Corrigendum to "Evaluating Ambiguous Random Variables from Choquet to Maxmin Expected Utility" [J. Econ. Theory 192 (2021) 105129]

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Gul and Pesendorfer (2021, GP henceforth) introduce a theory of belief updating for ambiguous random variables. In their model, the decision maker's uncertainty is described by a totally monotone capacity π, which is updated by first forming a so-called proxy capacity $\pi^{\mathcal{P}}$, which in general depends on the information partition $\mathcal{P}$. The proxy is then updated by Bayes' rule. GP note that this procedure can be interpreted as a modified version of prior-by-prior updating: Instead of updating every prior in the core of the capacity π, the decision maker considers and updates only a subset of the priors, namely those in the core of the proxy $\pi^{\mathcal{P}}$.

GP characterize the core of the proxy capacity $\pi^{\mathcal{P}}$ in terms of the Shapley value of the cooperative game corresponding to the capacity π. We show by counterexample that this characterization is incorrect. Thus, GP's Corollary 2 does not hold as

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stated. After presenting the example, we explain how the result should be corrected. We also give some simple characterizations of the core of $\pi^{\mathcal{P}}$, which show that the core is in general a subset of the set identified by GP.

1 Definitions

Let π be a totally monotone capacity and ρ_π the Shapley value for the "game" π. The additive extension of ρ_π to all events is $\rho_\pi(A) := \sum_{s\in A} \rho_\pi(s)$.

In an unnumbered display on page 10, GP define the set

$$\Delta^{\mathcal{P}}(\pi) := \{p \in \text{core}(\pi) : p(B) = \rho_\pi(B) \text{ for all } B \in \mathcal{P}\}. \tag{1}$$

In words, $\Delta^{\mathcal{P}}(\pi)$ consists of those priors in the core of the capacity π that assign the Shapley value of π to every event B in the partition $\mathcal{P}$ (but can differ from it on events that are not cells of $\mathcal{P}$).

GP claim following the display (and formally show in the proof of Corollary 2) that $\Delta^{\mathcal{P}}(\pi)$ is the core of the proxy capacity $\pi^{\mathcal{P}}$, which is also totally monotone and defined by its Möbius transform

$$\mu^{\mathcal{P}}(A) := \sum_{B\in\mathcal{P}} \sum_{\{D: D\cap B=A\}} \frac{|A|}{|D|} \cdot \mu(D), \tag{2}$$

where μ is the Möbius transform of π.

However, the following example shows that the core of $\pi^{\mathcal{P}}$ can be a strict subset of the set $\Delta^{\mathcal{P}}(\pi)$.

2 A Counterexample

Consider GP's "Prospect 1," where first a ball is drawn from an urn consisting of red and green balls of unknown proportions and the color of the ball is revealed to the decision maker. Then a fair coin is flipped. Formally, we have

- a set of states $S := \{rh, rt, gh, gt\}$;
- events $R := \{rh, rt\}$ ("ball is red"), $G := \{gh, gt\}$ ("ball is green"), $H := \{rh, gh\}$ ("coin comes up heads"), and $T := \{rt, gt\}$ ("coin comes up tails");
- a partition $\mathcal{P} := \{R, G\}$; and
- a capacity

$$\pi(A) := \begin{cases} 0.5 & \text{if } H \subseteq A \neq S, \\ 0.5 & \text{if } T \subseteq A \neq S, \\ 1 & \text{if } A = S, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that π's Möbius transform is given by

$$\mu(A) = \begin{cases} 0.5 & \text{if } A = H \text{ or } A = T, \\ 0 & \text{otherwise,} \end{cases}$$

and that the core of π is $\{p \in \Delta_S : p(H) = p(T) = 0.5\}$. Finally, by symmetry, the Shapley value is the uniform probability.

Thus, the set $\Delta^{\mathcal{P}}(\pi)$ defined in (1) is

$$\begin{aligned}\Delta^{\mathcal{P}}(\pi) &= \{p \in \text{core}(\pi) : p(R) = \rho_\pi(R) = 0.5 \text{ and } p(G) = \rho_\pi(G) = 0.5\} \\ &= \{p \in \Delta_S : p(R) = p(G) = p(H) = p(T) = 0.5\},\end{aligned}$$

which is represented in Table 1.

	H	**T**
R	α	$0.5-\alpha$
G	$0.5-\alpha$	α

Table 1: The core of $\pi^{\mathcal{P}}$; here $\alpha \in [0, 0.5]$.

As we will now show, in contradiction with GP's claim, the core of the proxy $\pi^{\mathcal{P}}$ is a strict subset of $\Delta^{\mathcal{P}}(\pi)$.

Claim 1. The proxy for the capacity in Example 1 is the uniform probability distribution and therefore its core is a singleton.

Proof. The claim follows by computing the Möbius transform for the proxy capacity using equation 2. Noting that $\{rh\} = R \cap H$ and recalling the Möbius transform from Example 1 gives

$$\mu^{\mathcal{P}}(rh) \geq \frac{1}{|H|} \cdot \mu(H) = \frac{1}{2} \cdot \frac{1}{2} = 0.25.$$

Analogous calculations show that $\mu^{\mathcal{P}}(s) \geq 0.25$ for all s. Thus, $\pi^{\mathcal{P}}(s) \geq 0.25$ for all s, which is possible only if $\pi^{\mathcal{P}}(s) = 0.25$ for all s. ■

3 The correct statement of GP's Corollary 2

Since GP's characterization does not hold, in the minimization problem in their Corollary 2 the set $\Delta^{\mathcal{P}}(\pi)$ needs to be replaced with the core of $\pi^{\mathcal{P}}$.

The mistake in GP's proof of Corollary 2 is in the last displayed equation of the proof (p. 23). Our Corollary 1 below shows that the second equality in that display is in general only a set inclusion; it holds as a strict inclusion in the above example.

4 The core of the proxy

The core of the proxy capacity $\pi^{\mathcal{P}}$ can be characterized as follows.

Proposition 1. The following are equivalent:

1. p is in the core of $\pi^{\mathcal{P}}$.

2. For all $B \in \mathcal{P}$ and $A \subseteq B$,

$$p(A) \geq \sum_{D:D\cap B\subseteq A} \frac{|D \cap B|}{|D|}\mu(D).$$

3. For all $A \subseteq S$,

$$p(A) \geq \sum_{B\in\mathcal{P}} \sum_{D:D\cap B\subseteq A} \frac{|D \cap B|}{|D|}\mu(D).$$

As we saw in the above example, the core of the proxy entails more restrictions than just the requirement that every element of the information partition receive their Shapley value. This is true in general.

Corollary 1. $\mathrm{core}(\pi^{\mathcal{P}}) \subseteq \Delta^{\mathcal{P}}(\pi) \subseteq \mathrm{core}(\pi)$.

5 Proofs

Proof of Proposition 1. ($1 \Leftrightarrow 3$): By definition, p is in the core of $\pi^{\mathcal{P}}$ iff for all $A \subseteq S$,

$$\begin{aligned} p(A) &\geq \pi^{\mathcal{P}}(A) \\ &= \sum_{E \subseteq A} \mu^{\mathcal{P}}(E) \\ &= \sum_{E \subseteq A} \sum_{B \in \mathcal{P}} \sum_{D \cap B = E} \frac{|E|}{|D|} \mu(D) \\ &= \sum_{B \in \mathcal{P}} \sum_{E \subseteq A} \sum_{D \cap B = E} \frac{|E|}{|D|} \mu(D) \\ &= \sum_{B \in \mathcal{P}} \sum_{E \subseteq A} \sum_{D \cap B = E} \frac{|D \cap B|}{|D|} \mu(D) \\ &= \sum_{B \in \mathcal{P}} \sum_{D \cap B \subseteq A} \frac{|D \cap B|}{|D|} \mu(D). \end{aligned}$$

($3 \Rightarrow 2$): If A is of the form $A \subseteq B$ for some $B \in \mathcal{P}$, then for any other event $B' \in \mathcal{P}$ the inner sum over D in statement 3 is over the empty set.

($2 \Rightarrow 3$): Let $A \subseteq S$ and partion it into events $A_B := A \cap B$, $B \in \mathcal{P}$. Then by 2,

$$p(A) = \sum_{B \in \mathcal{P}} p(A_B) \geq \sum_{B \in \mathcal{P}} \sum_{D : D \cap B \subseteq A_B} \frac{|D \cap B|}{|D|} \mu(D) = \sum_{B \in \mathcal{P}} \sum_{D : D \cap B \subseteq A} \frac{|D \cap B|}{|D|} \mu(D). \quad \blacksquare$$

Proof of Corollary 1. We first show that $\text{core}(\pi^{\mathcal{P}}) \subseteq \text{core}(\pi)$. To see that, take a subset of the sum in statement 3 of Proposition 1 by adding the condition $D : D \subseteq B$. This sub-sum equals $\sum_{E \subseteq A} \mu(E)$, which equals $\pi(A)$.

Now we prove that $\text{core}(\pi^{\mathcal{P}}) \subseteq \Delta^{\mathcal{P}}(\pi)$. We will use the fact that any totally monotone capacity π can be written as the convex combination

$$\pi = \sum_{D \subseteq S} \mu(D) \pi_D, \tag{3}$$

where each π_D is a simple capacity that assigns 1 to all supersets of the set D and zero to all other sets (i.e., $\pi_D(A) = 1$ if $D \supseteq A$ and $\pi_D(A) = 0$ otherwise).

Suppose that $p \in \text{core}(\pi^{\mathcal{P}})$. We just proved the first requirement in equation (1). It remains to show that p assigns the Shapley value to all sets in $\mathcal{P}$. Taking $A = B$ in statement 2 of Proposition 1 gives

$$p(B) \geq \sum_{D \cap B \subseteq B} \frac{|D \cap B|}{|D|} \mu(D) = \sum_{D \subseteq S} \frac{|D \cap B|}{|D|} \mu(D) = \sum_{D \subseteq S} \mu(D) \rho_D(B) = \rho_\pi(B),$$

where $\rho_D(B) := |D \cap B|(|D|)^{-1}$ is the Shapley value of B in the "unanimity game" π_D and the last equality is by equation (3) and linearity of the Shapley value. Summing the above inequality over the partition shows that it cannot be strict for any B:

$$1 = \sum_{B \in \mathcal{P}} p(B) \geq \sum_{B \in \mathcal{P}} \rho_\pi(B) = \pi(S) = 1.$$

We conclude that $p(B) = \rho_\pi(B)$ for all $B \in \mathcal{P}$, and thus $p \in \Delta(\pi^{\mathcal{P}})$. ■

Reference

Gul, F., and W. Pesendorfer (2021): "Evaluating Ambiguous Random Variables from Choquet to Maxmin Expectd Utility," *Journal of Economic Theory*, 192, 105129.