*; Y, B)$$

where for λ defined in (2.6) and \mathbf{Z} , a Gaussian Wigner matrix, as in (2.17)

$$\boldsymbol{T} = \boldsymbol{T}(\lambda) = \sqrt{\frac{\lambda}{n}} \boldsymbol{x}^* (\boldsymbol{x}^*)^\top + \boldsymbol{Z}.$$
 (2.19)

Asymptotic I-MMSE relation For Gaussian observation models, one has the famous I-MMSE relation [19]. For instance, for the Gaussian observation (T, B), define

$$\mathsf{GMMSE}_n(\lambda,\mu) = \frac{1}{n^2} \mathbb{E} \left\| \boldsymbol{x}^*(\boldsymbol{x}^*)^\top - \mathbb{E} [\boldsymbol{x}^*(\boldsymbol{x}^*)^\top \mid \boldsymbol{T}, \boldsymbol{B}] \right\|_F^2.$$
(2.20)

Then the I-MMSE relation refers to the identity

$$\frac{1}{n}\frac{d}{d\lambda}I\left(\boldsymbol{x}^{*}(\boldsymbol{x}^{*})^{\mathsf{T}};\boldsymbol{T},\boldsymbol{B}\right) = \frac{1}{4}\mathsf{GMMSE}_{n}(\lambda,\mu).$$
(2.21)

See Section C for a proof of (2.21). On the other hand, we shall derive the following asymptotic counterpart of (2.21) for the original observation model $\{G, B\}$:

$$\frac{1}{n}\frac{d}{d\lambda}I(\boldsymbol{x}^*;\boldsymbol{G},\boldsymbol{B}) - \frac{1}{4}\mathsf{MMSE}_n(\lambda,\mu) \to 0.$$
(2.22)

Furthermore, (263) of [15] implies for $\mathbf{K} = \mathbf{Y}$ and \mathbf{T}

$$\frac{1}{n}I(\boldsymbol{x}^*;\boldsymbol{K},\boldsymbol{B}) - \frac{1}{n}I\left(\boldsymbol{x}^*(\boldsymbol{x}^*)^\top;\boldsymbol{K},\boldsymbol{B}\right) \to 0.$$
(2.23)

Together with (2.18) and the fundamental theorem of calculus, (2.22) and (2.23) establish the asymptotic equivalence of the MMSE's in the two models. Hence, the proof of Theorem 2.2 reduces to finding the exact asymptotic limit of $\mathsf{GMMSE}_n(\lambda,\mu)$. To this end, we turn to Approximate Message Passing (AMP).

Approximate message passing and MMSE in the Gaussian observation model To obtain the "large n" limit of $\mathsf{GMMSE}_n(\lambda, \mu)$, we design the following orchestrated AMP algorithm where we extract information about \boldsymbol{x}^* from both the data sources.

Let $\boldsymbol{u}^0 = \boldsymbol{x}^0 = \boldsymbol{0}$, \boldsymbol{T} be as in (2.19) and \boldsymbol{B} be as in (2.2). Fix any $\varepsilon \in (0, 1)$. We define two companion AMP orbits $\{\boldsymbol{v}^t, \boldsymbol{u}^{t+1}\}$ and $\{\boldsymbol{x}^{t+1}\}$, for $t = 0, 1, 2, \ldots$, characterized by the sensing matrices \boldsymbol{T} and \boldsymbol{B} respectively, as follows

$$\boldsymbol{v}^{t} = \frac{\boldsymbol{B}}{\sqrt{p}} f_{t}(\boldsymbol{u}^{t}, \boldsymbol{x}^{t}, \boldsymbol{x}_{0}(\varepsilon)) - p_{t}g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)),$$

$$\boldsymbol{u}^{t+1} = \frac{\boldsymbol{B}^{\top}}{\sqrt{p}} g_{t}(\boldsymbol{v}^{t}, \boldsymbol{v}_{0}(\varepsilon)) - c_{t}f_{t}(\boldsymbol{u}^{t}, \boldsymbol{x}^{t}, \boldsymbol{x}_{0}(\varepsilon)),$$
(2.24)

and

$$\boldsymbol{x}^{t+1} = \frac{\boldsymbol{T}}{\sqrt{n}} f_t(\boldsymbol{u}^t, \boldsymbol{x}^t, \boldsymbol{x}_0(\varepsilon)) - d_t f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_0(\varepsilon)).$$
(2.25)

Here $\boldsymbol{x}_0(\varepsilon) = (x_{0,1}(\varepsilon), \dots, x_{0,n}(\varepsilon))^\top = (B_1 x_1^*, \dots, B_n x_n^*)^\top$ where B_i 's are i.i.d. Bern (ε) and $\boldsymbol{v}_0(\varepsilon) = (v_{0,1}(\varepsilon), \dots, v_{0,p}(\varepsilon))^\top = (\widetilde{B}_1 v_1^*, \dots, \widetilde{B}_p v_p^*)^\top$ where \widetilde{B}_j 's are i.i.d. Bern (ε) . Next, we define

$$g_t(\boldsymbol{v}^t, \boldsymbol{v}_0) = (g_t(v_1^t, v_{0,1}(\varepsilon)), \dots, g_t(v_p^t, v_{0,p}(\varepsilon)))^\top, f_t(\boldsymbol{u}^t, \boldsymbol{x}^t, \boldsymbol{x}_0) = (f_t(u_1^t, x_1^t, x_{0,1}(\varepsilon)), \dots, f_t(u_n^t, x_n^t, x_{0,n}(\varepsilon)))^\top$$

and

$$c_t = \frac{1}{p} \sum_{i=1}^p \frac{\partial g_t}{\partial v} (v_i^t, v_{0,i}(\varepsilon)), \quad p_t = \frac{c}{n} \sum_{i=1}^n \frac{\partial f_t}{\partial u} (u_i^t, x_i^t, x_{0,i}(\varepsilon)), \quad d_t = \frac{1}{n} \sum_{i=1}^n \frac{\partial f_t}{\partial x} (u_i^t, x_i^t, x_{0,i}(\varepsilon)),$$

where g_{-1} is the zero function, and $f_t : \mathbb{R}^3 \to \mathbb{R}$ and $g_t : \mathbb{R}^2 \to \mathbb{R}$ for $t \in \mathbb{N} \cup \{0\}$ are defined as follows:

$$f_t(x, y, z) = \mathbb{E} \left[X_0 | \alpha_{t-1} X_0 + \tau_{t-1} Z_1 = x, \mu_t X_0 + \sigma_t Z_2 = y, X_0(\varepsilon) = z \right],$$
(2.26)

$$g_t(x,z) = \mathbb{E}\left[V_0|\beta_t V_0 + \vartheta_t Z_3 = x, V_0(\varepsilon) = z\right].$$
(2.27)

In the above definitions (2.26) and (2.27)

$$X_0 \sim \text{Rademacher}, \ X_0(\varepsilon) = B(\varepsilon)X_0 \text{ with } B(\varepsilon) \sim \text{Bern}(\varepsilon),$$

$$V_0(\varepsilon) = \widetilde{B}(\varepsilon)V_0 \text{ with } \widetilde{B}(\varepsilon) \sim \text{Bern}(\varepsilon), \ V_0, Z_1, Z_2, Z_3 \sim N(0, 1), \text{ and}$$
(2.28)

$$X_0, V_0, B(\varepsilon), \widetilde{B}(\varepsilon), Z_1, Z_2 \text{ and } Z_3 \text{ are mutually independent.}$$

In addition, the quantities $\alpha_t, \tau_t, \mu_t, \sigma_t, \beta_t$ and ϑ_t 's are recursively defined as follows. Let $\mu_0 = \sigma_0 = \alpha_{-1} = \tau_{-1} = 0$ and mmse(·) be defined as in (2.13), then for all $t \ge 0$

$$\mu_{t+1} = \sqrt{\lambda} \left(1 - (1 - \varepsilon) \operatorname{mmse} \left(\frac{\alpha_{t-1}^2}{\tau_{t-1}^2} + \frac{\mu_t^2}{\sigma_t^2} \right) \right), \quad \sigma_{t+1}^2 = 1 - (1 - \varepsilon) \operatorname{mmse} \left(\frac{\alpha_{t-1}^2}{\tau_{t-1}^2} + \frac{\mu_t^2}{\sigma_t^2} \right);$$

$$\alpha_t = \sqrt{\frac{\mu}{c}} \left((1 - \varepsilon) \frac{\beta_t^2}{\beta_t^2 + \vartheta_t^2} \right), \quad \tau_t^2 = (1 - \varepsilon) \frac{\beta_t^2}{\beta_t^2 + \vartheta_t^2};$$

$$\beta_t = \sqrt{\frac{\mu}{c}} c \left(1 - (1 - \varepsilon) \operatorname{mmse} \left(\frac{\alpha_{t-1}^2}{\tau_{t-1}^2} + \frac{\mu_t^2}{\sigma_t^2} \right) \right), \quad \vartheta_t^2 = c \left(1 - (1 - \varepsilon) \operatorname{mmse} \left(\frac{\alpha_{t-1}^2}{\tau_{t-1}^2} + \frac{\mu_t^2}{\sigma_t^2} \right) \right).$$

$$(2.29)$$

Remark 2.5. These AMP iterations can be viewed as a corrected version of the power iteration to simultaneously estimate the leading eigenvector of T and the leading singular vectors of B. However, studying the asymptotics of the iterates of the power iteration is difficult because of the dependence introduced in each step. This difficulty is overcome by subtracting a so-called "Onsager term" in every iteration, which ensures that x_i^t for $i = 1, \ldots, n$ are "almost independent". Further, we deviate from using the linear version of the power iteration and specific non-linear functions f_t , g_t (tailored to the priors in the model) are applied componentwise/row-wise to previous iterates before post multiplying the iterates to the matrices A and B, so as to obtain asymptotically Bayes optimal estimates of x^* . One can refer to [9, 22] for further understanding of AMP in general.

Remark 2.6. The AMP iterates (2.24) and (2.25) are based on the ε -revelation of the truth \boldsymbol{x}^* and \boldsymbol{v}^* , which is adopted here to eliminate the degenerate case where all updates $\boldsymbol{u}^t = \boldsymbol{x}^t = \boldsymbol{v}^t = 0$ for $t \ge 0$. Alternatively, such degeneracy could potentially be avoided by considering spectral initialization (e.g., [30]). Since our primary goal here is to use AMP for bounding $\mathsf{GMMSE}_n(\lambda, \mu)$, we choose to work with the ε -revelation approach as its theoretical analysis is cleaner.

Remark 2.7. The major difficulty in designing the AMP iterates lies in the effective integration of information from multiple data sources. One possibility is to treat T as the main information and B as the side information, or vice versa. Although AMP with side information has been considered in [26] in the context of signal recovery from noisy observations, the generic approach in [26] does not work in the present context. In [26], the side information, contained in a set of random variables $\{S_1, \ldots, S_n\}$ where S_i contains the side information for node *i*, has the special property that they are mutually independent. In our case, the side information is in the form of $\{b_1, \ldots, b_n\}$ where b_i is the *i*-th column of the matrix B. They are not independent whenever $\mu > 0$, and hence the side information are not independent across nodes, and a direct application of the results in [8] as in [26] is impossible. An alternative approach is to construct a sequence of AMP recursions with matrix valued iterates as in (28)-(29) of [22]. However, we can verify that this leads to a version of the AMP that is not Bayes optimal. This is because x^* is essentially estimated in two separate iterations using T and B. This nonsynchronized iteration is the root of the sub-optimal performance. Our proposed iterates (2.24)-(2.25) are designed to resolve this issue by running two parallel AMP orbits with sensing matrices T and B, respectively, while sharing information between each other at each iteration. This is achieved by the use of a synchronized update function f_t which takes both u^t and x^t in its arguments.

Finally, we define a sequence of estimates of x^* based on the AMP iterates as

$$\widehat{\boldsymbol{x}}^{t} = f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)), \qquad (2.30)$$

As these estimates are functions of \boldsymbol{T} , \boldsymbol{B} , $\boldsymbol{x}_0(\varepsilon)$ and $\boldsymbol{v}_0(\varepsilon)$, the mean square errors of $\hat{\boldsymbol{x}}^t(\hat{\boldsymbol{x}}^t)^\top$ to estimate $\boldsymbol{x}^*(\boldsymbol{x}^*)^\top$ provide a sequence of upper bounds for

$$\mathsf{GMMSE}_n(\lambda,\mu,\varepsilon) = \frac{1}{n^2} \mathbb{E} \left\| \boldsymbol{x}^*(\boldsymbol{x}^*)^\top - \mathbb{E} [\boldsymbol{x}^*(\boldsymbol{x}^*)^\top \mid \boldsymbol{T}, \boldsymbol{B}, \boldsymbol{x}_0(\varepsilon), \boldsymbol{v}_0(\varepsilon)] \right\|_F^2$$

We shall show that the mean square errors of $\hat{x}^t(\hat{x}^t)^{\top}$ converge to the same limit as GMMSE_n in the "large n, large t' asymptotics. We analyze the asymptotics of the mean square errors of $\hat{x}^t(\hat{x}^t)^{\top}$ by analyzing the AMP defined in (2.24) and (2.25). To this end, we augment the techniques in [8] to handle multiple communicating orbits. Last but not least, we argue that as ε goes to 0, the mean square errors of $\hat{x}^t(\hat{x}^t)^{\top}$ approximate the limit of $\mathsf{GMMSE}_n(\lambda, \mu)$. Therefore, by showing that the "large n, large t, small ε " limit of mean square errors of the AMP iterates is exactly the same as that in Theorem 2.2, we complete the proof.

3 Gaussian Approximation and Asymptotic Per-Vertex Mutual Information

The results spelt out in this section closely follow the results of Section 5 in [15]. We list them here for the paper to be self contained.

3.1 Mutual Information in the Gaussian Model

Let us recall the Gaussian model given by \mathbf{Y} , the collection of Gaussian random matrices defined in (2.16); the SBM ensemble \mathbf{G} defined by (2.1); and the covariate matrix \mathbf{B} defined in (2.2). We shall show that as $n \to \infty$, the per-vertex mutual information between \mathbf{x}^* and the model $\{\mathbf{Y}, \mathbf{B}\}$ is asymptotically the same as the the per-vertex mutual information between \mathbf{x}^* and the model $\{\mathbf{G}, \mathbf{B}\}$.

To this end, we begin by defining the Hamiltonian function \mathcal{H} for m arbitrary $n \times n$ symmetric matrices $V^{(1)}, V^{(2)}, \ldots, V^{(m)}$:

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{V}, \boldsymbol{B}, \boldsymbol{\lambda}, \mu, n, p) := \mathcal{H}'(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{V}, \boldsymbol{\lambda}, n) - \frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v} \boldsymbol{x}^\top \right\|_F^2$$

where

$$\mathcal{H}'(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{V}, \boldsymbol{\lambda}, n) := \sum_{i=1}^m \sum_{k < l} V_{kl}^{(i)}(x_k x_l - x_k^* x_l^*) + \sum_{i=1}^m \sum_{k < l} \frac{\lambda^{(i)}}{n} x_k x_l x_k^* x_l^*$$

with $\boldsymbol{V} := (\boldsymbol{V}^{(1)}, \dots, \boldsymbol{V}^{(m)})$. Further, define

$$\phi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{V}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p) = \log \left\{ \sum_{\boldsymbol{x} \in \{\pm 1\}^n} \int_{\mathbb{R}^p} \exp(\mathcal{H}(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{V}, \boldsymbol{B}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p)) \exp\left(-\frac{\|\boldsymbol{v}\|^2}{2}\right) d\boldsymbol{v} \right\}.$$
(3.1)

Then we have the following lemma.

Lemma 3.1. Let us consider $\mathbf{Y} = {\mathbf{Y}^{(i)} : i \in [m]}$ defined in (2.16) and \mathbf{B} defined in (2.2). Then we have

$$\begin{split} I(\boldsymbol{x}^*; \boldsymbol{Y}, \boldsymbol{B}) &= n \log 2 + \frac{n-1}{2} \sum_{i=1}^m \lambda^{(i)} \\ &+ \mathbb{E} \log \left(\int_{\mathbb{R}^p} \exp\left(-\frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v}(\boldsymbol{x}^*)^\top \right\|_F^2 \right) \exp\left(-\frac{\|\boldsymbol{v}\|^2}{2} \right) d\boldsymbol{v} \right) \\ &- \mathbb{E}[\phi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{W}, \boldsymbol{\lambda}, \mu, n, p)] \end{split}$$

where $\phi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{W}, \boldsymbol{\lambda}, \mu, n, p)$ is defined in (3.1) and $\boldsymbol{W} = (\sqrt{\lambda^{(1)}/n} \boldsymbol{Z}^{(1)}, \dots, \sqrt{\lambda^{(m)}/n} \boldsymbol{Z}^{(m)}).$

Proof. See Section A.1.

Furthermore, if we consider the random matrix $T(\lambda)$ defined by (2.19), the following lemma shows that the mutual information between x^* and $\{Y, B\}$ is the same as the mutual information between x^* and $\{T, B\}$.

Lemma 3.2. If we consider $T(\lambda)$ defined in (2.19), Y defined in (2.16) and B defined in (2.2) then we have

$$I(\boldsymbol{x}^*; \boldsymbol{Y}, \boldsymbol{B}) = I(\boldsymbol{x}^*; \boldsymbol{T}(\lambda), \boldsymbol{B}).$$

Proof. See Section A.2.

This shows that it is equivalent to study the model $\{T, B\}$ or $\{Y, B\}$. It is easier to study the model $\{T, B\}$ as instead of dealing with an *n*-vector of parameters λ in $\{Y, B\}$, in $\{T, B\}$ we can study the model with respect to a single parameter λ .

3.2 Mutual Information in the Original Model

Next, we observe that the entries of the adjacency matrix $G_{kl}^{(i)}$ of the adjacency matrices $G^{(i)}$ are given by

$$G_{kl}^{(i)} := \begin{cases} 1 & \text{with probability } \bar{p}_n^{(i)} + \Delta_n^{(i)} x_k^* x_l^*, \\ 0 & \text{with probability } 1 - \bar{p}_n^{(i)} - \Delta_n^{(i)} x_k^* x_l^*. \end{cases}$$

We define the function \mathcal{H}_{SBM} , the Hamiltonian with respect to the multilayer SBM as follows.

$$\mathcal{H}_{SBM}(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{u}, \boldsymbol{G}, \boldsymbol{B}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{n}, \boldsymbol{p})) = \mathcal{H}_{SBM}'(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{G}, \boldsymbol{\lambda}, \boldsymbol{n}) - \frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v} \boldsymbol{x}^\top \right\|_F^2, \quad (3.2)$$

where

$$\mathcal{H}_{SBM}'(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{G}, \boldsymbol{\lambda}, n) := \sum_{i=1}^{m} \sum_{k < l} \left[G_{kl}^{(i)} \log \left(\frac{\bar{p}_n^{(i)} + \Delta_n^{(i)} x_k x_l}{\bar{p}_n^{(i)} + \Delta_n^{(i)} x_k^* x_l^*} \right) + \left(1 - G_{kl}^{(i)} \right) \log \left(\frac{1 - \bar{p}_n^{(i)} - \Delta_n^{(i)} x_k x_l}{1 - \bar{p}_n^{(i)} - \Delta_n^{(i)} x_k^* x_l^*} \right) \right].$$
(3.3)

Let us define

$$\psi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{G}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p) = \log \left\{ \sum_{\boldsymbol{x} \in \{\pm 1\}^n} \int_{\mathbb{R}^p} \exp(\mathcal{H}_{SBM}(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{G}, \boldsymbol{B}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p)) \exp\left(-\frac{\|\boldsymbol{v}\|^2}{2}\right) d\boldsymbol{v} \right\}.$$
(3.4)

Then we have the following lemma characterizing the mutual information between x^* and $\{G, B\}$.

Lemma 3.3. Let us consider **B** defined in (2.2) and $\mathbf{G} = {\mathbf{G}^{(i)} : i \in [m]}$ defined in (2.1). Then we have

$$I(\boldsymbol{x}^*; \boldsymbol{G}, \boldsymbol{B}) = n \log 2 + \mathbb{E} \log \left(\int_{\mathbb{R}^p} \exp\left(-\frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v}(\boldsymbol{x}^*)^\top \right\|_F^2 \right) \exp\left(-\frac{\|\boldsymbol{v}\|^2}{2}\right) d\boldsymbol{v} \right) \\ - \mathbb{E}[\psi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{G}, \boldsymbol{\lambda}, \mu, n, p)]$$

where $\psi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{G}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{n}, \boldsymbol{p})$ is defined in (3.4).

Proof. See Section A.3.

To connect $I(\mathbf{x}^*; \mathbf{G}, \mathbf{B})$ to $I(\mathbf{x}^*; \mathbf{Y}, \mathbf{B})$, we use Lindeberg's Interpolation Argument. For that purpose, let us define the auxiliary random matrices $\tilde{\mathbf{G}}^{(i)}$ where

$$\widetilde{G}_{kl}^{(i)} := \frac{\Delta_n^{(i)}}{\overline{p}_n^{(i)}(1 - \overline{p}_n^{(i)})} \left(G_{kl}^{(i)} - \overline{p}_n^{(i)} - \Delta_n^{(i)} x_k^* x_l^* \right).$$
(3.5)

By \widetilde{G} we refer to the collection of random matrices $\{\widetilde{G}^{(1)}, \ldots, \widetilde{G}^{(m)}\}$. The mutual information between x^* and $\{G, B\}$ is related to x^* and $\{\widetilde{G}, B\}$ in the following way.

Lemma 3.4. Let us consider \tilde{G} defined in (3.5). Then with $n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)}) \to \infty$ for $i = 1, \ldots, m$, we have the following identity

$$\begin{split} I(\boldsymbol{x}^*; \boldsymbol{G}, \boldsymbol{B}) &= n \log 2 + \frac{n-1}{2} \sum_{i=1}^n \lambda^{(i)} \\ &+ \mathbb{E} \log \left(\int_{\mathbb{R}^p} \exp\left(-\frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v}(\boldsymbol{x}^*)^\top \right\|_F^2 \right) \exp\left(-\frac{\|\boldsymbol{v}\|^2}{2}\right) d\boldsymbol{v} \right) \\ &- \mathbb{E}[\phi(\boldsymbol{x}^*, \boldsymbol{B}, \widetilde{\boldsymbol{G}}, \boldsymbol{\lambda}, \mu, n, p)] + O\left(\sum_{i=1}^m \frac{n(\lambda^{(i)})^{3/2}}{\sqrt{n\overline{p}_n^{(i)}(1 - \overline{p}_n^{(i)})}} \right). \end{split}$$

Proof. See Section A.4.

3.3 Gaussian Approximation

Next, we use Lindeberg's interpolation to approximate $\mathbb{E}[\phi(\boldsymbol{x}^*, \boldsymbol{B}, \tilde{\boldsymbol{G}}, \boldsymbol{\lambda}, \mu, n, p)]$ by $\mathbb{E}[\phi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{W}, \boldsymbol{\lambda}, \mu, n, p)]$.

Lemma 3.5. Suppose $n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)}) \to \infty$ for $i = 1, \ldots, m$. Then we have

$$\mathbb{E}[\phi(\boldsymbol{x}^*, \boldsymbol{B}, \widetilde{\boldsymbol{G}}, \boldsymbol{\lambda}, \mu, n, p)] = \mathbb{E}[\phi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{W}, \boldsymbol{\lambda}, \mu, n, p)] + O\left(\sum_{i=1}^m \frac{n(\lambda^{(i)})^{3/2}}{\sqrt{n\bar{p}_n^{(i)}(1 - \bar{p}_n^{(i)})}}\right).$$

Proof. See Section A.5.

Finally we get the following theorem showing the asymptotic equivalence of the per-vertex mutual information in the two models.

Theorem 3.1. Let us consider $\mathbf{Y} = {\mathbf{Y}^{(i)} : i \in [m]}$ defined in (2.16), \mathbf{B} defined in (2.2), and $\mathbf{G} = {\mathbf{G}^{(i)} : i \in [m]}$ defined in (2.1). If for all $i \in [m]$ we have $n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)}) \to \infty$ for i = 1, ..., m, then we have

$$\left|\frac{1}{n}I(\boldsymbol{x}^{*};\boldsymbol{Y},\boldsymbol{B}) - \frac{1}{n}I(\boldsymbol{x}^{*};\boldsymbol{G},\boldsymbol{B})\right| \leqslant O\left(\sum_{i=1}^{m} \frac{(\lambda^{(i)})^{3/2}}{\sqrt{n\bar{p}_{n}^{(i)}(1-\bar{p}_{n}^{(i)})}}\right)$$

Proof. The proof easily follows using Lemma 3.1, Lemma 3.4 and Lemma 3.5.

Remark 3.1. The above theorem shows that as $n \to \infty$ the asymptotic per-vertex mutual information about x^* obtained from (Y, B) is same as that obtained from (G, B).

An immediate corollary to the above theorem is as follows.

Corollary 3.1. Consider $T(\lambda)$ defined by (2.19) and $\lambda = \sum_{i=1}^{m} \lambda^{(i)}$. If for all $i \in [m]$ we have $n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)}) \to \infty$ for i = 1, ..., m, then we have the following inequality.

$$\left|\frac{1}{n}I(\boldsymbol{x}^*; \boldsymbol{T}(\lambda), \boldsymbol{B}) - \frac{1}{n}I(\boldsymbol{x}^*; \boldsymbol{G}, \boldsymbol{B})\right| \leqslant O\left(\sum_{i=1}^m \frac{(\lambda^{(i)})^{3/2}}{\sqrt{n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})}}\right)$$

Proof. This corollary immediately follows from Theorem 3.1 and Lemma 3.2.

4 An Asymptotic I-MMSE Relation

Let us begin by observing that the collection of SBM's $G^{(1)}, \ldots, G^{(m)}$ can be represented as the collection of random variables $\{G_{kl}^{(i)}: 1 \leq i \leq m, 1 \leq k < l \leq n\}$. Instead of considering $\{0, 1\}$ valued random variables $G_{kl}^{(i)}$, we shall consider $\{-1, 1\}$ valued random variables $L_{kl}^{(i)} = 2G_{kl}^{(i)} - 1$. This collection will be called \mathcal{L} , that is,

$$\mathcal{L} = \left\{ 2G_{kl}^{(i)} - 1 : 1 \leq i \leq m, 1 \leq k < l \leq n \right\}.$$

Since the elements of \mathcal{L} are linear transformations of the elements of $\{G_{kl}^{(i)} : i \in [m], 1 \leq k < l \leq n\}$, we have

$$H(\boldsymbol{x}^*|\boldsymbol{G},\boldsymbol{B}) = H(\boldsymbol{x}^*|\mathcal{L},\boldsymbol{B}), \qquad (4.1)$$

where $H(\mathbf{x}^*|\mathcal{L}, \mathbf{B})$ is the conditional entropy of \mathbf{x}^* given $(\mathcal{L}, \mathbf{B})$. This implies that

$$\frac{1}{n}I(\boldsymbol{x}^*;\boldsymbol{G},\boldsymbol{B}) = \frac{1}{n}I(\boldsymbol{x}^*;\boldsymbol{\mathcal{L}},\boldsymbol{B}).$$
(4.2)

Then an asymptotic I-MMSE identity for the differentiation of $I(\boldsymbol{x}^*; \mathcal{L}, \boldsymbol{B})$ is given by the following lemma.

Lemma 4.1. Let λ , λ be as defined in (2.6) and $r^{(i)}$ for $i = 1, \ldots, m$ be as defined in (2.7). If $n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)}) \to \infty$, then there is a positive constant C such that

$$\left|\frac{1}{n}\frac{dI(\boldsymbol{x}^*;\mathcal{L},\boldsymbol{B})}{d\lambda} - \frac{1}{4}\operatorname{\mathsf{MMSE}}_n(\lambda,\mu)\right| \leqslant C\left(\sum_{i=1}^m \sqrt{\frac{r^{(i)}\lambda}{n\overline{p}_n^{(i)}(1-\overline{p}_n^{(i)})}}\right).$$

Proof. See Section B.1.

Together with (4.2), the above lemma implies

$$\left|\frac{1}{n}\frac{dI(\boldsymbol{x}^*;\boldsymbol{G},\boldsymbol{B})}{d\lambda} - \frac{1}{4}\operatorname{\mathsf{MMSE}}_n(\lambda,\mu)\right| \to 0 \quad \text{as } n \to \infty.$$

5 Asymptotic MMSE in the Gaussian Model

In this section, we derive the asymptotic limit of the quantity $\mathsf{GMMSE}_n(\lambda,\mu)$ defined in (2.20). To this end, we shall show that, for the AMP iterate \hat{x}^t defined in (2.30), the mean square error in estimating $x^*(x^*)^\top$ by $\hat{x}^t(\hat{x}^t)^\top$ in the Gaussian observation model (2.19) is asymptotically the same as the limit of $\mathsf{GMMSE}_n(\lambda,\mu)$ as ε goes to zero and n, t goes to infinity. The matrix mean square error in estimating $x^*(x^*)^\top$ by $\hat{x}^t(\hat{x}^t)^\top$, referred to as $\mathsf{MSE}_n^{\mathsf{AMP}}(t;\lambda,\mu,\varepsilon)$, is defined by

$$\mathsf{MSE}_{n}^{\mathsf{AMP}}(t;\lambda,\mu,\varepsilon) = \frac{1}{n^{2}} \mathbb{E}\left[\|\boldsymbol{x}^{*}(\boldsymbol{x}^{*})^{\top} - \hat{\boldsymbol{x}}^{t}(\hat{\boldsymbol{x}}^{t})^{\top} \|_{F}^{2} \right].$$
(5.1)

We show that in the "large n, large t, small ε " limit this sequence of estimators is asymptotically Bayes optimal in the sense that $\mathsf{MSE}_n^{\mathsf{AMP}}(t;\lambda,\mu,\varepsilon)$ converges to the same limit as $\mathsf{GMMSE}_n(\lambda,\mu)$. Hence, from the properties of the AMP iterates that we shall derive in this section, we can characterize the precise limit of $\mathsf{GMMSE}_n(\lambda,\mu)$ (and hence of $\mathsf{MMSE}_n(\lambda,\mu)$) as $n \to \infty$.

As a byproduct, we obtain an explicit formula of the asymptotic limit of the per-vertex mutual information in the Gaussian observation model. By Corollary 3.1, it also gives the asymptotic limit of the per-vertex mutual information in the original model (2.1)-(2.2).

State evolution of the AMP iterates Recall AMP iterates u^t, x^t and v^t defined in (2.24) and (2.25), and state evolution (2.29). From (2.29), we obtain the following

$$\begin{split} &\frac{\mu_{t+1}^2}{\sigma_{t+1}^2} = \lambda \left(1 - (1 - \varepsilon) \text{mmse} \left(\frac{\alpha_{t-1}^2}{\tau_{t-1}^2} + \frac{\mu_t^2}{\sigma_t^2} \right) \right), \\ &\frac{\beta_t^2}{\vartheta_t^2} = \mu \left(1 - (1 - \varepsilon) \text{mmse} \left(\frac{\alpha_{t-1}^2}{\tau_{t-1}^2} + \frac{\mu_t^2}{\sigma_t^2} \right) \right), \\ &\frac{\alpha_t^2}{\tau_t^2} = (1 - \varepsilon) \frac{\mu}{c} \frac{\beta_t^2}{\beta_t^2 + \vartheta_t^2}. \end{split}$$

Define $\theta_t := \beta_t^2/\vartheta_t^2$ and $\gamma_t := \mu_t^2/\sigma_t^2$. Then we have the following

$$\gamma_{t+1} = \lambda \left(1 - (1 - \varepsilon) \text{mmse} \left(\gamma_t + (1 - \varepsilon) \frac{\mu}{c} \frac{\theta_{t-1}}{1 + \theta_{t-1}} \right) \right),$$
$$\theta_t = \mu \left(1 - (1 - \varepsilon) \text{mmse} \left(\gamma_t + (1 - \varepsilon) \frac{\mu}{c} \frac{\theta_{t-1}}{1 + \theta_{t-1}} \right) \right).$$

Further, define

$$z_t = \frac{\gamma_{t+1}}{\lambda} = \frac{\theta_t}{\mu}.$$
(5.2)

Then the state evolution recursion reduces to

$$z_{t+1} = 1 - (1 - \varepsilon) \operatorname{mmse}\left(\lambda z_t + (1 - \varepsilon)\frac{\mu^2}{c}\frac{z_t}{1 + \mu z_t}\right).$$
(5.3)

Since the function on the right side of (5.3) is concave, increasing monotonically and bounded as a function of z_t (as we shall show later in the proof of Theorem 5.1 in Section D.1), we have

$$z_t \to z_*(\lambda, \mu, \varepsilon), \quad \text{as} \quad t \to \infty.$$

This implies that $z_*(\lambda, \mu, \varepsilon)$ satisfies

$$z_*(\lambda,\mu,\varepsilon) = 1 - (1-\varepsilon) \operatorname{mmse}\left(\lambda z_*(\lambda,\mu,\varepsilon) + (1-\varepsilon)\frac{\mu^2}{c}\frac{z_*(\lambda,\mu,\varepsilon)}{1+\mu z_*(\lambda,\mu,\varepsilon)}\right).$$
(5.4)

Limit of MMSE As a first step, we have the following theorem that characterizes the asymptotics of $\mathsf{MSE}_n^{\mathsf{AMP}}(t;\lambda,\mu,\varepsilon)$.

Theorem 5.1. Let $MSE_n^{AMP}(t; \lambda, \mu, \varepsilon)$ be defined as in (5.1). Then we have

$$\lim_{n \to \infty} \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t; \lambda, \mu, \varepsilon) = 1 - z_t^2,$$

where z_t is defined by (5.2). Taking $t \to \infty$, we have

$$\lim_{t\to\infty}\lim_{n\to\infty}\mathsf{MSE}^{\mathsf{AMP}}_\mathsf{n}(t;\lambda,\mu,\varepsilon) = 1 - z^2_*(\lambda,\mu,\varepsilon),$$

where $z_*(\lambda, \mu, \varepsilon)$ is the largest non-negative solution to (5.4). As $\varepsilon \to 0$, we get

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \lim_{n \to \infty} \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t; \lambda, \mu, \varepsilon) = 1 - z_*^2(\lambda, \mu),$$

where $z_*(\lambda, \mu)$ is the largest non-negative solution to (2.14).

Proof. See Section D.1.

Next, we have the following theorem characterizing asymptotic $\mathsf{GMMSE}_n(\lambda, \mu)$ and pervertex mutual information.

Theorem 5.2. Consider $\mathsf{GMMSE}_n(\lambda, \mu)$ defined in (2.20). Then for all $\lambda, \mu \ge 0$,

$$\lim_{n \to \infty} \mathsf{GMMSE}_n(\lambda, \mu) = 1 - z_*^2(\lambda, \mu),$$

where $z_*(\lambda, \mu)$ is the largest non-negative solution to (2.14). Further,

$$\lim_{n \to \infty} \frac{1}{n} I(\boldsymbol{x}^*; \boldsymbol{T}(\lambda), \boldsymbol{B}) = \xi(z_*(\lambda, \mu), \lambda, \mu),$$

where $I(\cdot)$ is defined in (2.12) and

$$\xi(z,\lambda,\mu) = \frac{\lambda z^2}{4} - \frac{\lambda z}{2} + \frac{\lambda}{4} + \frac{1}{2c}\log(1+\mu z) + \frac{1}{2c}\frac{(1+\mu)}{(1+\mu z)} + I\left(\lambda z + \frac{\mu^2}{c}\frac{z}{1+\mu z}\right) - \frac{1}{2c}\log(1+\mu) - \frac{1}{2c}.$$
(5.5)

Proof. See Section D.2.

Finally, as an immediate corollary, we obtain the following limit for per-vertex mutual information in the original observation model.

Corollary 5.1. Consider G defined by (2.1). Then we have the following

$$\lim_{n \to \infty} \frac{1}{n} I(\boldsymbol{x}^*; \boldsymbol{G}, \boldsymbol{B}) = \xi(z_*(\lambda, \mu), \lambda, \mu),$$

where $z_*(\lambda,\mu)$ is the largest non-negative solution to (2.14), and $\xi(z,\lambda,\mu)$ is defined by (5.5).

Proof. Follows from Corollary 3.1 and Theorem 5.2.

6 Proof of Theorem 2.2

With all the ingredients collected in Sections 3–5, we now give a formal proof of Theorem 2.2 according to the outline laid out in Section 2.4.

Using (263) of [15], we get

$$\lim_{n \to \infty} \left[\frac{1}{n} I(\boldsymbol{x}^*; \boldsymbol{T}(\lambda), \boldsymbol{B}) - \frac{1}{n} I(\boldsymbol{x}^*(\boldsymbol{x}^*)^\top; \boldsymbol{T}(\lambda), \boldsymbol{B}) \right] = 0.$$

Now using the same arguments as those in Section C, we have

$$\lim_{n \to \infty} \frac{1}{n} \left(I(\boldsymbol{x}^*; \boldsymbol{T}(\lambda_1), \boldsymbol{B}) - I(\boldsymbol{x}^*; \boldsymbol{T}(\lambda_2), \boldsymbol{B}) \right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \left(I(\boldsymbol{x}^*(\boldsymbol{x}^*)^\top; \boldsymbol{T}(\lambda_1), \boldsymbol{B}) - I(\boldsymbol{x}^*(\boldsymbol{x}^*)^\top; \boldsymbol{T}(\lambda_2), \boldsymbol{B}) \right)$$

$$= \lim_{n \to \infty} \int_{\lambda_1}^{\lambda_2} \frac{1}{4} \mathsf{GMMSE}_n(\theta, \mu) d\theta$$
 (6.1)

where $\mathsf{GMMSE}_n(\theta, \mu) = \frac{1}{n^2} \mathbb{E} \| \boldsymbol{x}^*(\boldsymbol{x}^*)^\top - \mathbb{E} \left[\boldsymbol{x}^*(\boldsymbol{x}^*)^\top | \boldsymbol{T}(\theta), \boldsymbol{B} \right] \|_F^2$ and $\boldsymbol{T}(\theta)$ are defined in (2.19). Fix a set of $r^{(i)}$'s defined in (2.7). For any $\theta > 0$, we can write $\mathsf{MMSE}_n(\theta, \mu)$ for $\mathsf{MMSE}_n(\theta, \mu)$

Fix a set of $r^{(i)}$'s defined in (2.7). For any $\theta > 0$, we can write $\mathsf{MMSE}_n(\theta, \mu)$ for $\mathsf{MMSE}_n(\theta, \mu)$ where the *i*th element of θ is $\theta r^{(i)}$. Using Lemma 4.1 and (2.4), we get for all finite λ_1 and λ_2

$$\lim_{n \to \infty} \int_{\lambda_1}^{\lambda_2} \frac{1}{4} \mathsf{MMSE}_n(\theta, \mu) d\theta = \lim_{n \to \infty} \frac{1}{n} \left(I(\boldsymbol{x}^*; \boldsymbol{G}(\lambda_2), \boldsymbol{B}) - I(\boldsymbol{x}^*; \boldsymbol{G}(\lambda_1), \boldsymbol{B}) \right)$$

where $I(\boldsymbol{x}^*; \boldsymbol{G}(\lambda), \boldsymbol{B})$ refers to the mutual information between \boldsymbol{x}^* and $(\boldsymbol{G}, \boldsymbol{B})$. Then using Corollary 3.1 and (6.1) we get for all $\lambda \ge 0$ and $\mu \ge 0$

$$\lim_{n \to \infty} \mathsf{GMMSE}_n(\lambda, \mu) = \lim_{n \to \infty} \mathsf{MMSE}_n(\lambda, \mu).$$

Now using Theorem 5.2 we get

$$\lim_{n \to \infty} \mathsf{MMSE}_n(\lambda, \mu) = 1 - z_*^2(\lambda, \mu),$$

where $z_*(\lambda, \mu)$ is the largest non-negative solution to (2.14). Define

$$G(z) = 1 - \text{mmse}\left(\lambda z + \frac{\mu^2}{c} \frac{z}{1 + \mu z}\right),$$

and

$$S(z) = 1 - \operatorname{mmse}(z).$$

Then using Lemma 6.1 of [15] we get that G is increasing, concave, G(0) = 0 and there is an unique positive solution of (2.14) if and only if

$$G'(0) = \left(\lambda + \frac{\mu^2}{c}\right)S'(0) = \lambda + \frac{\mu^2}{c} > 1.$$

In this case, $z_*(\lambda, \mu) < 1$ by its definition in (2.14). Otherwise, if $\lambda + \mu^2/c \leq 1$, then the only non-negative solution to (2.14) is 0.

Since the foregoing arguments hold for any fixed $r^{(1)}, \ldots, r^{(m)}$, this implies if $\lambda + \mu^2/c \leq 1$

$$\lim_{n\to\infty}\mathsf{MMSE}_n(\boldsymbol{\lambda},\mu)=1,$$

and if $\lambda + \mu^2/c > 1$

$$\lim_{n\to\infty}\mathsf{MMSE}_n(\boldsymbol{\lambda},\mu) < 1.$$

This completes the proof.

7 Orchestrated Approximate Message Passing

This section collects the key results on the SLLN type behavior of the orchestrated AMP iterates with multiple parallel orbits. These results are used to derive the properties of the sequence of estimators \hat{x}^t in Section 5, and they are potentially of independent interest. Although we focus on the case of two orbits in this section, the arguments could be extended to more than two orbits.

To fully accommodate the ε -revelation approach we have taken in (2.24)–(2.25), we need to introduce some additional technicalities for the function classes that we establish convergence results on. The details are spelled out in Section 7.1. The SLLN-type behavior of the iterates in AMP without and with signal is established in Sections 7.2 and 7.3, respectively.

7.1 Partially Pseudo-Lipschitz and Partially Lipschitz Functions

Traditionally, while analyzing the convergence of the AMP iterates, one considers pseudo-Lipschitz functions [9, 22, 15]. However, many update functions which are intuitive may not belong to this function class. For example, in our case, while the sequence of update functions f_t (in (2.24) and (2.25)) are pseudo-Lipschitz, the functions g_t are not. Fortunately, the asymptotics of the AMP iterates that we have designed can be analyzed with a weaker requirement in the update functions.

To motivate our definition, observe that $g_t : \mathbb{R}^2 \to \mathbb{R}$ in (2.27) is given by

$$g_t(x,z) = \begin{cases} \frac{\beta_t}{\beta_t^2 + \vartheta_t^2} x & \text{if } z = 0, \\ z & \text{if } z \neq 0. \end{cases}$$
(7.1)

This function is discontinuous at (x, 0) for all $x \in \mathbb{R} \setminus \{0\}$. Hence, it is not pseudo-Lipschitz. However, if we fix the last argument and view the function as only a function of the remaining arguments, then it becomes pseudo-Lipschitz. Such functions are sufficiently smooth for the AMP iterates to behave properly in the asymptotic regime. In view of this we define the following *partially pseudo-Lipschitz* functions.

Definition 7.1. Let $\boldsymbol{a} = (a_1, \ldots, a_k)^{\top}$, $\boldsymbol{b} = (b_1, \ldots, b_k)^{\top}$, and $z \in \mathbb{R}$. A function $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}$ is called partially pseudo-Lipschitz if there is an absolute constant C > 0 such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$ and $z \in \mathbb{R}$,

$$|\varphi(\boldsymbol{a}, z) - \varphi(\boldsymbol{b}, z)| \leq C \left(1 + \sum_{i=1}^{k} |a_i| + \sum_{i=1}^{k} |b_i| + |z| \right) \|\boldsymbol{a} - \boldsymbol{b}\|.$$
 (7.2)

Further there exists $C_1 > 0$, such that for all $z \in \mathbb{R}$,

$$|\varphi(\mathbf{0}, z)\rangle| \leqslant C_1 \left(1 + |z|^2\right). \tag{7.3}$$

In a partially pseudo-Lipschitz function, the first k variables are the main variables, and the last is called the *offset variable*.

Similar to pseudo-Lipschitz functions, partially pseudo-Lipschitz functions also form a function class on which one has the desired SLLN type behavior. In the same spirit, we define *partially Lipschitz* functions as follows.

Definition 7.2. Consider $\boldsymbol{a} = (a_1, \ldots, a_k)^{\top}$, $\boldsymbol{b} = (b_1, \ldots, b_k)^{\top}$ and $z \in \mathbb{R}$. A function $f : \mathbb{R}^{k+1} \to \mathbb{R}$ is called partially Lipschitz if there is an absolute constant C > 0 such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$ and $z \in \mathbb{R}$,

$$|f(\boldsymbol{a}, z) - f(\boldsymbol{b}, z)| \leq C \|\boldsymbol{a} - \boldsymbol{b}\|.$$

Further there exists $C_1 > 0$, such that for all $z \in \mathbb{R}$,

$$|f(\mathbf{0},z)| \leq C_1(1+|z|).$$

Remark 7.1. All pseudo-Lipschitz functions are partially pseudo-Lipschitz. This implies that all Lipschitz functions are partially pseudo-Lipschitz. Furthermore, if $f(x_1, \ldots, x_k, z) : \mathbb{R}^{k+1} \to \mathbb{R}$, is Lipschitz, then the functions f^2 and $x_i f$ for $i \in [k]$ are partially pseudo-Lipschitz. For two Lipschitz functions $f, g : \mathbb{R}^{k+1} \to \mathbb{R}$, the function fg is partially pseudo-Lipschitz. Finally, by Lemma D.1, the sequence of functions $f_t(x, y, z)$ defined by (2.26) is Lipschitz. Hence, $f_t^2(x, y, z)$, $xf_t(x, y, z), yf_t(x, y, z)$ and $f_t(x, y, z)f_s(x, y, z)$ are all partially pseudo-Lipschitz. The same is true for $\partial f_t/\partial x$ and $\partial f_t/\partial y$ for all t, as they are Lipschitz continuous. Next we observe the following properties of partially Lipschitz and partially pseudo-Lipschitz functions.

Lemma 7.1. Consider two partially Lipschitz functions $f, g : \mathbb{R}^{k+1} \to \mathbb{R}$. Then they satisfy the following properties.

- 1. The function $h(x_1, \ldots, x_k, z) = f(x_1, \ldots, x_k, z)g(x_1, \ldots, x_k, z)$ is partially pseudo-Lipschitz.
- 2. Consider a random variable X with finite expectation. For any fixed $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k$, let

$$H(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k, z) = \mathbb{E}_X \{ \phi(x_1, \ldots, x_{r-1}, X, x_{r+1}, \ldots, x_k, z) \},\$$

where ϕ is partially pseudo-Lipschitz. Then the function $H : \mathbb{R}^k \to \mathbb{R}$ is partially pseudo-Lipschitz.

Remark 7.2. Recall that $g_t(x, z)$ defined in (2.27) satisfies (7.1). It is straightforward to check that g_t 's are partially Lipschitz. As all partially Lipschitz functions are partially pseudo-Lipschitz, g_t 's are partially pseudo-Lipschitz. Further, $\partial g_t/\partial x$'s are also partially Lipschitz for all t and hence partially pseudo-Lipschitz. Furthermore, $g_t^2(x, z)$, $xg_t(x, z)$, and $g_t(x, z)g_s(x, z)$ are all partially pseudo-Lipschitz.

7.2 Orchestrated AMP with Mean Zero Gaussian Sensing Matrices

Let \boldsymbol{L} be a $p \times n$ random matrix where

$$L_{ij} \stackrel{iid}{\sim} N(0, 1/p), \tag{7.4}$$

and let N be a scaled GOE(n) matrix where

$$N_{ii} \stackrel{iid}{\sim} N(0, 2/n) \text{ and } N_{ij} = N_{ji} \stackrel{iid}{\sim} N(0, 1/n) \text{ when } i \neq j.$$
 (7.5)

In addition, assume that L and N are mutually independent. We want to construct two orchestrated AMP orbits based on the matrices L and N with information sharing between them in each iteration.

Construction of orchestrated AMP orbits Consider a sequence of update functions $f_t : \mathbb{R}^4 \to \mathbb{R}$, where for all integers $t \ge 0$,

 f_t 's are partially Lipschitz and their partial derivatives with respect to the first two variables are also partially Lipschitz. (7.6)

Let us consider another sequence of update functions $\mathbf{g}_t : \mathbb{R}^3 \to \mathbb{R}$, where for any integer $t \ge 0$,

 \mathbf{g}_t 's are partially Lipschitz and their partial derivatives with respect to the first argument are also partially Lipschitz. (7.7)

In addition, let f_{-1} and g_{-1} be zero functions.

Starting with $h^0 = y^0 = 0$, we consider the following two AMP orbits:

$$\boldsymbol{b}^{t} = \boldsymbol{L} \boldsymbol{f}_{t}(\boldsymbol{h}^{t}, \boldsymbol{y}^{t}, \boldsymbol{\xi}_{0}, \boldsymbol{x}_{0}) - p_{t} \boldsymbol{g}_{t-1}(\boldsymbol{b}^{t-1}, \boldsymbol{\omega}_{0}, \boldsymbol{v}_{0}),$$

$$\boldsymbol{h}^{t+1} = \boldsymbol{L}^{\top} \boldsymbol{g}_{t}(\boldsymbol{b}^{t}, \boldsymbol{\omega}_{0}, \boldsymbol{v}_{0}) - c_{t} \boldsymbol{f}_{t}(\boldsymbol{h}^{t}, \boldsymbol{y}^{t}, \boldsymbol{\xi}_{0}, \boldsymbol{x}_{0}),$$

(7.8)

and

$$y^{t+1} = Nf_t(h^t, y^t, \xi_0, x_0) - d_t f_{t-1}(h^{t-1}, y^{t-1}, \xi_0, x_0),$$

where $\boldsymbol{\xi}_0 = (\xi_{0,1}, \dots, \xi_{0,n})^\top$ and $\boldsymbol{x}_0 = (x_{0,1}, \dots, x_{0,n})^\top$ with $(\xi_{0,i}, x_{0,i}) \stackrel{iid}{\sim} P_{\boldsymbol{\xi}, \boldsymbol{x}}$ which has a finite second moment. Similarly $\boldsymbol{\omega}_0 = (\boldsymbol{\omega}_{0,1}, \dots, \boldsymbol{\omega}_{0,p})^\top$ and $\boldsymbol{v}_0 = (v_{0,1}, \dots, v_{0,p})^\top$ with $(\boldsymbol{\omega}_{0,j}, v_{0,j}) \stackrel{iid}{\sim} P_{\boldsymbol{\omega}, \boldsymbol{v}}$ which also has finite second moment. We further assume that $(\boldsymbol{\xi}_0, \boldsymbol{x}_0)$, and $(\boldsymbol{\omega}_0, \boldsymbol{v}_0)$ are independent of \boldsymbol{L} and \boldsymbol{N} . Moreover, in (7.8)

$$\mathbf{g}_{t}(\boldsymbol{b}^{t}, \boldsymbol{\omega}_{0}, \boldsymbol{v}_{0}) = (\mathbf{g}_{t}(b_{1}^{t}, \omega_{0,1}, v_{0,1}), \dots, \mathbf{g}_{t}(b_{p}^{t}, \omega_{0,p}, v_{0,p}))^{\top}, \\ \mathbf{f}_{t}(\boldsymbol{h}^{t}, \boldsymbol{y}^{t}, \boldsymbol{\xi}_{0}, \boldsymbol{x}_{0}) = (\mathbf{f}_{t}(h_{1}^{t}, y_{1}^{t}, \xi_{0,1}, x_{0,1}), \dots, \mathbf{f}_{t}(h_{n}^{t}, y_{n}^{t}, \xi_{0,n}, x_{0,n}))^{\top},$$

and

$$c_{t} = \frac{1}{p} \sum_{i=1}^{p} \frac{\partial g_{t}}{\partial b} (b_{i}^{t}, \omega_{0,i}, v_{0,i}),$$

$$p_{t} = \frac{c}{n} \sum_{i=1}^{n} \frac{\partial f_{t}}{\partial h} (h_{i}^{t}, y_{i}^{t}, \xi_{0,i}, x_{0,i}),$$

$$d_{t} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial f_{t}}{\partial y} (h_{i}^{t}, y_{i}^{t}, \xi_{0,i}, x_{0,i}),$$

where $c = \lim_{n \to \infty} n/p$. Note that in construction of the above (partially) pseudo-Lipschitz functions, elements of \boldsymbol{x}_0 and \boldsymbol{v}_0 are offset variables, while those of $\boldsymbol{\xi}_0$ and $\boldsymbol{\omega}_0$ belong to the main variables. Finally, denote

$$\boldsymbol{m}^t = \boldsymbol{\mathsf{g}}_t(\boldsymbol{b}^t, \boldsymbol{\omega}_0, \boldsymbol{v}_0) \text{ and } \boldsymbol{q}^t = \boldsymbol{\mathsf{f}}_t(\boldsymbol{h}^t, \boldsymbol{y}^t, \boldsymbol{\xi}_0, \boldsymbol{x}_0).$$
 (7.9)

Remark 7.3. Compared to (2.24) and (2.25), for AMP iterations without signal (7.8), we have increased the number of arguments in both update function sequences by one to accommodate later analysis. Therefore, we change the notation to f_t and g_t to alert the readers that the number of arguments has increased. Their connection with the update functions $\{f_t, g_t : t \ge 0\}$ used when signal is present will be made explicit in Remark 7.6.

Remark 7.4. The use of the same update function while updating h^t and y^t is not necessary. We have considered this setup because it helps in the analysis of the estimate \hat{x}^t .

State evolution For notational simplicity, we define for any vector $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$,

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_m = \frac{1}{m} \sum_{i=1}^m u_i v_i \text{ and } \langle \boldsymbol{u} \rangle_m = \frac{1}{m} \sum_{i=1}^m u_i.$$

The asymptotics of the foregoing AMP can be analyzed by its state evolution described below. Let $\tau_{-1}^2 = \sigma_0^2 = 0$, $\vartheta_0^2 = c \lim_{n \to \infty} \langle \boldsymbol{q}^0, \boldsymbol{q}^0 \rangle_n$, and $\sigma_1^2 = \lim_{n \to \infty} \langle \boldsymbol{q}^0, \boldsymbol{q}^0 \rangle_n$. For all integer $t \ge 1$, we define recursively

$$\sigma_{t}^{2} = \mathbb{E}\{\mathbf{f}_{t-1}(\tau_{t-2}Z_{1}, \sigma_{t-1}Z_{2}, \widetilde{\Xi}_{0}, \widetilde{X}_{0})^{2}\},\$$

$$\tau_{t-1}^{2} = \mathbb{E}\{\mathbf{g}_{t-1}(\vartheta_{t-1}Z_{3}, \widetilde{\Omega}_{0}, \widetilde{V}_{0})^{2}\},\$$

$$\vartheta_{t}^{2} = c \mathbb{E}\{\mathbf{f}_{t}(\tau_{t-1}Z_{1}, \sigma_{t}Z_{2}, \widetilde{\Xi}_{0}, \widetilde{X}_{0})^{2}\}.$$
(7.10)

Here, $Z_1, Z_2, Z_3 \stackrel{iid}{\sim} N(0,1)$, $(\tilde{\Xi}_0, \tilde{X}_0) \sim P_{\xi,x}$ and $(\tilde{\Omega}_0, \tilde{V}_0) \sim P_{\omega,v}$, and they are mutually independent. As before, $c = \lim_{n \to \infty} n/p$.

With the foregoing definitions, the following theorem characterizes the SLLN type behavior of the "large n" averages of partially pseudo-Lipschitz functions applied on AMP iterates.

Theorem 7.1. Consider \boldsymbol{L} and \boldsymbol{N} defined in (7.4) and (7.5) that are mutually independent, and the AMP iterates (7.8) satisfying (7.6) and (7.7). Let $\boldsymbol{\xi}_0 = (\xi_{0,1}, \ldots, \xi_{0,n})^{\top}$ and $\boldsymbol{x}_0 = (x_{0,1}, \ldots, x_{0,n})^{\top}$ with $(\xi_{0,i}, x_{0,i}) \stackrel{iid}{\sim} P_{\boldsymbol{\xi}, \boldsymbol{x}}$ which has finite second moment. Similarly, $\boldsymbol{\omega}_0 = (\omega_{0,1}, \ldots, \omega_{0,p})^{\top}$ and $\boldsymbol{v}_0 = (v_{0,1}, \ldots, v_{0,p})^{\top}$ with $(\omega_{0,j}, v_{0,j}) \stackrel{iid}{\sim} P_{\omega,v}$ which also has finite second moment. In addition, suppose $(\boldsymbol{\xi}_0, \boldsymbol{x}_0)$ and $(\boldsymbol{\omega}_0, \boldsymbol{v}_0)$ are independent of both \boldsymbol{L} and \boldsymbol{N} . Furthermore, let $\tau_t^2, \vartheta_t^2, \sigma_t^2$ be defined by the recursions (7.10) with initializations $\boldsymbol{y}^0 = 0, \boldsymbol{h}^0 = 0,$ $\vartheta_0^2 = c \lim_{p \to \infty} \langle \boldsymbol{q}^0, \boldsymbol{q}^0 \rangle_n$, and $\sigma_1^2 = \lim_{n \to \infty} \langle \boldsymbol{q}^0, \boldsymbol{q}^0 \rangle_n$. For any partially pseudo-Lipschitz functions $\phi_h : \mathbb{R}^4 \to \mathbb{R}$ and $\psi_b : \mathbb{R}^3 \to \mathbb{R}$ in the sense of (7.2), we have

$$\frac{1}{n}\sum_{i=1}^{n}\phi_h(h_i^{t+1}, y_i^{t+1}, \xi_{0,i}, x_{0,i}) \xrightarrow{a.s.} \mathbb{E}\left\{\phi_h(\tau_t Z_1, \sigma_{t+1} Z_2, \widetilde{\Xi}_0, \widetilde{X}_0)\right\},\$$

and

$$\frac{1}{p} \sum_{i=1}^{p} \psi_b(b_i^t, \omega_{0,i}, v_{0,i}) \xrightarrow{a.s.} \mathbb{E} \left\{ \psi_b(\vartheta_t Z_3, \widetilde{\Omega}_0, \widetilde{V}_0) \right\}.$$

Here $Z_1, Z_2, Z_3 \stackrel{iid}{\sim} N(0,1), (\widetilde{\Xi}_0, \widetilde{X}_0) \sim P_{\xi,x}, (\widetilde{\Omega}_0, \widetilde{V}_0) \sim P_{\omega,v}$, and they are mutually independent.

Remark 7.5. The presence of two AMP orbits and the use of orchestrated iterates y^t and h^t in each f_t prevents us from directly using existing AMP convergence results in [9], [22] or [10]. To resolve this issue, we shall modify the proof of Lemma 1 in [9] by using the conditioning technique in [12] directly and prove an analogous lemma in Subsection E.2 (Lemma E.1) suitable for (7.8). The lemma will then be used to prove Theorem 7.1.

7.3 Orchestrated AMP with Rank-One Deformed Sensing Matrices

We now turn back to the AMP iterates u^t, v^t and x^t defined by (2.24) and (2.25) with some generic update functions f_t and g_t . With a slight abuse of notation, we define $\mu_0 = \sigma_0 = \alpha_{-1} = \tau_{-1} = 0$ and

$$\beta_0 = \sqrt{\frac{\mu}{c}} \mathbb{E} \left\{ X_0 f_0(0, 0, X_0(\varepsilon)) \right\}, \quad \vartheta_0^2 = c \lim_{n \to \infty} \langle f_0(\boldsymbol{u}^0, \boldsymbol{x}^0, \boldsymbol{x}_0), f_0(\boldsymbol{u}^0, \boldsymbol{x}^0, \boldsymbol{x}_0) \rangle_n.$$

Then we define for all $t \ge 1$

$$\mu_{t} = \sqrt{\lambda} \mathbb{E} \left\{ X_{0} f_{t-1} (\tau_{t-2} Z_{1} + \alpha_{t-2} X_{0}, \sigma_{t-1} Z_{2} + \mu_{t-1} X_{0}, X_{0}(\varepsilon)) \right\},$$

$$\alpha_{t-1} = \sqrt{\frac{\mu}{c}} \mathbb{E} \left\{ V_{0} g_{t-1} (\vartheta_{t-1} Z_{3} + \beta_{t-1} V_{0}, V_{0}(\varepsilon)) \right\},$$

$$\beta_{t} = c \sqrt{\frac{\mu}{c}} \mathbb{E} \left\{ X_{0} f_{t} (\tau_{t-1} Z_{1} + \alpha_{t-1} X_{0}, \sigma_{t} Z_{2} + \mu_{t} X_{0}, X_{0}(\varepsilon)) \right\},$$

$$\sigma_{t}^{2} = \mathbb{E} \left\{ \left[f_{t-1} (\tau_{t-2} Z_{1} + \alpha_{t-2} X_{0}, \sigma_{t-1} Z_{2} + \mu_{t-1} X_{0}, X_{0}(\varepsilon)) \right]^{2} \right\},$$

$$\tau_{t-1}^{2} = \mathbb{E} \left\{ \left[g_{t-1} (\vartheta_{t-1} Z_{3} + \beta_{t-1} V_{0}, V_{0}(\varepsilon)) \right]^{2} \right\},$$

$$\vartheta_{t}^{2} = c \mathbb{E} \left\{ \left[f_{t} (\tau_{t-1} Z_{1} + \alpha_{t-1} X_{0}, \sigma_{t} Z_{2} + \mu_{t} X_{0}, X_{0}(\varepsilon)) \right]^{2} \right\},$$
(7.11)

where $X_0, X_0(\varepsilon), V_0, V_0(\varepsilon), Z_1, Z_2$ and Z_3 satisfy (2.28).

Remark 7.6. Here we slightly abuse notation in the sense that we use $\alpha_t, \beta_t, \mu_t, \sigma_t^2, \tau_t^2$ and ϑ_t^2 for state evolution with generic f_t and g_t , whereas they were originally defined only for the specific f_t and g_t in (2.26)–(2.27). Effectively, we could think of these quantities as functions of $\{f_t, g_t : t \ge 0\}$. In this way, the notation could be unified. Furthermore, the notation σ_t^2 , τ_{t-1}^2 and ϑ_t^2 are in accordance with that of (7.10) by identifying $(X_0, X_0(\varepsilon))$ with $(\tilde{\Xi}_0, \tilde{X}_0)$, $(V_0, V_0(\varepsilon))$ with $(\tilde{\Omega}_0, \tilde{V}_0)$, $f_t(x_1, x_2, y, z)$ with $f_t(x_1 + \alpha_{t-1}y, x_2 + \mu_t y, z)$ and $\mathbf{g}_{t-1}(x, y, z)$ with $g_{t-1}(x + \beta_{t-1}y, z)$.

The following theorem establishes the SLLN type behavior for AMP iterates defined by (2.24) and (2.25) with generic f_t and g_t satisfying certain smoothness conditions.

Theorem 7.2. Consider partially pseudo-Lipschitz functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ in the sense of (7.2). Suppose that, in (2.24) and (2.25), the update functions f_t and its partial derivatives with respect to the first two variables are partially Lipschitz for all $t \ge 0$. Further for all $t \ge 0$, g_t and its partial derivative with respect to the first argument are also partially Lipschitz. In addition, let f_{-1} and g_{-1} be zero functions. Then for all $t \in \mathbb{N}$, and for $\mu_t, \alpha_{t-1}, \beta_t, \sigma_t^2, \tau_{t-1}^2$ and ϑ_t^2 defined in (7.11), we have the following identities:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi(u_{i}^{t}, x_{i}^{t}, x_{0,i}) \stackrel{a.s.}{=} \mathbb{E} \left\{ \phi\left(\alpha_{t-1}X_{0} + \tau_{t-1}Z_{1}, \mu_{t}X_{0} + \sigma_{t}Z_{2}, X_{0}(\varepsilon)\right) \right\},\$$

and

$$\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \psi(v_i^t, v_{0,i}) \stackrel{a.s.}{=} \mathbb{E} \left\{ \psi \left(\beta_t V_0 + \vartheta_t Z_3, V_0(\varepsilon) \right) \right\}.$$

Here $X_0, X_0(\varepsilon), V_0, V_0(\varepsilon), Z_1, Z_2$ and Z_3 satisfy (2.28).

8 Numerical Experiments

The AMP algorithm defined by the recursions (2.24) and (2.25) is asymptotically Bayes optimal for estimating $\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top}$ in the Gaussian model. However, its dependence on the partial revelation of the truth \boldsymbol{x}^* and \boldsymbol{v}^* makes it impractical. In this section, we investigate the empirical performance of a practically implementable variant of (2.24)– (2.25): we initialize with a spectral estimator and force $\varepsilon = 0$ in the AMP iterates (2.24)– (2.25).

To this end, we propose to initialize both \mathbf{x}^0 and \mathbf{u}^0 with $\sqrt{n}\,\mathbf{\bar{e}}$, where $\mathbf{\bar{e}}$ is the leading eigenvector of $\mathbf{T} + a_0 \mathbf{B} \mathbf{B}^{\top}$ for some constant a_0 defined below. Throughout this section, we define a_0 as the unique solution to the following equation:

$$\frac{\mu}{c\lambda} = \frac{-\lambda + (ca^2 + \mu a^2) + \sqrt{(\lambda + ca^2 + \mu a^2)^2 - 4\lambda ca^2}}{2\mu}.$$
(8.1)

It can be shown that if $\lambda + \mu^2/c > 1$, then the leading eigenvector of $\mathbf{T} + a_0 \mathbf{B} \mathbf{B}^{\top}$ is asymptotically correlated with \mathbf{x}^* , and they are asymptotically orthogonal if $\lambda + \mu^2/c \leq 1$. Rigorous proofs of the properties of this initializer and related issues are beyond the scope of the present paper, and they are being investigated in [31].

In the rest of this section, we first conduct a simulation study of the above algorithm under the Gaussian observation model (2.19) and (2.2). Next, we study its performance under the original multilayer network plus covariate model with one and two layers. In both cases, the empirical performance of this practical variant agrees well with the theoretical predictions in Theorems 5.2 and 2.2, respectively.

8.1 The Gaussian Observation Model

We take n = 1500 and p = 900 and consider two settings: "fixed μ varying λ " and "fixed λ varying μ ".

In the first setting, we fix $\mu \in \{0.5, 0.7, 0.9\}$, respectively. At each fixed value μ , we vary λ across 25 equally-spaced values in the interval [0.5, 4.5]. For each combination (λ, μ) , we generate 25 i.i.d. copies of (\mathbf{T}, \mathbf{B}) pairs. For each (\mathbf{T}, \mathbf{B}) pair, we run the iterates in (2.24) and (2.25) with $\varepsilon = 0$ for 100 iterations after initializing $\mu_0, \sigma_0, \alpha_{-1}$ and τ_{-1} randomly in the interval [4, 10] and using the spectral initialization for \mathbf{u}^0 and \mathbf{x}^0 . We construct the AMP estimate $\hat{\mathbf{x}}^{100}$ as in (2.30). The upper panel of Figure 2 reports the average and spread of the empirical MMSEs on 25 replications at each λ for all three fixed values of μ and compares the average with the theoretical prediction (2.15). These plots show that the MMSEs of the spectral initialized AMP iterates agree well with the theoretical limits across all (λ, μ) value pairs.

In the second setting, we fix $\lambda \in \{0.3, 0.6, 0.9\}$, respectively. At each fixed value λ , we vary μ across 25 equally-spaced values in the interval [0.5, 4.5]. For each combination (λ, μ) , the other simulation details are the same as in the first setting. The lower panel of Figure 2 reports the average and spread of the empirical MMSEs over 25 replications at each μ for the three fixed values of λ and compares the average with the theoretical prediction (2.15). As in the previous setting, the empirical MMSEs agree well with theoretical predictions.

8.2 The Original Observation Model with One Layer

We now consider the original observation model (2.1)–(2.2) with m = 1. This special case is also known as the contextual SBM. Although the AMP algorithm defined by (2.24) and (2.25) is designed for a Gaussian sensing matrix, but arguments of [36] show that the same state evolution limits can be obtained if instead of T, one considers the matrix

$$oldsymbol{A} = rac{oldsymbol{G} - ar{p}_n oldsymbol{1} oldsymbol{1}^{dash}}{\sqrt{n ar{p}_n (1 - ar{p}_n)}}$$

where G is the adjacency matrix of the one layer network. Thus, we could apply the practical algorithm presented at the beginning of this section with T replaced with A.

We take n = 2000 and p = 3000 and $\bar{p}_n = 0.7/\sqrt{n}$, and all other simulation details are identical to those used in Section 8.1. In the upper panel of Figure 3 we plot the average and the spread of the empirical MMSE of the estimator defined by (2.30) over 25 iterates for each value of λ at three fixed values of μ and compare it against the theoretical prediction given by (2.15). In the lower panel of Figure 3, we repeat the experiment across different μ values at three fixed λ values. In both settings, we see the same pattern as in the Gaussian observation model: the empirical MMSEs of the practical algorithm approximate the theoretical predictions well across all (λ, μ) combinations that we consider.

8.3 The Original Observation Model with Three Layers

In our last set of simulations, we turn to the original observation model (2.1)-(2.2) with m = 3. We take n = 2000, p = 3000, and consider three SBMs with $\bar{p}_n^{(1)} = 0.7/\sqrt{n}$, $\bar{p}_n^{(2)} = 0.4/\sqrt{n}$ and



Figure 2: Upper Panel: Empirical MMSE plots of the AMP estimator versus λ for different fixed μ 's. Lower Panel: Empirical MMSE plots of the AMP estimator versus μ for different fixed λ 's.

 $\bar{p}_n^{(3)} = 0.3/\sqrt{n}$. In addition, we keep the SNR fractions $r^{(1)}$, $r^{(2)}$ and $r^{(3)}$ in (2.7) at 0.6, 0.2 and 0.2, respectively. The adjacency matrices of the SBMs are denoted by G_1 and G_2 . To find the counterpart for T to be used in the practical algorithm, we first define the centered and scaled adjacency matrices:

$$m{A}_i = rac{m{G}_i - ar{p}_n^{(i)} m{1} m{1}^{ op}}{\sqrt{n ar{p}_n^{(i)} (1 - ar{p}_n^{(i)})}}, \qquad i = 1, 2, 3.$$

Simple algebra suggests that we should replace T in the Gaussian model with

$$oldsymbol{A} := \sqrt{\lambda_1/\lambda}oldsymbol{A}_1 + \sqrt{\lambda_2/\lambda}oldsymbol{A}_2 + \sqrt{\lambda_3/\lambda}oldsymbol{A}_3$$

Other than the foregoing modification, the other experiment details are identical to what we have used in the previous two subsections.

In the upper panel of Figure 4 we plot the average and the spread of the empirical MMSE of the estimator defined by (2.30) over 25 iterations at each value of λ for three fixed values of μ . We compare empirical MMSEs with the theoretical prediction (2.15). In the lower panel of Figure 4 we switch the roles of λ and μ , that is, we fix λ and vary μ . In both settings, we see the same pattern as in the Gaussian observation model and in the contextual SBM: the empirical MMSEs of the practical algorithm approximate the theoretical predictions well across all (λ, μ) value pairs that we consider.



Figure 3: Upper Panel: Empirical MMSE plots of the AMP estimator based on Graph Adjacency matrix versus λ for different fixed μ 's. Lower Panel: Empirical MMSE plots of the AMP estimator based on Graph Adjacency matrix versus μ for different fixed λ 's.

9 Concluding Remarks

In this paper, we have designed an orchestrated AMP algorithm with two orbits and ε -revelation to establish the exact asymptotic limit of MMSE for estimating \boldsymbol{x}^* . The theoretically justified version is not practical due to its dependence on the true parameter values through ε -revelation. In Section 8, a practical variant with spectral initialization leads to empirical estimation errors that closely approximate the theoretically predicted optimal values over all simulated examples. To fully establish its generality, it is of great interest to show mathematically that this practical algorithm indeed reaches the asymptotic MMSE. An alternative practical algorithm for community detection in single layer Contextual SBM has been described in Section 4 of [17]. This algorithm can potentially be modified to handle multilayer networks and covariates. However, this is beyond the scope of the current manuscript and we leave it for future research.

We have focused exclusively on the balanced two-block setting, which is the simplest nontrivial case for community detection. In addition, we have considered Gaussian covariates. These assumptions could be relaxed. The extension of our results to general sub-Gaussian sensing matrices can be derived by directly using the techniques described in [36]. Therefore, we do not describe it in detail. In addition, we could consider the balanced k block setting with k > 2, in which case the signal would be encoded by a matrix of rank k - 1. See, for instance, the multiple spiked models (1.1) and (1.2) in [30]. The AMP algorithms for handling such cases can be developed by following the principle in the present paper and generalizing our techniques along the line of [30]. Furthermore, we could consider detection threshold in a sparse setting



Figure 4: Upper Panel: Empirical MMSE plots of the AMP estimator based on three Graph Adjacency matrices versus λ for different fixed μ 's. Lower Panel: Empirical MMSE plots of the AMP estimator based on two Graph Adjacency matrices versus μ for different fixed λ 's.

with non-diverging average degrees. We could also consider the optimal rate of community detection in these models under a Hamming loss as in [39] in the regime of weak consistency as opposed to detection. In these settings, different techniques from AMP are expected to be needed to achieve the information-theoretically optimal performance. That being said, the idea of developing orchestrated parallel estimation sequences with information sharing at each step for different data sources could still be useful. We leave the aforementioned potential extensions for future research.

In the present work, we have considered an orchestrated AMP algorithm with two orbits as we have two sources of information about the estimand. It could be generalized to more than two orbits when additional sources of information are present. For example, suppose that there are *n* vertices in total. In addition to network and covariate information for all vertices as we have considered in this work, there may be an additional network on a subset of vertices and some additional covariates on a different subset, represented by an $n_1 \times n_1$ adjacency matrix and an $n_2 \times p'$ covariate matrix, where $n_1, n_2 < n$. To pool all the information together, we anticipate that an orchestrated AMP algorithm with four orbits would be needed to achieve an information-theoretically optimal estimation error. We think that the study of orchestrated AMP algorithms in more general settings would be an interesting future research topic.

Acknowledgment

The authors would like to thank Fan Yang for helpful discussions and communication that lead to (8.1) and Galen Reeves for interesting discussions on the connection of the present work to estimation under the multi-view spiked matrix models.

A Proof of Results in Section 3

A.1 Proof of Lemma 3.1

From the definition of mutual information we have

$$I(\boldsymbol{x}^*; \boldsymbol{Y}, \boldsymbol{B}) = \mathbb{E}\left[\log \frac{dp_{\boldsymbol{Y}, \boldsymbol{B} \mid \boldsymbol{x}^*}(\boldsymbol{Y}, \boldsymbol{B} \mid \boldsymbol{x}^*)}{dp_{\boldsymbol{Y}, \boldsymbol{B}}(\boldsymbol{Y}, \boldsymbol{B})}\right].$$

Let

$$\mathcal{E}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{Y}, \boldsymbol{B}) = \exp\left(\sum_{i=1}^{m} -\frac{1}{4} \left\| \boldsymbol{Y}^{(i)} - \sqrt{\frac{\lambda^{(i)}}{n}} \boldsymbol{x} \boldsymbol{x}^{\mathsf{T}} \right\|_{F}^{2} - \frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v} \boldsymbol{x}^{\mathsf{T}} \right\|_{F}^{2} \right).$$

Then using the property of Gaussian channel we get

$$\begin{split} I(\boldsymbol{x}^{*};\boldsymbol{Y},\boldsymbol{B}) \\ &= \mathbb{E}\log\left\{\frac{\int_{\mathbb{R}^{p}}\mathcal{E}(\boldsymbol{x}^{*},\boldsymbol{v},\boldsymbol{Y},\boldsymbol{B})\exp(-\frac{1}{2}\|\boldsymbol{v}\|^{2})d\boldsymbol{v}}{\sum_{\boldsymbol{x}\in\{\pm1\}^{n}}\int_{\mathbb{R}^{p}}2^{-n}\mathcal{E}(\boldsymbol{x},\boldsymbol{v},\boldsymbol{Y},\boldsymbol{B})\exp(-\frac{1}{2}\|\boldsymbol{v}\|^{2})d\boldsymbol{v}}\right\} \\ &= n\log2 + \mathbb{E}\log\left(\int_{\mathbb{R}^{p}}\exp\left(-\frac{1}{2}\left\|\boldsymbol{B}-\sqrt{\frac{\mu}{n}}\boldsymbol{v}(\boldsymbol{x}^{*})^{\top}\right\|_{F}^{2}\right)\exp\left(-\frac{\|\boldsymbol{v}\|^{2}}{2}\right)d\boldsymbol{v}\right) \\ &- \mathbb{E}\log\left\{\sum_{\boldsymbol{x}\in\{\pm1\}^{n}}\int_{\mathbb{R}^{p}}\exp\left(\sum_{i=1}^{m}\left[\sum_{k$$

Furthermore, if we note that

$$\sum_{k < l} (x_k x_l - x_k^* x_l^*)^2 = n(n-1) - 2 \sum_{k < l} x_k x_l x_k^* x_l^*,$$

then we easily get

$$\begin{split} I(\boldsymbol{x}^*; \boldsymbol{Y}, \boldsymbol{B}) &= n \log 2 + \frac{n-1}{2} \sum_{i=1}^m \lambda^{(i)} \\ &+ \mathbb{E} \log \left(\int_{\mathbb{R}^p} \exp\left(-\frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v}(\boldsymbol{x}^*)^\top \right\|_F^2 \right) \exp\left(-\frac{\|\boldsymbol{v}\|^2}{2} \right) d\boldsymbol{v} \right) \\ &- \mathbb{E}[\phi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{W}, \boldsymbol{\lambda}, \mu, n, p)]. \end{split}$$

A.2 Proof of Lemma 3.2

A careful inspection of the expression of $I(x^*; Y, B)$ in Lemma 3.1 shows that the mutual information depends on $\{\lambda^{(i)}\}$ and $\{Z^{(i)}\}$ only through

$$\lambda = \sum_{i=1}^{m} \lambda^{(i)} \quad \text{and} \quad \sum_{i=1}^{m} \sqrt{\frac{\lambda^{(i)}}{n}} Z_{kl}^{(i)}, \quad \text{for all } k < l.$$

The proof is simply completed by noting that

$$\sum_{i=1}^{m} \sqrt{\frac{\lambda^{(i)}}{n}} Z_{kl}^{(i)} \stackrel{d}{=} \sqrt{\frac{\lambda}{n}} Z_{kl}, \quad \text{for all } k < l.$$

A.3 Proof of Lemma 3.3

Let us define

$$\mathcal{F}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{G}, \boldsymbol{B}) = \left[\prod_{i=1}^{m} \prod_{k < l} (\bar{p}_{n}^{(i)} + \Delta_{n}^{(i)} x_{k} x_{l})^{G_{kl}^{(i)}} (1 - \bar{p}_{n}^{(i)} - \Delta_{n}^{(i)} x_{k} x_{l})^{1 - G_{kl}^{(i)}} \right] \\ \exp\left(-\frac{1}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v} \boldsymbol{x}^{\top} \right\|_{F}^{2} \right).$$

Then from the definition of mutual information we have

$$\begin{split} I(\boldsymbol{x}^*;\boldsymbol{G},\boldsymbol{B}) &= \mathbb{E}\log\left\{\frac{\int_{\mathbb{R}^p} \mathcal{F}(\boldsymbol{x}^*,\boldsymbol{v},\boldsymbol{G},\boldsymbol{B})\exp(-\frac{1}{2}\|\boldsymbol{v}\|^2)d\boldsymbol{v}}{\sum_{\boldsymbol{x}\in\{\pm1\}^n}\int_{\mathbb{R}^p} 2^{-n}\mathcal{F}(\boldsymbol{x},\boldsymbol{v},\boldsymbol{G},\boldsymbol{B})\exp(-\frac{1}{2}\|\boldsymbol{v}\|^2)d\boldsymbol{v}}\right\} \\ &= n\log 2 + \mathbb{E}\log\left(\int_{\mathbb{R}^p}\exp\left(-\frac{1}{2}\left\|\boldsymbol{B}-\sqrt{\frac{\mu}{n}}\boldsymbol{v}(\boldsymbol{x}^*)^{\top}\right\|_F^2\right)\exp\left(-\frac{\|\boldsymbol{v}\|^2}{2}\right)d\boldsymbol{v}\right) \\ &- \mathbb{E}\log\left\{\sum_{\boldsymbol{x}\in\{\pm1\}^n}\int_{\mathbb{R}^p}\exp\left(\sum_{i=1}^m\sum_{\boldsymbol{k}$$

A.4 Proof of Lemma 3.4

We begin by noting that if $x \in \{\pm 1\}$ then

$$\log (c + dx) = \frac{1}{2} \log (c + d)(c - d) + \frac{x}{2} \log \frac{c + d}{c - d}.$$

Now from (3.3) we get

$$\begin{aligned} \mathcal{H}_{SBM}'(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{G}, \boldsymbol{\lambda}, n) &= \sum_{k < l} (x_k x_l - x_k^* x_l^*) \left[\sum_{i=1}^m \frac{G_{kl}^{(i)}}{2} \log \left(\frac{1 + \Delta_n^{(i)} / \bar{p}_n^{(i)}}{1 - \Delta_n^{(i)} / \bar{p}_n^{(i)}} \right) \right] \\ &+ \sum_{k < l} (x_k x_l - x_k^* x_l^*) \left[\sum_{i=1}^m \frac{1 - G_{kl}^{(i)}}{2} \log \left(\frac{1 - \Delta_n^{(i)} / \left(1 - \bar{p}_n^{(i)} \right)}{1 + \Delta_n^{(i)} / \left(1 - \bar{p}_n^{(i)} \right)} \right) \right]. \end{aligned}$$

For large values of n, there exists some sufficiently small $c_0 < \frac{1}{2}$ such that for all $i \in [m]$

$$\max\left(\frac{\Delta_n^{(i)}}{\overline{p}_n^{(i)}}, \frac{\Delta_n^{(i)}}{1 - \overline{p}_n^{(i)}}\right) \leqslant c_0.$$

Using Taylor approximation for $z \in [0, c_0]$ we have

$$\left|\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) - z\right| \leqslant z^3.$$

By triangle inequality

$$\mathcal{H}_{SBM}'(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{G}, \boldsymbol{\lambda}, n) = \sum_{i=1}^m \left[\sum_{k < l} \left((x_k x_l - x_k^* x_l^*) \left(\frac{\Delta_n^{(i)} G_{kl}^{(i)}}{\bar{p}_n^{(i)}} - \frac{\Delta_n^{(i)} (1 - G_{kl}^{(i)})}{1 - \bar{p}_n^{(i)}} \right) \right) + \operatorname{err}_n^{(i)} \right]$$

where $\operatorname{err}_{n}^{(i)}$ satisfies, for all $i \in [m]$

$$\operatorname{err}_{n}^{(i)} \bigg| \leq C_{1} \left(\frac{\Delta_{n}^{(i)}}{\overline{p}_{n}^{(i)}} \right)^{3} \left(\left| \boldsymbol{x}^{\top} \boldsymbol{G}^{(i)} \boldsymbol{x} \right| + \left| (\boldsymbol{x}^{*})^{\top} \boldsymbol{G}^{(i)} \boldsymbol{x}^{*} \right| \right) + C_{2} \left(\frac{\Delta_{n}^{(i)}}{1 - \overline{p}_{n}^{(i)}} \right)^{3} \left(\left| \boldsymbol{x}^{\top} (\boldsymbol{1} \boldsymbol{1}^{\top} - \boldsymbol{G}^{(i)}) \boldsymbol{x} \right| + \left| (\boldsymbol{x}^{*})^{\top} (\boldsymbol{1} \boldsymbol{1}^{\top} - \boldsymbol{G}^{(i)}) \boldsymbol{x}^{*} \right| \right)$$
(A.1)

with \mathcal{C}_1 and \mathcal{C}_2 absolute positive constants. Furthermore

$$\sum_{k
= $\sum_{k
= $-\frac{n-1}{2} \lambda^{(i)} + \sum_{k$$$$

This implies

$$\mathcal{H}_{SBM}'(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{G}, \boldsymbol{\lambda}, n) = -\frac{n-1}{2} \sum_{i=1}^m \lambda^{(i)} + \mathcal{H}'(\boldsymbol{x}, \boldsymbol{x}^*, \widetilde{\boldsymbol{G}}, \boldsymbol{\lambda}, n) + \sum_{i=1}^m \operatorname{err}_n^{(i)}.$$

where $\operatorname{err}_{n}^{(i)}$ satisfies (A.1) for all $i \in [m]$. Using (3.2) we get

$$\mathcal{H}_{SBM}(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{G}, \boldsymbol{B}, \boldsymbol{\lambda}, \mu, n, p)) = -\frac{n-1}{2} \sum_{i=1}^m \lambda^{(i)} + \mathcal{H}'(\boldsymbol{x}, \boldsymbol{x}^*, \widetilde{\boldsymbol{G}}, \boldsymbol{\lambda}, n) - \frac{p}{2} \left\| \boldsymbol{B} - \sqrt{\frac{\mu}{n}} \boldsymbol{v} \boldsymbol{x}^\top \right\|_F^2 + \sum_{i=1}^m \operatorname{err}_n^{(i)}.$$

Furthermore, by Remark 5.4 of [15] and (3.4), we obtain the following.

$$\begin{split} & \mathbb{E}\psi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{G}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p) \\ &= -\frac{n-1}{2} \sum_{i=1}^m \lambda^{(i)} + \mathbb{E}\log\left\{\sum_{\boldsymbol{x} \in \{\pm 1\}^n} \int_{\mathbb{R}^p} \exp(\mathcal{H}(\boldsymbol{x}, \boldsymbol{x}^*, \boldsymbol{\lambda}, \boldsymbol{v}, \tilde{\boldsymbol{G}}, \boldsymbol{B}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p)) \exp\left(-\frac{\|\boldsymbol{v}\|^2}{2}\right) d\boldsymbol{v}\right\} \\ & + O\left(\sum_{i=1}^m \frac{n\left(\lambda^{(i)}\right)^{3/2}}{\sqrt{n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})}}\right) \end{split}$$

We complete the proof by applying Lemma 3.3.

A.5 Proof of Lemma 3.5

We begin by observing that

$$\mathbb{E}\left[\widetilde{G}_{kl}^{(i)} \middle| \, \boldsymbol{x}^*\right] = 0 = \mathbb{E}\left[Z_{kl}^{(i)} \sqrt{\frac{\lambda^{(i)}}{n}} \, \middle| \, \boldsymbol{x}^*\right].$$
(A.2)

Following the arguments of Lemma 5.5 of [15], we obtain

$$\begin{aligned} \left| \mathbb{E} \Big[(\tilde{G}_{kl}^{(i)})^2 - (Z_{kl}^{(i)})^2 \frac{\lambda^{(i)}}{n} \, \Big| \, \boldsymbol{x}^* \Big] \right| \\ &= \left| \frac{(\Delta_n^{(i)})^2}{(\bar{p}_n^{(i)})^2 (1 - \bar{p}_n^{(i)})^2} (\bar{p}_n^{(i)} + \Delta_n^{(i)} x_i^* x_j^*) (1 - \bar{p}_n^{(i)} + \Delta_n^{(i)} x_i^* x_j^*) - \frac{\lambda^{(i)}}{n} \right| \\ &\leqslant \frac{\lambda^{(i)}}{n} \left(\sqrt{\frac{\lambda^{(i)}}{n \bar{p}_n^{(i)} (1 - \bar{p}_n^{(i)})}} + \frac{\lambda^{(i)}}{n} \right), \end{aligned}$$
(A.3)

and

$$\left| \mathbb{E} \left[(\tilde{G}_{kl}^{(i)})^3 \,|\, \boldsymbol{x}^* \right] \right| \leqslant \frac{3(\lambda^{(i)})^{3/2}}{n^{3/2} \sqrt{\bar{p}_n^{(i)}(1 - \bar{p}_n^{(i)})}} \,, \tag{A.4}$$

$$\mathbb{E}\left[(\widetilde{G}_{kl}^{(i)})^4 \,|\, \boldsymbol{x}^* \right] \leqslant \frac{2(\lambda^{(i)})^2}{n^2 \overline{p}_n^{(i)} (1 - \overline{p}_n^{(i)})} \,. \tag{A.5}$$

Now we shall use Lindeberg's generalization theorem (Theorem 5.6 of [15]) to establish the desired result. We consider the collections of random variables $\{\widetilde{G}_{kl}^{(i)}\}$ and $\{Z_{kl}^{(i)}\sqrt{\lambda^{(i)}/n}\}$. To this end, we regard the function $\phi(\boldsymbol{x}^*, \boldsymbol{B}, \boldsymbol{V}, \boldsymbol{\lambda}, \mu, n, p)$ as a function of $\boldsymbol{V} = (\boldsymbol{V}^{(1)}, \boldsymbol{V}^{(2)}, ..., \boldsymbol{V}^{(m)})$. Define

$$\partial_{k,l;i}^r \phi := rac{\partial^r}{\partial (V_{kl}^{(i)})^r} \phi(oldsymbol{x}^*, oldsymbol{B}, oldsymbol{V}, oldsymbol{\lambda}, \mu, n, p).$$

Let us consider the measure on $\{\pm 1\}^n$ given by

$$\boldsymbol{m}(\boldsymbol{x}) = \frac{\int_{\mathbb{R}^p} \exp(\mathcal{H}(\boldsymbol{v}, \boldsymbol{x}^*, \boldsymbol{v}, \boldsymbol{W}, \boldsymbol{B}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p)) \exp(-\|\boldsymbol{v}\|^2/2) d\boldsymbol{v}}{\sum_{\boldsymbol{x} \in \{\pm 1\}^n} \int_{\mathbb{R}^p} \exp(\mathcal{H}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{V}, \boldsymbol{B}, \boldsymbol{\lambda}, \boldsymbol{\mu}, n, p)) \exp(-\|\boldsymbol{v}\|^2/2) d\boldsymbol{v}}, \quad \boldsymbol{x} \in \{\pm 1\}^n.$$

Then it is easy to verify using induction that for all i, k, l, and any r > 1

$$\partial_{k,l;i}^{1}\phi = \mathbb{E}_{\boldsymbol{m}}\left[x_{k}x_{l} - x_{k}^{*}x_{l}^{*}\right],\\ \partial_{k,l;i}^{r}\phi = \mathbb{E}_{\boldsymbol{m}}\left[(x_{k}x_{l} - x_{k}^{*}x_{l}^{*}) - \mathbb{E}_{\boldsymbol{m}}\left[x_{k}x_{l} - x_{k}^{*}x_{l}^{*}\right]\right]^{r}.$$

We note that in this case also the expressions of the partial derivatives are equivalent to the polynomial representations p_r mentioned in Lemma 5.5 of [15]. Since $|x_k x_l - x_k^* x_l^*| \leq 2$ for all i, k, l there exists a constant C such that

$$\left|\partial_{k,l;i}^{r}\phi\right| \leq C,$$

for all $r \leq 4$. Then, using Theorem 5.6 of [15] and (A.2), (A.3), (A.4) and (A.5) we get

$$\mathbb{E}\phi(\boldsymbol{x}^*,\boldsymbol{B},\tilde{\boldsymbol{G}},\boldsymbol{\lambda},\mu,n,p) = \mathbb{E}\phi(\boldsymbol{x}^*,\boldsymbol{B},\boldsymbol{W},\boldsymbol{\lambda},\mu,n,p) + O\left(\sum_{i=1}^m \frac{n(\lambda^{(i)})^{3/2}}{\sqrt{n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})}}\right).$$

B Proofs of Results in Section 4

We start with some definitions. For $i \in [m]$, we let $E(\mathbf{G}^{(i)})$ denote all $\binom{n}{2}$ unordered pairs of nodes in the *i*th graph. For a pair of vertices e = (k, l), consider the following random variables.

$$x_e := x_k^* x_l^*$$
 and $\ell_e^{(i)} := L_{k,l}^{(i)}$,

where $L_{k,l}^{(i)} = 2G_{k,l}^{(i)} - 1$. Further, we define

$$\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e) := \begin{cases} \bar{p}_n^{(i)} + x_e \Delta_n^{(i)} & \text{if } \ell_e^{(i)} = +1\\ 1 - (\bar{p}_n^{(i)} + x_e \Delta_n^{(i)}) & \text{if } \ell_e^{(i)} = -1. \end{cases}$$
(B.1)

and

$$p(y_e) = \mathbb{P}_{x_e} \left(x_e = y_e \right). \tag{B.2}$$

By definitions in Section 2.1

$$\Delta_n^{(i)} = \sqrt{\frac{\lambda r^{(i)} \overline{p}_n^{(i)} (1 - \overline{p}_n^{(i)})}{n}},$$

then it is easy to show that

$$\frac{d\pi^{(i)}(\lambda;\ell_e^{(i)},x_e)}{d\lambda} = \frac{1}{2}\sqrt{\frac{\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})r^{(i)}}{n\lambda}} x_e \ell_e^{(i)}.$$

By definition, it is immediate that $\mathsf{MMSE}_n(\lambda, \mu)$ defined in (2.9) is a function of λ and μ only. For the purpose of this lemma, we consider $\mu \in (0, \infty)$ to be a fixed number and study the function as a function of λ only. We note that

$$H(\boldsymbol{x}^*|\boldsymbol{G},\boldsymbol{B}) = -\mathbb{E}_{\boldsymbol{x}^*,\boldsymbol{G},\boldsymbol{B}}\left[\log \pi\left(\boldsymbol{x}^*|\boldsymbol{G},\boldsymbol{B}\right)\right],$$

where $\pi(x^*|G, B)$ is the posterior density of x^* given G and B. By the definition of mutual information,

$$I(\boldsymbol{x}^*;\boldsymbol{G},\boldsymbol{B}) = H(\boldsymbol{x}^*) - H(\boldsymbol{x}^*|\boldsymbol{G},\boldsymbol{B}),$$

where $H(\mathbf{x}^*)$ is the entropy of \mathbf{x}^* . It is easy to observe that $H(\mathbf{x}^*)$ equals $n \log 2$ as \mathbf{x}^* is a n vector made of i.i.d Rademacher random variables. Therefore

$$rac{dI\left(oldsymbol{x}^{*};oldsymbol{G},oldsymbol{B}
ight)}{d\lambda}=-rac{dH(oldsymbol{x}^{*}|oldsymbol{G},oldsymbol{B})}{d\lambda}.$$

By (4.1) we have

$$H(\boldsymbol{x}^*|\boldsymbol{G},\boldsymbol{B}) = H(\boldsymbol{x}^*|\boldsymbol{\mathcal{L}},\boldsymbol{B}),$$

where $H(\mathbf{x}^*|\mathcal{L}, \mathbf{B})$ is the conditional entropy of \mathbf{x}^* given \mathcal{L} and \mathbf{B} . Finally, for e = (k, l) we define

$$\mathcal{L}_{-e}^{(i)} = \mathcal{L} \setminus \{L_{k,l}^{(i)}\}.$$

B.1 Proof of Lemma 4.1

Let us define $\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B}) = \mathbb{E}[x_e | \mathcal{L}_{-e}^{(i)}, \mathbf{B}]$ and

$$\mathcal{P}_{e}^{(i)}(\mathcal{L}^{(i)}, \boldsymbol{B}, y_{e}, \ell_{e}^{(i)}) = \mathbb{P}(x_{e} = y_{e} | \mathcal{L}_{-e}^{(i)}, \boldsymbol{B}) \log\left(\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; \ell_{e}^{(i)}, x) p(x | \mathcal{L}_{-kl}^{(i)}, \boldsymbol{B})\right).$$
(B.3)

Using Lemma B.1 below in Section B.2, we have

$$\begin{split} \frac{dH(\boldsymbol{x}^*|\mathcal{L},\boldsymbol{B})}{d\lambda} \\ &= \frac{1}{2} \sum_{i=1}^m \sqrt{\frac{\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})r^{(i)}}{n\lambda}} \sum_{e \in E(G^{(i)})} \sum_{\substack{y_e^{(i)} \in \{\pm 1\} \\ \ell_e^{(i)} \in \{\pm 1\}}} \ell_e^{(i)} y_e^{(i)} \mathbb{E}_{\mathcal{L}_{-e}^{(i)},\boldsymbol{B}} \left[\mathcal{P}_e^{(i)}(\mathcal{L}^{(i)},\boldsymbol{B}, y_e^{(i)}, \ell_e^{(i)}) \right] \\ &- \frac{1}{2} \sum_{i=1}^m \sqrt{\frac{\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})r^{(i)}}{n\lambda}} \sum_{e \in E(G^{(i)})} \sum_{\substack{y_e^{(i)} \in \{\pm 1\} \\ \ell_e^{(i)} \in \{\pm 1\}}} \ell_e^{(i)} y_e^{(i)} p(y_e^{(i)}) \log \pi^{(i)}(\lambda; \ell_e^{(i)}, y_e) \\ &= \frac{1}{2} \sum_{i=1}^m \sqrt{\frac{\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})r^{(i)}}{n\lambda}} \sum_{e \in E(G^{(i)})} \mathbb{E}_{\mathcal{L}_{-e}^{(i)},\boldsymbol{B}} \left\{ \hat{x}_e(\mathcal{L}_{-e}^{(i)},\boldsymbol{B}) \log \left[\frac{\sum_x \pi^{(i)}(\lambda; 1,x)p(x|\mathcal{L}_{-e}^{(i)},\boldsymbol{B})}{\sum_x \pi^{(i)}(\lambda; -1,x)p(x|\mathcal{L}_{-e}^{(i)},\boldsymbol{B})} \right] \right\} \\ &- \frac{1}{4} \sum_{i=1}^m \sqrt{\frac{\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})r^{(i)}}{n\lambda}} \binom{n}{2} \log \left\{ \frac{\pi^{(i)}(\lambda; -1, -1)\pi^{(i)}(\lambda; +1, +1)}{\pi^{(i)}(\lambda; +1, -1)\pi^{(i)}(\lambda; -1, +1)} \right\}. \end{split}$$

Next we observe

$$\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; 1, x) p(x | \mathcal{L}_{-e}^{(i)}, \mathbf{B}) = p_n^{(i)} p(1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B}) + q_n^{(i)} p(-1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B})$$

$$= p_n^{(i)} p(1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B}) + q_n^{(i)} [1 - p(1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B})]$$

$$= q_n^{(i)} + (p_n^{(i)} - q_n^{(i)}) p(1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B}).$$
(B.4)

We also have

$$\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; 1, x) p(x | \mathcal{L}_{-e}^{(i)}, \mathbf{B}) = p_n^{(i)} [1 - p(-1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B})] + q_n^{(i)} p(-1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B})$$

$$= p_n^{(i)} - (p_n^{(i)} - q_n^{(i)}) p(-1 | \mathcal{L}_{-e}^{(i)}, \mathbf{B}).$$
(B.5)

Combining (B.4) and (B.5) we get

$$2\sum_{x\in\{\pm1\}}\pi^{(i)}(\lambda;1,x)p(x|\mathcal{L}_{-e}^{(i)},\boldsymbol{B}) = (p_n^{(i)} + q_n^{(i)}) + \hat{x}_e(\mathcal{L}_{-e}^{(i)},\boldsymbol{B})(p_n^{(i)} - q_n^{(i)}).$$
(B.6)

By similar calculations, we also get

$$2\sum_{x\in\{\pm1\}}\pi^{(i)}(\lambda;-1,x)p(x|\mathcal{L}_{-e}^{(i)},\boldsymbol{B}) = (2-p_n^{(i)}-q_n^{(i)}) - \hat{x}_e(\mathcal{L}_{-e}^{(i)},\boldsymbol{B})(p_n^{(i)}-q_n^{(i)}).$$
(B.7)

From (B.6) and (B.7) we get

$$\frac{\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; 1, x) p(x | \mathcal{L}_{-e}^{(i)}, \mathbf{B})}{\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; -1, x) p(x | \mathcal{L}_{-e}^{(i)}, \mathbf{B})} = \frac{(p_n^{(i)} + q_n^{(i)}) + \hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})(p_n^{(i)} - q_n^{(i)})}{(2 - p_n^{(i)} - q_n^{(i)}) - \hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})(p_n^{(i)} - q_n^{(i)})} = \frac{\bar{p}_n^{(i)}}{1 - \bar{p}_n^{(i)}} \left[\frac{1 + (\Delta_n^{(i)} / \bar{p}_n^{(i)}) \hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})}{1 - (\Delta_n^{(i)} / (1 - \bar{p}_n^{(i)})) \hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})} \right].$$

In addition, it is easy to observe that

$$\frac{\pi^{(i)}(\lambda;-1,-1)\pi^{(i)}(\lambda;+1,+1)}{\pi^{(i)}(\lambda;+1,-1)\pi^{(i)}(\lambda;-1,+1)} = \frac{(1+(\Delta_n^{(i)}/\bar{p}_n^{(i)}))(1+(\Delta_n^{(i)}/(1-\bar{p}_n^{(i)})))}{(1-(\Delta_n^{(i)}/\bar{p}_n^{(i)}))(1-(\Delta_n^{(i)}/(1-\bar{p}_n^{(i)})))}$$

Since we have $|\Delta_n^{(i)}/\bar{p}_n^{(i)}|, |\Delta_n^{(i)}/(1-\bar{p}_n^{(i)})| \leq \sqrt{\lambda r^{(i)}/(n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)}))} \to 0$, and $|\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})| \leq 1$, for each $\lambda_{max} \in \mathbb{R}$ there exists a $n_0(\lambda_{max})$ such that for $n \geq n_0(\lambda_{max})$ we have for all $1 \leq i \leq m$ and $e \in E(G^{(i)})$

$$\left| \log \left[\frac{\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; 1, x) p(x | \mathcal{L}_{-e}^{(i)}, \mathbf{B})}{\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; -1, x) p(x | \mathcal{L}_{-e}^{(i)}, \mathbf{B})} \right] - B_0^{(i)} - \frac{\Delta_n^{(i)} \hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})}{\bar{p}_n^{(i)}(1 - \bar{p}_n^{(i)})} \right| \leq C_1 \frac{\lambda r^{(i)}}{n \bar{p}_n^{(i)}(1 - \bar{p}_n^{(i)})}$$

and

$$\log\left[\frac{\pi^{(i)}(\lambda;-1,-1)\pi^{(i)}(\lambda;+1,+1)}{\pi^{(i)}(\lambda;+1,-1)\pi^{(i)}(\lambda;-1,+1)}\right] - \frac{2\Delta_n^{(i)}}{\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})} \leqslant C_2 \frac{\lambda r^{(i)}}{n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})},$$

where $B_0^{(i)} = \log(\bar{p}_n^{(i)}/(1-\bar{p}_n^{(i)}))$ and C_1, C_2 are positive constants depending on λ_{max} . We observe that $\mathbb{E}[\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})] = \mathbb{E}[x_e] = 0$. This implies

$$\frac{1}{n} \frac{dH(\boldsymbol{x}^* | \mathcal{L}, \boldsymbol{B})}{d\lambda} + \frac{1}{2n^2} \sum_{i=1}^m \sum_{e \in E(G^{(i)})} r^{(i)} \left(1 - \mathbb{E}[\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \boldsymbol{B})]^2 \right) \leqslant C_2 \sum_{i=1}^m \sqrt{\frac{\lambda r^{(i)}}{n \bar{p}_n^{(i)} (1 - \bar{p}_n^{(i)})}}.$$

Let $\hat{x}_e(\mathcal{L}, \mathbf{B}) = \mathbb{E}[x_e | \mathcal{L}, \mathbf{B}]$. Recall that $L_e^{(i)} = L_{k,l}^{(i)}$ for e = (k, l), then by Bayes' formula we have

$$p^{(i)}(y_e|\mathcal{L}, \mathbf{B}) = \frac{p^{(i)}(L_e^{(i)}, \mathcal{L}_{-e}^{(i)}, \mathbf{B}, x_e = y_e)}{\sum_{x \in \{\pm 1\}} p^{(i)}(L_e^{(i)}, \mathcal{L}_{-kl}^{(i)}, \mathbf{B}, x_e = x)}$$
$$\stackrel{(1)}{=} \frac{\pi^{(i)}(\lambda; L_e^{(i)}, y_e) p^{(i)}(y_e|\mathcal{L}_{-e}^{(i)}, \mathbf{B})}{\sum_{x \in \{\pm 1\}} \pi^{(i)}(\lambda; L_e^{(i)}, x) p^{(i)}(x|\mathcal{L}_{-e}^{(i)}, \mathbf{B})}.$$

Here, equality (1) follows as conditional on x_e , $L_e^{(i)}$ is independent of $\mathcal{L}_{-e}^{(i)}$. Let us define

$$b^{(i)}(L_e^{(i)}) = \frac{\pi^{(i)}(\lambda; L_e^{(i)}, +1) - \pi^{(i)}(\lambda; L_e^{(i)}, -1)}{\pi^{(i)}(\lambda; L_e^{(i)}, +1) + \pi^{(i)}(\lambda; L_e^{(i)}, -1)}.$$

We note that

$$\frac{\widehat{x}_{e}(\mathcal{L}_{-e}^{(i)}, \mathbf{B}) + b^{(i)}(L_{e}^{(i)})}{2} = \frac{\pi^{(i)}(\lambda; L_{e}^{(i)}, +1)p^{(i)}(+1|\mathcal{L}_{-e}^{(i)}, \mathbf{B}) - \pi^{(i)}(\lambda; L_{e}^{(i)}, -1)p^{(i)}(-1|\mathcal{L}_{-e}^{(i)}, \mathbf{B})}{\pi^{(i)}(\lambda; L_{e}^{(i)}, +1) + \pi^{(i)}(\lambda; L_{e}^{(i)}, -1)}$$
(B.8)

and

$$\frac{1+b^{(i)}(L_e^{(i)})\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})}{2} = \frac{\pi^{(i)}(\lambda; L_e^{(i)}, +1)p^{(i)}(+1|\mathcal{L}_{-e}^{(i)}, \mathbf{B}) + \pi^{(i)}(\lambda; L_e^{(i)}, -1)p^{(i)}(-1|\mathcal{L}_{-e}^{(i)}, \mathbf{B})}{\pi^{(i)}(\lambda; L_e^{(i)}, +1) + \pi^{(i)}(\lambda; L_e^{(i)}, -1)}$$
(B.9)

Then from (B.8) and (B.9) we get

$$\hat{x}_e(\mathcal{L}, \mathbf{B}) = \frac{\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B}) + b^{(i)}(L_e^{(i)})}{1 + b^{(i)}(L_e^{(i)})\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \mathbf{B})}$$

From the definition of $\pi^{(i)}(\lambda; L_e^{(i)}, x_e)$ it follows that

$$b^{(i)}(L_e^{(i)}) = \begin{cases} \sqrt{(1 - \bar{p}_n^{(i)})\lambda r^{(i)}/(n\bar{p}_n^{(i)})} & \text{if } L_e^{(i)} = 1, \\ -\sqrt{\bar{p}_n^{(i)}\lambda r^{(i)}/(n(1 - \bar{p}_n^{(i)}))} & \text{if } L_e^{(i)} = -1. \end{cases}$$

This in particular gives us $|b^{(i)}(L_e^{(i)})| \leq \sqrt{\lambda r^{(i)}/(n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)}))} \rightarrow 0$. In addition, $|\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \boldsymbol{B})| \leq 1$. Thus

$$\begin{aligned} \left| \hat{x}_{e}(\mathcal{L}, \mathbf{B}) - \hat{x}_{e}(\mathcal{L}_{-e}^{(i)}, \mathbf{B}) \right| &= \left| \frac{b^{(i)}(L_{e}^{(i)})(1 - \hat{x}_{e}(\mathcal{L}_{-e}^{(i)}, \mathbf{B})^{2})}{1 + b^{(i)}(L_{e}^{(i)}) \hat{x}_{e}(\mathcal{L}_{-e}^{(i)}, \mathbf{B})^{2}} \right| \\ &\leq |b^{(i)}(L_{e}^{(i)})| \leq \sqrt{\frac{\lambda r^{(i)}}{n\bar{p}_{n}^{(i)}(1 - \bar{p}_{n}^{(i)})}} \,. \end{aligned}$$
(B.10)

Using $|\hat{x}_e(\mathcal{L}_{-e}^{(i)}, \boldsymbol{B})| \leq 1$ and (B.10) we get

$$\left|\frac{1}{n}\frac{dH(\boldsymbol{x}^*|\mathcal{L},\boldsymbol{B})}{d\lambda} + \frac{1}{2n^2}\sum_{i=1}^m\sum_{e\in E(G^{(i)})}r^{(i)}\left(1 - \mathbb{E}\left[\hat{x}_e(\mathcal{L},\boldsymbol{B})\right]^2\right)\right| \leqslant C_2\sum_{i=1}^m\left(\sqrt{\frac{\lambda r^{(i)}}{n\bar{p}_n^{(i)}(1-\bar{p}_n^{(i)})}}\right)$$

It is easy to observe that for e = (k, l)

$$\left(1 - \mathbb{E}\left[\hat{x}_e(\mathcal{L}, \boldsymbol{B})\right]^2\right) = \mathbb{E}\left[\left(\boldsymbol{x}_k^* \boldsymbol{x}_l^* - \mathbb{E}(\boldsymbol{x}_k^* \boldsymbol{x}_l^* | \mathcal{L}, \boldsymbol{B})\right)^2\right]$$

Then using $\sum_{i=1}^{m} r^{(i)} = 1$ we get

$$\left|\frac{1}{n} \frac{dH(\boldsymbol{x}^* | \mathcal{L}, \boldsymbol{B})}{d\lambda} + \frac{1}{4} \mathsf{MMSE}_n(\lambda, \mu)\right| \leq C_2 \left(\sum_{i=1}^m \sqrt{\frac{\lambda r^{(i)}}{n \overline{p}_n^{(i)}(1 - \overline{p}_n^{(i)})}}\right).$$

As $H(\boldsymbol{x}^*) = n \log 2$ we get using definition of conditional entropy and mutual information

$$\frac{1}{n} \frac{dI(\boldsymbol{x}^*; \mathcal{L}, \boldsymbol{B})}{d\lambda} - \frac{1}{4} \mathsf{MMSE}_n(\lambda, \mu) \bigg| \leqslant C_2 \left(\sum_{i=1}^m \sqrt{\frac{\lambda r^{(i)}}{n \overline{p}_n^{(i)} (1 - \overline{p}_n^{(i)})}} \right),$$

which implies the lemma.

B.2 Results Used to Prove Lemma 4.1

Lemma B.1. Let $\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e)$, $p(y_e^{(i)})$, $\mathcal{P}_e^{(i)}(\mathcal{L}^{(i)}, \boldsymbol{B}, y_e^{(i)}, \ell_e^{(i)})$ be defined in (B.1), (B.2) and (B.3) respectively, then

$$\begin{split} \frac{dH(\boldsymbol{x}^* | \mathcal{L}^{(i)}, \boldsymbol{B})}{d\lambda} &= -\sum_{i=1}^m \sum_{e \in E(G^{(i)})} \sum_{\substack{y_e^{(i)} \in \{\pm 1\}\\ \ell_e^{(i)} \in \{\pm 1\}}} p(y_e^{(i)}) \log \pi^{(i)}(\lambda; \ell_e^{(i)}, y_e^{(i)}) \frac{d\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e)}{d\lambda}}{d\lambda} \\ &+ \sum_{i=1}^m \sum_{\substack{e \in E(G^{(i)})\\ \ell_e \in \{\pm 1\}}} \sum_{\substack{y_e^{(i)} \in \{\pm 1\}\\ \ell_e \in \{\pm 1\}}} \frac{d\pi^{(i)}(\lambda; \ell_e^{(i)}, y_e^{(i)})}{d\lambda} \mathbb{E}_{\mathcal{L}_{-e}^{(i)}, \boldsymbol{B}} \left[\mathcal{P}_e^{(i)}(\mathcal{L}^{(i)}, \boldsymbol{B}, y_e^{(i)}, \ell_e^{(i)}) \right] \end{split}$$

Proof. We begin by observing that $H(\mathbf{x}^*|\mathcal{L}^{(i)}, \mathbf{B})$ is a function of λ through $\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e)$ for $e \in E(G^{(i)})$ and $1 \leq i \leq m$.

By the chain rule and linearity of differentiation, it suffices to assume that only $\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e)$ depends on λ . Then, for e = (k, l), we have

$$H(\boldsymbol{x}^*|\mathcal{L}, \boldsymbol{B}) + H(L_e^{(i)}|\mathcal{L}_{-e}^{(i)}, \boldsymbol{B}) = H(\boldsymbol{x}^*; L_e^{(i)}|\mathcal{L}_{-e}^{(i)}, \boldsymbol{B})$$

$$\stackrel{(1)}{=} H(\boldsymbol{x}^*|\mathcal{L}_{-e}^{(i)}, \boldsymbol{B}) + H(L_e^{(i)}|\boldsymbol{x}^*, \mathcal{L}_{-e}^{(i)}, \boldsymbol{B})$$

$$\stackrel{(2)}{=} H(\boldsymbol{x}^*|\mathcal{L}_{-e}^{(i)}, \boldsymbol{B}) + H(L_e^{(i)}|\boldsymbol{x}_e).$$

Here, equality (1) follows by writing the entropy in two different forms using the chain rule, and equality (2) follows from observing that given x_e , $L_e^{(i)}$ is independent of everything else. This implies

$$\frac{dH(\boldsymbol{x}^*|\mathcal{L},\boldsymbol{B})}{d\lambda} = \frac{dH(L_e^{(i)}|x_e)}{d\lambda} - \frac{dH(L_e^{(i)}|\mathcal{L}_{-e}^{(i)},\boldsymbol{B})}{d\lambda},$$

because $H(\boldsymbol{x}^*|\mathcal{L}_{-e}^{(i)}, \boldsymbol{B})$ does not depend on $\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e)$, hence not on λ under the simplifying assumption that only $\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e)$ depends on λ .

Next let us observe

$$\frac{dH(L_{e}^{(i)}|x_{e})}{d\lambda} = -\frac{d}{d\lambda} \sum_{\ell_{e}^{(i)} \in \{\pm 1\}} \mathbb{E}_{\boldsymbol{x}^{*}} \left[\pi^{(i)}(\lambda; \ell_{e}^{(i)}, x_{e}) \log \pi^{(i)}(\lambda; \ell_{e}^{(i)}, x_{e}) \right] \\
= -\sum_{\ell_{e}^{(i)} \in \{\pm 1\}} \mathbb{E}_{\boldsymbol{x}^{*}} \left[\frac{d\pi^{(i)}(\lambda; \ell_{e}^{(i)}, x_{e})}{d\lambda} \log \pi^{(i)}(\lambda; \ell_{e}^{(i)}, x_{e}) + \frac{d\pi^{(i)}(\lambda; \ell_{e}^{(i)}, x_{e})}{d\lambda} \right] \\
= -\sum_{\ell_{e}^{(i)} \in \{\pm 1\}} \mathbb{E}_{\boldsymbol{x}^{*}} \left[\frac{d\pi^{(i)}(\lambda; \ell_{e}^{(i)}, x_{e})}{d\lambda} \log \pi^{(i)}(\lambda; \ell_{e}^{(i)}, x_{e}) \right] \\
= -\sum_{\substack{y_{e}^{(i)} \in \{\pm 1\}\\\ell_{e}^{(i)} \in \{\pm 1\}}} \mathbb{P}_{x_{e}} \left(x_{e} = y_{e}^{(i)} \right) \frac{d\pi^{(i)}(\lambda; \ell_{e}^{(i)}, y_{e}^{(i)})}{d\lambda} \log \pi^{(i)}(\lambda; \ell_{e}^{(i)}, y_{e}^{(i)}).$$
(B.11)

Here the third equality holds since

$$\sum_{\substack{\ell_e^{(i)} \in \{\pm 1\}}} \frac{d\pi^{(i)}(\lambda; \ell_e^{(i)}, x_e)}{d\lambda} = 0, \qquad x_e \in \{\pm 1\}.$$

Next, we note that

$$H(L_e^{(i)}|\mathcal{L}_{-e}^{(i)}, \boldsymbol{B}) = -\sum_{\ell_e^{(i)} \in \{\pm 1\}} \mathbb{E}_{\mathcal{L}_{-e}^{(i)}, \boldsymbol{B}} \left[\sum_{y_e^{(i)} \in \{\pm 1\}} \pi^{(i)}(\lambda; \ell_e^{(i)}, y_e^{(i)}) \mathcal{P}_e^{(i)}(\mathcal{L}, \boldsymbol{B}, y_e^{(i)}, \ell_e^{(i)}) \right].$$

So we have

$$\frac{dH(L_{e}^{(i)}|\mathcal{L}_{-e}^{(i)}, \mathbf{B})}{d\lambda} = -\sum_{\substack{y_{e}^{(i)} \in \{\pm 1\} \\ \ell_{e}^{(i)} \in \{\pm 1\} \\ \ell_{e}^{(i)$$

Here, the second equality holds since

$$\begin{split} &\sum_{\substack{y_e^{(i)} \in \{\pm 1\} \\ \ell_e^{(i)} \in \{\pm 1\}}} \mathbb{E}_{\mathcal{L}_{-e}^{(i)}, B} \left[\frac{\pi^{(i)}(\lambda; \ell_e^{(i)}, y_e^{(i)}) p(y_e^{(i)} | \mathcal{L}_{-e}^{(i)}, B)}{p(\ell_e^{(i)} | \mathcal{L}_{-e}^{(i)}, B)} \sum_{x \in \{\pm 1\}} \frac{d\pi^{(i)}(\lambda; \ell_e^{(i)}, x)}{d\lambda} p(x | \mathcal{L}_{-e}^{(i)}, B) \right] \\ &= \sum_{\ell_e^{(i)} \in \{\pm 1\}} \mathbb{E}_{\mathcal{L}_{-e}^{(i)}, B} \left[\sum_{x \in \{\pm 1\}} \frac{d\pi^{(i)}(\lambda; \ell_e^{(i)}, x)}{d\lambda} p(x | \mathcal{L}_{-e}^{(i)}, B) \right] \\ &= \mathbb{E}_{\mathcal{L}_{-e}^{(i)}, B} \left[\frac{d}{d\lambda} \left(\sum_{x \in \{\pm 1\}} \sum_{\ell_e^{(i)} \in \{\pm 1\}} \pi^{(i)}(\lambda; \ell_e^{(i)}, x) p(x | \mathcal{L}_{-e}^{(i)}, B) \right) \right] \\ &= \mathbb{E}_{\mathcal{L}_{-e}^{(i)}, B} \left[\frac{d}{d\lambda} 1 \right] = 0. \end{split}$$

Combining (B.11) and (B.12), we complete the proof of the lemma.

C Proof of I-MMSE Identity in the Gaussian Model

Let us consider the vector t containing $\{T_{i,j}\}_{i < j}$ and the vector r containing $\{x_i^* x_j^*\}_{i < j}$. Then from the definition of T we have

$$oldsymbol{t} = \sqrt{rac{\lambda}{n}}oldsymbol{r} + oldsymbol{q},$$

where $\boldsymbol{q} \sim N_{C_2^n}(0, \boldsymbol{I}_{C_2^n})$ $(C_2^n = \binom{n}{2})$. As the diagonal entries of $\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top}$ are all 1, let us consider

$$oldsymbol{s} = \sqrt{rac{\lambda}{n}} + oldsymbol{q}_1,$$

where $\boldsymbol{q}_1 \sim N_n(0, 2\boldsymbol{I}_n)$. Let $\mathcal{Y} = (\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{B})$ and $\boldsymbol{B} = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_p)$ where $\boldsymbol{b}_i = (\boldsymbol{I} + \frac{\mu}{n} \boldsymbol{x}^* (\boldsymbol{x}^*)^\top)^{-1} \boldsymbol{\tilde{z}}_i$, where $\boldsymbol{\tilde{z}}_i \sim N_n(\boldsymbol{0}, \boldsymbol{I}_n)$. From the definition, it is clear that,

$$I(\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top}; \boldsymbol{T}, \boldsymbol{B}) = I(\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top}; \boldsymbol{\mathcal{Y}})$$

Next we have

$$I(\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top}; \mathcal{Y}) = H(\mathcal{Y}) - H(\mathcal{Y}|\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top}).$$

Note that

$$H(\mathcal{Y}|\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top}) = H(\boldsymbol{q}, \boldsymbol{q}_1, \{\widetilde{\boldsymbol{z}}_i\}_{i=1}^p)$$

Observe that the right hand side is free of λ and hence we get

$$\frac{d}{d\lambda}I(\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top};\mathcal{Y}) = \frac{d}{d\lambda}H(\mathcal{Y}).$$

Since the density of \mathcal{Y} can be written as

$$f_{\mathcal{Y}}(\mathcal{Y}) = \sum_{\boldsymbol{x} \in \{\pm 1\}^n} \frac{1}{2^n} \left[\left\{ \prod_{i < j} \phi\left(T_{i,j} - \sqrt{\frac{\lambda}{n}} x_i x_j\right) \right\} \left\{ \prod_{i=1}^n \phi\left(\frac{T_{i,i} - \sqrt{\lambda/n}}{\sqrt{2}}\right) \right\} f(\boldsymbol{B}|\boldsymbol{x}) \right],$$

we have

$$\begin{split} \frac{d}{d\lambda} H(\mathcal{Y}) &= \frac{d}{d\lambda} \Biggl\{ - \mathbb{E}_{\mathcal{Y}, \boldsymbol{x}^*} [\log f_{\mathcal{Y}}(\mathcal{Y})] \Biggr\} \\ &= \frac{1}{2\sqrt{\lambda n}} \mathbb{E}_{\mathcal{Y}, \boldsymbol{x}^*} \Biggl[\frac{1}{2^n} \sum_{\boldsymbol{x} \in \{\pm 1\}^n} \sum_{i < j} \frac{(T_{i,j} - \sqrt{\lambda/n} x_i x_j)(x_i^* x_j^* - x_i x_j) f_{\mathcal{Y}}(\mathcal{Y}|\boldsymbol{x})}{f_{\mathcal{Y}}(\mathcal{Y})} \Biggr] \\ &= \frac{1}{2\sqrt{\lambda n}} \mathbb{E}_{\mathcal{Y}, \boldsymbol{x}^*} \Biggl[\sum_{\boldsymbol{x} \in \{\pm 1\}^n} \sum_{i < j} (T_{i,j} - \sqrt{\lambda/n} x_i x_j)(x_i^* x_j^* - x_i x_j) f(\boldsymbol{x}|\mathcal{Y}) \Biggr] \\ &= \frac{1}{2\sqrt{\lambda n}} \mathbb{E}_{\mathcal{Y}, \boldsymbol{x}^*} \Biggl[\sum_{i < j} \Biggl\{ T_{i,j} x_i^* x_j^* - T_{i,j} \mathbb{E}[x_i^* x_j^*|\mathcal{Y}] - \sqrt{\frac{\lambda}{n}} x_i^* x_j^* \mathbb{E}[x_i^* x_j^*|\mathcal{Y}] \\ &+ \sqrt{\frac{\lambda}{n}} \mathbb{E}[(x_i^* x_j^*)^2 |\mathcal{Y}] \Biggr] \\ &= \frac{1}{2n} \sum_{i < j} \mathbb{E} \Biggl[(x_i^* x_j^* - \mathbb{E}[x_i^* x_j^*|\mathcal{Y}])^2 \Biggr] \\ &= \frac{1}{4n^2} \mathbb{E} \| \boldsymbol{x}^* (\boldsymbol{x}^*)^\top - \mathbb{E}[\boldsymbol{x}^* (\boldsymbol{x}^*)^\top |\mathcal{Y}] \|_F^2 \\ &= \frac{1}{4} \mathsf{GMMSE}_n(\lambda, \mu). \end{split}$$

This implies

$$\frac{1}{n}\frac{d}{d\lambda}I(\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top};\boldsymbol{T},\boldsymbol{B}) = \frac{1}{4}\mathsf{GMMSE}_n(\lambda,\mu).$$

D Proof of Results in Section 5

D.1 Proof of Theorem 5.1

We start by observing that

$$\begin{aligned} \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t;\lambda,\mu,\varepsilon) \\ &= 1 - 2 \ \mathbb{E}\langle \hat{\boldsymbol{x}}^{t}, \boldsymbol{x}^{*} \rangle_{n}^{2} + \frac{1}{n^{2}} \mathbb{E} \| \hat{\boldsymbol{x}}^{t} \|^{4} \\ &= 1 - 2 \ \mathbb{E}\langle f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{y}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)), \boldsymbol{x}^{*} \rangle_{n}^{2} + \frac{1}{n^{2}} \mathbb{E} \| f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{y}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) \|^{4}. \end{aligned}$$
(D.1)

Since f_{t-1} is Lipschitz (by Lemma D.1), $(x, y, w, z) \mapsto wf_{t-1}(x, y, z)$ is partially pseudo-Lipschitz. This implies using Theorem 7.2, we get

$$\lim_{n \to \infty} \langle \hat{\boldsymbol{x}}^t, \boldsymbol{x}^* \rangle_n^2 = \mathbb{E} \left[X_0 \mathbb{E} \left[X_0 | \alpha_{t-2} X_0 + \tau_{t-2} Z_0, \mu_{t-1} X_0 + \sigma_{t-1} \widetilde{Z}_0, X_0(\varepsilon) \right] \right],$$

where $Z_0, \tilde{Z}_0 \stackrel{iid}{\sim} N(0,1), X_0 \sim$ Rademacher and $X_0(\varepsilon) = B_0 X_0$ where $B_0 \sim \text{Bern}(\varepsilon)$ is independent of all other random variables. Similarly, $(x, y, w, z) \mapsto f_{t-1}^2(x, y, z)$ is partially pseudo-Lipschitz, and Theorem 7.2 implies

$$\lim_{n \to \infty} \frac{1}{n} \| \boldsymbol{f}_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{y}^{t-1}, \boldsymbol{x}_0(\varepsilon)) \|^2 = \mathbb{E} \left[\left(\mathbb{E} \left[X_0 | \alpha_{t-2} X_0 + \tau_{t-2} Z_0, \mu_{t-1} X_0 + \sigma_{t-1} \widetilde{Z}_0, X_0(\varepsilon) \right] \right)^2 \right].$$

Then using the dominated convergence theorem, property of conditional expectation and (D.1), we obtain the following.

$$\lim_{n \to \infty} \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t; \lambda, \mu, \varepsilon) = 1 - \left(\mathbb{E} \Big[\big(\mathbb{E} \Big[X_0 | \alpha_{t-2} X_0 + \tau_{t-2} Z_0, \mu_{t-1} X_0 + \sigma_{t-1} \widetilde{Z}_0, X_0(\varepsilon) \Big] \big)^2 \Big] \right)^2.$$

Now using (2.29), and (5.2) we get

$$\lim_{n \to \infty} \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t; \lambda, \mu, \varepsilon) = 1 - z_t^2.$$

It is immediate to show that

$$G_{\varepsilon}(z) = 1 - (1 - \varepsilon) \operatorname{mmse}\left(\lambda z + (1 - \varepsilon) \frac{\mu^2}{c} \frac{z}{1 + \mu z}\right)$$

is continuous on $[0, \infty)$, $\lim_{z\to\infty} G_{\varepsilon}(z) = 1$ and $G_{\varepsilon}(0) = \varepsilon$. Using the fact that the function $t \mapsto \mathsf{mmse}(t)$ is monotone decreasing and $\varepsilon \in [0, 1]$, it is easy to show that $G_{\varepsilon}(z)$ is monotone increasing in z. Further, using Lemma 6.1 of [15], it can be concluded that $G_{\varepsilon}(z)$ is strictly concave in $[0, \infty)$. From these observations we have

$$\lim_{t \to \infty} \lim_{n \to \infty} \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t; \lambda, \mu, \varepsilon) = 1 - z_*^2(\lambda, \mu, \varepsilon),$$

where $z_*(\lambda, \mu, \varepsilon)$ is the largest non-negative solution to (5.4). Note that

$$G_{\varepsilon}(z) = 1 + (\varepsilon - 1) \mathsf{mmse}\left(\lambda z + (1 - \varepsilon)\frac{\mu^2}{c}\frac{z}{1 + \mu z}\right)$$

As $\varepsilon \mapsto \mathsf{mmse}\left(\lambda z + (1-\varepsilon)(\mu^2/c)(z/(1+\mu z))\right)$ is increasing in ε , we have $G_{\varepsilon}(z)$ is increasing as a function of ε . From this observation and boundedness of $G_{\varepsilon}(z)$, we have

$$z_*(\lambda,\mu,\varepsilon) \to z_*(\lambda,\mu)$$

as $\varepsilon \to 0$, where $z_*(\lambda, \mu)$ satisfies (2.14) and hence

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \lim_{n \to \infty} \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t; \lambda, \mu, \varepsilon) = 1 - z_*^2(\lambda, \mu).$$

D.2 Proof of Theorem 5.2

Begin by noting that

$$\begin{aligned} \left| \frac{1}{n} I\left(\boldsymbol{x}^*; \boldsymbol{T}(\lambda), \boldsymbol{B}, \boldsymbol{x}_0(\varepsilon), \boldsymbol{w}_0(\varepsilon) \right) - \frac{1}{n} I\left(\boldsymbol{x}^*; \boldsymbol{T}(\lambda), \boldsymbol{B} \right) \right| &= \frac{1}{n} I\left(\boldsymbol{x}^*; \boldsymbol{x}_0(\varepsilon), \boldsymbol{w}_0(\varepsilon) | \boldsymbol{T}(\lambda), \boldsymbol{B} \right) \\ &\leqslant \frac{1}{n} H(\boldsymbol{x}_0(\varepsilon), \boldsymbol{w}_0(\varepsilon)) \leqslant \varepsilon \log 2 + \frac{p}{2n} \varepsilon \log(2\pi e) \to 0, \end{aligned}$$

as $\varepsilon \to 0$. Further, using techniques similar to the proof of Remark 6.5 in [15], we can show

$$\lim_{\lambda \to \infty} \lim_{n \to \infty} \frac{1}{n} I\left(\boldsymbol{x}^* (\boldsymbol{x}^*)^\top; \boldsymbol{T}(\lambda), \boldsymbol{B} \right) = \log 2,$$
(D.2)

where $T(\lambda)$ is defined in (2.19). From Lemma D.2, we also have

$$\lim_{n \to \infty} \frac{1}{n} I\left(\boldsymbol{x}^* (\boldsymbol{x}^*)^\top; \boldsymbol{T}(0), \boldsymbol{B} \right)$$
(D.3)
= $\frac{1}{2c} \log(1 + \mu \gamma_*) + \frac{1}{2c} \frac{(1 + \mu)}{(1 + \mu \gamma_*)} + I\left(\frac{\mu^2}{c} \frac{\gamma_*}{1 + \mu \gamma_*}\right) - \frac{1}{2c} \log(1 + \mu) - \frac{1}{2c}$
= $\kappa(\mu, \gamma_*),$

where γ_* satisfies

$$\gamma_* = 1 - \text{mmse}\left(\frac{\mu^2}{c} \frac{\gamma_*}{1 + \mu\gamma_*}\right). \tag{D.4}$$

From (263) of [15] we have for all $\lambda \ge 0$

$$\lim_{n \to \infty} \left[\frac{1}{n} I(\boldsymbol{x}^*; \boldsymbol{T}(\lambda), \boldsymbol{B}, \boldsymbol{x}_0(\varepsilon), \boldsymbol{w}_0(\varepsilon)) - \frac{1}{n} I(\boldsymbol{x}^*(\boldsymbol{x}^*)^\top; \boldsymbol{T}(\lambda), \boldsymbol{B}, \boldsymbol{x}_0(\varepsilon), \boldsymbol{w}_0(\varepsilon)) \right] = 0.$$

Next observe that for all $\lambda,\mu,\varepsilon>0$

$$\mathsf{GMMSE}_n(\lambda,\mu,\varepsilon) \leqslant \mathsf{MSE}_{\mathsf{n}}^{\mathsf{AMP}}(t;\lambda,\mu,\varepsilon), \tag{D.5}$$

where

$$\mathsf{GMMSE}_n(\lambda,\mu,\varepsilon) = \frac{1}{n^2} \mathbb{E} \left\| \boldsymbol{x}^*(\boldsymbol{x}^*)^\top - \mathbb{E} [\boldsymbol{x}^*(\boldsymbol{x}^*)^\top \mid \boldsymbol{T}, \boldsymbol{B}, \boldsymbol{x}_0(\varepsilon), \boldsymbol{w}_0(\varepsilon)] \right\|_F^2.$$

With the same techniques used to prove (2.21), we have

$$\frac{1}{n}\frac{d}{d\lambda}I\left(\boldsymbol{x}^{*}(\boldsymbol{x}^{*})^{\top};\boldsymbol{T}(\lambda),\boldsymbol{B},\boldsymbol{x}_{0}(\varepsilon),\boldsymbol{w}_{0}(\varepsilon)\right) = \frac{1}{4}\mathsf{GMMSE}_{n}(\lambda,\mu,\varepsilon). \tag{D.6}$$

Using Lemma D.3, we further have

$$\xi(z_*(\lambda,\mu),\lambda,\mu) = \xi(\gamma_*,0,\mu) + \int_0^\lambda \frac{1}{4} \left(1 - z_*^2(t,\mu)\right) dt.$$
(D.7)

Then using Theorem 5.1, (D.2), (D.3), (D.5), (D.6), (D.7)

This implies that all inequalities in (D.8) are equalities, which, in turn, implies

$$\lim_{n \to \infty} \mathsf{GMMSE}_n(\lambda, \mu) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \lim_{n \to \infty} \mathsf{MSE}_n^{\mathsf{AMP}}(t; \lambda, \mu, \varepsilon) = 1 - z_*^2(\lambda, \mu).$$

By the definition of ξ

$$\lim_{n \to \infty} \frac{1}{n} I\left(\boldsymbol{x}^* (\boldsymbol{x}^*)^\top; \boldsymbol{T}(0), \boldsymbol{B} \right) = \xi(\gamma_*, 0, \mu).$$

Finally using Theorem 5.1 and Lemma D.3, we have

$$\lim_{n \to \infty} \frac{1}{n} I\left(\boldsymbol{x}^*(\boldsymbol{x}^*)^\top; \boldsymbol{T}(\lambda), \boldsymbol{B}\right) = \xi(\gamma_*, 0, \mu) + \lim_{n \to \infty} \int_0^\lambda \frac{1}{4} \mathsf{GMMSE}_n(t, \mu) \, dt$$
$$= \xi(\gamma_*, 0, \mu) + \lim_{n \to \infty} \int_0^\lambda \frac{1}{4} \mathsf{MSE}_n^{\mathsf{AMP}}(t; \lambda, \mu) \, dt$$
$$= \xi(\gamma_*, 0, \mu) + \int_0^\lambda \frac{1}{4} \left(1 - z_*^2(t, \mu)\right) \, dt$$
$$= \xi(z_*(\lambda, \mu), \lambda, \mu).$$

This completes the proof.

D.3 Lemmas Used to Prove Results in Section 5

Lemma D.1. Consider f_t defined in (2.26). Then f_t and its partial derivatives with respect to the first and second arguments are Lipschitz for all $t \ge 0$.

Proof. Let $\boldsymbol{x} = (x, y)$ and $\boldsymbol{a} = (a, b)$. We begin by observing that

$$f_t(x, y, z) = \begin{cases} 1 & \text{if } z = 1\\ -1 & \text{if } z = -1\\ \tanh\left(-\frac{\alpha_{t-1}}{\tau_{t-1}^2}x - \frac{\mu_t}{\sigma_t^2}y\right) & \text{if } z = 0 \end{cases}$$
(D.9)

Using (D.9), we get

$$|f_t(x, y, 1) - f_t(a, b, 1)| = 0 \le ||(x, 1) - (a, 1)||,$$

and

$$|f_t(x, y, -1) - f_t(a, b, -1)| = 0 \le ||(x, 1) - (a, 1)||$$

Further, since $|\tanh'(x)| \leq 1$ for all x, using multivariate mean-value theorem, we have

$$|f_t(x, y, 0) - f_t(a, b, 0)| \leq (\alpha_{t-1}/\tau_{t-1}^2 + \mu_t/\sigma_t^2)^{1/2} ||(x, 0) - (a, 0)||.$$

Again as $|\tanh(x)| \leq 1$, it can be easily shown that

$$|f_t(x, y, -1) - f_t(a, b, 0)| \leq C ||(x, -1) - (a, 0)||,$$

and

$$|f_t(x, y, 1) - f_t(a, b, 0)| \leq C ||(x, 1) - (a, 0)||.$$

Again, by definition

$$|f_t(x, y, 1) - f_t(a, b, -1)| = 2 \leq C ||(x, 1) - (a, -1)||.$$

Next observe that

$$\frac{\partial f_t(x, y, z)}{\partial x} = \begin{cases} 0 & \text{if } z = 1, \\ 0 & \text{if } z = -1, \\ -\frac{\alpha_{t-1}}{\tau_{t-1}^2} \text{sech}^2 \left(-\left(\frac{\alpha_{t-1}}{\tau_{t-1}^2} x + \frac{\mu_t}{\sigma_t^2} y\right) \right) & \text{if } z = 0; \end{cases}$$

Observing that $|\operatorname{sech}(x)| \leq 1$ and $|\operatorname{sech}(x) \tanh(x)| \leq 1$, and using arguments similar to those previously used, we can show that $\frac{\partial f_t(x,y,z)}{\partial x}$ is Lipschitz. Similarly, we can also show $\frac{\partial f_t(x,y,z)}{\partial y}$ is Lipschitz.

Lemma D.2. We have

$$\lim_{n \to \infty} \frac{1}{n} I(\boldsymbol{x}^*; \boldsymbol{B}) = \kappa(\mu, \gamma_*),$$

where $\kappa(\mu, \gamma_*)$ is as defined in (D.3).

Proof. Let us observe that if $\lambda = 0$, we have **B** is the transpose of the matrix **Y** described in (1) of [28] with $U = x^*$, $V = v^*$, $\lambda = \mu$ and $\alpha = 1/c$. Since

$$\frac{1}{n}I(\boldsymbol{v}^*(\boldsymbol{x}^*)^{\top}; \boldsymbol{B}) = \frac{1}{n}I((\boldsymbol{v}^*, \boldsymbol{x}^*); \boldsymbol{B}) = \frac{1}{n}I((\boldsymbol{v}^*, \boldsymbol{x}^*); \boldsymbol{B}^{\top}),$$

the result of [28] directly apply in our case. Let us consider the scalar model

$$Y = \sqrt{\gamma} X + Z,$$

where $Z \sim N(0,1)$ and $X \in \{U, V\}$ where $V \sim N(0,1)$ and $U \sim$ Rademacher. Recall

$$\mathcal{Z}(Y) = \int dP_X e^{\gamma x X + \sqrt{\gamma} x Z - \frac{\gamma x^2}{2}}$$

and the function $F_{P_X}(\gamma)$ defined in (9) of [28] given by

$$F_{P_X}(\gamma) = \mathbb{E}\left[\frac{1}{\mathcal{Z}(Y)}\int xXe^{\gamma xX + \sqrt{\gamma}xZ - \frac{\gamma x^2}{2}}dP_X\right].$$

For X = V it can be easily verified that

$$F_{P_X}(\gamma) = \frac{\gamma}{1+\gamma}.$$

By definition of $\Gamma(\mu, c)$ as in (11) of [28], in our case we have

$$\Gamma(\mu, c) = \left\{ \left(q, \frac{\mu q}{1 + \mu q}\right) : q \ge 0 \right\}.$$

If we recall the definition of $\psi_{P_X}(\gamma)$ as in (10) of [28], that is

$$\psi_{P_X}(\gamma) = \mathbb{E} \log \left(\int e^{\gamma x X + \sqrt{\gamma} x Z - \frac{\gamma x^2}{2}} dP_X \right),$$

then it is easy to show that

$$\psi_{P_V}(\gamma) = \frac{\gamma}{2} - \frac{1}{2}\log(1+\gamma),$$

and

$$\psi_{P_U}(\gamma) = \frac{\gamma}{2} - \mathsf{I}(\gamma),$$

where $I(\gamma)$ is defined in (2.12). Let us define

$$\begin{aligned} \mathcal{F}(q) &= \psi_{P_U} \left(\frac{\mu^2}{c} \frac{q}{1+\mu q} \right) + \frac{1}{c} \psi_{P_V}(\mu q) - \frac{\mu^2}{2c} \frac{q^2}{1+\mu q} \\ &= \frac{\mu}{2c} + \frac{1}{2c} - \frac{\mu+1}{2c(1+\mu q)} - \mathsf{I}\left(\frac{\mu^2}{c} \frac{q}{1+\mu q}\right) - \frac{1}{2c} \log(1+\mu q). \end{aligned}$$

Now if γ^* is the supremum of $\mathcal{F}(q)$, then it must satisfy, $\mathcal{F}'(\gamma^*) = 0$, which further implies

$$\gamma^* = 1 - \text{mmse}\left(\frac{\mu^2}{c} \frac{\gamma^*}{1 + \mu\gamma^*}\right).$$

Using Corollary 1 of [28], we get

$$\begin{split} \lim_{n \to \infty} \frac{1}{n} I(\boldsymbol{v}^*(\boldsymbol{x}^*)^\top; \boldsymbol{B}) &= \frac{\mu}{2c} - \frac{\mu}{2c} - \frac{1}{2c} + \frac{\mu + 1}{2c(1 + \mu\gamma^*)} + \mathsf{I}\left(\frac{\mu^2}{c}\frac{\gamma^*}{1 + \mu\gamma^*}\right) + \frac{1}{2c}\log(1 + \mu\gamma^*) \\ &= \frac{\mu + 1}{2c(1 + \mu\gamma^*)} + \mathsf{I}\left(\frac{\mu^2}{c}\frac{\gamma^*}{1 + \mu\gamma^*}\right) + \frac{1}{2c}\log(1 + \mu\gamma^*) - \frac{1}{2c} \\ &= \kappa(\mu, \gamma_*) + \frac{1}{2c}\log(1 + \mu). \end{split}$$

We note that given $\boldsymbol{v}^*(\boldsymbol{x}^*)^{\top}$, \boldsymbol{B} is equal in distribution to $(\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_p)$, where $\boldsymbol{y}_i \sim \mathbf{N}_n \left(\sqrt{\mu/n} \boldsymbol{v}^*(\boldsymbol{x}^*)^{\top}, \boldsymbol{I}_n \right)$. This implies

$$H(\boldsymbol{B}|\boldsymbol{v}^*(\boldsymbol{x}^*)^{\top}) = \frac{p}{2}\log\det\left(2\pi e\mathbf{I}_n\right).$$

Also note that, given \boldsymbol{x}^* , $\boldsymbol{B} \stackrel{d}{=} (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_p)$, where $\boldsymbol{b}_i \sim \mathbf{N}_n (\mathbf{0}, (\mu/n) \boldsymbol{x}^* (\boldsymbol{x}^*)^\top + \boldsymbol{I}_n)$. This implies

$$H(\boldsymbol{B}|\boldsymbol{v}^*(\boldsymbol{x}^*)^{\top}) = \frac{p}{2}\log\det\left(2\pi e\left(\frac{\mu}{n}\boldsymbol{x}^*(\boldsymbol{x}^*)^{\top} + \boldsymbol{I}_n\right)\right).$$

Next we get

$$\frac{1}{n} \left[I(\boldsymbol{x}^*; \boldsymbol{B}) - I(\boldsymbol{v}^*(\boldsymbol{x}^*)^\top; \boldsymbol{B}) \right] = \frac{1}{n} \left[H(\boldsymbol{B} | \boldsymbol{v}^*(\boldsymbol{x}^*)^\top) - H(\boldsymbol{B} | \boldsymbol{x}^*) \right] = -\frac{p}{2n} \log(1 + \mu).$$

Thus, we have

$$\lim_{n \to \infty} \frac{1}{n} I(\boldsymbol{x}^*; \boldsymbol{B}) = \lim_{n \to \infty} \frac{1}{n} \left[I(\boldsymbol{x}^*; \boldsymbol{B}) - I(\boldsymbol{v}^*(\boldsymbol{x}^*)^\top; \boldsymbol{B}) \right] + \lim_{n \to \infty} \frac{1}{n} I(\boldsymbol{v}^*(\boldsymbol{x}^*)^\top; \boldsymbol{B})$$
$$= \kappa(\mu, \gamma_*).$$

Lemma D.3. Let us consider the function ξ defined in (5.5). Then for all $\lambda, \mu > 0$

$$\xi(z_*(\lambda,\mu),\lambda,\mu) = \xi(\gamma_*,0,\mu) + \int_0^\lambda \frac{1}{4} \left(1 - z_*^2(t,\mu)\right) \, dt.$$

Proof. From (5.4) it is easy to see that $z_*(\lambda, \mu, \varepsilon) = \gamma_*$ where γ_* is the unique non-negative solution to (D.4). Then we have

$$\frac{\partial \xi(\gamma,\lambda,\mu)}{\partial \gamma} \Big|_{(z_*(\lambda,\mu),\lambda)} = \frac{1}{2} \left(\lambda + \frac{\mu^2}{c} \frac{1}{(\mu z_*(\lambda,\mu) + 1)^2} \right) \left\{ z_*(\lambda,\mu) - 1 + \operatorname{mmse}\left(\lambda z_*^2(\lambda,\mu) + \frac{\mu^2}{c} \frac{z_*(\lambda,\mu)}{\mu z_*(\lambda,\mu) + 1} \right) \right\}$$

also

$$\frac{\partial \xi(\gamma,\lambda,\mu)}{\partial \lambda} \bigg|_{(z_*(\lambda,\mu),\lambda)} = \frac{1}{4} \left(1 - z_*^2(\lambda,\mu) \right).$$

This implies using (5.4)

$$\frac{d\xi(\gamma,\lambda,\mu)}{d\lambda}\Big|_{(z_*(\lambda,\mu),\lambda)} = \frac{1}{4}\left(1 - z_*^2(\lambda,\mu)\right).$$

E Proof of Results in Section 7

E.1 Proof of Lemma 7.1

Let us consider two $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$, where $\boldsymbol{x} = (x_1, \dots, x_k)$ and $\boldsymbol{y} = (y_1, \dots, y_k)$. Then

$$|f(x,z)| \leq |f(x,z) - f(0,z)| + |f(0,z)|.$$

Since f is partially Lipschitz, we have using (7.2)

$$|f(\boldsymbol{x}, z) - f(\boldsymbol{0}, z)| \leq C \|\boldsymbol{x}\|,$$

for some constant C > 0. Again, using (7.3)

$$|f(\mathbf{0}, z)| \leq C(1+|z|).$$

Hence, we have

$$|f(x,z)| \leq C(1 + ||x|| + |z|)$$

1. Now let us observe that

$$\begin{aligned} &|f(\boldsymbol{x}, z)g(\boldsymbol{x}, z) - f(\boldsymbol{y}, z)g(\boldsymbol{y}, z)| \\ &= |f(\boldsymbol{x}, z)g(\boldsymbol{x}, z) - f(\boldsymbol{y}, z)g(\boldsymbol{x}, z) + f(\boldsymbol{y}, z)g(\boldsymbol{x}, z) - f(\boldsymbol{y}, z)g(\boldsymbol{y}, z)| \\ &\leqslant |f(\boldsymbol{x}, z) - f(\boldsymbol{y}, z)||g(\boldsymbol{x}, z)| + |g(\boldsymbol{x}, z) - g(\boldsymbol{y}, z)||f(\boldsymbol{y}, z)| \\ &\leqslant C(1 + \|\boldsymbol{x}\| + \|\boldsymbol{y}\| + |z|)\|\boldsymbol{x} - \boldsymbol{y}\|. \end{aligned}$$

Also

$$|f(\mathbf{0},z)g(\mathbf{0},z)| \leq C^2(1+|z|)^2 \leq C_1(1+|z|^2).$$

2. Let us denote $\mathbf{x}' = (x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_k)$ and $\mathbf{y}' = (y_1, \dots, y_{r-1}, y_{r+1}, \dots, y_k)$. Next note that

$$\begin{aligned} |H(\mathbf{x}', z) - H(\mathbf{y}', z)| \\ &\leq \mathbb{E}_{X} \Big[\Big| \phi(x_{1}, \dots, x_{r-1}, X, x_{r+1}, \dots, x_{k}, z) - \phi(y_{1}, \dots, y_{r-1}, X, y_{r+1}, \dots, y_{k}, z) \Big| \Big] \\ &\leq C \mathbb{E}_{X} \Big[(1 + \|\mathbf{x}'\| + \|\mathbf{y}'\| + |X| + |z|) \|\mathbf{x}' - \mathbf{y}'\| \Big] \\ &\leq C_{1} (1 + \|\mathbf{x}'\| + \|\mathbf{y}'\| + |z|) \|\mathbf{x}' - \mathbf{y}'\|. \end{aligned}$$

Next, note that

$$|H(\mathbf{0}, z)| \leq \mathbb{E}_{X} \Big[\Big| \phi(0, \dots, 0, X, 0, \dots, 0, z) - \phi(0, \dots, 0, 0, 0, \dots, 0, z) \Big| \\ + \Big| \phi(0, \dots, 0, 0, 0, \dots, 0, z) \Big| \\ \leq C \mathbb{E}_{X} \Big[(1 + |X| + |z|) |X| \Big] + C(1 + |z|^{2}) \\ \leq C_{1}(1 + |z| + |z|^{2}) \\ \leq C_{2}(1 + |z|^{2}).$$

E.2 Proof of Theorem 7.1, Conditioning Technique and the Main Technical Lemma

To prove Theorem 7.1, we apply the same device used in [9]. We begin by observing that for the Gaussian matrix \boldsymbol{L} defined at the beginning of Section 7.2 and a fixed vector \boldsymbol{v} , $\boldsymbol{L}\boldsymbol{v}$ is a centered Gaussian vector with i.i.d. entries and variance $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_p$. Similarly, $\boldsymbol{N}\boldsymbol{v}$ is a centered Gaussian vector with the covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{I}_n + \frac{1}{n}\boldsymbol{v}\boldsymbol{v}^{\top}$. However, $\boldsymbol{L}^{\top}\boldsymbol{m}^t$ is not a centered Gaussian by the previous argument as \boldsymbol{m}^t is not independent of \boldsymbol{L} . We can argue similarly for the other terms. To resolve this problem we adopt the conditioning technique developed in [12] and later used in [9], [22] and [10].

E.2.1 Conditioning Technique

With q^t and m^t defined in (7.9), the AMP orbits can be written as

$$b^{t} = Lq^{t} - p_{t}m^{t-1},$$

$$h^{t+1} = L^{\top}m^{t} - c_{t}q^{t},$$

and

$$\boldsymbol{y}^{t+1} = \boldsymbol{N}\boldsymbol{q}^t - d_t\boldsymbol{q}^{t-1}.$$

The Asymmetric Orbit. Let us observe that, to construct h^{t+1} we need to know $\mathcal{A}_{h,t+1} = \{h^1, \ldots, h^t, y^1, \ldots, y^t, b^0, \ldots, b^t, m^0, \ldots, m^t, q^0, \ldots, q^t, \xi_0, x_0, \omega_0, v_0\}$. Let the sigma algebra generated by these random variables be denoted by $\mathcal{G}_{t+1,t}$. Since m^j 's and q^j 's are functions of $h^j, y^j, q^j, \xi_0, x_0, \omega_0$ and v_0 ; $\mathcal{G}_{t+1,t}$ is the sigma-algebra generated by $\{h^1, \ldots, h^t, y^1, \ldots, y^t, b^0, \ldots, b^t, \xi_0, x_0, \omega_0, v_0\}$. Further, since h^{t+1} depends on y^1, \ldots, y^t through m^0, \ldots, m^t and q^0, \ldots, q^t , the conditional distribution of L given $\mathcal{G}_{t+1,t}$ is equal to the conditional distribution of L given

$$\underbrace{[\boldsymbol{h}^1 + c_0 \boldsymbol{q}^0| \dots | \boldsymbol{h}^t + c_{t-1} \boldsymbol{q}^{t-1}]}_{= \overline{H}_t} = \boldsymbol{L}^\top \underbrace{[\boldsymbol{m}^0| \dots | \boldsymbol{m}^{t-1}]}_{= M_t},$$

and

$$\underbrace{[\boldsymbol{b}^0|\ldots|\boldsymbol{b}^t+p_t\boldsymbol{m}^{t-1}]}_{=\bar{\boldsymbol{B}}_{t+1}}=\boldsymbol{L}\underbrace{[\boldsymbol{q}^0|\ldots|\boldsymbol{q}^t]}_{=\boldsymbol{Q}_{t+1}}.$$

Using Lemma 11 and Lemma 12 of [9], we get

$$\boldsymbol{L}|_{\mathcal{G}_{t+1,t}} \stackrel{d}{=} \mathcal{E}_{t+1,t} + \mathcal{P}_{t+1,t}(\widetilde{\boldsymbol{L}}),$$

where

$$\begin{aligned} \mathcal{E}_{t+1,t} &= \bar{\boldsymbol{B}}_{t+1} (\boldsymbol{Q}_{t+1}^{\top} \boldsymbol{Q}_{t+1})^{-1} \boldsymbol{Q}_{t+1}^{\top} + \boldsymbol{M}_t (\boldsymbol{M}_t^{\top} \boldsymbol{M}_t)^{-1} \overline{\boldsymbol{H}}_t^{\top} \\ &- \boldsymbol{M}_t (\boldsymbol{M}_t^{\top} \boldsymbol{M}_t)^{-1} \boldsymbol{M}_t^{\top} \bar{\boldsymbol{B}}_{t+1} (\boldsymbol{Q}_{t+1}^{\top} \boldsymbol{Q}_{t+1})^{-1} \boldsymbol{Q}_{t+1}^{\top} \end{aligned}$$

and

$$\mathcal{P}_{t+1,t}(\widetilde{\boldsymbol{L}}) = P_{\boldsymbol{M}_t}^{\perp} \widetilde{\boldsymbol{L}} P_{\boldsymbol{Q}_{t+1}}^{\perp}.$$

Here $\tilde{\boldsymbol{L}}$ is an independent copy of \boldsymbol{L} and $P_{\boldsymbol{M}_t}^{\perp}, P_{\boldsymbol{Q}_{t+1}}^{\perp}$ are the orthogonal projectors on to the orthogonal complements of the column spaces of \boldsymbol{M}_t and \boldsymbol{Q}_{t+1} , respectively.

Next, if we consider \boldsymbol{b}^t we need to know $\mathcal{A}_{b,t} = \{\boldsymbol{h}^1, \dots, \boldsymbol{h}^t, \boldsymbol{y}^1, \dots, \boldsymbol{y}^t, \boldsymbol{b}^0, \dots, \boldsymbol{b}^{t-1}, \boldsymbol{m}^0, \dots, \boldsymbol{m}^{t-1}, \boldsymbol{q}^0, \dots, \boldsymbol{q}^t, \boldsymbol{\xi}_0, \boldsymbol{x}_0, \boldsymbol{\omega}_0, \boldsymbol{v}_0\}$. Let the sigma algebra generated by the above mentioned variables be denoted by $\mathcal{G}_{t,t}$. The conditional distribution of \boldsymbol{L} given $\mathcal{G}_{t,t}$ is equal to the conditional distribution of \boldsymbol{L} given $\boldsymbol{H}_t = \boldsymbol{L}^\top \boldsymbol{M}_t$ and $\boldsymbol{B}_t = \boldsymbol{L}\boldsymbol{Q}_t$. By Lemmas 11 and 12 of [9] we get

$$\boldsymbol{L}|_{\mathcal{G}_{t,t}} \stackrel{d}{=} \mathcal{E}_{t,t} + \mathcal{P}_{t,t}(\boldsymbol{\widetilde{L}}),$$

where

$$\mathcal{E}_{t,t} = \bar{\boldsymbol{B}}_t (\boldsymbol{Q}_t^\top \boldsymbol{Q}_t)^{-1} \boldsymbol{Q}_t^\top + \boldsymbol{M}_t (\boldsymbol{M}_t^\top \boldsymbol{M}_t)^{-1} \overline{\boldsymbol{H}}_t^\top - \boldsymbol{M}_t (\boldsymbol{M}_t^\top \boldsymbol{M}_t)^{-1} \boldsymbol{M}_t^\top \bar{\boldsymbol{B}}_t (\boldsymbol{Q}_t^\top \boldsymbol{Q}_t)^{-1} \boldsymbol{Q}_t^\top,$$

and

$$\mathcal{P}_{t,t}(\widetilde{\boldsymbol{L}}) = P_{\boldsymbol{M}_t}^{\perp} \widetilde{\boldsymbol{L}} P_{\boldsymbol{Q}_t}^{\perp}.$$

The Symmetric Orbit Now, we consider the second orbit characterized by B. Observe that the distribution of y^{t+1} depends on the sigma algebra generated by $A_{y,t+1} = \{h^1, \ldots, h^t, y^1, \ldots, y^t, b^0, \ldots, b^t, m^0, \ldots, m^{t-1}, q^0, \ldots, q^t, \xi_0, x_0, \omega_0, v_0\}$. This implies that we must consider the distribution of N given $\mathcal{G}_{t+1,t}$, or equivalently given

$$\underbrace{[\boldsymbol{y}^1|\ldots|\boldsymbol{y}^t+d_{t-1}\boldsymbol{q}^{t-2}]}_{=\bar{\boldsymbol{Y}}_t}=\boldsymbol{N}\underbrace{[\boldsymbol{q}^0|\ldots|\boldsymbol{q}^{t-1}]}_{=\boldsymbol{Q}_t}.$$

Now, using Lemma 3 of [22], we get

$$\boldsymbol{N}|_{\mathcal{G}_{t+1,t}} \stackrel{d}{=} \mathcal{F}_{t+1,t} + \mathcal{P}_{t+1,t}(\widetilde{\boldsymbol{N}}), \tag{E.1}$$

where

$$\mathcal{F}_{t+1,t} = \bar{\boldsymbol{Y}}_t(\boldsymbol{Q}_t^{\top}\boldsymbol{Q}_t)^{-1}\boldsymbol{Q}_t^{\top} + \boldsymbol{Q}_t(\boldsymbol{Q}_t^{\top}\boldsymbol{Q}_t)^{-1}\bar{\boldsymbol{Y}}_t^{\top} - \boldsymbol{Q}_t(\boldsymbol{Q}_t^{\top}\boldsymbol{Q}_t)^{-1}\boldsymbol{Q}_t^{\top}\bar{\boldsymbol{Y}}_t(\boldsymbol{Q}_t^{\top}\boldsymbol{Q}_t)^{-1}\boldsymbol{Q}_t^{\top}, \quad (E.2)$$

and

$$\mathcal{P}_{t+1,t}(\widetilde{\boldsymbol{N}}) = P_{\boldsymbol{Q}_t}^{\perp} \widetilde{\boldsymbol{N}} P_{\boldsymbol{Q}_t}^{\perp}.$$

Here \widetilde{N} is an independent copy of N and $P_{Q_t}^{\perp}$ is the orthogonal projector to the orthogonal complement of the column space of Q_t . Using the above conditioning technique and the following main technical lemma (that is, Lemma E.1), the proof of Theorem 7.1 is immediate.

E.2.2 Main Technical Lemma

Let us denote the projection of \boldsymbol{m}^t on the column space of \boldsymbol{M}_t by $\boldsymbol{m}_{\parallel}^t$ and its ortho-complement by \boldsymbol{m}_{\perp}^t . Similarly $\boldsymbol{q}_{\parallel}^t$ denotes the projection of \boldsymbol{q}^t onto the column space of \boldsymbol{Q}_t and \boldsymbol{q}_{\perp}^t be its ortho-complement. This implies, if we define

$$\boldsymbol{\alpha}_t = (\boldsymbol{\alpha}_0^t, \dots, \boldsymbol{\alpha}_{t-1}^t) = \left[\frac{\boldsymbol{M}_t^\top \boldsymbol{M}_t}{p}\right]^{-1} \frac{\boldsymbol{M}_t^\top \boldsymbol{m}^t}{p},$$

and

$$\boldsymbol{\beta}_t = (\boldsymbol{\beta}_0^t, \dots, \boldsymbol{\beta}_{t-1}^t) = \left[\frac{\boldsymbol{Q}_t^\top \boldsymbol{Q}_t}{n}\right]^{-1} \frac{\boldsymbol{Q}_t^\top \boldsymbol{q}^t}{n},$$

then we have

$$oldsymbol{m}^t_{\parallel} = \sum_{i=0}^{t-1} lpha^t_i oldsymbol{m}^i, \qquad oldsymbol{m}^t_{\perp} = oldsymbol{m}^t - oldsymbol{m}^t_{\parallel};$$

and

$$oldsymbol{q}^t_{\parallel} = \sum_{i=0}^{t-1}eta^t_ioldsymbol{q}^i, \qquad oldsymbol{q}^t_{\perp} = oldsymbol{q}^t - oldsymbol{q}^t_{\parallel}.$$

Finally, for two sequences of random vectors $\boldsymbol{x}_n, \boldsymbol{y}_n$, by $\boldsymbol{x}_n \stackrel{P}{\simeq} \boldsymbol{y}_n$ we mean $\boldsymbol{x}_n - \boldsymbol{y}_n \stackrel{P}{\rightarrow} 0$. With all these defined, let us now state the following general result.

Lemma E.1. Suppose that the conditions of Theorem 7.1 hold. Then for all $t \in \mathbb{N} \cup \{0\}$, we get (a)

$$egin{aligned} m{h}^{t+1}|_{\mathcal{G}_{t+1,t}} &\stackrel{d}{=} \sum_{i=0}^{t-1} lpha_i^t m{h}^{i+1} + \widetilde{m{L}}^ op m{m}_\perp^t + \widetilde{m{Q}}_{t+1} o(1), \ m{b}^t|_{\mathcal{G}_{t,t}} &\stackrel{d}{=} \sum_{i=0}^{t-1} eta_i^t m{b}^i + \widetilde{m{L}} m{q}_\perp^t + \widetilde{m{M}}_t o(1), \end{aligned}$$

and

$$\boldsymbol{y}^{t+1}|_{\mathcal{G}_{t+1,t}} \stackrel{d}{=} \sum_{i=0}^{t-1} \beta_i^t \boldsymbol{y}^{i+1} + \widetilde{\boldsymbol{N}} \boldsymbol{q}_{\perp}^t + \widetilde{\boldsymbol{Q}}_t o(1),$$

where \widetilde{Q}_t (alternatively, \widetilde{M}_t) is a matrix whose columns form an orthogonal basis of Q_t (respectively, M_t), and $\widetilde{Q}_t^{\top} \widetilde{Q}_t = n I_t$ ($\widetilde{M}_t^{\top} \widetilde{M}_t = p I_t$).

(b) For all partially pseudo-Lipschitz functions $\phi_h : \mathbb{R}^{2t+4} \to \mathbb{R}, \phi_b : \mathbb{R}^{t+3} \to \mathbb{R}$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi_h(h_i^1, \dots, h_i^{t+1}, y_i^1, \dots, y_i^{t+1}, \xi_{0,i}, x_{0,i})$$

$$\stackrel{a.s.}{=} \mathbb{E} \left\{ \phi_h(\tau_0 Z_0, \dots, \tau_t Z_t, \sigma_1 \widetilde{Z}_1, \dots, \sigma_{t+1} \widetilde{Z}_{t+1}, \widetilde{\Xi}_0, \widetilde{X}_0) \right\},$$

$$\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \phi_b(b_i^0, \dots, b_i^t, \omega_{0,i}, v_{0,i}) \stackrel{a.s.}{=} \mathbb{E} \left\{ \phi_b(\vartheta_0 \check{Z}_0, \dots, \vartheta_t \check{Z}_t, \widetilde{\Omega}_0, \widetilde{V}_0) \right\},\,$$

where (Z_0, \ldots, Z_t) , $(\widetilde{Z}_1, \ldots, \widetilde{Z}_t)$, $(\check{Z}_1, \ldots, \check{Z}_t)$, (Ξ_0, \widetilde{X}_0) and $(\widetilde{\Omega}_0, \widetilde{V}_0)$ are mutually independent random vectors. Marginally, $Z_i, \widetilde{Z}_i, \check{Z}_i \sim N(0, 1)$, $(\Xi_0, \widetilde{X}_0) \sim P_{\xi,x}$ and $(\widetilde{\Omega}_0, \widetilde{V}_0) \sim P_{\omega,v}$.

(c) For all $0 \le k, \ell \le t$, the following equations hold and all the limits exist, are bounded and have degenerate distributions:

$$\begin{split} \lim_{n \to \infty} \langle \boldsymbol{h}^{k+1}, \boldsymbol{h}^{\ell+1} \rangle_n &\stackrel{a.s.}{=} \lim_{p \to \infty} \langle \boldsymbol{m}^k, \boldsymbol{m}^\ell \rangle_p, \\ \lim_{p \to \infty} \langle \boldsymbol{b}^k, \boldsymbol{b}^\ell \rangle_p &\stackrel{a.s.}{=} c \lim_{n \to \infty} \langle \boldsymbol{q}^k, \boldsymbol{q}^\ell \rangle_n, \end{split}$$

and

$$\lim_{n\to\infty} \langle \boldsymbol{y}^{k+1}, \boldsymbol{y}^{\ell+1} \rangle_n \stackrel{a.s.}{=} \lim_{n\to\infty} \langle \boldsymbol{q}^k, \boldsymbol{q}^\ell \rangle_n.$$

(d) For all 0 ≤ k, ℓ ≤ t and for any partially Lipschitz functions φ : ℝ⁴ → ℝ, ψ : ℝ³ → ℝ, with φ'₁ being the derivative of φ with respect to the first coordinate, φ'₂ being the derivative of φ with respect to the second coordinate and ψ' being the derivative of ψ with respect to the first coordinate, where φ'₁, φ'₂ and ψ' being partially Lipschitz, the following equations hold and all the limits exist, are bounded and have degenerate distributions:

$$\lim_{n \to \infty} \langle \boldsymbol{h}^{k+1}, \varphi(\boldsymbol{h}^{\ell+1}, \boldsymbol{y}^{\ell+1}, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \rangle_n \stackrel{a.s.}{=} \lim_{n \to \infty} \langle \boldsymbol{h}^{k+1}, \boldsymbol{h}^{\ell+1} \rangle_n \langle \varphi_1'(\boldsymbol{h}^{\ell+1}, \boldsymbol{y}^{\ell+1}, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \rangle_n, \\
\lim_{p \to \infty} \langle \boldsymbol{b}^r, \psi(\boldsymbol{b}^s, \boldsymbol{\omega}_0, \boldsymbol{v}_0) \rangle_p \stackrel{a.s.}{=} \lim_{p \to \infty} \langle \boldsymbol{b}^r, \boldsymbol{b}^s \rangle_p \langle \psi'(\boldsymbol{b}^s, \boldsymbol{\omega}_0, \boldsymbol{v}_0) \rangle_p,$$

and

$$\lim_{n\to\infty} \langle \boldsymbol{y}^{r+1}, \varphi(\boldsymbol{h}^{s+1}, \boldsymbol{y}^{s+1}, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \rangle_n \stackrel{a.s.}{=} \lim_{n\to\infty} \langle \boldsymbol{y}^{r+1}, \boldsymbol{y}^{s+1} \rangle_n \langle \varphi_2'(\boldsymbol{h}^{s+1}, \boldsymbol{y}^{s+1}, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \rangle_n.$$

(e) The following relations hold almost surely

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^n(h_i^{t+1})^2<\infty,\quad \limsup_{p\to\infty}\frac{1}{p}\sum_{i=1}^n(b_i^t)^2<\infty,\quad \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^n(y_i^{t+1})^2<\infty.$$

(f) For all $0 \leq r \leq t$:

$$\lim_{n\to\infty} \langle \boldsymbol{h}^{t+1}, \boldsymbol{q}^0 \rangle_n \stackrel{a.s.}{=} 0.$$

(g) For all $0 \le s \le t - 1$, $0 \le k \le t$, the following limits exist, and there exist strictly positive constants ρ_k, \varkappa_s such that the following relations hold almost surely:

$$\lim_{n\to\infty} \langle \boldsymbol{q}_{\perp}^k, \boldsymbol{q}_{\perp}^k \rangle_n > \rho_k, \quad \lim_{p\to\infty} \langle \boldsymbol{m}_{\perp}^s, \boldsymbol{m}_{\perp}^s \rangle_p > \varkappa_s.$$

E.2.3 Proof of Theorem 7.1

Proof. The desired result is a direct consequence of Lemma E.1 claim (b).

E.3 Proof of Lemma E.1

We shall prove this lemma by induction. We shall first show that the statements (a)-(g) hold true for b^0 , h^1 and y^1 . Then assuming that the result holds true for $0 \le s \le t-1$, we shall show that the statements (a)-(g) hold b^t , h^{t+1} and y^{t+1} .

First, let us observe that if $f_t(x, y, \xi_0, x_0)$ is free of x, y almost surely with respect to ξ_0 and x_0 or if $g_t(u, \omega_0, v_0)$ is free of u almost surely with respect to ω_0 and v_0 , then the lemma is immediate. So we assume that these degenerate cases do not arise in the rest of this proof.

The Base Case $(b^0, h^1 \text{ and } y^1)$. The proofs of the assumptions (a) - (g) for b^0 follows using exactly the same arguments as in \mathcal{B}_0 of [9]. So we skip the details. Next, let us observe that as $y^1 = Nq^0$, and Q_0 is empty matrix, we have $q_{\perp}^0 = q^0$. Thus using the definition of $\mathcal{G}_{1,0}$, we conclude

$$oldsymbol{y}^1|_{\mathcal{G}_{1,0}} \stackrel{d}{=} oldsymbol{N}oldsymbol{q}^0_\perp$$

Again, the assertions (a), (c), (e) and (f) for h^1 follows immediately using the techniques of \mathcal{H}_1 of [9]. Now, let us consider these assertions for y^1 . For the assertion (a), let us observe that, as $y^1 = Nq^0$, and Q_0 is empty matrix, we have $q_{\perp}^0 = q^0$. Thus using the definition of $\mathcal{G}_{1,0}$, we conclude

$$oldsymbol{y}^1ert_{\mathcal{G}_{1,0}} \stackrel{d}{=} oldsymbol{N}oldsymbol{q}^0_{ot}.$$

Next, for the assertion (c), using Lemma E.2(c), we have

$$\lim_{n\to\infty} \langle \boldsymbol{y}^1, \boldsymbol{y}^1\rangle_n |_{\mathcal{G}_{1,0}} \stackrel{d}{=} \lim_{n\to\infty} \frac{1}{n} \|\boldsymbol{N}\boldsymbol{q}^0\|^2 \stackrel{a.s.}{=} \lim_{n\to\infty} \langle \boldsymbol{q}^0, \boldsymbol{q}^0\rangle_n \stackrel{a.s.}{=} \sigma_1^2.$$

Then for assertion (e), by Lemma E.2 (d), we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n (y_i^1)^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n [\mathbf{N} \mathbf{q}^0]_i^2 \stackrel{a.s.}{=} \lim_{n \to \infty} \langle \mathbf{q}^0, \mathbf{q}^0 \rangle_n \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n z_i^2,$$

which implies the assertion. Next, we prove the assertion (b) for the pair $(\mathbf{h}^1, \mathbf{y}^1)$. Let us observe that, as ϕ_h is partially pseudo-Lipschitz, by Lemma E.2 (d) with r = 1 and m = 2, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left[\phi_h(h_i^1, y_i^1, \xi_{0,i}, x_{0,i}) - \phi_h(h_i^1, z_i, \xi_{0,i}, x_{0,i}) \right] \stackrel{a.s.}{=} 0,$$

where $\boldsymbol{z} \sim N_n(0, (\|\boldsymbol{q}^0\|^2/n)\boldsymbol{I}_n)$. Now conditional on $\mathcal{G}_{1,0}$, we have

$$\frac{1}{n}\sum_{i=1}^{n}\phi_{h}(h_{i}^{1},z_{i},\xi_{0,i},x_{0,i})\Big|_{\mathcal{G}_{1,0}}\stackrel{d}{=}\frac{1}{n}\sum_{i=1}^{n}\phi_{h}([\widetilde{\boldsymbol{L}}^{\top}\boldsymbol{m}^{0}]_{i}+o(1)q_{i}^{0},z_{i},\xi_{0,i},x_{0,i}).$$

Then using the techniques used to prove $\mathcal{H}_1(b)$ of [9], we can show

$$\frac{1}{n}\sum_{i=1}^{n} \left[\mathbb{E}\left\{ \phi_h([\widetilde{\boldsymbol{L}}^{\top}\boldsymbol{m}^0]_i, z_i, \xi_{0,i}, x_{0,i}) | \mathcal{G}_{1,0} \right\} - \mathbb{E}\left\{ \phi_h(\tau_0 Z_i, \sigma_1 \widetilde{Z}_i, \xi_{0,i}, x_{0,i}) | \mathcal{G}_{1,0} \right\} \right] \stackrel{a.s.}{=} 0.$$

It is easy to see that

$$\mathbb{E}\left\{\phi_h(\tau_0 Z_i, \sigma_1 \widetilde{Z}_i, \xi_{0,i}, x_{0,i}) | \mathcal{G}_{1,0}\right\} = \mathbb{E}_{Z_i, \widetilde{Z}_i}\left\{\phi_h(\tau_0 Z_i, \sigma_1 \widetilde{Z}_i, \xi_{0,i}, x_{0,i})\right\},\$$

where the expectation is taken with respect to Z_i, \widetilde{Z}_i , treating $\xi_{0,i}, x_{0,i}$ as constants. Now using SLLN for i.i.d $\psi(\xi_{0,i}, x_{0,i}) = \mathbb{E}_{Z,\widetilde{Z}} \left\{ \phi_h(\tau_0 Z, \sigma_1 \widetilde{Z}, \xi_{0,i}, x_{0,i}) \right\}$ we get

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left\{\phi_{h}([\widetilde{\boldsymbol{L}}^{\top}\boldsymbol{m}^{0}]_{i}, z_{i}, \xi_{0,i}, x_{0,i})|\mathcal{G}_{1,0}\right\} \stackrel{a.s.}{=} \mathbb{E}\left\{\phi_{h}(\tau_{0}Z, \sigma_{1}\widetilde{Z}, \widetilde{\Xi}_{0}, \widetilde{X}_{0})\right\}.$$

On the right side of the last display, we have that Z, \tilde{Z} and $(\tilde{\Xi}_0, \tilde{X}_0)$ are mutually independent. By our foregoing arguments, the randomness of Z comes from that of \tilde{L} , the randomness of \tilde{Z} comes from that of N, and both are independent of $(\boldsymbol{\xi}_0, \boldsymbol{x}_0)$. To prove the assertion (d) for \boldsymbol{h}^1 and \boldsymbol{y}^1 , we use part (b) for partially pseudo-Lipschitz function $\phi_h(x, z, \xi_{0,i}, x_{0,i}) = x\varphi(x, z, \xi_{0,i}, x_{0,i})$ (by Lemma 7.1(1)) to obtain $\lim_{n\to\infty} \langle \boldsymbol{h}^1, \varphi(\boldsymbol{h}^1, \boldsymbol{x}^1, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \rangle_n \stackrel{a.s.}{=} \mathbb{E} \left\{ \tau_0 Z \varphi(\tau_0 Z, \sigma_1 \widetilde{Z}, \widetilde{\Xi}_0, \widetilde{X}_0) \right\}$. Now, we have using Lemma 4 of [9]

$$\mathbb{E}\left\{\tau_{0}Z\varphi(\tau_{0}Z,\sigma_{1}\widetilde{Z},\widetilde{\Xi}_{0},\widetilde{X}_{0})\right\} = \mathbb{E}\left\{\mathbb{E}\left\{\tau_{0}Z\varphi(\tau_{0}Z,\sigma_{1}\widetilde{Z},\widetilde{\Xi}_{0},\widetilde{X}_{0})|\widetilde{Z},\widetilde{\Xi}_{0},\widetilde{X}_{0}\right\}\right\}$$
$$= \mathbb{E}\left\{\mathbb{E}\left\{\tau_{0}^{2}\varphi_{1}'(\tau_{0}Z,\sigma_{1}\widetilde{Z},\widetilde{\Xi}_{0},\widetilde{X}_{0})|\widetilde{Z},\widetilde{\Xi}_{0},\widetilde{X}_{0}\right\}\right\}$$
$$= \tau_{0}^{2}\mathbb{E}\left\{\varphi_{1}'(\tau_{0}Z,\sigma_{1}\widetilde{Z},\widetilde{\Xi}_{0},\widetilde{X}_{0})\right\}.$$

The second equality holds since Z is independent of $\widetilde{Z}, \widetilde{\Xi}_0$ and \widetilde{X}_0 . Note that $\tau_0^2 = \lim_{n \to \infty} \langle \boldsymbol{h}^1, \boldsymbol{h}^1 \rangle_n$. Using part (b) and the fact that φ_1' is partially Lipschitz we get $\lim_{n \to \infty} \langle \varphi_1'(\boldsymbol{h}^1, \boldsymbol{x}^1, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \rangle_n \stackrel{a.s.}{=} \mathbb{E} \left\{ \varphi_1'(\tau_0 Z, \sigma_1 \widetilde{Z}, \widetilde{\Xi}_0, \widetilde{X}_0) \right\}$. The assertion about

 $\lim_{n\to\infty} \langle \boldsymbol{y}^1, \varphi(\boldsymbol{h}^1, \boldsymbol{y}^1, \boldsymbol{\xi}_0, \boldsymbol{x}_0) \rangle_n$ follows similarly. Finally, since t = 0 and $\boldsymbol{q}^0 = \boldsymbol{q}_{\perp}^0$, the assertion (g) follows from

$$\lim_{p\to\infty} \langle \boldsymbol{q}^0, \boldsymbol{q}^0 \rangle_n = \frac{1}{c} \vartheta_0^2 < \infty.$$

Inductive Step. Let us assume that the assertions (a)-(g) for b^s , h^{s+1} and y^{s+1} for $0 \le s \le t-1$. We shall show that the assertions hold for s = t. The assertions for b^t follows exactly using the same arguments used to prove \mathcal{B}_t , (a)-(g) of [9].

Next, we consider the assertion (g) for h^{t+1} and y^{t+1} . Applying the induction hypothesis to the partially pseudo-Lipschitz function $\phi_b(h_i^r, y_i^r, \xi_{0,i}, x_{0,i}) = f_r(h_i^r, y_i^r, \xi_{0,i}, x_{0,i})$ and $f_s(h_i^s, y_i^s, \xi_{0,i}, x_{0,i})$ (by Lemma 7.1(1)) such that for $1 \leq r, s \leq t$, we have almost surely

$$\lim_{n \to \infty} \langle \boldsymbol{q}^r, \boldsymbol{q}^s \rangle_n = \mathbb{E} \left\{ \mathsf{f}_r(\tau_{r-1} Z_{r-1}, \sigma_r \widetilde{Z}_r, \widetilde{\Xi}_0, \widetilde{X}_0) \mathsf{f}_s(\tau_{s-1} Z_{s-1}, \sigma_s \widetilde{Z}_s, \widetilde{\Xi}_0, \widetilde{X}_0) \right\}.$$

Further

$$\langle \boldsymbol{q}_{\perp}^{t}, \boldsymbol{q}_{\perp}^{t} \rangle_{n} = \langle \boldsymbol{q}^{t}, \boldsymbol{q}^{t} \rangle_{n} - \frac{(\boldsymbol{q}^{t})^{\top} \boldsymbol{Q}_{t}}{n} \left[\frac{\boldsymbol{Q}_{t}^{\top} \boldsymbol{Q}_{t}}{n} \right]^{-1} \frac{\boldsymbol{Q}_{t}^{\top} \boldsymbol{q}^{t}}{n}$$

Using induction hypotheses part (g), we have $\lim_{p\to\infty} \langle \boldsymbol{q}_{\perp}^r, \boldsymbol{q}_{\perp}^r \rangle_n > \rho_r$ for all $r \leq t-1$. Now using Lemma 9 of [9], for large enough *n* the smallest eigenvalue of matrix $\boldsymbol{Q}_t^\top \boldsymbol{Q}_t/n$ is larger than positive constant *c'* independent of *n*. By Lemma 10 of [9], $\boldsymbol{Q}_t^\top \boldsymbol{Q}_t/n$ converges to an invertible limit. Hence, we have

$$\lim_{n \to \infty} \langle \boldsymbol{q}_{\perp}^t, \boldsymbol{q}_{\perp}^t \rangle_n \stackrel{a.s.}{=} \mathbb{E} \left\{ [\mathbf{f}_t(\tau_{t-1} Z_{t-1}, \sigma_t \widetilde{Z}_t, \widetilde{\Xi}_0, \widetilde{X}_0)]^2 \right\} - u^{\top} C^{-1} u$$

with $u \in \mathbb{R}^t$ and $C \in \mathbb{R}^t \times \mathbb{R}^t$ such that $1 \leq r, s \leq t$:

$$u_r = \mathbb{E}\left\{\mathsf{f}_r(\tau_{r-1}Z_{r-1}, \sigma_r\widetilde{Z}_r, \widetilde{\Xi}_0, \widetilde{X}_0)\mathsf{f}_t(\tau_{t-1}Z_{t-1}, \sigma_t\widetilde{Z}_t, \widetilde{\Xi}_0, \widetilde{X}_0)\right\},\,$$

and

$$C_{r,s} = \mathbb{E}\left\{\mathsf{f}_r(\tau_{r-1}Z_{r-1}, \sigma_r\widetilde{Z}_r, \widetilde{\Xi}_0, \widetilde{X}_0)\mathsf{f}_s(\tau_{s-1}Z_{s-1}, \sigma_s\widetilde{Z}_s, \widetilde{\Xi}_0, \widetilde{X}_0)\right\}.$$

If we show $\operatorname{Var}[\tau_{r-1}Z_{r-1}|\tau_0Z_0,\ldots,\tau_{r-2}Z_{r-2},\sigma_1\widetilde{Z}_1,\ldots,\sigma_{r-1}\widetilde{Z}_{r-1}]$ and $\operatorname{Var}[\sigma_r\widetilde{Z}_r|\tau_0Z_0,\ldots,\tau_{r-2}Z_{r-2},\sigma_1\widetilde{Z}_1,\ldots,\sigma_{r-1}\widetilde{Z}_{r-1}]$ are strictly positive for $1 \leq r \leq t$, then using Lemma E.3 the

result follows. Using the induction hypotheses, part (b), and the techniques similar to the proof of \mathcal{B}_t (g) of [9], we have for all $1 \leq r, s \leq t$:

$$\lim_{n \to \infty} \langle \boldsymbol{h}_{\perp}^{r}, \boldsymbol{h}_{\perp}^{r} \rangle_{n} = \lim_{n \to \infty} \left(\langle \boldsymbol{h}^{r}, \boldsymbol{h}^{r} \rangle_{n} - \frac{(\boldsymbol{h}^{r})^{\top} \boldsymbol{H}_{r}}{n} \left[\frac{(\boldsymbol{H}_{r})^{\top} \boldsymbol{H}_{r}}{n} \right]^{-1} \frac{\boldsymbol{H}_{r}^{\top} \boldsymbol{h}^{r}}{n} \right)$$
$$= \operatorname{Var}[\tau_{r-1} Z_{r-1} | \tau_{0} Z_{0}, \dots, \tau_{r-2} Z_{r-2}, \sigma_{1} \widetilde{Z}_{1}, \dots, \sigma_{r-1} \widetilde{Z}_{r-1}].$$

Next, using part (c) of the induction hypotheses, we have almost surely

$$\lim_{n \to \infty} \langle \boldsymbol{h}_{\perp}^{r}, \boldsymbol{h}_{\perp}^{r} \rangle_{n} = \lim_{n \to \infty} \left(\langle \boldsymbol{h}^{r}, \boldsymbol{h}^{r} \rangle_{n} - \frac{(\boldsymbol{h}^{r})^{\top} \boldsymbol{H}_{r}}{n} \left[\frac{(\boldsymbol{H}_{r})^{\top} \boldsymbol{H}_{r}}{n} \right]^{-1} \frac{\boldsymbol{H}_{r}^{\top} \boldsymbol{h}^{r}}{n} \right)$$
$$= \lim_{p \to \infty} \left(\langle \boldsymbol{m}^{r-1}, \boldsymbol{m}^{r-1} \rangle_{p} - \frac{(\boldsymbol{m}^{r-1}) \boldsymbol{M}_{r-1}^{\top}}{p} \left[\frac{(\boldsymbol{M}_{r-1})^{\top} \boldsymbol{M}_{r-1}}{p} \right]^{-1} \frac{\boldsymbol{M}_{r-1}^{\top} \boldsymbol{m}^{r-1}}{p} \right)$$
$$= \lim_{p \to \infty} \langle \boldsymbol{m}_{\perp}^{r-1}, \boldsymbol{m}_{\perp}^{r-1} \rangle_{p}.$$

Using part (g) of the induction hypotheses, we have $\lim_{p\to\infty} \langle \boldsymbol{m}_{\perp}^{r-1}, \boldsymbol{m}_{\perp}^{r-1} \rangle_p > \varkappa_{r-1} > 0$, and hence the result follows.

The assertion (a) for h^{t+1} follows using the techniques used to prove $\mathcal{H}_{t+1}(a)$ of [9]. To prove the same for y^{t+1} , we consider $\mathcal{F}_{t+1,t}$ defined in (E.2), we get

$$\mathcal{F}_{t+1,t} \boldsymbol{q}_{\perp}^t = \boldsymbol{Q}_t (\boldsymbol{Q}_t^\top \boldsymbol{Q}_t)^{-1} \bar{\boldsymbol{Y}}_t^\top \boldsymbol{q}_{\perp}^t.$$

Further, note that using $\boldsymbol{Q}_t^{\top} \bar{\boldsymbol{Y}}_t = \bar{\boldsymbol{Y}}_t^{\top} \boldsymbol{Q}_t$ we get

$$\mathcal{F}_{t+1,t} \boldsymbol{q}_{\parallel}^t = ar{\boldsymbol{Y}}_t (\boldsymbol{Q}_t^\top \boldsymbol{Q}_t)^{-1} \boldsymbol{Q}_t^\top \boldsymbol{q}_{\parallel}^t$$

Combining these two equations we get

$$\mathcal{F}_{t+1,t} oldsymbol{q}^t = oldsymbol{Q}_t (oldsymbol{Q}_t^ op oldsymbol{Q}_t)^{-1} oldsymbol{ar{Y}}_t^ op oldsymbol{q}_\perp^t + oldsymbol{ar{Y}}_t (oldsymbol{Q}_t^ op oldsymbol{Q}_t)^{-1} oldsymbol{Q}_t^ op oldsymbol{q}_\parallel^ op$$

Then using (E.1) we get

$$\boldsymbol{y}^{t+1}|_{\mathcal{G}_{t+1,t}} \stackrel{d}{=} \boldsymbol{Q}_t(\boldsymbol{Q}_t^\top \boldsymbol{Q}_t)^{-1} \bar{\boldsymbol{Y}}_t^\top \boldsymbol{q}_\perp^t + \bar{\boldsymbol{Y}}_t(\boldsymbol{Q}_t^\top \boldsymbol{Q}_t)^{-1} \boldsymbol{Q}_t^\top \boldsymbol{q}_\parallel^t + P_{\boldsymbol{Q}_t}^\perp \widetilde{\boldsymbol{N}} P_{\boldsymbol{Q}_t}^\perp \boldsymbol{q}^t - d_t \boldsymbol{q}^{t-1}.$$

Now $\bar{\boldsymbol{Y}}_t = \boldsymbol{Y}_t + [0|\boldsymbol{Q}_{t-1}]\boldsymbol{D}_t$, where $\boldsymbol{Y}_t = [\boldsymbol{y}^1|\dots|\boldsymbol{y}^t]$ and $\boldsymbol{D}_t = \text{diag}(d_0,\dots,d_{t-1})$. As $\bar{\boldsymbol{Y}}_t^\top \boldsymbol{q}_\perp^t = \boldsymbol{Y}_t^\top \boldsymbol{q}_\perp^t$, so we need to show

$$\boldsymbol{Q}_t(\boldsymbol{Q}_t^{\top}\boldsymbol{Q}_t)^{-1}\boldsymbol{Y}_t^{\top}\boldsymbol{q}_{\perp}^t + [0|\boldsymbol{Q}_{t-1}]\boldsymbol{D}_t(\boldsymbol{Q}_t^{\top}\boldsymbol{Q}_t)^{-1}\boldsymbol{Q}_t^{\top}\boldsymbol{q}^t - d_t\boldsymbol{q}^{t-1} = \boldsymbol{Q}_t\boldsymbol{o}(1),$$

or equivalently

$$[0|\boldsymbol{Q}_{t-1}]\boldsymbol{D}_t\boldsymbol{\beta}^t + \boldsymbol{Q}_t(\boldsymbol{Q}_t^{\top}\boldsymbol{Q}_t)^{-1}\boldsymbol{Y}_t^{\top}\boldsymbol{q}_{\perp}^t - d_t\boldsymbol{q}^{t-1} = \boldsymbol{Q}_to(1),$$

We need to show that the coefficients of $q^{\ell-1}$ converge to zero for $\ell = 1, \ldots, t$. Now the coefficient of $q^{\ell-1}$ is given by

$$\left[\boldsymbol{Q}_{t}(\boldsymbol{Q}_{t}^{\top}\boldsymbol{Q}_{t})^{-1}\boldsymbol{Y}_{t}^{\top}\boldsymbol{q}_{\perp}^{t}\right]_{\ell} - d_{l}(-\beta_{\ell}^{t})^{\mathbb{I}_{\ell\neq t}} = \sum_{k=1}^{t} \left[\left(\frac{\boldsymbol{Q}_{t}^{\top}\boldsymbol{Q}_{t}}{n}\right)^{-1} \right]_{\ell,k} \langle \boldsymbol{y}^{k}, \boldsymbol{q}^{t} - \sum_{s=0}^{t-1} \beta_{s}^{t}\boldsymbol{q}^{s} \rangle_{n} - d_{\ell}(-\beta_{\ell}^{t})^{\mathbb{I}_{\ell\neq t}}.$$

Denoting $\boldsymbol{Q}_t^\top \boldsymbol{Q}_t / n$ by \boldsymbol{G} , we get

$$\lim_{n \to \infty} \text{Coefficient of } \boldsymbol{q}^{\ell-1} = \lim_{n \to \infty} \left\{ \sum_{k=1}^t (G^{-1})_{\ell,k} \langle \boldsymbol{y}^k, \boldsymbol{q}^t - \sum_{s=0}^{t-1} \beta_s^t \boldsymbol{q}^s \rangle_n - d_\ell (-\beta_\ell^t)^{\mathbb{I}_{\ell \neq t}} \right\}.$$

Using parts (c) and (d) of the induction hypotheses, for k = 1, ..., t, we get

$$\lim_{n \to \infty} \text{Coefficient of } \boldsymbol{q}^{\ell-1} \stackrel{a.s.}{=} \lim_{n \to \infty} \left\{ \sum_{k=1}^{t} (G^{-1})_{\ell,k} \langle \boldsymbol{y}^{k}, d_{t} \boldsymbol{x}^{t} - \sum_{s=0}^{t-1} \beta_{s}^{t} d_{s} \boldsymbol{y}^{s} \rangle_{n} - d_{\ell} (-\beta_{\ell}^{t})^{\mathbb{I}_{\ell \neq t}} \right\}$$
$$\stackrel{a.s.}{=} \lim_{n \to \infty} \left\{ \sum_{k=1}^{t} (G^{-1})_{\ell,k} [G_{k,t} d_{t} - \sum_{s=0}^{t-1} \beta_{s}^{t} G_{k,s} d_{s}] - d_{\ell} (-\beta_{\ell}^{t})^{\mathbb{I}_{\ell \neq t}} \right\}$$
$$= \lim_{n \to \infty} \left\{ d_{t} \mathbb{I}_{t=\ell} - \sum_{s=0}^{t-1} \beta_{s}^{t} d_{s} \mathbb{I}_{\ell=s} - d_{\ell} (-\beta_{\ell}^{t})^{\mathbb{I}_{\ell \neq t}} \right\}$$
$$= 0.$$

This implies

$$\boldsymbol{y}^{t+1}|_{\mathcal{G}_{t+1,t}} \stackrel{d}{=} \sum_{i=0}^{t-1} \beta_i^t \boldsymbol{y}^{i+1} + \widetilde{\boldsymbol{N}} \boldsymbol{q}_{\perp}^t - P_{\boldsymbol{Q}_t} \widetilde{\boldsymbol{N}} \boldsymbol{q}_{\perp}^t + \boldsymbol{Q}_t o(1).$$

Using the fact that

$$\widetilde{\boldsymbol{N}} = rac{1}{2} \left(\boldsymbol{N}_1^\top + \boldsymbol{N}_1
ight),$$

where all entries of N_1 are distributed as N(0, 1/n). Hence

$$P_{\boldsymbol{Q}_t} \widetilde{\boldsymbol{N}} \boldsymbol{q}_{\perp}^t = \frac{1}{2} \left(\sum_{i=1}^t \langle \widetilde{\boldsymbol{q}}_i, \boldsymbol{N}_1^\top \boldsymbol{q}_{\perp}^t \rangle_n \widetilde{\boldsymbol{q}}_i \right) + \frac{1}{2} \left(\sum_{i=1}^t \langle \widetilde{\boldsymbol{q}}_i, \boldsymbol{N}_1 \boldsymbol{q}_{\perp}^t \rangle_n \widetilde{\boldsymbol{q}}_i \right),$$

where \tilde{q}_i are columns of \tilde{Q}_t . Using Lemma E.2 (b) along with arguments similar to the proof of $\mathcal{H}_{t+1}(\mathbf{a})$ of [9] and $\langle \boldsymbol{q}_{\perp}^t, \boldsymbol{q}_{\perp}^t \rangle_n < \infty$, we get

$$P_{\boldsymbol{Q}_t} \widetilde{\boldsymbol{N}} \boldsymbol{q}_{\perp}^t = \widetilde{\boldsymbol{Q}}_t o(1).$$

Hence we have the result.

Using the induction hypotheses and the proof of $\mathcal{H}_{t+1}(d)$ of [9], one can prove the assertion (d) for both h^{t+1} and y^{t+1} .

The assertion (e) for h^{t+1} follows using the proof of $\mathcal{H}_{t+1}(e)$ of [9]. To show the same for y^{t+1} we condition on $\mathcal{G}_{t+1,t}$, and using the assertion (a) we get

$$\frac{1}{n}\sum_{i=1}^{n}(y_{l}^{t+1})^{2} \leq \frac{C}{n}\sum_{i=1}^{n}\left(\sum_{r=0}^{t-1}\beta_{r}^{t}\boldsymbol{y}_{i}^{r+1}\right)^{2} + \frac{C}{n}\sum_{i=1}^{n}\left([P_{\boldsymbol{Q}_{t}}^{\perp}\widetilde{\boldsymbol{N}}^{\top}\boldsymbol{q}_{\perp}^{t}]_{i}\right)^{2} + o(1)\frac{C}{n}\sum_{r=0}^{t-1}\sum_{i=1}^{n}\left([\boldsymbol{q}^{r}]_{i}\right)^{2}.$$

Now using the techniques described in \mathcal{B}_t (e) of [9], we can show

$$\frac{C}{n}\sum_{i=1}^{n}\left(\sum_{r=0}^{t-1}\boldsymbol{\beta}_{r}^{t}\boldsymbol{y}_{i}^{r+1}\right)^{2}<\infty,$$

and

$$\frac{C}{n}\sum_{r=0}^{t-1}\sum_{i=1}^{n}\left([\boldsymbol{q}^{r}]_{i}\right)^{2}<\infty.$$

Finally

$$\frac{C}{n}\sum_{i=1}^{n}\left([P_{\boldsymbol{Q}_{t}}^{\perp}\widetilde{\boldsymbol{N}}^{\top}\boldsymbol{q}_{\perp}^{t}]_{i}\right)^{2} \leqslant O\left(\frac{C}{n}\sum_{i=1}^{n}\left([\widetilde{\boldsymbol{N}}^{\top}\boldsymbol{q}_{\perp}^{t}]_{i}\right)^{2}\right) + O\left(\frac{C}{n}\sum_{i=1}^{n}\left([P_{\boldsymbol{Q}_{t}}\widetilde{\boldsymbol{N}}^{\top}\boldsymbol{q}_{\perp}^{t}]_{i}\right)^{2}\right).$$

Using Lemma E.2 (d) with $\varphi_n(\boldsymbol{x}) = (\|\boldsymbol{x}\|^2)/n$ and $\langle \boldsymbol{q}_{\perp}^t, \boldsymbol{q}_{\perp}^t \rangle_n < \langle \boldsymbol{q}^t, \boldsymbol{q}^t \rangle_n < \infty$, we show that the first term is finite. We show that the second term is finite using Lemma E.2 (b). Hence, we have

$$\frac{1}{n}\sum_{i=1}^n(y_l^{t+1})^2<\infty$$

To show part (b) for \boldsymbol{h}^{t+1} and \boldsymbol{y}^{t+1} we use part (a) to get

$$\begin{split} \phi_h(h_i^1, \dots, h_i^{t+1}, y_i^1, \dots, y_i^{t+1}, \xi_{0,i}, x_{0,i}) \big|_{\mathcal{G}_{t+1,t}} \\ \stackrel{d}{=} \phi_h\Big(h_i^1, \dots, h_i^t, \left[\sum_{i=0}^{t-1} \alpha_i^t \boldsymbol{h}^{i+1} + \widetilde{\boldsymbol{L}}^\top \boldsymbol{m}_{\perp}^t + \widetilde{\boldsymbol{Q}}_{t+1} o(1)\right]_i, y_i^1, \dots, y_i^t, \\ & \left[\sum_{i=0}^{t-1} \beta_i^t \boldsymbol{y}^{i+1} + \widetilde{N} \boldsymbol{q}_{\perp}^t + \widetilde{\boldsymbol{Q}}_t o(1)\right]_i, \xi_{0,i}, x_{0,i}\Big). \end{split}$$

Firstly, using Lemma E.2 (d) with r = 2t + 1 and m = 2, we get

$$\frac{1}{n}\sum_{i=1}^{n}\phi_{h}\left(h_{i}^{1},\ldots,h_{i}^{t},\left[\sum_{i=0}^{t-1}\alpha_{i}^{t}\boldsymbol{h}^{i+1}+\widetilde{\boldsymbol{L}}^{\top}\boldsymbol{m}_{\perp}^{t}+\widetilde{\boldsymbol{Q}}_{t+1}\boldsymbol{o}(1)\right]_{i},y_{i}^{1},\ldots,y_{i}^{t},\\
\left[\sum_{i=0}^{t-1}\beta_{i}^{t}\boldsymbol{y}^{i+1}+\widetilde{N}\boldsymbol{q}_{\perp}^{t}+\widetilde{\boldsymbol{Q}}_{t}\boldsymbol{o}(1)\right]_{i},\xi_{0,i},x_{0,i}\right)\\
-\frac{1}{n}\sum_{i=1}^{n}\phi_{h}\left(h_{i}^{1},\ldots,h_{i}^{t},\left[\sum_{i=0}^{t-1}\alpha_{i}^{t}\boldsymbol{h}^{i+1}+\widetilde{\boldsymbol{L}}^{\top}\boldsymbol{m}_{\perp}^{t}+\widetilde{\boldsymbol{Q}}_{t+1}\boldsymbol{o}(1)\right]_{i},y_{i}^{1},\ldots,y_{i}^{t}\right)\\
\left[\sum_{i=0}^{t-1}\beta_{i}^{t}\boldsymbol{y}^{i+1}+\boldsymbol{z}\frac{\|\boldsymbol{q}_{\perp}^{t}\|}{\sqrt{n}}+\widetilde{\boldsymbol{Q}}_{t}\boldsymbol{o}(1)\right]_{i},\xi_{0,i},x_{0,i}\right)\\
\xrightarrow{a.s.}0,$$

where $\boldsymbol{z} \sim N_n(0, \boldsymbol{I}_n)$ is independent of everything else. Now using the techniques used in \mathcal{B}_t (b) of [9] we can remove the terms $\tilde{\boldsymbol{Q}}_t o(1)$ and $\tilde{\boldsymbol{Q}}_{t+1} o(1)$. So it is enough to consider

$$\widetilde{X}_{n,i} = \phi_h \left(h_i^1, \dots, h_i^t, \left[\sum_{i=0}^{t-1} \alpha_i \boldsymbol{h}^{i+1} + \widetilde{\boldsymbol{L}}^\top \frac{\|\boldsymbol{m}_{\perp}^t\|}{\sqrt{p}} \right]_i, y_i^1, \dots, y_i^t, \left[\sum_{i=0}^{t-1} \beta_i^t \boldsymbol{y}^{i+1} + \boldsymbol{z} \frac{\|\boldsymbol{q}_{\perp}^t\|}{\sqrt{n}} \right]_i, \xi_{0,i}, x_{0,i} \right).$$

It is easy to verify the conditions of Theorem 3 of [9] conditionally on $\mathcal{G}_{t+1,t}$ and hence we get

for $\boldsymbol{z}_1, \boldsymbol{z}_2 \stackrel{iid}{\sim} N_n(0, \boldsymbol{I}_n)$, given $\mathcal{G}_{t+1,t}$

$$\frac{1}{n}\sum_{i=1}^{n}\left\{\phi_{h}\left(h_{i}^{1},\ldots,h_{i}^{t},\left[\sum_{i=0}^{t-1}\alpha_{i}^{t}\boldsymbol{h}^{i+1}+\tilde{\boldsymbol{L}}^{\top}\boldsymbol{m}_{\perp}^{t}\right]_{i},y_{i}^{1},\ldots,y_{i}^{t},\left[\sum_{i=0}^{t-1}\beta_{i}^{t}\boldsymbol{y}^{i+1}+\boldsymbol{z}\frac{\|\boldsymbol{q}_{\perp}^{t}\|}{\sqrt{n}}\right]_{i},\xi_{0,i},x_{0,i}\right)\right.\\\left.-\mathbb{E}_{\boldsymbol{z}_{1},\boldsymbol{z}_{2}}\left[\phi_{h}\left(h_{i}^{1},\ldots,h_{i}^{t},\left[\sum_{i=0}^{t-1}\alpha_{i}^{t}\boldsymbol{h}^{i+1}+\boldsymbol{z}_{1}\frac{\|\boldsymbol{m}_{\perp}^{t}\|}{\sqrt{p}}\right]_{i},y_{i}^{1},\ldots,y_{i}^{t},\left[\sum_{i=0}^{t-1}\beta_{i}^{t}\boldsymbol{y}^{i+1}+\boldsymbol{z}_{2}\frac{\|\boldsymbol{q}_{\perp}^{t}\|}{\sqrt{n}}\right]_{i},\xi_{0,i},x_{0,i}\right)\right]\right\}\xrightarrow{a.s.}0.$$

It follows that we also have this marginally. Let $\delta_t = \lim_{n \to \infty} \frac{\|\boldsymbol{m}_{\perp}^t\|}{\sqrt{n}}$ and $\rho_t = \lim_{n \to \infty} \frac{\|\boldsymbol{q}_{\perp}^t\|}{\sqrt{n}}$. Then by partially pseudo-Lipschitz property of ϕ_h

$$\frac{1}{n}\sum_{i=1}^{n}\left\{\mathbb{E}_{\boldsymbol{z}_{1},\boldsymbol{z}_{2}}\left[\phi_{h}\left(h_{i}^{1},\ldots,h_{i}^{t},\left[\sum_{i=0}^{t-1}\alpha_{i}^{t}\boldsymbol{h}^{i+1}+\boldsymbol{z}_{1}\frac{\|\boldsymbol{m}_{\perp}^{t}\|}{\sqrt{p}}\right]_{i},\right.\right.$$
$$\left.y_{i}^{1},\ldots,y_{i}^{t},\left[\sum_{i=0}^{t-1}\beta_{i}^{t}\boldsymbol{y}^{i+1}+\boldsymbol{z}_{2}\frac{\|\boldsymbol{q}_{\perp}^{t}\|}{\sqrt{n}}\right]_{i},\xi_{0,i},x_{0,i}\right)\right]$$
$$\left.-\mathbb{E}_{\boldsymbol{z}_{1},\boldsymbol{z}_{2}}\left[\phi_{h}\left(h_{i}^{1},\ldots,h_{i}^{t},\left[\sum_{i=0}^{t-1}\alpha_{i}^{t}\boldsymbol{h}^{i+1}+\boldsymbol{z}_{1}\delta_{t}\right]_{i},y_{i}^{1},\ldots,y_{i}^{t},\left[\sum_{i=0}^{t-1}\beta_{i}^{t}\boldsymbol{y}^{i+1}+\boldsymbol{z}_{2}\rho_{t}\right]_{i},\right.$$
$$\left.\xi_{0,i},x_{0,i}\right)\right]\right\}\xrightarrow{a.s.}0.$$

Now consider the partially pseudo-Lipschitz function

$$\hat{\phi}_{h}(h_{i}^{1},\ldots,h_{i}^{t},y_{i}^{1},\ldots,y_{i}^{t},\xi_{0,i},x_{0,i})$$

$$= \mathbb{E}_{\boldsymbol{z}_{1},\boldsymbol{z}_{2}} \left\{ \phi_{h} \left(h_{i}^{1},\ldots,h_{i}^{t}, \left[\sum_{i=0}^{t-1} \alpha_{i}^{t} \boldsymbol{h}^{i+1} + \delta_{t} \boldsymbol{z}_{1} \right]_{i}, y_{i}^{1},\ldots,y_{i}^{t}, \left[\sum_{i=0}^{t-1} \beta_{i}^{t} \boldsymbol{y}^{i+1} + \rho_{t} \boldsymbol{z}_{2} \right]_{i}, \xi_{0,i}, x_{0,i} \right) \right\}.$$

That it is partially pseudo-Lipschitz follows by Lemma 7.1(2). By the induction hypothesis part (b), we get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \hat{\phi}_h(h_i^1, \dots, h_i^t, y_i^1, \dots, y_i^t, \xi_{0,i}, x_{0,i})$$

= $\mathbb{E} \left\{ \phi_h \left(\tau_0 Z_0, \dots, \tau_{t-1} Z_{t-1}, \sum_{i=0}^{t-1} \alpha_i^t \tau_i Z_i + \delta_t Z, \sigma_1 \widetilde{Z}_1, \dots, \sigma_t \widetilde{Z}_t, \sum_{i=0}^{t-1} \beta_i^t \sigma_{i+1} \widetilde{Z}_{i+1} + \rho_t \widetilde{Z}, \widetilde{\Xi}_0, \widetilde{X}_0 \right) \right\},$

As both $\tau_t Z_t = \sum_{i=0}^{t-1} \alpha_i^t \tau_i Z_i + \delta_t Z$ and $\sigma_{t+1} \widetilde{Z}_{t+1} = \sum_{i=0}^{t-1} \beta_i^t \sigma_{i+1} \widetilde{Z}_{i+1} + \rho_t \widetilde{Z}$ are centered Gaussians, it is enough to show their variances are τ_t^2 and σ_{t+1}^2 respectively. Proceeding as in \mathcal{B}_t (b)

of [9]

$$\mathbb{E}\left\{\sum_{i=0}^{t-1} \boldsymbol{\alpha}_{i}^{t} \tau_{i} Z_{i} + \boldsymbol{\delta}_{t} Z\right\}^{2} \stackrel{a.s.}{=} \lim_{n \to \infty} \langle \boldsymbol{h}^{t+1}, \boldsymbol{h}^{t+1} \rangle_{n}$$
$$\stackrel{a.s.}{=} \lim_{n \to \infty} \langle \boldsymbol{m}^{t}, \boldsymbol{m}^{t} \rangle_{p}$$
$$\stackrel{a.s.}{=} \mathbb{E}\left\{g_{t}(\vartheta_{t} \boldsymbol{\check{Z}}_{t}, \widetilde{\Omega}_{0}, \boldsymbol{\check{V}}_{0})^{2}\right\}$$
$$= \tau_{t}^{2}.$$

Similarly we have

$$\mathbb{E}\left\{\sum_{i=0}^{t-1}\beta_{i}^{t}\sigma_{i+1}\widetilde{Z}_{i+1} + \rho_{t}\widetilde{Z}\right\}^{2} \stackrel{a.s.}{=} \lim_{n \to \infty} \langle \boldsymbol{y}^{t+1}, \boldsymbol{y}^{t+1} \rangle_{n}$$
$$\stackrel{a.s.}{=} \lim_{n \to \infty} \langle \boldsymbol{q}^{t}, \boldsymbol{q}^{t} \rangle_{n}$$
$$\stackrel{a.s.}{=} \mathbb{E}\left\{f_{t}(\tau_{t-1}Z_{t-1}, \sigma_{t}\widetilde{Z}_{t}, \widetilde{\Xi}_{0}, \widetilde{X}_{0})^{2}\right\}$$
$$= \sigma_{t+1}^{2}.$$

Last but not least, using induction hypotheses, and the fact that Z and \widetilde{Z} in the definition of Z_t and \widetilde{Z}_t are independent of everything else, we obtain that (Z_0, \ldots, Z_t) , $(\widetilde{Z}_1, \ldots, \widetilde{Z}_{t+1})$ and $(\widetilde{\Xi}_0, \widetilde{X}_0)$ are mutually independent. This completes the proof of part (b). Finally, the assertion (d) for \mathbf{h}^{t+1} and \mathbf{y}^{t+1} follows using the arguments similar to the base case.

Proof of Theorem 7.2 **E.4**

Let us define the following AMP which is easier to analyze. We shall show that the iterates of this AMP is asymptotically close to the iterates of the original AMP given by (2.24) and (2.25)with generic $\{f_t, g_t : t \ge 0\}$ satisfying the condition of Theorem 7.2. Let us define $\widetilde{u}^0 = \widetilde{y}^0 = 0$. Then for $t \in \mathbb{N} \cup \{0\}$, we define

$$\widetilde{\boldsymbol{v}}^{t} = \frac{\boldsymbol{R}}{\sqrt{p}} f_{t}(\alpha_{t-1}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{u}}^{t}, \mu_{t}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t}, \boldsymbol{x}_{0}(\varepsilon)) - \widetilde{p}_{t}g_{t-1}(\beta_{t-1}\boldsymbol{v}^{*} + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_{0}(\varepsilon))), \qquad (E.3)$$

$$\widetilde{\boldsymbol{u}}^{t+1} = \frac{\boldsymbol{R}^{\top}}{\sqrt{p}} g_{t}(\beta_{t}\boldsymbol{v}^{*} + \widetilde{\boldsymbol{v}}^{t}, \boldsymbol{v}_{0}(\varepsilon)) - \widetilde{c}_{t}f_{t}(\alpha_{t-1}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{u}}^{t}, \mu_{t}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t}, \boldsymbol{x}_{0}(\varepsilon)),$$

$$\widetilde{\boldsymbol{x}}^{t+1} = \frac{\boldsymbol{Z}}{\sqrt{n}} f_{t}(\alpha_{t-1}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{u}}^{t}, \mu_{t}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t}, \boldsymbol{x}_{0}(\varepsilon)) - \widetilde{d}_{t}f_{t-1}(\alpha_{t-2}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{u}}^{t-1}, \mu_{t-1}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)),$$

where

$$\begin{aligned} \widetilde{c}_t &= \frac{1}{p} \sum_{i=1}^p \frac{\partial g_t}{\partial v} (\beta_t v_i^* + \widetilde{v}_i^t, v_{0,i}(\varepsilon)), \\ \widetilde{p}_t &= \frac{c}{n} \sum_{i=1}^n \frac{\partial f_t}{\partial u} (\alpha_{t-1} x_i^* + \widetilde{u}_i^t, \mu_t x_i^* + \widetilde{x}_i^t, x_{0,i}(\varepsilon)), \\ \widetilde{d}_t &= \frac{1}{n} \sum_{i=1}^n \frac{\partial f_t}{\partial y} (\alpha_{t-1} x_i^* + \widetilde{u}_i^t, \mu_t x_i^* + \widetilde{x}_i^t, x_{0,i}(\varepsilon)). \end{aligned}$$

Let us observe that $f(u, v, x, y) = f_t(\alpha_{t-1}x + u, \mu_t x + v, y)$ and $g(u, x, y) = g_{t-1}(\beta_{t-1}x + u, y)$ are partially Lipschitz functions for all t. The iterates defined in (E.3) is of the form (7.8). Hence using Theorem 7.1 for any partially pseudo-Lipschitz functions $\hat{\phi}$ and $\hat{\psi}$, we get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \widehat{\phi}(\widetilde{u}_{i}^{t}, \widetilde{x}_{i}^{t}, x_{i}^{*}, x_{0,i}(\varepsilon)) \stackrel{a.s.}{=} \mathbb{E}\left\{\widehat{\phi}\left(\tau_{t-1}Z_{1}, \sigma_{t}Z_{2}, X_{0}, X_{0}(\varepsilon)\right)\right\},\tag{E.4}$$

and

$$\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \widehat{\psi}(\widetilde{v}_{i}^{t}, v_{i}^{*}, v_{0,i}(\varepsilon)) \stackrel{a.s.}{=} \mathbb{E}\left\{\widehat{\psi}\left(\vartheta_{t} Z_{3}, V_{0}, V_{0}(\varepsilon)\right)\right\}.$$
(E.5)

Define $\hat{\phi}(x, y, z, w) = \phi(\alpha_{t-1}z + x, \mu_t z + y, z, w)$ and $\hat{\psi}(x, y, r) = \psi(\beta_t y + x, y, r)$. It is not hard to observe that these functions are partially pseudo-Lipschitz. Then we obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi(\alpha_{t-1} x_{i}^{*} + \widetilde{u}_{i}^{t}, \mu_{t} x_{i}^{*} + \widetilde{x}_{i}^{t}, x_{0,i}(\varepsilon)) \stackrel{a.s.}{=} \mathbb{E} \left\{ \phi\left(\alpha_{t-1} X_{0} + \tau_{t-1} Z_{1}, \mu_{t} X_{0} + \sigma_{t} Z_{2}, X_{0}(\varepsilon)\right) \right\},\$$

and

$$\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \psi(\beta_t v_i^* + \widetilde{v}_i^t, v_{0,i}(\varepsilon)) \stackrel{a.s.}{=} \mathbb{E} \left\{ \psi\left(\beta_t V_0 + \vartheta_t Z_3, V_0(\varepsilon)\right) \right\}.$$

Hence, it is enough to show that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left[\phi(\alpha_{t-1} x_i^* + \widetilde{u}_i^t, \mu_t x_i^* + \widetilde{x}_i^t, x_{0,i}(\varepsilon)) - \phi(u_i^t, x_i^t, x_{0,i}(\varepsilon)) \right] \stackrel{a.s.}{=} 0,$$

and

$$\lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} \left[\psi(\beta_t v_i^* + \widetilde{v}_i^t, v_{0,i}(\varepsilon)) - \psi(v_i^t, v_{0,i}(\varepsilon)) \right] \stackrel{a.s.}{=} 0.$$

We shall prove the last two displays by induction on the following hypotheses:

1.
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[\phi(\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}, \mu_{t}x_{i}^{*} + \widetilde{x}_{i}^{t}, x_{0,i}(\varepsilon)) - \phi(u_{i}^{t}, x_{i}^{t}, x_{0,i}(\varepsilon)) \right]^{a.s.} 0,$$
2.
$$\lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} \left[\psi(\beta_{t}v_{i}^{*} + \widetilde{v}_{i}^{t}, v_{0,i}(\varepsilon)) - \psi(v_{i}^{t}, v_{0,i}(\varepsilon)) \right]^{a.s.} 0,$$
3.
$$\lim_{n\to\infty} \frac{\|\Delta_{1}^{t}\|^{2}}{n} \stackrel{a.s.}{=} 0,$$
4.
$$\lim_{n\to\infty} \frac{\|\Delta_{2}^{t}\|^{2}}{n} \stackrel{a.s.}{=} 0,$$
5.
$$\lim_{p\to\infty} \frac{\|\Delta_{2}^{t}\|^{2}}{n} \stackrel{a.s.}{=} 0,$$
6.
$$\lim_{n\to\infty} \frac{\|\alpha_{t-1}x^{*} + \widetilde{u}^{t}\|^{2}}{n} < \infty \text{ a.s.},$$
7.
$$\lim_{n\to\infty} \frac{\|\mu_{t}x^{*} + \widetilde{v}^{t}\|^{2}}{n} < \infty \text{ a.s.},$$
8.
$$\lim_{p\to\infty} \frac{\|\beta_{t}v^{*} + \widetilde{v}^{t}\|^{2}}{p} < \infty \text{ a.s.},$$
where
$$\Delta_{1}^{t} = x^{t} - \mu_{t}x^{*} - \widetilde{x}^{t}, \ \Delta_{2}^{t} = u^{t} - \alpha_{t-1}x^{*} - \widetilde{u}^{t}, \text{ and } \Delta_{3}^{t} = v^{t} - \beta_{t}v^{*} - \widetilde{v}^{t}.$$

Step 1: The t = 0 case Using $\alpha_{-1} = \mu_0 = 0$ and $u^0 = \widetilde{u}^0 = y^0 = \widetilde{y}^0 = 0$, hypotheses (1), (3), (4), (6) and (7) follows. Now note that

$$oldsymbol{v}^0 = \sqrt{rac{\mu}{np}}oldsymbol{v}^*(oldsymbol{x}^*)^{ op}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{R}) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{R}) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{0},oldsymbol{R}) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{R}) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{0},oldsymbol{R}) + rac{oldsymbol{R}}{\sqrt{p}}f_0(oldsymbol{R},oldsymbol{R}) + rac{oldsymbol{R}}{\sqrt$$

and

$$\widetilde{oldsymbol{v}}^0 = rac{oldsymbol{R}}{\sqrt{p}} f_0(oldsymbol{0},oldsymbol{0},oldsymbol{x}_0(arepsilon)).$$

We have for $i \in [p]$

$$\widetilde{v}_i^0 = \|f_0(\mathbf{0}, \mathbf{0}, \boldsymbol{x}_0(\varepsilon))\| \frac{z_i}{\sqrt{p}},$$

where z_1, \ldots, z_p are i.i.d N(0, 1) and $\boldsymbol{z} = (z_1, \ldots, z_p)^{\top}$. Then we get

$$\frac{\|\beta_0 \boldsymbol{v}^* + \widetilde{\boldsymbol{v}}^0\|^2}{p} = \beta_0^2 \frac{\|\boldsymbol{v}^*\|^2}{p} + \frac{\|f_0(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{x}_0(\varepsilon))\|^2}{p} \frac{\|\boldsymbol{z}\|^2}{p} + 2\beta_0 \frac{\|f_0(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{x}_0(\varepsilon))\|}{\sqrt{p}} \langle \boldsymbol{z}, \boldsymbol{v}^* \rangle_p.$$

By SLLN, we get that all the terms are finite. Hence Hypothesis (8) follows. We further note that

$$\lim_{p \to \infty} \frac{1}{p} \|\boldsymbol{\Delta}_3^0\|^2 = \lim_{p \to \infty} \left(\sqrt{\frac{\mu}{np}} (\boldsymbol{x}^*)^\top f_0(\boldsymbol{u}^0, \boldsymbol{y}^0, \boldsymbol{x}_0(\varepsilon)) - \beta_0 \right)^2 \frac{\|\boldsymbol{v}^*\|^2}{p}.$$

Using SLLN, definition of β_0 and $p/n \to 1/c$, we get Hypothesis (5). Again note that

$$\frac{\|\boldsymbol{v}^0\|}{\sqrt{p}} \leq \left|\sqrt{\frac{\mu}{np}}(\boldsymbol{x}^*)^\top f_0(\boldsymbol{0},\boldsymbol{0},\boldsymbol{x}_0(\varepsilon))\right| \frac{\|\boldsymbol{v}^*\|}{\sqrt{p}} + \frac{\|\boldsymbol{z}\|}{\sqrt{p}} \frac{\|f_0(\boldsymbol{0},\boldsymbol{0},\boldsymbol{x}_0(\varepsilon))\|}{\sqrt{p}}$$

Then using SLLN we get $\lim_{p\to\infty} \|\boldsymbol{v}^0\|/\sqrt{p} < \infty$ almost surely. Now using the partially pseudo-Lipschitz property of ψ , we get

$$\begin{aligned} \left| \frac{1}{p} \sum_{i=1}^{p} \left[\psi(\beta_{0} \boldsymbol{v}_{0,i}^{*} + \widetilde{\boldsymbol{v}}_{i}^{0}, \boldsymbol{v}_{0,i}(\varepsilon)) - \psi(\boldsymbol{v}_{i}^{0}, \boldsymbol{v}_{0,i}(\varepsilon)) \right] \right| \\ & \leq \left(1 + \frac{\|\beta_{0} \boldsymbol{v}^{*} + \widetilde{\boldsymbol{v}}^{0}\|}{\sqrt{p}} + \frac{\|\boldsymbol{v}^{0}\|}{\sqrt{p}} + \frac{\|\boldsymbol{v}_{0}(\varepsilon)\|}{\sqrt{p}} \right) \frac{\|\boldsymbol{v}^{*}\|}{\sqrt{p}} \left| \sqrt{\frac{\mu}{np}} (\boldsymbol{x}^{*})^{\top} f_{0}(\boldsymbol{u}^{0}, \boldsymbol{x}^{0}, \boldsymbol{x}_{0}(\varepsilon)) - \beta_{0} \right| \stackrel{a.s.}{\to} 0, \end{aligned}$$

by SLLN, definition of β_0 and $p/n \to 1/c$. This shows Hypothesis (2).

Let the hypotheses hold for $\ell = 0, ..., t - 1$. Now we show the hypotheses for $\ell = t$ to complete the induction.

Step 2: Hypothesis (6), (7) and (8) First consider Hypothesi (6) for $\ell = t$. If we consider the partially pseudo-Lipschitz function $\hat{\phi}(x, y, z, w) = (\alpha_{t-1}z + x)^2$, then using (E.4) this Hypothesis follows. Similarly using $\hat{\phi}(x, y, z, w) = (\mu_t z + y)^2$, Hypothesis (7) follows. Finally using $\hat{\psi}(x, y, r) = (\beta_t y + x)^2$ and (E.5), Hypothesis (8) follows. **Step 3: Hypothesis (3)** Consider Hypothesis (3) for $\ell = t$. It can be observed that

$$\begin{aligned} \Delta_{1,i}^{t} &= \left(\sqrt{\lambda} \langle \boldsymbol{x}^{*}, f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) \rangle_{n} - \mu_{t}\right) x_{i}^{*} \\ &+ \widetilde{d}_{t-1} f_{t-2}(\alpha_{t-3} x_{i}^{*} + \widetilde{u}_{i}^{t-2}, \mu_{t-2} x_{i}^{*} + \widetilde{x}_{i}^{t-2}, \boldsymbol{x}_{0,i}(\varepsilon)) \\ &+ \frac{1}{\sqrt{n}} \langle \boldsymbol{Z}_{i,*}, f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t-1}(\alpha_{t-2} \boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t-1}, \mu_{t-1} \boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) \rangle_{n} \\ &- d_{t-1} f_{t-2}(u_{i}^{t-2}, x_{i}^{t-2}, \boldsymbol{x}_{0,i}(\varepsilon)). \end{aligned}$$

Thus using the Jensen's inequality, we have for constant $L_1 > 0$

$$\begin{split} &\frac{1}{n} \|\Delta_{1}^{t}\|^{2} \\ &\leq L_{1} \left(\sqrt{\lambda} \langle \boldsymbol{x}^{*}, f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) \rangle_{n} - \mu_{t} \right)^{2} \\ &+ L_{1} |\tilde{d}_{t-1} - d_{t-1}|^{2} \frac{1}{n} \sum_{i=1}^{n} f_{t-2}^{2} (\alpha_{t-3} \boldsymbol{x}_{i}^{*} + \tilde{\boldsymbol{u}}_{i}^{t-2}, \mu_{t-2} \boldsymbol{x}_{i}^{*} + \tilde{\boldsymbol{x}}_{i}^{t-2}, \boldsymbol{x}_{0,i}(\varepsilon)) \\ &+ \frac{L_{1}}{n^{2}} \|\boldsymbol{Z}\|_{\text{op}}^{2} \|f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t-1}(\alpha_{t-2} \boldsymbol{x}^{*} + \tilde{\boldsymbol{u}}^{t-1}, \mu_{t-1} \boldsymbol{x}^{*} + \tilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) \|^{2} \\ &+ \frac{L_{1}}{n} |d_{t-1}|^{2} \|f_{t-2}(\boldsymbol{u}^{t-2}, \boldsymbol{x}^{t-2}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t-2}(\alpha_{t-3} \boldsymbol{x}^{*} + \tilde{\boldsymbol{u}}^{t-2}, \mu_{t-2} \boldsymbol{x}^{*} + \tilde{\boldsymbol{x}}^{t-2}, \boldsymbol{x}_{0}(\varepsilon)) \|^{2}. \end{split}$$

Using the partially pseudo-Lipschitz function $\hat{\phi}(x, y, z, w) = z f_{t-1}(\alpha_{t-1}z + x, \mu_t z + y, w)$ (by Lemma 7.1(1)), definition of μ_t and Hypothesis (1) for $\ell = t - 1$ we have

$$\sqrt{\lambda} \langle \boldsymbol{x}^*, f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_0(\varepsilon)) \rangle_n - \mu_t \stackrel{a.s.}{\to} 0.$$
 (E.6)

From [3], we have

$$\limsup_{n\to\infty}\frac{1}{n}\|\boldsymbol{Z}\|_{\scriptscriptstyle\rm op}^2<\infty\quad\text{ a.s.}$$

Using the partially Lipschitz property of f_{t-1} , we have $\|f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_0(\varepsilon)) - f_{t-1}(\alpha_{t-2}\boldsymbol{x}^* + \tilde{\boldsymbol{u}}^{t-1}, \mu_{t-1}\boldsymbol{x}^* + \tilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_0(\varepsilon))\|^2 \leq L_1(\|\Delta_1^{t-1}\|^2 + \|\Delta_2^{t-1}\|^2).$ By induction hypothesis (3) and (4) for $\ell = t-1$ we get

$$\frac{1}{n} \|f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_0(\varepsilon)) - f_{t-1}(\alpha_{t-2}\boldsymbol{x}^* + \widetilde{\boldsymbol{u}}^{t-1}, \mu_{t-1}\boldsymbol{x}^* + \widetilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_0(\varepsilon))\|^2 \\ \leqslant L_1\left(\frac{\|\Delta_1^{t-1}\|^2}{n} + \frac{\|\Delta_2^{t-1}\|^2}{n}\right) \xrightarrow{a.s.} 0.$$

This implies

$$\frac{1}{n^2} \|\boldsymbol{Z}\|_{\text{op}}^2 \|f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_0(\varepsilon)) - f_{t-1}(\alpha_{t-2}\boldsymbol{x}^* + \widetilde{\boldsymbol{u}}^{t-1}, \mu_{t-1}\boldsymbol{x}^* + \widetilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_0(\varepsilon))\|^2 \xrightarrow{a.s.} 0. \quad (E.7)$$

Since $\hat{\psi}(x, y, r) = f_{t-2}^2(x, y, r)$ is partially pseudo-Lipschitz (by Lemma 7.1(1)), using (E.4), we have almost surely

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f_{t-2}^2(\alpha_{t-3}x_i^* + \widetilde{u}_i^{t-2}, \mu_{t-2}x_i^* + \widetilde{x}_i^{t-2}, x_{0,i}(\varepsilon)) < \infty.$$

As $f_{t-1}^{(2)}(x, y, r) = \frac{\partial f_{t-1}}{\partial y}(x, y, r)$ is partially Lipschitz, we have using Hypothesis (1)

$$|\widetilde{d}_{t-1} - d_{t-1}| = \left| \frac{1}{n} \sum_{i=1}^{n} [f_{t-1}^{(2)}(\alpha_{t-2}x_i^* + \widetilde{u}_i^{t-1}, \mu_{t-1}x_i^* + \widetilde{x}_i^{t-1}, x_{0,i}(\varepsilon)) - f_{t-1}^{(2)}(u_i^{t-1}, x_i^{t-1}, x_{0,i}(\varepsilon))] \right| \overset{a.s}{\to} 0.$$

This implies

$$|\widetilde{d}_{t-1} - d_{t-1}|^2 \frac{1}{n} \sum_{i=1}^n f_{t-2}^2(\alpha_{t-3}x_i^* + \widetilde{u}_i^{t-2}, \mu_{t-2}x_i^* + \widetilde{x}_i^{t-2}, \boldsymbol{x}_{0,i}(\varepsilon)) \xrightarrow{a.s.} 0.$$
(E.8)

Again, using similar arguments, we can show that

 $|d_{t-1}|^2 < \infty \quad \text{a.s.}$

Again using the induction hypothesis (3) and (4) for $\ell = t - 2$ we get

$$\begin{split} &\frac{1}{n} \| f_{t-2}(\boldsymbol{u}^{t-2}, \boldsymbol{x}^{t-2}, \boldsymbol{x}_0) - f_{t-2}(\alpha_{t-3}\boldsymbol{x}^* + \widetilde{\boldsymbol{u}}^{t-2}, \mu_{t-2}\boldsymbol{x}^* + \widetilde{\boldsymbol{x}}^{t-2}, \boldsymbol{x}_0(\varepsilon)) \|^2 \\ & \leqslant L_1 \left(\frac{\| \Delta_1^{t-2} \|^2}{n} + \frac{\| \Delta_2^{t-2} \|^2}{n} \right) \stackrel{a.s.}{\to} 0. \end{split}$$

Thus, we have the following.

$$\|d_{t-1}\|^{2} \frac{1}{n} \|f_{t-2}(\boldsymbol{u}^{t-2}, \boldsymbol{x}^{t-2}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t-2}(\alpha_{t-3}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{u}}^{t-2}, \mu_{t-2}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t-2}, \boldsymbol{x}_{0}(\varepsilon))\|^{2} \xrightarrow{a.s.} 0.$$
(E.9)

Using (E.6), (E.7), (E.8), and (E.9) we have

$$\lim_{n \to \infty} \frac{\|\Delta_1^t\|^2}{n} \stackrel{a.s.}{=} 0$$

Step 4: Hypothesis (4) Next we try to prove Hypothesis (4) for $\ell = t$. We observe that

$$\begin{split} \Delta_{2,i}^{t} &= \left((\sqrt{p\mu/n}) \langle \boldsymbol{v}^{*}, g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) \rangle_{p} - \alpha_{t-1} \right) x_{i}^{*} \\ &+ \widetilde{c}_{t-1} f_{t-1}(\alpha_{t-2} x_{i}^{*} + \widetilde{u}_{i}^{t-1}, \mu_{t-1} x_{i}^{*} + \widetilde{x}_{i}^{t-1}, x_{0,i}(\varepsilon)) \\ &+ \langle \boldsymbol{L}_{*,i}, g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) - g_{t-1}(\beta_{t-1} \boldsymbol{v}^{*} + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) \rangle \\ &- c_{t-1} f_{t-1}(u_{i}^{t-1}, x_{i}^{t-1}, x_{0,i}(\varepsilon)). \end{split}$$

If $\boldsymbol{L} = \boldsymbol{V}/\sqrt{n}$, then by the Jensen's inequality, we have for constant $L_2 > 0$

$$\frac{\|\Delta_{2}^{t}\|^{2}}{n} \leq L_{2} \left(\sqrt{\frac{\mu p}{n}} \langle \boldsymbol{v}^{*}, g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) \rangle_{p} - \alpha_{t-1} \right)^{2} \\ + L_{2} |\tilde{c}_{t-1} - c_{t-1}|^{2} \frac{1}{n} \sum_{i=1}^{n} f_{t-1}^{2} (\alpha_{t-2} \boldsymbol{x}_{i}^{*} + \tilde{\boldsymbol{u}}_{i}^{t-1}, \mu_{t-1} \boldsymbol{x}_{i}^{*} + \tilde{\boldsymbol{x}}_{i}^{t-1}, \boldsymbol{x}_{0,i}(\varepsilon)) \\ + \frac{L_{2}}{n p} \lambda_{\max}(\boldsymbol{V}\boldsymbol{V}^{\top}) \|g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) - g_{t-1}(\beta_{t-1}\boldsymbol{v}^{*} + \tilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) \|^{2} \\ + \frac{L_{2}}{n} |c_{t-1}|^{2} \|f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t-1}(\alpha_{t-1}\boldsymbol{x}^{*} + \tilde{\boldsymbol{u}}^{t-1}, \mu_{t-1}\boldsymbol{x}^{*} + \tilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_{0}(\varepsilon)) \|^{2}.$$

Using the partially pseudo-Lipschitz function $\hat{\psi}(x, y, r) = yg_{t-1}(\beta_{t-1}y + x, r)$ (by Lemma 7.1(1)), definition of α_{t-1} and Hypothesis (2) for $\ell = t - 1$, we have

$$\sqrt{\frac{\mu p}{n}} \langle \boldsymbol{v}^*, g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_0(\varepsilon)) \rangle_p - \alpha_{t-1} \xrightarrow{a.s.} 0.$$
(E.10)

Since $p/n \to 1/c$, using Corollary 5.35 of [35], we have

$$\limsup_{n \to \infty} \frac{\lambda_{\max}(\boldsymbol{V}\boldsymbol{V}^{\top})}{n} < \infty \quad \text{a.s}$$

Using the partially Lipschitz property of g_{t-1} , we have

$$\|g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_0(\varepsilon)) - g_{t-1}(\beta_{t-1}\boldsymbol{v}^* + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_0(\varepsilon))\|^2 \leq L_2 \|\Delta_3^{t-1}\|^2$$

By induction hypothesis (5) for $\ell = t - 1$, we get

$$\frac{1}{p} \|g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_0(\varepsilon)) - g_{t-1}(\beta_{t-1}\boldsymbol{v}^* + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_0(\varepsilon))\|^2 \leqslant L_2\left(\frac{\|\Delta_3^{t-1}\|^2}{p}\right) \stackrel{a.s.}{\to} 0.$$

This implies

$$\limsup_{n \to \infty} \frac{\lambda_{\max}(\boldsymbol{V}\boldsymbol{V}^{\top})}{np} \|g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_0(\varepsilon)) - g_{t-1}(\beta_{t-1}\boldsymbol{v}^* + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_0(\varepsilon))\|^2 \stackrel{a.s.}{=} 0.$$
(E.11)

Since $\hat{\psi}(x, y, r) = f_{t-1}^2(x, y, r)$ is partially pseudo-Lipschitz (by Lemma 7.1 (1)), using (E.4), we have almost surely

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f_{t-1}^2(\alpha_{t-2}x_i^* + \widetilde{u}_i^{t-1}, \mu_{t-1}x_i^* + \widetilde{x}_i^{t-1}, x_{0,i}(\varepsilon)) < \infty.$$

As $g'_{t-1}(x,y) = \frac{\partial g_{t-1}}{\partial x}(x,y)$ is partially Lipschitz, we have using Hypothesis (2) for $\ell = t-1$

$$\widetilde{c}_{t-1} - c_{t-1} = \left| \frac{1}{p} \sum_{i=1}^{p} \left[g'_{t-1}(\beta_{t-1}v_i^* + \widetilde{v}_i^{t-1}, v_{0,i}(\varepsilon)) - g'_{t-1}(v_i^{t-1}, v_{0,i}(\varepsilon)) \right] \right| \stackrel{a.s}{\to} 0.$$

This implies

$$|\tilde{c}_{t-1} - c_{t-1}|^2 \frac{1}{n} \sum_{i=1}^n f_{t-1}^2(\alpha_{t-2}x_i^* + \tilde{u}_i^{t-1}, \mu_{t-1}x_i^* + \tilde{x}_i^{t-1}, x_{0,i}(\varepsilon)) \xrightarrow{a.s.} 0.$$
(E.12)

Using similar arguments, we can show

$$|c_{t-1}|^2 < \infty$$
 a.s

Using the induction hypotheses (3) and (4) for $\ell = t - 1$ we get

$$\frac{1}{n} \| f_{t-1}(\boldsymbol{u}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{x}_0(\varepsilon)) - f_{t-1}(\alpha_{t-2}\boldsymbol{x}^* + \widetilde{\boldsymbol{u}}^{t-1}, \mu_{t-1}\boldsymbol{x}^* + \widetilde{\boldsymbol{x}}^{t-1}, \boldsymbol{x}_0(\varepsilon)) \|^2 \\
\leqslant L_1 \left(\frac{\|\Delta_1^{t-2}\|^2}{n} + \frac{\|\Delta_2^{t-2}\|^2}{n} \right) \xrightarrow{a.s.} 0.$$

Thus we have

$$\|c_{t-1}\|^{2} \frac{1}{n} \|f_{t-2}(\boldsymbol{u}^{t-2}, \boldsymbol{x}^{t-2}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t-2}(\alpha_{t-3}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{u}}^{t-2}, \mu_{t-2}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t-2}, \boldsymbol{x}_{0}(\varepsilon))\|^{2} \xrightarrow{\text{a.s.}} 0. \quad (E.13)$$

Using (E.10), (E.11), (E.12), and (E.13), we have

$$\lim_{n \to \infty} \frac{\|\Delta_2^t\|^2}{n} \stackrel{a.s.}{=} 0.$$

Step 5: Hypothesis (1) Now observe that using the partially pseudo-Lipschitz property of ϕ , we have for a constant $C_1 > 0$

$$\begin{aligned} \left| \phi(\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}, \mu_{t}x_{i}^{*} + \widetilde{x}_{i}^{t}, x_{0,i}(\varepsilon)) - \phi(u_{i}^{t}, x_{i}^{t}, x_{0,i}(\varepsilon)) \right| \\ &\leq C_{1}(\left|\Delta_{1,i}^{t}\right| + \left|\Delta_{2,i}^{t}\right|)(1 + \left|u_{i}^{t}\right| + \left|x_{i}^{t}\right| + \left|\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}\right| + \left|\mu_{t}x_{i}^{*} + \widetilde{y}_{i}^{t}\right| + \left|x_{0,i}(\varepsilon)\right|) \\ &\leq 2C_{1}(\left|\Delta_{1,i}^{t}\right| + \left|\Delta_{2,i}^{t}\right|)(1 + \left|\Delta_{1,i}^{t}\right| + \left|\Delta_{2,i}^{t}\right| + \left|\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}\right| + \left|\mu_{t}x_{i}^{*} + \widetilde{x}_{i}^{t}\right| + \left|x_{0,i}(\varepsilon)\right|). \end{aligned}$$
(E.14)

Using (E.14) and the Cauchy Schwarz inequality, we get

$$\begin{split} \frac{1}{n} \sum_{i=1}^{n} \left| \phi(\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}, \mu_{t}x_{i}^{*} + \widetilde{x}_{i}^{t}, x_{0,i}(\varepsilon)) - \phi(u_{i}^{t}, x_{i}^{t}, x_{0,i}(\varepsilon)) \right| \\ & \leq \frac{2L}{n} \sum_{i=1}^{n} \{ |\Delta_{1,i}^{t}| + |\Delta_{1,i}^{t}|^{2} + |\Delta_{1,i}^{t}| |\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}| + |\Delta_{1,i}^{t}| |\Delta_{2,i}^{t}| \\ & + |\Delta_{1,i}^{t}| |\mu_{t}x_{i}^{*} + \widetilde{x}_{i}^{t}| + |\Delta_{1,i}^{t}| |x_{0,i}(\varepsilon)| + |\Delta_{2,i}^{t}| + |\Delta_{2,i}^{t}|^{2} \\ & + |\Delta_{2,i}^{t}| |\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}| + |\Delta_{1,i}^{t}| |\Delta_{2,i}^{t}| + |\Delta_{2,i}^{t}| |\mu_{t}x_{i}^{*} + \widetilde{x}_{i}^{t}| + |\Delta_{2,i}^{t}| |x_{0,i}(\varepsilon)| \} \\ & \leq \frac{2L}{n} \{ \sqrt{n} \|\Delta_{1}^{t}\| + \|\Delta_{1}^{t}\|^{2} + \|\Delta_{1}^{t}\| \|\alpha_{t-1}x^{*} + \widetilde{u}^{t}\| + 2\|\Delta_{1}^{t}\| \|\Delta_{2}^{t}\| \\ & + \|\Delta_{1}^{t}\| \|\mu_{t}x^{*} + \widetilde{x}^{t}\| + \|\Delta_{1}^{t}\| \|x_{0}(\varepsilon)\| + \sqrt{n} \|\Delta_{2}^{t}\| + \|\Delta_{2}^{t}\| \|x_{0}(\varepsilon)\| \} \end{split}$$

Thus, using Hypotheses (3) and (4) for $\ell = t$, we have

$$\frac{1}{n}\sum_{i=1}^{n} \left| \phi(\alpha_{t-1}x_{i}^{*} + \widetilde{u}_{i}^{t}, \mu_{t}x_{i}^{*} + \widetilde{x}_{i}^{t}, x_{0,i}(\varepsilon)) - \phi(u_{i}^{t}, x_{i}^{t}, x_{0,i}(\varepsilon)) \right| \stackrel{a.s.}{\to} 0.$$

.

Step 6: Hypothesis (5) Now we try to prove Hypothesis (5) for $\ell = t$. We first observe that

$$\Delta_{3,i}^{t} = \left((\sqrt{n\mu/p}) \langle \boldsymbol{x}^{*}, f_{t}(\boldsymbol{u}^{t}, \boldsymbol{x}^{t}, \boldsymbol{x}_{0}(\varepsilon)) \rangle_{n} - \beta_{t} \right) v_{i}^{*} + \widetilde{p}_{t} g_{t-1}(\beta_{t-1}v_{i}^{*} + \widetilde{v}_{i}^{t-1}, v_{0,i}(\varepsilon)) \\ + \langle \boldsymbol{L}_{i,*}, f_{t}(\boldsymbol{u}^{t}, \boldsymbol{x}^{t}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t}(\alpha_{t-1}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{u}}^{t-1}, \mu_{t}\boldsymbol{x}^{*} + \widetilde{\boldsymbol{x}}^{t}, \boldsymbol{x}_{0}(\varepsilon)) \rangle - p_{t} g_{t-1}(v_{i}^{t-1}, v_{0,i}(\varepsilon)).$$

By Jensen's inequality, there exists a constant $L_3 > 0$ such that

$$\begin{split} \frac{1}{p} \|\Delta_{3}^{t}\|^{2} &\leq L_{3} \left(\sqrt{\frac{n\mu}{p}} \langle \boldsymbol{x}^{*}, f_{t}(\boldsymbol{u}^{t}, \boldsymbol{x}^{t}, \boldsymbol{x}_{0}(\varepsilon)) \rangle_{n} - \beta_{t} \right)^{2} \left(\frac{1}{p} \sum_{i=1}^{p} v_{0,i}^{2}(\varepsilon) \right) \\ &+ L_{3} \frac{\lambda_{\max}(\boldsymbol{V}^{\top} \boldsymbol{V})}{p^{2}} \| f_{t}(\boldsymbol{u}^{t}, \boldsymbol{x}^{t}, \boldsymbol{x}_{0}(\varepsilon)) - f_{t}(\alpha_{t-1}\boldsymbol{x}^{*}_{i} + \widetilde{\boldsymbol{u}}^{t}, \mu_{t}\boldsymbol{x}^{*}_{i} + \widetilde{\boldsymbol{x}}^{t}, \boldsymbol{x}_{0}(\varepsilon)) \|^{2} \\ &+ \frac{L_{3}}{p} |p_{t}|^{2} \| g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) - g_{t-1}(\beta_{t-1}\boldsymbol{v}^{*} + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_{0}(\varepsilon)) \|^{2} \\ &+ L_{3} |\widetilde{p}_{t} - p_{t}|^{2} \frac{1}{p} \sum_{i=1}^{n} g_{t-1}^{2}(\beta_{t-1}\boldsymbol{v}^{*}_{i} + \widetilde{\boldsymbol{v}}^{t-1}_{i}, \boldsymbol{v}_{0,i}(\varepsilon)). \end{split}$$

Using the partially pseudo-Lipschitz function $\hat{\phi}(x, y, z, w) = z f_t(\alpha_{t-1}z + x, \mu_t z + y, w)$ (by Lemma 7.1(2)), definition of β_t and Hypothesis (1) for $\ell = t - 1$ we have

$$\sqrt{\frac{n\mu}{p}} \langle \boldsymbol{x}^*, f_t(\boldsymbol{u}^t, \boldsymbol{x}^t, \boldsymbol{x}_0(\varepsilon)) \rangle_n - \beta_t \stackrel{a.s.}{\to} 0.$$
(E.15)

Since $p/n \rightarrow 1/c$, using Corollary 5.35 of [35], we have

$$\limsup_{n \to \infty} \frac{\lambda_{\max}(\boldsymbol{V}^{\top} \boldsymbol{V})}{p} < \infty \quad \text{ a.s}$$

Using the partially Lipschitz property of f_t , we have

$$\|f_t(\boldsymbol{u}^t, \boldsymbol{x}^t, \boldsymbol{x}_0(\varepsilon)) - f_t(\alpha_{t-1}\boldsymbol{x}^* + \widetilde{\boldsymbol{u}}^t, \mu_t \boldsymbol{x}^* + \widetilde{\boldsymbol{x}}^t, \boldsymbol{x}_0(\varepsilon))\|^2 \leq L_3 \left(\|\Delta_1^t\|^2 + \|\Delta_2^t\|^2\right)$$

By induction hypotheses (3) and (4) for $\ell = t$ we get

$$\frac{1}{n} \|f_t(\boldsymbol{u}^t, \boldsymbol{x}^t, \boldsymbol{x}_0(\varepsilon)) - f_t(\alpha_{t-1}\boldsymbol{x}^* + \widetilde{\boldsymbol{u}}^t, \mu_t \boldsymbol{x}^* + \widetilde{\boldsymbol{x}}^t, \boldsymbol{x}_0(\varepsilon))\|^2 \leq L_3 \left(\frac{\|\Delta_1^t\|^2}{n} + \frac{\|\Delta_2^t\|^2}{n}\right) \stackrel{a.s.}{\to} 0.$$

This implies

$$\limsup_{n \to \infty} \frac{\lambda_{\max}(\boldsymbol{V}^{\top} \boldsymbol{V})}{p^2} \| f_t(\boldsymbol{u}^t, \boldsymbol{x}^t, \boldsymbol{x}_0(\varepsilon)) - f_t(\alpha_{t-1} \boldsymbol{x}^* + \widetilde{\boldsymbol{u}}^t, \mu_t \boldsymbol{x}^* + \widetilde{\boldsymbol{x}}^t, \boldsymbol{x}_0(\varepsilon)) \|^2 \stackrel{a.s.}{=} 0.$$
(E.16)

Since, $\hat{\psi}(x, y, r) = g_{t-1}^2(\beta_{t-1}y + x, r)$ is partially pseudo-Lipschitz (by Lemma 7.1(1)), using (E.4), we have almost surely

$$\limsup_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} g_{t-1}^2(\beta_{t-1}v_i^* + \widetilde{v}_i^{t-1}, v_{0,i}(\varepsilon)) < \infty.$$

As $f_t^{(1)}(x, y, r) = \frac{\partial f_t}{\partial x}(x, y, r)$ is partially Lipschitz, we have using Hypothesis (1) for $\ell = t$

$$|\widetilde{p}_t - p_t| = \left| \frac{1}{n} \sum_{i=1}^n [f_t^{(1)}(\alpha_{t-1}x_i^* + \widetilde{u}_i^t, \mu_t x_i^* + \widetilde{x}_i^t, x_{0,i}(\varepsilon)) - f_t^{(1)}(u_i^t, x_i^t, x_{0,i}(\varepsilon))] \right| \stackrel{a.s}{\to} 0.$$
(E.17)

This implies

$$|\widetilde{p}_{t} - p_{t}|^{2} \frac{1}{p} \sum_{i=1}^{p} g_{t-1}^{2} (\beta_{t-1} v_{i}^{*} + \widetilde{v}_{i}^{t-1}, v_{0,i}(\varepsilon)) \xrightarrow{a.s.} 0.$$
(E.18)

Using arguments similar to (E.17), we can show that

$$\limsup_{n \to \infty} |p_t|^2 < \infty \quad \text{a.s.}$$

Then, using the induction hypothesis (5) for $\ell = t - 1$, we get

$$\frac{1}{p} \|g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_0(\varepsilon)) - g_{t-1}(\beta_{t-1}\boldsymbol{v}^* + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_0(\varepsilon))\|^2 \leq L_3 \frac{\|\Delta_3^{t-1}\|^2}{p} \stackrel{a.s.}{\to} 0$$

Thus, we have the following.

$$|p_t|^2 \frac{1}{p} \|g_{t-1}(\boldsymbol{v}^{t-1}, \boldsymbol{v}_0(\varepsilon)) - g_{t-1}(\beta_{t-1}\boldsymbol{v}^* + \widetilde{\boldsymbol{v}}^{t-1}, \boldsymbol{v}_0(\varepsilon))\|^2 \xrightarrow{a.s.} 0.$$
(E.19)

Using (E.15), (E.16), (E.18), and (E.19) we have

$$\lim_{n \to \infty} \frac{\|\Delta_3^t\|^2}{n} \stackrel{a.s.}{=} 0$$

Step 7: Hypothesis (2) Again using the partially pseudo-Lipschitz property of ψ and the Cauchy–Schwarz inequality we get

$$\begin{split} \frac{1}{p} \sum_{i=1}^{p} \left| \psi(\beta^{t} v_{i}^{*} + \widetilde{v}_{i}^{t}, v_{0,i}(\varepsilon)) - \psi(v_{i}^{t}, v_{0,i}(\varepsilon)) \right| \\ & \leq \frac{2L}{p} \{ \sqrt{p} \| \Delta_{3}^{t} \| + \| \Delta_{3}^{t} \|^{2} + \| \Delta_{3}^{t} \|^{2} + \| \Delta_{3}^{t} \| \| \beta_{t} \boldsymbol{v}^{*} + \widetilde{\boldsymbol{v}}^{t} \| + \| \Delta_{1}^{t} \| \| \boldsymbol{v}_{0}(\varepsilon) \|. \end{split}$$

Using Hypothesis (5) for $\ell = t$ gives us

$$\frac{1}{p}\sum_{i=1}^{p} \left|\psi(\beta^{t}v_{i}^{*}+\widetilde{v}_{i}^{t},v_{0,i}(\varepsilon))-\psi(v_{i}^{t},v_{0,i}(\varepsilon))\right| \stackrel{a.s.}{\to} 0.$$

E.5 Lemmas Used to Prove Results in Section 7

Variants of the following lemmas have previously appeared in [8]. We include their statement and proof here mainly for the manuscript to be self-contained.

Lemma E.2. Consider a sequence of matrices $\mathbf{A} \sim GOE(n)$ and two sequences of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that $\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{n}$.

- (a) $\langle \boldsymbol{v}, \boldsymbol{A}\boldsymbol{u} \rangle_n \stackrel{a.s.}{\rightarrow} 0$,
- (b) Let $\mathbf{P} \in \mathbb{R}^{n \times n}$ be a sequence of projection matrices such that there exists a constant t that satisfies for all n, rank $(\mathbf{P}) \leq t$. Then $\frac{1}{n} \|\mathbf{P}\mathbf{A}\mathbf{u}\|_2^2 \xrightarrow{a.s.} 0$,
- (c) $\frac{1}{n} \|\boldsymbol{A}\boldsymbol{u}\|_2^2 \xrightarrow{a.s.} 1$,
- (d) There exists a sequence of random vectors $\boldsymbol{z} \sim N(0, \boldsymbol{I}_n)$ such that for any sequence of functions $\varphi_n : (\mathbb{R}^n)^r \times \mathbb{R}^n \times (\mathbb{R}^n)^m \to \mathbb{R}, n \ge 1$ satisfying

$$\begin{aligned} |\varphi_n(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_r,\boldsymbol{x},\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m) - \varphi_n(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_r,\boldsymbol{y},\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)| \\ &\leqslant L\left(1+\sum_{i=1}^r \frac{\|\boldsymbol{h}_i\|}{\sqrt{n}} + \frac{\|\boldsymbol{x}\|}{\sqrt{n}} + \frac{\|\boldsymbol{y}\|}{\sqrt{n}} + \sum_{j=1}^m \frac{\|\boldsymbol{\xi}_j\|}{\sqrt{n}}\right) \frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\sqrt{n}}, \end{aligned}$$

where for all $i \in [r]$ and $j \in [m]$,

$$\limsup_{n\to\infty}\frac{\|\boldsymbol{h}_i\|}{\sqrt{n}}<\infty\qquad\limsup_{n\to\infty}\frac{\|\boldsymbol{\xi}\|}{\sqrt{n}}<\infty.$$

Then we have

$$\varphi_n(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_r,\boldsymbol{A}\boldsymbol{u},\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)-\varphi_n(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_r,\boldsymbol{z},\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_m)\stackrel{a.s.}{\rightarrow} 0.$$

Proof. First note that for any fixed k > 0, if we have a sequence of random variables X_n defined on the same probability space such that $X_n \sim N(0, k/n)$. Then we have the following inequality

$$\mathbb{P}\left(|X_n| \ge \frac{1}{n^{1/4}}\right) \le 2\exp\left(-\frac{\sqrt{n}}{2k}\right).$$

As $\sum_{n=1}^{\infty} \exp\left(-\frac{\sqrt{n}}{2k}\right) < \infty$, using the Borel Cantelli Lemma we have

$$|X_n| \stackrel{a.s.}{\to} 0.$$

(a) Recall that $\mathbf{A} = \mathbf{G} + \mathbf{G}^{\top}$ where $G_{i,j}$ are i.i.d N(0, 1/(2n)) random variables, thus

$$\frac{1}{n} \langle \boldsymbol{v}, \boldsymbol{A} \boldsymbol{u} \rangle = \frac{1}{n} \langle \boldsymbol{v}, \boldsymbol{G} \boldsymbol{u} \rangle + \frac{1}{n} \langle \boldsymbol{v}, \boldsymbol{G}^\top \boldsymbol{u} \rangle$$

The random variable $\frac{1}{n} \langle \boldsymbol{v}, \boldsymbol{G} \boldsymbol{u} \rangle$ is a centered Gaussian random variable with variance 1/2n. Thus $\frac{1}{n} \langle \boldsymbol{v}, \boldsymbol{G} \boldsymbol{u} \rangle \xrightarrow{a.s.} 0$. Similarly we can show $\frac{1}{n} \langle \boldsymbol{v}, \boldsymbol{G}^{\top} \boldsymbol{u} \rangle \xrightarrow{a.s.} 0$.

(b) Suppose v_1, \ldots, v_k , an orthogonal basis of the image P, such that $||v_i|| = \sqrt{n}$. As k is bounded by t, by part (a)

$$\frac{1}{n} \| \boldsymbol{P} \boldsymbol{A} \boldsymbol{u} \|_{2}^{2} = \frac{1}{n} \sum_{i=1}^{k} \left(\frac{\langle \boldsymbol{v}, \boldsymbol{A} \boldsymbol{u} \rangle}{\| \boldsymbol{v}_{j} \|} \right)^{2} = \sum_{i=1}^{k} \left(\frac{1}{n} \langle \boldsymbol{v}, \boldsymbol{A} \boldsymbol{u} \rangle \right)^{2} \xrightarrow{a.s.} 0.$$

(c) By (d), we have a sequence of random vectors $\boldsymbol{z} \sim N(0, \boldsymbol{I}_n)$

$$\frac{1}{n} \|\boldsymbol{A}\boldsymbol{u}\|_2^2 - \frac{1}{n} \|\boldsymbol{z}\|_2^2 \stackrel{a.s.}{\to} 0.$$

As $\frac{1}{n} \|\boldsymbol{z}\|_2^2 \xrightarrow{a.s.} 1$, we can show part (c).

(d) We shall show this for r = 1 and m = 1, the case for higher r and m follows. It is easy to check that Au is a centered Gaussian vector with covariance matrix $\Sigma = I_n + \frac{1}{n} u u^{\top}$. Thus there exists a Gaussian vector $z \sim N(0, I_n)$ such that $Au = \Sigma^{1/2} z = z + (\sqrt{2} - 1) \frac{1}{n} u u^{\top} z$. By the property of φ_n we have

$$|arphi_n(m{h},m{A}m{u},m{\xi}) - arphi_n(m{h},m{z},m{\xi})| \leqslant L \left(1 + rac{\|m{h}\|}{\sqrt{n}} + rac{\|m{A}m{u}\|}{\sqrt{n}} + rac{\|m{z}\|}{\sqrt{n}}
ight) rac{\|m{A}m{u} - m{z}\|}{\sqrt{n}}.$$

The law of large numbers imply, $||z||_2/\sqrt{n} \xrightarrow{a.s.} 1$, and we have $||Au||/\sqrt{n} \leq ||\Sigma^{1/2}||_{\text{op}} ||z||/\sqrt{n} \leq \sqrt{2} ||z||/\sqrt{n} \xrightarrow{a.s.} \sqrt{2}$. Further

$$\frac{\|\boldsymbol{A}\boldsymbol{u}-\boldsymbol{z}\|}{\sqrt{n}} = \frac{\|\left(\boldsymbol{\Sigma}^{1/2}-\boldsymbol{I}_n\right)\boldsymbol{z}\|}{\sqrt{n}} = \frac{1}{n^{3/2}}(\sqrt{2}-1)\|\boldsymbol{u}\boldsymbol{u}^{\top}\boldsymbol{z}\| = (\sqrt{2}-1)\frac{1}{n}|\boldsymbol{u}^{\top}\boldsymbol{z}| \stackrel{a.s.}{\to} 0.$$

The last assertion follows as $\frac{1}{n} | \boldsymbol{u}^{\top} \boldsymbol{z} |$ is a centered Gaussian with variance 1/n.

Lemma E.3. Let (Z_1, \ldots, Z_t) , $(\tilde{Z}_1, \ldots, \tilde{Z}_t)$ be sequences Gaussian random variables, where the two sequences are independent. Let c_1, \ldots, c_t and $\tilde{c}_1, \ldots, \tilde{c}_t$ be strictly positive constants such that for all $i = 1, \ldots, t$:

 $Var(Z_i|Z_1,\ldots,Z_{i-1},\widetilde{Z}_1,\ldots,\widetilde{Z}_{i-1}) > c_i$

and

$$Var(\widetilde{Z}_i|Z_1,\ldots,Z_{i-1},\widetilde{Z}_1,\ldots,\widetilde{Z}_{i-1})>\widetilde{c}_i.$$

Further assume $\mathbb{E}\left\{Z_i^2\right\} \leq K$ for all i and $\mathbb{E}\left\{\widetilde{Z}_i^2\right\} \leq L$. Let Y be a random variable in the same probability space.

Finally let $\ell : \mathbb{R}^3 \to \mathbb{R}$ be a Lipschitz function, with $(z, y) \mapsto \ell(z, y, Y)$ non-constant with positive probability (with respect to Y). Then there exists a positive constant c'_t such that

$$\mathbb{E}\left\{\left[\ell(Z_t, \widetilde{Z}_t, Y)^2\right]\right\} - u^\top C^{-1} u > c'_t,$$

where $u \in \mathbb{R}^{t-1}$ is given by $u_i = \mathbb{E}\left\{\ell(Z_t, \widetilde{Z}_t, Y)\ell(Z_i, \widetilde{Z}_i, Y)\right\}$, and $C \in \mathbb{R}^{t-1} \times \mathbb{R}^{t-1}$ satisfies $C_{i,j} = \mathbb{E}\left\{\ell(Z_i, \widetilde{Z}_i, Y)\ell(Z_j, \widetilde{Z}_j, Y)\right\}$ for all $1 \leq i, j \leq t-1$.

Proof. Let us denote by Q the covariance of the Gaussian vectors Z_1, \ldots, Z_t , and Q' the covariance of the Gaussian vectors $\tilde{Z}_1, \ldots, \tilde{Z}_t$. The set of matrices Q, Q' satisfying the constraints with constants c_1, \ldots, c_t, K is compact. So if the thesis does not hold then there must exist a covariance matrix

$$\mathbb{E}\left\{\left[\ell(Z_t, \widetilde{Z}_t, Y)\right]^2\right\} - u^{\top} C^{-1} u = 0.$$
(E.20)

Let $S \in \mathbb{R}^{t \times t}$ be the matrix with the entries $S_{i,j} = \mathbb{E}\left\{\ell(Z_i, \widetilde{Z}_i, Y)\ell(Z_j, \widetilde{Z}_j, Y)\right\}$. Then (E.20) implies that S is not invertible by the Schur Complement Formula. Therefore, there exist non-vanishing constants a_1, \ldots, a_ℓ such that

$$a_1\ell(Z_1,\widetilde{Z}_1,Y) + \ldots + a_t\ell(Z_t,\widetilde{Z}_t,Y) \stackrel{a.s.}{=} 0.$$

The function $(z_1, \ldots, z_t) \mapsto a_1 \ell(z_1, \tilde{z}_1, Y) + \ldots + a_t \ell(z_t, \tilde{z}_t, Y)$ is Lipschitz and non-constant. Hence there is a set $\mathcal{A} \subseteq \mathbb{R}^t$ of positive Lebesgue Measure such that it is non-vanishing on \mathcal{A} . Therefore, \mathcal{A} must have zero measure under the law of (Z_1, \ldots, Z_t) and $(\tilde{Z}_1, \ldots, \tilde{Z}_t)$, i.e., $\lambda_{\min}(Q) = 0$ and $\lambda_{\min}(Q') = 0$. This implies there exists a'_1, \ldots, a'_t and b'_1, \ldots, b'_t such that

$$a'_{1}Z_{1} + \ldots + a'_{t}Z_{t} \stackrel{a.s.}{=} 0$$
, and $b'_{1}\widetilde{Z}_{1} + \ldots + b'_{t}\widetilde{Z}_{2} \stackrel{a.s.}{=} 0$

If $t_* = \max\{i \in \{1, \dots, t\} : a'_i \neq 0\}$ and $s_* = \max\{i \in \{1, \dots, t\} : b'_i \neq 0\}$, this implies

$$Z_{t_*} \stackrel{a.s.}{=} \sum_{i=1}^{t_*-1} (-a'_i/a'_{t_*}) Z_i, \text{ and } \widetilde{Z}_{t_*} \stackrel{a.s.}{=} \sum_{i=1}^{t_*-1} (-b'_i/b'_{t_*}) \widetilde{Z}_i.$$

This violates the assumption of the hypothesis.

Lemma E.4. The vectors

$$\boldsymbol{\alpha}_t = (\boldsymbol{\alpha}_1^t, \dots, \boldsymbol{\alpha}_{t-1}^t) = \left[\frac{M_t^{\top} M_t}{p}\right]^{-1} \frac{M_t^{\top} \boldsymbol{m}^t}{p}$$

and

$$\boldsymbol{\beta}_t = (\boldsymbol{\beta}_1^t, \dots, \boldsymbol{\beta}_{t-1}^t) = \left[\frac{Q_t^\top Q_t}{n}\right]^{-1} \frac{Q_t^\top q^t}{n},$$

have finite limits as $p, n \to \infty$.

Proof. Applying Lemma 9 of [8] and $\mathcal{B}_t(g), \mathcal{H}_t(g), \mathcal{X}_t(g)$ we can obtain that for large *n* the smallest eigenvalues of $(M_t^{\top} M_t)/n$, $(Q_t^{\top} Q_t)/n$ are all strictly positive. By Lemma 10 of [8] this implies they converge to invertible limits. Then using $\mathcal{H}_t(c), \mathcal{X}_t(c)$ and $\mathcal{B}_{t-1}(c)$ we have the result.

References

- [1] Abbe, E. (2017). Community detection and stochastic block models: recent developments. The Journal of Machine Learning Research, 18(1):6446–6531.
- [2] Abbe, E., Fan, J., and Wang, K. (2020). An ℓ_p theory of pca and spectral clustering. *arXiv:* Statistics Theory.
- [3] Anderson, G. W., Guionnet, A., and Zeitouni, O. (2009). An Introduction to Random Matrices. Cambridge Studies in Advanced Mathematics. Cambridge University Press.
- [4] Baik, J., Ben Arous, G., and Péché, S. (2005). Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. Annals of Probability, 33(5):1643–1697.
- [5] Baik, J. and Silverstein, J. W. (2006). Eigenvalues of large sample covariance matrices of spiked population models. *Journal of Multivariate Analysis*, 97(6):1382–1408.
- [6] Barbier, J. and Macris, N. (2019). The adaptive interpolation method: a simple scheme to prove replica formulas in bayesian inference. *Probability Theory and Related Fields*, 174(3):1133–1185.
- [7] Barbier, J. and Reeves, G. (2020). Information-theoretic limits of a multiview low-rank symmetric spiked matrix model. In 2020 IEEE International Symposium on Information Theory (ISIT), pages 2771–2776.
- [8] Bayati, M. and Montanari, A. (2010). The dynamics of message passing on dense graphs, with applications to compressed sensing. arXiv preprint arXiv:1001.3448v4.
- [9] Bayati, M. and Montanari, A. (2011). The dynamics of message passing on dense graphs, with applications to compressed sensing. *IEEE Transactions on Information Theory*, 57(2):764– 785.
- [10] Berthier, R., Montanari, A., and Nguyen, P.-M. (2019). State evolution for approximate message passing with non-separable functions. *Information and Inference: A Journal of the IMA*, 9(1):33–79.
- [11] Binkiewicz, N., Vogelstein, J. T., and Rohe, K. (2017). Covariate-assisted spectral clustering. *Biometrika*, 104(2):361–377.
- [12] Bolthausen, E. (2014). An iterative construction of solutions of the tap equations for the sherrington-kirkpatrick model. *Communications in Mathematical Physics*, 325(1):333–366.
- [13] Chen, H.-B., Mourrat, J.-C., and Xia, J. (2021). Statistical inference of finite-rank tensors.
- [14] De Las Rivas, J. and Fontanillo, C. (2010). Protein-protein interactions essentials: key concepts to building and analyzing interactome networks. *PLoS Computational Biology*, 6(6):e1000807.
- [15] Deshpande, Y., Abbe, E., and Montanari, A. (2016). Asymptotic mutual information for the balanced binary stochastic block model. *Information and Inference: A Journal of the IMA*, 6(2):125–170.

- [16] Deshpande, Y. and Montanari, A. (2014). Information-theoretically optimal sparse PCA. In 2014 IEEE International Symposium on Information Theory, pages 2197–2201.
- [17] Deshpande, Y., Sen, S., Montanari, A., and Mossel, E. (2018). Contextual stochastic block models. In Advances in Neural Information Processing Systems 31, pages 8581–8593.
- [18] Gao, C. and Ma, Z. (2021). Minimax rates in network analysis: Graphon estimation, community detection and hypothesis testing. *Statistical Science*, 36(1):16–33.
- [19] Guo, D., Shamai, S., and Verdú, S. (2005). Mutual information and minimum mean-square error in gaussian channels. *IEEE Transactions on Information Theory*, 51(4):1261–1282.
- [20] Han, Q., Xu, K. S., and Airoldi, E. M. (2014). Consistent estimation of dynamic and multi-layer networks. arXiv preprint arXiv:1410.8597.
- [21] Holland, P. W., Laskey, K. B., and Leinhardt, S. (1983). Stochastic blockmodels: First steps. Social Networks, 5(2):109–137.
- [22] Javanmard, A. and Montanari, A. (2013). State evolution for general approximate message passing algorithms, with applications to spatial coupling. *Information and Inference: A Journal of the IMA*, 2(2):115–144.
- [23] Jog, V. and Loh, P.-L. (2015). Information-theoretic bounds for exact recovery in weighted stochastic block models using the renyi divergence. arXiv preprint arXiv:1509.06418.
- [24] Johnstone, I. M. (2001). On the distribution of the largest eigenvalue in principal components analysis. Annals of statistics, 29(2):295–327.
- [25] Li, X., Chen, Y., and Xu, J. (2021). Convex relaxation methods for community detection. Statistical Science, 36(1):2–15.
- [26] Liu, H., Rush, C., and Baron, D. (2019). An analysis of state evolution for approximate message passing with side information. In 2019 IEEE International Symposium on Information Theory (ISIT), pages 2069–2073.
- [27] Lu, C. and Sen, S. (2020). Contextual stochastic block model: Sharp thresholds and contiguity. arXiv preprint arXiv:2011.09841.
- [28] Miolane, L. (2017). Fundamental limits of low-rank matrix estimation: The non-symmetric case. arXiv preprint arXiv:1702.00473.
- [29] Montanari, A. and Tse, D. (2006). Analysis of belief propagation for non-linear problems: The example of CDMA (or: How to prove Tanaka's formula). In 2006 IEEE Information Theory Workshop - ITW '06 Punta del Este, pages 160–164.
- [30] Montanari, A. and Venkataramanan, R. (2021). Estimation of low-rank matrices via approximate message passing. Annals of Statistics, 49(1):321–345.
- [31] Nandy, S., Yang, F., and Ma, Z. (2022). Spectral methods for community detection in contextual stochastic block models (unpublished manuscript).
- [32] Paul, D. (2007). Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statistica Sinica*, 17(4):1617–1642.

- [33] Paul, S. and Chen, Y. (2016). Consistent community detection in multi-relational data through restricted multi-layer stochastic blockmodel. *Electronic Journal of Statistics*, 10(2):3807–3870.
- [34] Reeves, G. (2020). Information-theoretic limits for the matrix tensor product. *IEEE Journal* on Selected Areas in Information Theory, 1(3):777–798.
- [35] Vershynin, R. (2010). Introduction to the non-asymptotic analysis of random matrices. arXiv preprint arXiv:1011.3027.
- [36] Wang, T., Zhong, X., and Fan, Z. (2022). Universality of approximate message passing algorithms and tensor networks.
- [37] Xu, M., Jog, V., and Loh, P.-L. (2020). Optimal rates for community estimation in the weighted stochastic block model. Annals of Statistics, 48(1):183–204.
- [38] Yan, B. and Sarkar, P. (2020). Covariate regularized community detection in sparse graphs. Journal of the American Statistical Association, 116:1–29.
- [39] Zhang, A. Y. and Zhou, H. H. (2016). Minimax rates of community detection in stochastic block models. *The Annals of Statistics*, 44(5):2252 – 2280.