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Dynamic stochastic games with random moves

Ulrich Doraszelski[1] · **Kenneth L. Judd**[2]

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Abstract We reformulate the quality ladder model of Pakes and McGuire, *Rand Journal of Economics*, *25*(4), 555–589 (1994) as a dynamic stochastic game with random moves in which each period one firm is picked at random to make an investment decision. Contrasting this model to the standard version with simultaneous moves illustrates the computational advantages of random moves. In particular, the quality ladder model with random moves avoids the curse of dimensionality in computing firms' expectations over all possible future states and is therefore orders of magnitude faster to solve than its counterpart with simultaneous moves when there are more than just a few firms. Perhaps unexpectedly, the equilibria of the quality ladder model with random moves are practically indistinguishable from those of the model with simultaneous moves.

Keywords Dynamic stochastic games · Markov perfect equilibrium · Curse of dimensionality · Protocol of moves

JEL Classification C63 · C73 · L13

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✉ Ulrich Doraszelski
doraszelski@wharton.upenn.edu

Kenneth L. Judd
judd@hoover.stanford.edu

1 Wharton School, University of Pennsylvania, Pennsylvania, PA 19104, USA

2 Hoover Institution, Stanford University, Stanford, CA 94305-6010, USA

1 Introduction

Dynamic stochastic games have been applied widely to study the strategic interactions among forward-looking players in dynamic environments. The model as written down by Ericson and Pakes (1995), Pakes and McGuire (1994, 2001), and in standard textbook treatments of dynamic stochastic games (e.g., Filar and Vrieze 1997; Basar and Olsder 1999) assumes that the players choose their actions simultaneously each period and that the state then changes accordingly.

Besides simultaneous moves, there are many other ways to formulate dynamic stochastic games. In applied work the timing of decisions within periods is typically not observable in the data, thus giving the researcher considerable latitude in specifying a protocol of moves. As a consequence, a number of recent papers have experimented with alternatives to the standard assumption of simultaneous moves (Igami 2017, 2018; Iskhakov 2017).

Alternative protocols of moves are often defended on the grounds of computational convenience. As Pakes and McGuire (2001) and Doraszelski and Judd (2011) observe, models with simultaneous moves suffer from a “curse of dimensionality” as the burden of computing players’ expectations over all possible future states under widely-used laws of motion increases exponentially in the number of players and state variables.[1] Other formulations of dynamic stochastic games may be computationally more tractable.

However, computational convenience may come at the high cost of altering the model’s predictions and implications. From the basically static models in Cournot (1838) and von Stackelberg (1934) to the genuinely dynamic models in Cyert and DeGroot (1970) and Maskin and Tirole (1987, 1988a, b), a long literature has pointed out cases where the protocol of moves matters crucially for equilibrium behavior.

In this paper, we explore the implications of the protocol of moves for the computational burden and equilibrium behavior in the quality ladder model of Pakes and McGuire (1994). In the Pakes and McGuire (1994) model, forward-looking oligopolistic firms compete with each other in the product market and through their investment, entry, and exit decisions. By investing a firm aims to increase the quality of its product—and ultimately its profit from product market competition—over time. Investment, entry, and exit decisions are thus both dynamic and strategic.

The Pakes and McGuire (1994) model has been widely used as a template for dynamic models of investment in the Markov perfect equilibrium framework of Ericson and Pakes (1995). It has been adapted to study mergers (Gowrisankaran 1999; Gowrisankaran and Holmes 2004; Mermelstein et al. 2014); capacity accumulation (Besanko and Doraszelski 2004; Besanko et al. 2010); advertising (Doraszelski and Markovich 2007; Dube et al. 2005); network effects (Markovich 2008; Markovich and Moenius 2009; Chen et al. 2009); research joint ventures (Song 2011); durable

[1] Another curse of dimensionality arises if the number of states increases exponentially in the number of players. Applications of the Ericson and Pakes (1995) framework therefore routinely impose symmetry and anonymity restrictions that ensure that the number of states grows polynomially rather than exponentially.

goods (Goettler and Gordon 2011); investment in both vertical and horizontal product differentiation (Narayanan and Manchanda 2009); the timing of version releases (Borkovsky 2017a); and brand equity (Borkovsky et al. 2017b, c).

We reformulate the quality ladder model of Pakes and McGuire (1994) as a dynamic stochastic game with random moves. Each period one firm is picked at random to make an investment decision and the firm's product quality changes accordingly. Contrasting this model to the standard version with simultaneous moves illustrates the computational advantages of random moves. In particular, the quality ladder model with random moves avoids the curse of dimensionality in computing the expectation over successor states and is therefore orders of magnitude faster to solve when there are more than just a few firms.

Perhaps unexpectedly, the equilibria of the quality ladder model with random moves are practically indistinguishable from those of the model with simultaneous moves. Hence, the computational advantage of the model with random moves does not come at the cost of altering equilibrium behavior and the industry dynamics implied by it. This finding exemplifies the protocol-invariance theorem in Doraszelski and Escobar (2017). We argue that the quality ladder model of Pakes and McGuire (1994) fits into the class of separable dynamic games with noisy transitions defined by Doraszelski and Escobar (2017), so that the protocol of moves ceases to matter provided that periods are sufficiently short and moves are therefore sufficiently frequent. Our computations illustrate that protocol invariance extends some distance from this limit.

The remainder of this paper is organized as follows. Section 2 reviews the Pakes and McGuire (1994) model and reformulates it with random moves. Section 3 compares the computational burden of the models. Section 4 discusses the implications of the protocol of moves for equilibrium behavior and Section 5 concludes.

2 Quality ladder model

We first review the Pakes and McGuire (1994) model and then reformulate it with random moves. To focus on contrasting random with simultaneous moves and simplify the exposition, we abstract from entry and exit. Time is discrete and the horizon is infinite. At each point in time, the industry consists of N firms with potentially different product qualities $\omega = (\omega^1, \ldots \omega^N) \in \Omega = \{1, \ldots, M\}^N$, where $\omega^i \in \{1, \ldots, M\}$ is the quality of firm i's product.[2] We refer to ω^i as the state of firm i, to ω as the state (of the industry), and to Ω as the state space. A firm strives to maximize the expected net present value of its stream of cash flows and discounts future cash flows using a discount factor $\beta \in [0, 1)$. The solution concept is symmetric and

[2]To account for entry and exit, we add an extra state, say $\omega^i = 0$, that designates firm i as a potential entrant. Entry is a transition from state $\omega^i = 0$ to state $\left(\omega'\right)^i > 0$ and exit a transition from state $\omega^i > 0$ to state $\left(\omega'\right)^i = 0$. In this setting, N is the number of incumbent firms with $\omega^i > 0$ plus the number of potential entrants with $\omega^i = 0$. See Appendix D for a simple example and Doraszelski and Satterthwaite (2010) and Borkovsky et al. (2012) for further discussion.

anonymous Markov perfect equilibrium.

Per-period profit We follow the literature and treat the per-period profit $\pi^i(\omega)$ of firm i from product market competition in state ω as a reduced-form input into the model. We provide details on product market competition in Appendix A.

State-to-state transitions Firm i can invest to improve the quality of its product over time. Let $x^i \geq 0$ denote firm i's investment in quality improvements. If the investment is successful, then quality increases by one level. The probability of success is $\frac{\alpha x^i}{1+\alpha x^i}$, where $\alpha > 0$ is a measure of the effectiveness of investment. With probability $\delta \in [0, 1]$ the firm is hit by a depreciation shock and quality decreases by one level.[3]

Combining investment decisions and depreciation shocks, the probability that the quality of firm i's product changes from $\omega^i \in \{2, \ldots, M-1\}$ in the current period to $\left(\omega'\right)^i$ in the subsequent period is

$$\Pr(\left(\omega'\right)^i | \omega^i, x^i) = \begin{cases} \frac{(1-\delta)\alpha x^i}{1+\alpha x^i} & \text{if } \left(\omega'\right)^i = \omega^i + 1, \\ \frac{1-\delta+\delta\alpha x^i}{1+\alpha x^i} & \text{if } \left(\omega'\right)^i = \omega^i, \\ \frac{\delta}{1+\alpha x^i} & \text{if } \left(\omega'\right)^i = \omega^i - 1. \end{cases}$$

To ensure $\left(\omega'\right)^i \in \{1, \ldots, M\}$ we further set

$$\Pr(\left(\omega'\right)^i | 1, x^i) = \begin{cases} \frac{(1-\delta)\alpha x^i}{1+\alpha x^i} & \text{if } \left(\omega'\right)^i = 2, \\ \frac{1+\delta\alpha x^i}{1+\alpha x^i} & \text{if } \left(\omega'\right)^i = 1, \end{cases}$$

$$\Pr(\left(\omega'\right)^i | M, x^i) = \begin{cases} \frac{1-\delta+\alpha x^i}{1+\alpha x^i} & \text{if } \left(\omega'\right)^i = M, \\ \frac{\delta}{1+\alpha x^i} & \text{if } \left(\omega'\right)^i = M - 1. \end{cases}$$

Parameterization As in Pakes and McGuire (1994), the discount factor is $\beta = 0.925$, the effectiveness of investment is $\alpha = 3$, and the depreciation probability is $\delta = 0.7$. The number of quality levels per firm is $M = 18$, but we also examine $M = 9$ in Section 3.

2.1 Simultaneous moves

Bellman equation Let $V^i(\omega)$ denote the expected net present value of the cash flows accruing to firm i starting from state ω. The Bellman equation of firm i is

$$V^i(\omega) = \max_{x^i \geq 0} \pi^i(\omega) - x^i + \beta \mathrm{E}_{\omega'} \left\{ V^i\left(\omega'\right) | \omega, x^i, X^{-i}(\omega) \right\}, \tag{1}$$

[3]Whereas Pakes and McGuire (1994) assume an industry-wide depreciation shock, we assume that the depreciation shocks are independent across firms.

where $X^{-i}(\omega) = \left(X^1(\omega), \ldots X^{i-1}(\omega), X^{i+1}(\omega), \ldots, X^N(\omega)\right)$ denotes the investment strategies of firm i's rivals and the expectation is taken over all possible successor states ω':

$$\mathrm{E}_{\omega'}\left\{V^i\left(\omega'\right)|\omega, x^i, X^{-i}(\omega)\right\}$$
$$= \sum_{(\omega')^1 \in \{\omega^1-1,\omega^1,\omega^1+1\}} \cdots \sum_{(\omega')^N \in \{\omega^N-1,\omega^N,\omega^N+1\}} V^i(\omega') \prod_{j \neq i} \Pr(\left(\omega'\right)^j |\omega^j, X^j(\omega)) \Pr(\left(\omega'\right)^i |\omega^i, x^i).$$

This N-dimensional expectation consists of 3^N terms. Because of this exponential growth in the number of firms N, the model with simultaneous moves suffers from a curse of dimensionality in computing the expectation over successor states (Pakes and McGuire 2001; Doraszelski and Judd 2011).

Investment strategy The investment strategy of firm i is

$$X^i(\omega) = \arg\max_{x^i \geq 0} \pi^i(\omega) - x^i + \beta \mathrm{E}_{\omega'}\left\{V^i\left(\omega'\right)|\omega, x^i, X^{-i}(\omega)\right\}. \tag{2}$$

We provide a closed-form expression for $X^i(\omega)$ in Appendix B.

Equilibrium. The system of nonlinear equations (1) and (2) for all firms $i \in \{1, \ldots, N\}$ and all states $\omega \in \Omega$ defines a Markov perfect equilibrium. We provide details on how we impose symmetry and anonymity in Appendix C.

2.2 Random moves

To reformulate the Pakes and McGuire (1994) model, we assume that each period one firm is picked at random to make an investment decision. The product quality of the firm with the move then changes in response to its investment decision. Then another random draw is taken to pick a firm and so on.

Note that in the model with random moves a firm's state can change once in every N periods *on average*, whereas it can change once in every period in the model with simultaneous moves. To make the models comparable, we shorten the length of a period by a factor of N in the model with random moves. To this end, we replace the discount factor β by $\sqrt[N]{\beta}$ and the per-period profit $\pi^i(\omega)$ by $\frac{1}{N}\pi^i(\omega)$. This ensures comparability of the frequency of changes in a firm's state and of the cash flows accruing to the firm over an interval of time of fixed length.

Bellman equation Let $V^{i,j}(\omega)$ denote the expected net present value of the cash flows accruing to firm i starting from state ω if firm j has the move. Note that the value function depends on the identity of the firm with the move. If firm i has the move ($j = i$), then its Bellman equation is

$$V^{i,i}(\omega) = \max_{x^i \geq 0} \frac{1}{N}\pi^i(\omega) - x^i + \sqrt[N]{\beta}\mathrm{E}_{k',(\omega')^i}\left\{V^{i,k'}\left(\left(\omega'\right)^i, \omega^{-i}\right)|\omega^i, x^i\right\}, \tag{3}$$

where k' denotes the next firm to move, $\omega^{-i} = \left(\omega^1, \ldots, \omega^{i-1}, \omega^{i+1}, \ldots, \omega^N\right)$ denotes the product qualities of firm i's rivals, and the expectation is

$$\begin{aligned}&\mathrm{E}_{k',(\omega')^i}\left\{V^{i,k'}\left(\left(\omega'\right)^i, \omega^{-i}\right)|\omega^i, x^i\right\}\\ &= \frac{1}{N}\sum_{k'\in\{1,\ldots,N\}}\ \sum_{(\omega')^i\in\{\omega^i-1,\omega^i,\omega^i+1\}} V^{i,k'}\left(\left(\omega'\right)^i, \omega^{-i}\right)\Pr(\left(\omega'\right)^i|\omega^i, x^i).\end{aligned}$$

Because the identity of the next firm to move is a random variable, firm i forms an expectation over it. If firm i does not have the move ($j \neq i$), then its Bellman equation is

$$V^{i,j}(\omega) = \frac{1}{N}\pi^i(\omega) + \sqrt[N]{\beta}\mathrm{E}_{k',(\omega')^j}\left\{V^{i,k'}\left(\left(\omega'\right)^j, \omega^{-j}\right)|\omega^j, X^j(\omega)\right\}. \tag{4}$$

To consolidate the N Bellman equations (3) and (4) for all $j \neq i$, we define

$$\overline{V}^i(\omega) = \frac{1}{N}\sum_{j=1}^{N} V^{i,j}(\omega). \tag{5}$$

Intuitively, while $V^{i,j}(\omega)$ is the value function *after* it is known that firm j has the move, $\overline{V}^i(\omega)$ is the expected value function *before* it is known who is next to move. Substituting (5) into (3) and (4) yields

$$V^{i,i}(\omega) = \max_{x^i\geq 0}\frac{1}{N}\pi^i(\omega) - x^i + \sqrt[N]{\beta}\mathrm{E}_{(\omega')^i}\left\{\overline{V}^i\left(\left(\omega'\right)^i, \omega^{-i}\right)|\omega^i, x^i\right\}, \tag{6}$$

$$V^{i,j}(\omega) = \frac{1}{N}\pi^i(\omega) + \sqrt[N]{\beta}\mathrm{E}_{(\omega')^j}\left\{\overline{V}^i\left(\left(\omega'\right)^j, \omega^{-j}\right)|\omega^j, X^j(\omega)\right\}. \tag{7}$$

Adding (6) and (7) for all $j \neq i$ and dividing by N yields

$$\begin{aligned}\overline{V}^i(\omega) = \frac{1}{N}\Bigg\{&\max_{x^i\geq 0}\pi^i(\omega) - x^i + \sqrt[N]{\beta}\mathrm{E}_{(\omega')^i}\left\{\overline{V}^i\left(\left(\omega'\right)^i, \omega^{-i}\right)|\omega^i, x^i\right\}\\ &+\sum_{j\neq i}\sqrt[N]{\beta}\mathrm{E}_{(\omega')^j}\left\{\overline{V}^i\left(\left(\omega'\right)^j, \omega^{-j}\right)|\omega^j, X^j(\omega)\right\}\Bigg\}\end{aligned} \tag{8}$$

as the Bellman equation of firm i.

In contrast to the N-dimensional expectation consisting of 3^N terms in Eq. 1, Eq. 8 contains N one-dimensional expectations that each consist of 3 terms. This yields a total of $3N$ terms. As the number of terms grows linearly rather than exponentially in the number of firms N, the model with random moves avoids the curse of dimensionality in computing the expectation over successor states.

Investment strategy The investment strategy of firm i is

$$\begin{aligned}X^i(\omega) &= \arg\max_{x^i\geq 0}\frac{1}{N}\pi^i(\omega) - x^i + \sqrt[N]{\beta}\mathrm{E}_{k',(\omega')^i}\left\{V^{i,k'}\left(\left(\omega'\right)^i, \omega^{-i}\right)|\omega^i, x^i\right\}\\ &= \arg\max_{x^i\geq 0}\pi^i(\omega) - x^i + \sqrt[N]{\beta}\mathrm{E}_{(\omega')^i}\left\{\overline{V}^i\left(\left(\omega'\right)^i, \omega^{-i}\right)|\omega^i, x^i\right\},\end{aligned} \tag{9}$$

where we again substitute (5). We provide a closed-form expression for $X^i(\omega)$ in Appendix B.

Equilibrium The system of nonlinear equations (8) and (9) for all firms $i \in \{1, \dots, N\}$ and all states $\omega \in \Omega$ defines a Markov perfect equilibrium.

3 Computational burden

We use the block Gauss-Seidel version of the Pakes and McGuire (1994) algorithm in Section 3.1 of Doraszelski and Judd (2011) to compute Markov perfect equilibria.[4]

The time to convergence in the left panel of Table 1 demonstrates the computational advantage of the model with random moves for the case of $M = 9$ quality levels per firm. The model with random moves is faster to solve than the model with simultaneous moves, and this advantage grows substantially with the number of firms.

To better understand the source of the computational advantage of the model with random moves, we decompose the time to convergence into the time per iteration and the number of iterations. The right panel of Table 1 shows the ratio of the time to convergence and its components for the model with simultaneous moves relative to the model with random moves. On the one hand, an iteration of the Gauss-Seidel algorithm is orders of magnitude faster for the model with random moves than for the model with simultaneous moves when there are more than just a few firms. This reflects the fact that the model with random moves avoids the curse of dimensionality in computing the expectation over successor states.

On the other hand, the model with random moves suffers an "iteration penalty." This is expected as the discount factor is a key determinant of the rate of convergence of any Gaussian algorithm, and the discount factor is $\sqrt[N]{\beta} > \beta$ in the model with random moves. However, the loss in the number of iterations is small when compared to the gain in the time per iteration from avoiding the curse of dimensionality. The model with random moves is thus much faster to solve than the model with simultaneous moves.

4 Equilibrium and dynamics

It is not obvious how firms' behavior in equilibrium and the industry dynamics implied by that behavior change as the standard version of the Pakes and McGuire (1994) model is recast as a model with random moves. The model with random moves can be thought of as a "random-leadership Stackelberg game" in the sense that when a firm makes its investment decision it knows that it is the leader at this point

[4]For the model with random moves, we further exploit the smaller number of successor states by using pre-computed addresses (Doraszelski and Judd 2011, Section 3.3). The stopping rule is "distance to truth $< 10^{-4}$" (Doraszelski and Judd 2011, Section 5.2).

Table 1 Time to convergence of model with simultaneous moves and model with random moves and ratio of time per iteration, number of iterations, and time to convergence of model with simultaneous moves relative to model with random moves

	Time to convergence		Ratio simultaneous to random moves		
Number of firms	simult. moves (mins.)	random moves (mins.)	time per iteration	number of iterations	time to convergence
2	1.41(-4)	1.00(-4)	2.55	0.55	1.40
3	1.33(-3)	8.11(-4)	4.58	0.36	1.64
4	1.24(-2)	4.18(-3)	10.72	0.28	2.96
5	1.02(-1)	1.99(-2)	22.61	0.23	5.13
6	7.74(-1)	6.89(-2)	57.84	0.19	11.23
7	5.19(0)	2.03(-1)	150.00	0.17	25.54
8	3.21(1)	5.99(-1)	352.17	0.15	53.66
9	1.94(2)	1.48(0)	960.78	0.14	131.12
10	*1.10(3)*	3.43(0)	2,642.86	*0.12*	*320.59*
11	*5.80(3)*	7.37(0)	7,134.15	*0.11*	*786.81*
12	*2.94(4)*	1.53(1)	19,008.97	*0.10*	*1,923.53*
13	*1.51(5)*	2.97(1)	54,527.14	*0.09*	*5,101.20*
14	*7.08(5)*	5.62(1)	144,763.30	*0.09*	*12,592.71*

(k) is shorthand for $\times 10^k$. Entries in italics are based on an estimated 119 iterations to convergence in model with simultaneous moves. Quality ladder model with $M = 9$ quality levels per firm

in time and that its rivals are the followers. This may have repercussions for firms' behavior. For example, in a preemption race (Fudenberg et al. 1983; Harris and Vickers 1987) an early mover has a head start over a late mover, so that handing one firm the move may well be decisive for the outcome of the race.

It turns out that in the quality ladder model the differences between random and simultaneous moves are very small. Figure 1 illustrates this point by plotting the value and policy functions for the case of $N = 2$ firms and $M = 18$ quality levels per firm. There are no visible differences between the model with simultaneous moves in the upper panels and the model with random moves in the lower panels.

From the policy function we construct the probability distribution over next period's state ω' given this period's state ω, i.e., the transition probability matrix that characterizes the Markov process of industry dynamics. We compute the transient distribution over states in period t, $\mu^t(\cdot)$, starting from state $(1, 1)$. This tells us how likely each possible industry structure is in period t, given that both firms began the game at the minimal quality level. In addition, we compute the limiting (or ergodic) distribution over states, $\mu^\infty(\cdot)$.[5] The transient distribution captures short-run dynamics and the limiting distribution captures long-run (or steady-state) dynamics.

[5] Let P be the $M^2 \times M^2$ transition probability matrix. The transient distribution in period t is given by $\mu^t = \mu^0 P^t$, where μ^0 is the $1 \times M^2$ initial distribution and P^t the t^{th} matrix power of P. The Markov process turns out to be irreducible. That is, all its states belong to a single closed communicating class and the $1 \times M^2$ limiting distribution μ^∞ solves the system of linear equations $\mu^\infty = \mu^\infty P$.

Figure 1 also depicts the limiting distribution. As can be seen, the differences between the models with random and simultaneous moves are again very small. Table 2 summarizes the dynamics of the industry. In the left panel, we list the most likely industry structure (modal state) and its probability at various points in time. Note that, because we shorten the length of a period in the model with random moves by a factor of $N = 2$, the transient distribution in period t in the model with simultaneous moves is comparable to the transient distribution in period $2t$ in the model with random moves. As can be seen, in the short run the industry evolves either in a symmetric or an asymmetric fashion. However, even if a firm is able to gain the upper hand over its rival in the short run, in the long run the most likely industry structure is symmetric and the limiting distribution leaves little probability mass on asymmetric industry structures (see again Fig. 1).

In the middle and right panels of Table 2, we additionally report a firm's expected profit from product market competition and its expected investment in quality improvements along with their standard deviations. Again the differences between the models with random and simultaneous moves are very small.

This finding exemplifies the protocol-invariance theorem in Doraszelski and Escobar (2017). Protocol invariance means that the set of Markov perfect equilibria is nearly the same irrespective of the order in which players are assumed to move within a period, including—and extending beyond—simultaneous, random, and alternating moves.

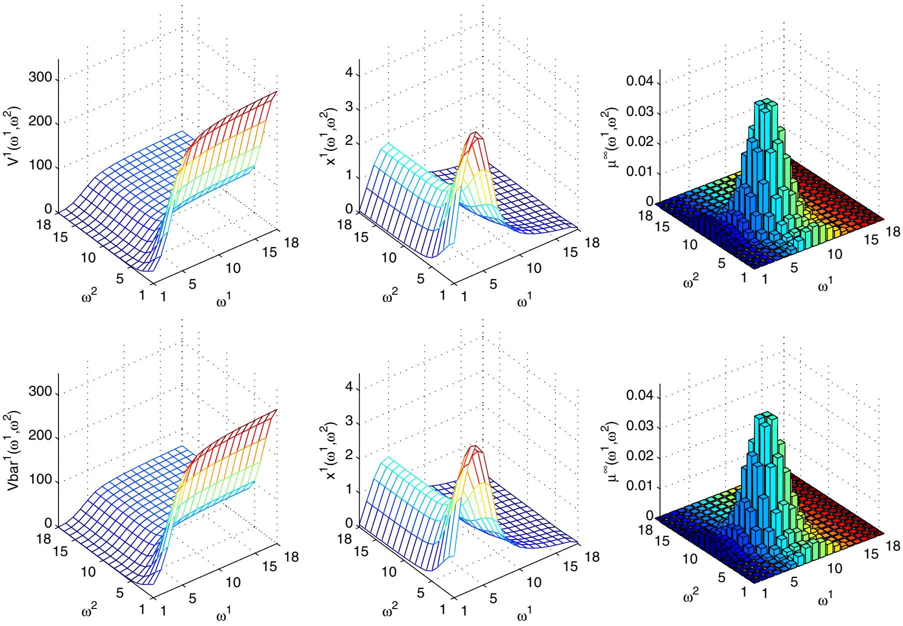

Fig. 1 Value and policy functions (left and middle panels) and limiting distribution (right panels). Model with simultaneous moves (upper panels) and model with random moves (lower panels). Quality ladder model with $N = 2$ firms and $M = 18$ quality levels per firm

Table 2 Most likely industry structure and its probability, expected profit and investment and their standard deviations

period	most likely industry structure	prob.	profit mean	profit std. dev.	investment mean	investment std. dev.
5	(2,1), (1,2)	0.1177	1.20	3.02	2.20	1.12
10	(3,3)	0.0510	4.57	7.12	2.32	1.19
25	(7,1), (1,7)	0.0301	8.60	9.32	1.46	0.93
50	(7,7)	0.0307	7.78	8.18	0.97	0.61
100	(8,7), (7,8)	0.0360	6.74	6.56	0.90	0.53
∞	(8,7), (7,8)	0.0404	6.14	5.27	0.91	0.51

period	most likely industry structure	prob.	profit mean	profit std. dev.	investment mean	investment std. dev.
10	(2,1), (1,2)	0.1160	1.43	3.68	2.25	1.15
20	(3,3)	0.0488	4.86	7.56	2.35	1.21
50	(7,1), (1,7)	0.0300	8.75	9.46	1.46	0.94
100	(7,7)	0.0305	7.78	8.17	0.98	0.61
200	(8,7), (7,8)	0.0363	6.66	6.39	0.91	0.53
∞	(8,7), (7,8)	0.0406	6.08	5.10	0.91	0.50

Model with simultaneous moves (upper panel) and model with random moves (lower panel). Quality ladder model with $N = 2$ firms and $M = 18$ quality levels per firm

Doraszelski and Escobar (2017) define a class of infinite-horizon dynamic stochastic games with finite states and actions that they call separable dynamic games with noisy transitions. In a separable dynamic game, per-period payoffs and state-to-state transitions are assumed to depend on players' actions in an additive manner: to a first-order approximation, per-period payoffs and state-to-state transitions are built from parts that depend on the actions taken by individual players. To the extent that there are complementarities between players' actions and other non-separabilities in per-period payoffs and state-to-state transitions, they must vanish as periods become short. Noisy transitions preclude that there is an action that a player can take to guarantee a change in the state. Doraszelski and Escobar (2017) establish that separable dynamic games with noisy transitions are protocol invariant provided that periods are sufficiently short.

Along with many other models, the quality ladder model of Pakes and McGuire (1994) satisfies the assumptions of separability and noisy transitions.[6] The quality ladder model is separable because a firm's investment decision affects its rivals only through changing the state and transitions are noisy because it is uncertain if the investment is successful and because of depreciation shocks. Hence, we expect protocol invariance to obtain in the limit as the period length goes to zero. Our computations illustrate that we come close to protocol invariance even though the period length—as determined by the discount factor β—is far from zero.

[6]Doraszelski and Escobar (2017) discuss in Section 4.3 how their protocol-invariance theorem extends to continuous actions such as the investment decision in the quality ladder model.

We emphasize that separability and noisy transitions are assumptions on the functional forms of per-period payoffs and state-to-state transitions. As such, protocol invariance does not hinge on particular parameterizations of the Pakes and McGuire (1994) model. Indeed, we expect protocol invariance to obtain in most dynamic models of investment in the Markov perfect equilibrium framework of Ericson and Pakes (1995).

5 Concluding remarks

There are many ways to formulate dynamic stochastic games, and some of them are computationally more tractable than others. Doraszelski and Judd (2011) and Arcidiacono et al. (2016) propose continuous-time models in order to avoid the curse of dimensionality in the expectation over successor states. This paper contributes by studying discrete-time models with random moves.

Our key assumption is that each period one player is picked at random to choose an action and that the player's state then changes accordingly. We show that reformulating the standard version of the Pakes and McGuire (1994) model in this way substantially decreases the computational burden when there are more than just a few firms. Importantly, the computational advantage of the model with random moves does not come at the cost of altering equilibrium behavior and the industry dynamics implied by it.

The computational advantage of the model with random moves does not stem directly from the protocol of moves. Indeed, one can write down a model with random moves that suffers from the curse of dimensionality in computing the expectation over successor states. Suppose, for example, that firm i's investment x^i in quality improvements increases the quality of firm k's product with probability $\frac{\gamma x^i}{1+\gamma x^i}$, where $\gamma \geq 0$, and that these spillovers occur independently across firm i's rivals. Then the expectation over successor states contains $3 \cdot 2^{N-1}$ terms.[7] By contrast, in the Pakes and McGuire (1994) model a firm's investment affects its product quality but not that of its rivals. Reformulating the model with random moves therefore avoids the curse of dimensionality in computing the expectation over successor states.

While the Pakes and McGuire (1994) model is a template for dynamic models of investment, not all applications and extensions of the Ericson and Pakes (1995) framework are equally well-suited to be formulated as models with random moves. For example, in a learning-by-doing model such as Benkard (2004), a firm's output decision affects how much the firm receives in the product market and how far it moves down its learning curve. Per-period profits and state-to-state transitions depend directly on the output decisions of all firms, whereas competitive interactions are channeled through the state in the quality ladder model. This makes it difficult to reformulate a learning-by-doing model with random moves. Moreover, because the

[7] On the other hand, the expectation over successor states contains 5 instead of $3 \cdot 2^{N-1}$ terms if an investment success of firm i spills over to all its rivals with probability $\eta \in [0, 1]$.

protocol-invariance theorem in Doraszelski and Escobar (2017) relies on the assumption of separability, reformulating the model may well alter how firms behave in equilibrium.

The assumption of noisy transitions is equally important for protocol invariance. In Appendix D, we develop an entry game between $N = 2$ firms. The state space is $\Omega = \{0, 1\}^2$. In state $\omega^i = 0$, firm i is a potential entrant that decides whether to enter an industry by incurring a setup cost $K > 0$. In state $\omega^i = 1$, firm i is an incumbent firm with no further decisions to make. An incumbent firm receives a profit $\pi^M > 0$ from product market competition if the industry is a monopoly and a profit $\pi^D = 0$ if the industry is a duopoly. We formulate this game first as a model with simultaneous moves and then as a model with random moves. We show that equilibrium behavior differs greatly between the models. In the model with simultaneous moves, there are two asymmetric equilibria in which in state (0, 0) one of the firms enters the industry for sure and one symmetric equilibrium in which both firms mix between entering and not entering. In the model with random moves, in contrast, there is a symmetric equilibrium in which in state (0, 0) the firm that first has the move enters the industry for sure. Protocol invariance fails in this game because state-to-state transitions are deterministic.[8]

In sum, dynamic stochastic games with random moves *can* have computational advantages over the standard assumption of simultaneous moves. Moreover, equilibrium behavior *can* be invariant to reformulating a game with random moves. Whether computational advantages and/or protocol invariance arise depends on the specific primitives of the application and has to be checked on a case-by-case basis.

Appendix A: Product market competition

Demand Each consumer purchases at most one unit of one product. The utility consumer k derives from purchasing product i is $g(\omega^i) - p^i + \epsilon^{ik}$, where

$$g(\omega^i) = \begin{cases} 3\omega^i - 4 & \text{if } \omega^i \leq 5, \\ 12 + \ln\left(2 - \exp\left(16 - 3\omega^i\right)\right) & \text{if } \omega^i > 5 \end{cases}$$

maps the quality of the product into the consumer's valuation for it and ϵ^{ik} represents taste differences among consumers. There is a no-purchase alternative, product 0, which has utility ϵ^{0k}. We assume that the idiosyncratic shocks $\epsilon^{0k}, \epsilon^{1k}, \ldots, \epsilon^{Nk}$ are independently and identically extreme value distributed across products and consumers; therefore, the demand for firm i 's product is

$$q^i(p^1, \ldots, p^N; \omega) = m \frac{\exp\left(g(\omega^i) - p^i\right)}{1 + \sum_{j=1}^N \exp\left(g(\omega^j) - p^j\right)},$$

where $m > 0$ is the size of the market (the measure of consumers).

[8] Doraszelski and Judd (2011) show how to model entry and exit to satisfy the assumption of noisy transitions (see also Doraszelski and Escobar 2017).

Per-period profit Firm i observes the quality of its and its rivals' products ω and chooses the price p^i of product i to maximize profit, thereby solving

$$\max_{p^i \geq 0} q^i(p^1, \ldots, p^N; \omega)\left(p^i - c\right),$$

where $c \geq 0$ is the marginal cost of production. There exists a unique Nash equilibrium $\left(p^1(\omega), \ldots, p^N(\omega)\right)$ of the product market game in state ω (Caplin and Nalebuff 1991). It is found easily by numerically solving the system of first-order conditions

$$1 + \sum_{j=1}^{N} \exp\left(g(\omega^j) - p^j\right) - \left(1 + \sum_{j \neq i} \exp\left(g(\omega^j) - p^j\right)\right)(p^i - c) = 0$$

for all firms $i \in \{1, \ldots, N\}$. The per-period profit of firm i in state ω is derived from the Nash equilibrium of the product market game as

$$\pi^i(\omega) = q^i(p^1(\omega), \ldots, p^N(\omega); \omega)(p^i(\omega) - c).$$

Note that we implicitly assume that all firms are able adjust their prices after a change in the state irrespective of whether they currently have the move or not.

Parameterization As in Pakes and McGuire (1994), the marginal cost of production is $c = 5$. The size of the market is $m = 5$.

Appendix B: Investment strategy

Simultaneous moves If $\omega^i \in \{2, \ldots, M-1\}$, then

$$X^i(\omega) = \frac{-1+\sqrt{\max\left\{1, \beta\alpha\left((1-\delta)(W^i(\omega^i+1)-W^i(\omega^i))+\delta(W^i(\omega^i)-W^i(\omega^i-1))\right)\right\}}}{\alpha},$$

where

$$W^i(\omega^i) = \sum_{(\omega')^1} \cdots \sum_{(\omega')^{i-1}} \sum_{(\omega')^{i+1}} \cdots \sum_{(\omega')^N} V^i(\omega^i, (\omega')^{-i}) \prod_{j \neq i} \Pr((\omega')^j | \omega^j, X^j(\omega))$$

Moreover, if $\omega^i \in \{1, M\}$, then

$$X^i(1, \omega^{-i}) = \frac{-1 + \sqrt{\max\left\{1, \beta\alpha(1-\delta)\left(W^i(2) - W^i(1)\right)\right\}}}{\alpha},$$

$$X^i(M, \omega^{-i}) = \frac{-1 + \sqrt{\max\left\{1, \beta\alpha\delta\left(W^i(M) - W^i(M-1)\right)\right\}}}{\alpha}.$$

Random moves If $\omega^i \in \{2, \ldots, M-1\}$, then

$$X^i(\omega) = \frac{-1+\sqrt{\max\left\{1, \sqrt[N]{\beta}\alpha\left((1-\delta)(W^i(\omega^i+1)-W^i(\omega^i))+\delta(W^i(\omega^i)-W^i(\omega^i-1))\right)\right\}}}{\alpha},$$

where $W^i(\omega^i) = \overline{V}^i(\omega^i, \omega^{-i})$. Moreover, if $\omega^i \in \{1, M\}$, then

$$X^i(1, \omega^{-i}) = \frac{-1 + \sqrt{\max\left\{1, \sqrt[N]{\beta}\alpha(1-\delta)\left(W^i(2) - W^i(1)\right)\right\}}}{\alpha},$$

$$X^i(M, \omega^{-i}) = \frac{-1 + \sqrt{\max\left\{1, \sqrt[N]{\beta}\alpha\delta\left(W^i(M) - W^i(M-1)\right)\right\}}}{\alpha}.$$

Appendix C: Symmetry and anonymity

Symmetry allows us to focus on firm 1 and anonymity—also called exchangeability—says that firm 1 does not care about the identity of its rivals, only about the distribution of their states. We refer the reader to Doraszelski and Satterthwaite (2010) for a formalization. In practice, symmetry and anonymity are imposed by limiting the computation of firms' values and policies to states in the set $\tilde{\Omega} = \left\{\omega \in \Omega : \omega^1 \leq \omega^2 \leq \ldots \leq \omega^N\right\}$. Some additional restrictions are needed. If $N = 2$, for example, symmetry and anonymity require that $V^1(1, 1) = V^2(1, 1)$ and $X^1(1, 1) = X^2(1, 1)$; if $N = 4$, they require that $V^2(1, 2, 2, 4) = V^3(1, 2, 2, 4)$ and $X^2(1, 2, 2, 4) = X^3(1, 2, 2, 4)$.

Appendix D: Entry game

Simultaneous moves Let $V^i(\omega)$ denote the expected net present value of the cash flows accruing to firm i starting from state ω and $\xi^i(\omega) \in [0, 1]$ the probability that firm i enters the industry in state ω. The Bellman equations defining a Markov perfect equilibrium are:

$$V^1(1, 1) = \pi^D + \beta V^1(1, 1), \tag{10}$$

$$V^2(1, 1) = \pi^D + \beta V^2(1, 1), \tag{11}$$

$$V^1(0, 1) = \max_{\xi^1 \in [0,1]} \xi^1 \left\{-K + \beta V^1(1, 1)\right\} + (1 - \xi^1)\beta V^1(0, 1), \tag{12}$$

$$V^2(0, 1) = \pi^M + \xi^1(0, 1)\beta V^2(1, 1) + (1 - \xi^1(0, 1))\beta V^2(0, 1), \tag{13}$$

$$V^1(1, 0) = \pi^M + \xi^2(1, 0)\beta V^1(1, 1) + (1 - \xi^2(1, 0))\beta V^1(1, 0), \tag{14}$$

$$V^2(1, 0) = \max_{\xi^2 \in [0,1]} \xi^2 \left\{-K + \beta V^2(1, 1)\right\} + (1 - \xi^2)\beta V^2(1, 0), \tag{15}$$

$$\begin{aligned} V^1(0, 0) = \max_{\xi^1 \in [0,1]} \xi^1 &\left\{-K + \xi^2(0, 0)\beta V^1(1, 1) + (1 - \xi^2(0, 0))\beta V^1(1, 0)\right\} \\ &+ (1 - \xi^1)\left\{\xi^2(0, 0)\beta V^1(0, 1) + (1 - \xi^2(0, 0))\beta V^1(0, 0)\right\}, \end{aligned} \tag{16}$$

$$\begin{aligned} V^2(0, 0) = \max_{\xi^2 \in [0,1]} \xi^2 &\left\{-K + \xi^1(0, 0)\beta V^2(1, 1) + (1 - \xi^1(0, 0))\beta V^2(0, 1)\right\} \\ &+ (1 - \xi^2)\left\{\xi^1(0, 0)\beta V^2(1, 0) + (1 - \xi^1(0, 0))\beta V^2(0, 0)\right\}. \end{aligned} \tag{17}$$

To facilitate the exposition, we assume $\pi^D = 0$ from hereon.

Proposition 1 *Suppose* $\frac{\beta\pi^M}{1-\beta} > K$. *There are three Markov perfect equilibria. In the first equilibrium,* $\xi^1(0,0) = 0$ *and* $\xi^2(0,0) = 1$*; in the second equilibrium,* $\xi^1(0,0) = 1$ *and* $\xi^2(0,0) = 0$*; and in the third equilibrium,* $\xi^1(0,0) = \xi^2(0,0) = \frac{\beta\pi^M-(1-\beta)K}{\beta\pi^M}$. *In any Markov perfect equilibrium,* $\xi^1(0,1) = \xi^2(1,0) = 0$.

Proof Equations 10 and 11 imply $V^1(1,1) = V^2(1,1) = 0$.

Consider (12) and (13) and plug in from above. There are three cases to consider:

1. *Firm 1 does not enter for sure.* The value and policy functions are:

$$V^1(0,1) = 0, \quad \xi^1(0,1) = 0.$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$V^1(0,1) \geq -K.$$

This holds.

2. *Firm 1 enters for sure.* The value and policy functions are:

$$V^1(0,1) = -K, \quad \xi^1(0,1) = 1.$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$V^1(0,1) \geq -\beta K.$$

This does not hold.

3. *Firm 1 mixes between entering and not entering.* The Bellman equation and indifference condition are:

$$V^1(0,1) = -\xi^1(0,1)K + (1-\xi^1(0,1))\beta V^1(0,1),$$
$$-K = \beta V^1(0,1).$$

This does not hold.

It follows that in any Markov perfect equilibrium $\xi^1(0,1) = 0$, $V^1(0,1) = 0$, and $V^2(0,1) = \frac{\pi^M}{1-\beta}$.

Consider (14) and (15). Similar to above, $\xi^2(1,0) = 0$, $V^1(1,0) = \frac{\pi^M}{1-\beta}$ and $V^2(1,0) = 0$ in any Markov perfect equilibrium.

Consider (16) and (17) and plug in from above. There are five cases to consider:

1. *Firm 1 does not enter for sure, firm 2 does not enter for sure.* The value and policy functions are:

$$V^1(0,0) = 0, \quad \xi^1(0,0) = 0,$$
$$V^2(0,0) = 0, \quad \xi^2(0,0) = 0.$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$V^1(0,0) \geq -K + \tfrac{\beta\pi^M}{1-\beta},$$
$$V^2(0,0) \geq -K + \tfrac{\beta\pi^M}{1-\beta}.$$

This does not hold.

2. *Firm 1 does not enter for sure, firm 2 enters for sure.* The value and policy functions are:

$$V^1(0,0) = 0, \quad \xi^1(0,0) = 0,$$
$$V^2(0,0) = -K + \tfrac{\beta\pi^M}{1-\beta}, \quad \xi^2(0,0) = 1.$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$V^1(0,0) \geq -K,$$
$$V^2(0,0) \geq \beta\left(-K + \tfrac{\beta\pi^M}{1-\beta}\right).$$

This holds.

3. *Firm 1 enters for sure, firm 2 does not enter for sure.* Similar to above.
4. *Firm 1 enters for sure, firm 2 enters for sure.* The value and policy functions are:

$$V^1(0,0) = -K, \quad \xi^1(0,0) = 1,$$
$$V^2(0,0) = -K, \quad \xi^2(0,0) = 1.$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$V^1(0,0) \geq 0,$$
$$V^2(0,0) \geq 0.$$

This does not hold.

5. *At least one firm mixes between entering and not entering.* The Bellman equations and indifference conditions are:

$$V^1(0,0) = \xi^1(0,0)\left\{-K + (1-\xi^2(0,0))\tfrac{\beta\pi^M}{1-\beta}\right\}$$
$$+(1-\xi^1(0,0))(1-\xi^2(0,0))\beta V^1(0,0),$$
$$-K + (1-\xi^2(0,0))\tfrac{\beta\pi^M}{1-\beta} = (1-\xi^2(0,0))\beta V^1(0,0),$$
$$V^2(0,0) = \xi^2(0,0)\left\{-K + (1-\xi^1(0,0))\tfrac{\beta\pi^M}{1-\beta}\right\}$$
$$+(1-\xi^2(0,0))(1-\xi^1(0,0))\beta V^2(0,0),$$
$$-K + (1-\xi^1(0,0))\tfrac{\beta\pi^M}{1-\beta} = (1-\xi^1(0,0))\beta V^2(0,0).$$

Solving for the value and policy functions yields:

$$V^1(0,0) = V^2(0,0) = 0, \quad \xi^1(0,0) = \xi^2(0,0) = \frac{\beta\pi^M - (1-\beta)K}{\beta\pi^M}.$$

For this to be part of a Markov perfect equilibrium, it must be that:

$$\xi^1(0,0) \in [0,1], \quad \xi^2(0,0) \in [0,1].$$

This holds.

□

Random moves Let $V^{i,j}(\omega)$ denote the expected net present value of the cash flows accruing to firm i starting from state ω if firm j has the move and $\xi^i(\omega) \in [0, 1]$ the probability that firm i enters the industry in state ω if it has the move. The Bellman equations defining a Markov perfect equilibrium are:

$$
\begin{aligned}
V^{1,1}(1,1) &= \frac{1}{2}\pi^D + \sqrt{\beta}\overline{V}^1(1,1),\\
V^{1,2}(1,1) &= \frac{1}{2}\pi^D + \sqrt{\beta}\overline{V}^1(1,1),\\
V^{2,2}(1,1) &= \frac{1}{2}\pi^D + \sqrt{\beta}\overline{V}^2(1,1),\\
V^{2,1}(1,1) &= \frac{1}{2}\pi^D + \sqrt{\beta}\overline{V}^2(1,1),\\
V^{1,1}(0,1) &= \max_{\xi^1\in[0,1]} \xi^1\left\{-K + \sqrt{\beta}\overline{V}^1(1,1)\right\} + (1-\xi^1)\sqrt{\beta}\overline{V}^1(0,1),\\
V^{1,2}(0,1) &= \sqrt{\beta}\overline{V}^1(0,1),\\
V^{2,2}(0,1) &= \frac{1}{2}\pi^M + \sqrt{\beta}\overline{V}^2(0,1),\\
V^{2,1}(0,1) &= \frac{1}{2}\pi^M + \xi^1(0,1)\sqrt{\beta}\overline{V}^2(1,1) + (1-\xi^1(0,1))\sqrt{\beta}\overline{V}^2(0,1),\\
V^{1,1}(1,0) &= \frac{1}{2}\pi^M + \sqrt{\beta}\overline{V}^1(1,0),\\
V^{1,2}(1,0) &= \frac{1}{2}\pi^M + \xi^2(1,0)\sqrt{\beta}\overline{V}^1(1,1) + (1-\xi^2(1,0))\sqrt{\beta}\overline{V}^1(1,0),\\
V^{2,2}(1,0) &= \max_{\xi^2\in[0,1]} \xi^2\left\{-K + \sqrt{\beta}\overline{V}^2(1,1)\right\} + (1-\xi^2)\sqrt{\beta}\overline{V}^2(1,0),\\
V^{2,1}(1,0) &= \sqrt{\beta}\overline{V}^2(1,0),\\
V^{1,1}(0,0) &= \max_{\xi^1\in[0,1]} \xi^1\left\{-K + \sqrt{\beta}\overline{V}^1(1,0)\right\} + (1-\xi^1)\sqrt{\beta}\overline{V}^1(0,0),\\
V^{1,2}(0,0) &= \xi^2(0,0)\sqrt{\beta}\overline{V}^1(0,1) + (1-\xi^2(0,0))\sqrt{\beta}\overline{V}^1(0,0),\\
V^{2,2}(0,0) &= \max_{\xi^2\in[0,1]} \xi^2\left\{-K + \sqrt{\beta}\overline{V}^2(0,1)\right\} + (1-\xi^2)\sqrt{\beta}\overline{V}^2(0,0),\\
V^{2,1}(0,0) &= \xi^1(0,0)\sqrt{\beta}\overline{V}^2(1,0) + (1-\xi^1(0,0))\sqrt{\beta}\overline{V}^2(0,0),
\end{aligned}
$$

where

$$\overline{V}^i(\omega) = \frac{1}{2}\sum_{j=1}^{2} V^{i,j}(\omega)$$

is the expected value function of firm i. Adding equations and dividing by 2 yields

$$\overline{V}^1(1,1) = \frac{1}{2}\left\{\pi^D + 2\sqrt{\beta}\overline{V}^1(1,1)\right\}, \tag{18}$$

$$\overline{V}^2(1,1) = \frac{1}{2}\left\{\pi^D + 2\sqrt{\beta}\overline{V}^2(1,1)\right\}, \tag{19}$$

$$\overline{V}^1(0,1) = \frac{1}{2}\left\{\max_{\xi^1\in[0,1]} \xi^1\left\{-K + \sqrt{\beta}\overline{V}^1(1,1)\right\} + (2-\xi^1)\sqrt{\beta}\overline{V}^1(0,1)\right\}, \tag{20}$$

$$\overline{V}^2(0,1) = \frac{1}{2}\left\{\pi^M + \xi^1(0,1)\sqrt{\beta}\overline{V}^2(1,1) + (2-\xi^1(0,1))\sqrt{\beta}\overline{V}^2(0,1)\right\}, \tag{21}$$

$$\overline{V}^1(1,0) = \frac{1}{2}\left\{\pi^M + \xi^2(1,0)\sqrt{\beta}\overline{V}^1(1,1) + (2-\xi^2(1,0))\sqrt{\beta}\overline{V}^1(1,0)\right\}, \tag{22}$$

$$\overline{V}^2(1,0) = \frac{1}{2}\left\{\max_{\xi^2\in[0,1]} \xi^2\left\{-K + \sqrt{\beta}\overline{V}^2(1,1)\right\} + (2-\xi^2)\sqrt{\beta}\overline{V}^2(1,0)\right\}, \tag{23}$$

$$\begin{aligned}\overline{V}^1(0,0) = \frac{1}{2}\Bigg\{&\max_{\xi^1\in[0,1]} \xi^1\left\{-K + \sqrt{\beta}\overline{V}^1(1,0)\right\} + (2-\xi^1-\xi^2(0,0))\sqrt{\beta}\overline{V}^1(0,0)\\ &+ \xi^2(0,0)\sqrt{\beta}\overline{V}^1(0,1)\Bigg\},\end{aligned} \tag{24}$$

$$\begin{aligned}\overline{V}^2(0,0) = \frac{1}{2}\Bigg\{&\max_{\xi^2\in[0,1]} \xi^2\left\{-K + \sqrt{\beta}\overline{V}^2(0,1)\right\} + (2-\xi^1(0,0)-\xi^2)\sqrt{\beta}\overline{V}^2(0,0)\\ &+ \xi^1(0,0)\sqrt{\beta}\overline{V}^2(1,0)\Bigg\}.\end{aligned} \tag{25}$$

Proposition 2 *Suppose* $\frac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})} > K$. *There is a Markov perfect equilibrium with* $\xi^1(0,0) = \xi^2(0,0) = 1$ *and* $\xi^1(0,1) = \xi^2(1,0) = 0$.

Proof Equations 18 and 19 imply $\overline{V}^1(1,1) = \overline{V}^2(1,1) = 0$.

Consider (20) and (21) and plug in from above. There are three cases to consider:

1. *Firm 1 does not enter for sure.* The value and policy functions are:

$$\overline{V}^1(0,1) = 0, \quad \xi^1(0,1) = 0.$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$\overline{V}^1(0,1) \geq -\frac{K}{2}.$$

This holds.

2. *Firm 1 enters for sure.* The value and policy functions are:

$$\overline{V}^1(0,1) = -\frac{K}{2-\sqrt{\beta}}, \quad \xi^1(0,1) = 1.$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$\overline{V}^1(0,1) \geq \frac{\sqrt{\beta}K}{2-\sqrt{\beta}}.$$

This does not hold.

3. *Firm 1 mixes between entering and not entering.* The Bellman equation and indifference condition are:

$$\begin{aligned}\overline{V}^1(0,1) &= \tfrac{1}{2}\left\{-\xi^1(0,1)K + (2-\xi^1(0,1))\sqrt{\beta}\overline{V}^1(0,1)\right\},\\ -K &= \sqrt{\beta}\overline{V}^1(0,1).\end{aligned}$$

This does not hold.

It follows that in any Markov perfect equilibrium $\xi^1(0,1) = 0$, $\overline{V}^1(0,1) = 0$ and $\overline{V}^2(0,1) = \frac{\pi^M}{2(1-\sqrt{\beta})}$.

Consider (22) and (23). Similar to above, $\xi^2(1,0) = 0$, $\overline{V}^1(1,0) = \frac{\pi^M}{2(1-\sqrt{\beta})}$ and $\overline{V}^2(1,0) = 0$.

Consider (24) and (25) and plug in from above. There are five cases to consider:

1. *Firm 1 does not enter for sure, firm 2 does not enter for sure.* The value and policy functions are:

$$\begin{aligned}\overline{V}^1(0,0) &= 0, \quad \xi^1(0,0) = 0,\\ \overline{V}^2(0,0) &= 0, \quad \xi^2(0,0) = 0.\end{aligned}$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$\begin{aligned}\overline{V}^1(0,0) &\geq \tfrac{1}{2}\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\},\\ \overline{V}^2(0,0) &\geq \tfrac{1}{2}\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\}.\end{aligned}$$

This does not hold.

2. *Firm 1 does not enter for sure, firm 2 enters for sure.* The value and policy functions are:

$$\begin{aligned}\overline{V}^1(0,0) &= 0, \quad \xi^1(0,0) = 0,\\ \overline{V}^2(0,0) &= \tfrac{1}{2-\sqrt{\beta}}\left(-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right), \quad \xi^2(0,0) = 1.\end{aligned}$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$\begin{aligned}V^1(0,0) &\geq \tfrac{1}{2}\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\},\\ V^2(0,0) &\geq \tfrac{\sqrt{\beta}}{2-\sqrt{\beta}}\left(-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right).\end{aligned}$$

This does not hold.

3. *Firm 1 enters for sure, firm 2 does not enter for sure.* Similar to above.

4. *Firm 1 enters for sure, firm 2 enters for sure.* The value and policy functions are:

$$\begin{aligned}\overline{V}^1(0,0) &= \tfrac{1}{2}\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\}, \quad \xi^1(0,0) = 1,\\ \overline{V}^2(0,0) &= \tfrac{1}{2}\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\}, \quad \xi^2(0,0) = 1.\end{aligned}$$

For this to be part of a Markov perfect equilibrium, no unilateral one-shot deviation can be profitable:

$$\begin{aligned}\overline{V}^1(0,0) &\geq \tfrac{\sqrt{\beta}}{4}\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\},\\ \overline{V}^2(0,0) &\geq \tfrac{\sqrt{\beta}}{4}\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\}.\end{aligned}$$

This holds.

5. *At least one firm mixes between entering and not entering.* The Bellman equations and indifference conditions are:

$$\begin{aligned}\overline{V}^1(0,0) &= \tfrac{1}{2}\left\{\xi^1(0,0)\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\} + (2-\xi^1(0,0)-\xi^2(0,0))\sqrt{\beta}\overline{V}^1(0,0)\right\},\\ -K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})} &= \sqrt{\beta}\overline{V}^1(0,0),\\ \overline{V}^2(0,0) &= \tfrac{1}{2}\left\{\xi^2(0,0)\left\{-K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})}\right\} + (2-\xi^1(0,0)-\xi^2(0,0))\sqrt{\beta}\overline{V}^2(0,0)\right\},\\ -K + \tfrac{\sqrt{\beta}\pi^M}{2(1-\sqrt{\beta})} &= \sqrt{\beta}\overline{V}^2(0,0).\end{aligned}$$

This does not hold.

□

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