Online Appendix: Latent Indices in Assortative Matching Models

(Not for publication)

William Diamond, Harvard University
Nikhil Agarwal, MIT and NBER

February 21, 2017

C Proofs: Identification

C.1 Non Identification in Data from One-to-One Matches

In this section, we show that a model with unobservables on one side of the market can rationalize any data from a large one-to-one matching market under the following condition:

Assumption C.1.7 *The primitives h, g, $F_{X,\varepsilon}$, $F_{Z,\eta}$ are such that*

(i) $F_{h(X)|V}(\bar{h}, v) = \mathbb{P}\left(h(X) \leq \bar{h} | h(X) + \varepsilon = v\right) = \gamma(\kappa\bar{h} - v)$ for some function γ and constant κ, (ii) $F_V^{-1} \circ F_U$ is a linear function, (iii) The functions h, g and $f_{h(X)|V}$ are twice continuously differentiable, and (iv) ε and η are independent of X and Z respectively.

As is evident, these conditions are satisfied in Example 2. The joint distribution F_{XZ} produced by the model in the example is identical to one produced by the following transformation model, $X = \frac{1}{\kappa^{1/2}} Z + \eta_1$, where $\eta_1 \sim N\left(0, 1 - \frac{1}{\kappa}\right)$.

Proposition C.1.6 *Under Assumptions C.1.7 and 2, any joint distribution F_{XZ} can be rationalized in a matching model with $\varepsilon \equiv 0$.*

Proof. The proof proceeds by rewriting the matching model with $\varepsilon \equiv 0$ in terms of the transformation model of Chiappori and Komunjer (2008). We will then use Chiappori and Komunjer (2008) Proposition 2, which states that the transformation model is correctly specified.

In the matching model, quantiles of $h(X) + \varepsilon$ are matched with quantiles of $g(Z) + \eta$. We will use Proposition 2 of Chiappori and Komunjer (2008) to show that there exist increasing functions $\bar{\Gamma}$, $\bar{g}$, $F_{\bar{\eta}}$ such that the the transformation model

$$h(X) = \bar{\Gamma}\left(\bar{g}(Z) + \bar{\eta}\right)$$

rationalizes any joint distribution F_{XZ} from a matching model satisfying Assumptions 1 - 2 and Condition C.1.7. This is model is equivalent to a matching model with $\bar{h} = \bar{\Gamma}^{-1} \circ h$, $\varepsilon \equiv 0$, and $F_{\bar{\eta}}$, $\bar{g}$. In what follows, we will treat X and Z as known scalars with $h(\cdot)$ and $g(\cdot)$ as increasing functions of them respectively. This simplification is without loss of generality since our we show that a positive monotone transformation $h(\cdot)$ and $g(\cdot)$ exists that yields an identical joint distribution F_{XZ}.

Since X and Z are unidimensional, Assumption A3 of Chiappori and Komunjer (2008) is then equivalent to independence of ε and η from X and Z respectively, as maintained under the hypotheses of Proposition C.1.6.

Let the probability that a firm with observable trait z is matched with workers with $h(X)$ at most $\bar{h}$ be denoted $\Phi\left(\bar{h}, z\right) = F_{h(X)|Z}\left(\bar{h}, z\right)$. Note that

$$\begin{aligned} \Phi\left(\bar{h}, z\right) &= \int F_{h(X)|V}\left(\bar{h}, F_V^{-1} F_U\left(g(z) + \eta\right)\right) dF_\eta \\ &= \int \gamma\left(\kappa\bar{h} - A\left(g(z) + \eta\right)\right) dF_\eta, \end{aligned}$$

for some constant A. The first equality is derived from the quantile-quantile matching of workers and firms and the second equality follows from Conditions C.1.7 (i) and C.1.7 (ii).

First, we ensure that Φ has continuous third order partial derivatives $\partial^3\Phi(\bar{h},z)/\partial\bar{h}\partial^2 z$ and $\partial^3\Phi(\bar{h},z)/\partial^2\bar{h}\partial z$, and that $\partial\Phi(\bar{h},z)/\partial\bar{h} > 0$. Conditions C.1.7 (iii) guarantees the existence of the required partial derivatives. Further, since $F_{h(X)|V}\left(\bar{h},v\right)$ is strictly increasing in $\bar{h}$, we have that $\partial\Phi(\bar{h},z)/\partial\bar{h} > 0$.

We now verify that $\Phi\left(\bar{h},z\right)$ satisfies Condition C in Chiappori and Komunjer (2008), i.e.

$$\frac{\partial^2}{\partial\bar{h}\partial z_k}\left(\log\left|\frac{\partial\Phi\left(\bar{h},z\right)/\partial\bar{h}}{\partial\Phi\left(\bar{h},z\right)/\partial z_1}\right|\right)=0.$$

The partial derivatives of $\Phi\left(\bar{h},z\right)$ with respect to $\bar{h}$ and z_1 are given by:

$$\begin{aligned}\frac{\partial\Phi\left(\bar{h},z\right)}{\partial\bar{h}} &= \kappa\int\gamma'\left(\kappa\bar{h}-A\left(g\left(z\right)+\eta\right)\right)dF_\eta\\ \frac{\partial\Phi\left(\bar{h},z\right)}{\partial z_1} &= -A\frac{\partial g\left(z\right)}{\partial z_1}\int\gamma'\left(\bar{h}-A\left(g\left(z\right)+\eta\right)\right)dF_\eta.\end{aligned}$$

Note that γ' exists since the existence of densities f_X, f_ε and differentiability of $h\left(\cdot\right)$ implies that the derivatives of $F_{h(X)|V}\left(\bar{h},v\right)$ exist.

Using the expressions above, rewrite

$$\frac{\partial^2}{\partial\bar{h}\partial z_k}\left(\log\left|\frac{\partial\Phi\left(\bar{h},z\right)/\partial\bar{h}}{\partial\Phi\left(\bar{h},z\right)/\partial z_1}\right|\right)=\frac{\partial^2}{\partial\bar{h}\partial z_k}\left(\log|\kappa|-\log\left|A\frac{\partial g\left(z\right)}{\partial z_1}\right|\right)=0.$$

The last equality follows since $\log\left|\frac{\partial g(z)}{\partial z_1}\right|$ and $\log|\kappa|$ do not depend on $\bar{h}$.

We now show that equations (4) and (5) in Chiappori and Komunjer (2008) are satisfied. Since $F_{h(X)|V}\left(\bar{h},A\left(g\left(z\right)+\eta\right)\right)$ is a cdf, it is bounded, $\lim_{\bar{h}\to-\infty}F_{X|V}\left(\bar{h},A\left(g\left(z\right)+\eta\right)\right)=0$ and $\lim_{\bar{h}\to\infty}F_{h(X)|V}\left(\bar{h},A\left(g\left(z\right)+\eta\right)\right)=1$ for each z and η. Hence, $\lim_{\bar{h}\to-\infty}\Phi\left(\bar{h},z\right)=0$ and $\lim_{\bar{h}\to\infty}\Phi\left(\bar{h},z\right)=1$.

To verify (5), note that

$$\begin{aligned}&\int_0^{\bar{h}}\frac{\partial\Phi\left(a,z\right)/\partial x}{\partial\Phi\left(a,z\right)/\partial z_1}\frac{\partial\Phi\left(0,z\right)/\partial z_1}{\partial\Phi\left(0,z\right)/\partial x}da\\ =&\int_0^{\bar{h}}\frac{\kappa}{-A\partial g\left(z\right)/\partial z_1}\frac{-A\partial g\left(z\right)/\partial z_1}{\kappa}da\\ =&\int_0^{\bar{h}}1da=\bar{h}.\end{aligned}$$

Equation (5) of Chiappori and Komunjer (2008) follows since $h\left(X\right)$ has full support on $\mathbb{R}$.

By Proposition 2 of Chiappori and Komunjer (2008), there exist $\bar{\Gamma}$, $\bar{g}$, $F_{\bar{\eta}}$ that rationalize Φ. ■

C.2 Preliminaries

Since $h\left(X\right)$ and $g\left(Z\right)$ admit bounded continuous densities and are identified up to positive monotone transformation, it is without loss to treat x and z as single dimensional variable that are uniformly distributed on $[0,1]$. Proposition 1 implies that this simplification is without loss of generality.

Let $v=h\left(x\right)+\varepsilon$, where $h\left(x\right)$ is strictly increasing with $h\left(\bar{x}\right)=0$, $h'\left(\bar{x}\right)=1$ and let ε be median zero with density f_ε. For quantile $\tau \in [0,1]$, let $f_{\tau|X}(\tau,x) = \frac{f_\varepsilon\left(F_V^{-1}(\tau)-h(x)\right)}{f_V\left(F_V^{-1}(\tau)\right)}$ be the density on $v = F_V^{-1}(\tau)$ given x, where $F_V(v) = \int F_\varepsilon(v-h(x))\,dF_X$.

Lemma C.2.2 *The function $h(x)$ and the density f_ε are identified from $f_{\tau|x}(\tau)$ if $h(x)$ is differentiable and ε has full support on $\mathbb{R}$.*

Proof. Let $\phi(x,x')$ be the probability that $h(x)+\varepsilon > h(x')+\varepsilon'$ given x and x'. $\phi(x,x')$ is identified from $f_{\tau|x}(\tau)$ since it can be written as

$$\phi(x,x') = \int_0^1 \int_{\tau>\tau'} f_{\tau|X}(\tau,x)\, f_{\tau|X}(\tau',x')\, d\tau d\tau'.$$

However, $\phi(x,x')$ can also be written in terms of the primitives $h(\cdot)$ and f_ε as

$$\phi(x,x') = \int F_\varepsilon(h(x)+\varepsilon-h(x'))\, f_\varepsilon(\varepsilon)\, d\varepsilon.$$

Taking the derivative with respect to x and x', we get

$$\begin{aligned}\frac{\partial\phi(x,x')}{\partial x} &= h'(x)\int f_\varepsilon(h(x)+\varepsilon-h(x'))\, f_\varepsilon(\varepsilon)\, d\varepsilon \\ \frac{\partial\phi(x,x')}{\partial x'} &= -h'(x')\int f_\varepsilon(h(x)+\varepsilon-h(x'))\, f_\varepsilon(\varepsilon)\, d\varepsilon.\end{aligned}$$

The ratio $\frac{\partial\phi(x,x')}{\partial x}/\frac{\partial\phi(x,x')}{\partial x'}$ is identified and is equal to $-\frac{h'(x)}{h'(x')}$.Since $h'(\bar{x}) = 1$, $h'(x)$ can be determined everywhere. The boundary condition $h(\bar{x}) = 0$ provides the unique solution to the resulting differential equation determining $h(\cdot)$.

We now need to show that F_ε is identified. Let $R_x(t) = \mathbb{P}\left(h(x)+\varepsilon \le F_V^{-1}(t)\,|x\right)$. $R_x(t)$ is known since it is equal to $\int_0^t f_{\tau|x}(\tau)\,d\tau$. Since F_V^{-1} is continuous and ε admits a full support density, $R_x(t)$ is continuous and strictly increasing in t. Let τ^* be the median rank of $\bar{x}$, i.e. $R_{\bar{x}}(t^*) = \frac{1}{2}$. Since ε is median-zero, $h(\bar{x}) = 0$ and $\mathbb{P}\left(h(\bar{x})+\varepsilon \le F_V^{-1}(t^*)\,|\bar{x}\right) = \frac{1}{2}$, we have that $F_V^{-1}(t^*) = 0$. For any x, $R_x(t^*) \in (0,1)$ is therefore the probability that $h(x)+\varepsilon \le 0$ given x, i.e. $R_x(t^*) = F_\varepsilon(-h(x))$. Since $h(x)$ and $R_x(t^*)$ are known and have full support on $\mathbb{R}$, F_ε is identified. ■

Lemma C.2.3 *Suppose f_ε has a non-vanishing characteristic function and $h(X)$ has full support on $\mathbb{R}$. For any function $m(v)$, we have that $\int f_\varepsilon(v-h(x))\,m(v)\,dv = 0$ for all x implies that $m(v) = 0$ a.e. Further, if $h(\cdot)$ is differentiable and strictly increasing, then for any function $m(x)$, $\int f_\varepsilon(v-h(x))\,m(x)\,dx = 0$ for all v implies that $m(x) = 0$ a.e.*

Proof. Note that

$$\int f_\varepsilon(v-h(x))\,m(v)\,dv = \int f_\varepsilon(\varepsilon)\,m(h(x)+\varepsilon)\,d\varepsilon$$

is a convolution of $m(\cdot)$ with $-\varepsilon$. Since f_ε has a non-vanishing characteristic, so does $f_{-\varepsilon}$. Therefore, completeness follows from Mattner (1993), Theorem 2.1. Similarly, by a change of variables, $h(x) = h$, we have that

$$\int f_\varepsilon(v-h(x))\,m(x)\,dx = \int f_\varepsilon(v-h)\,M(h)\,dh,$$

where $M(h) = \frac{m\left(h^{-1}(h)\right)}{h'(h^{-1}(h))}$. Since f_ε has a non-vanishing characteristic, $\int f_\varepsilon(v-h)\,M(h)\,dh = 0$ implies that $M(h) = 0$ for all h. Since h is strictly increasing, this implies that $m(x) = 0$ for all x. ■

For a function m, define the operator $L_{x|q1} : L_1([0,1]) \to L_1([0,1])$ as $L_{x|q}(m) = \int f(x|q)\, m(q)\, dq$ where $f(x|q)$ is the conditional density of X given $Q = q$.

Lemma C.2.4 *$L_{x_1|q}$ is injective if (i) f_ε has a non-vanishing characteristic function (ii) F_V is continuous and strictly increasing and (iii) $h(X)$ has full support on $\mathbb{R}$. Further, $L_{x|q}$ is bounded, and $L_{x|q}^{-1}$ is exists and is densely defined.*

Proof. We first rewrite the operator $L_{x|q}$ as a convolution:

$$\begin{aligned}\int_0^1 f(x|q)\, m(q)\, dq &= \int f(x|v)\, m(F_V(v))\, f_V(v)\, dv \\ &= \int f(v|x)\, m(F_V(v))\, dv \\ &= \int f_\varepsilon(v - h(x))\, M(v)\, dv\end{aligned}$$

where the first equality follows from a change of variables, the second equality uses Bayes' rule $f_{x|v}(x|v) = \frac{f_{v|x}(v|x) f_x(x)}{f_V(v)}$ and the fact that the distribution of x is normalized to uniform $[0,1]$, and the third equality uses the fact that $f_{v|x}(v|x) = f_\varepsilon(v - h(x))$ and sets $M(v) = m(F_V(v))$. By Lemma C.2.3, $M(v) = 0$ for almost all v. Since F_V is bijective, we have that $m(q) = 0$ for almost all q. Therefore, $f(x|q)$ is complete, and as noted in Hu and Schennach (2008), hence that $L_{x|q}$ is injective.

Note that the (operator) norm of $L_{x|q}$ is at most

$$\begin{aligned}&\sup_{m \in \mathcal{L}_1([0,1])} \frac{1}{\|m\|_1} \int_0^1 \int \frac{f_\varepsilon(v - h(x))}{f_V(v)} |m(F_V(v))|\, f_V(v)\, dv dx \\ =\ &\sup_{m \in \mathcal{L}_1([0,1])} \frac{1}{\|m\|_1} \int \left(\int_0^1 \frac{f_\varepsilon(v - h(x))}{f_V(v)} dx \right) |m(F_V(v))|\, f_V(v)\, dv \\ =\ &\frac{1}{\|m\|_1} \int_0^1 |m(q)|\, dq = 1\end{aligned}$$

where we use a change in the order of integration by Fubini's theorem, the identity $f_V(v) = \int_0^1 f_\varepsilon(v - h(x))\, dx$, and a change in variables $q = F_V(v)$.

As argued in the proof of HS, Lemma 1, to show that $L_{x|q}^{-1}$ is densely defined, it is sufficient to show that the adjoint $L_{x|q}^\dagger$ is injective. Note that for any y in the space of bounded functions (dual of the domain of $L_{x|q}$), it must be that

$$\begin{aligned}\left\langle m, L_{x|q}^\dagger y \right\rangle = \langle L_{x|q} m, y \rangle &= \int_0^1 \int_0^1 f_{x|q}(x|q)\, m(q)\, dq y(x)\, dx \\ &= \int_0^1 \int_0^1 f_{x|q}(x|q)\, y(x)\, dx m(q)\, dq,\end{aligned}$$

where we changed the order of integration using Fubini's theorem. Therefore, $L_{x|q}^\dagger y(q) = \int_0^1 f_{x|q}(x|q)\, y(x)\, dx$. Since $f_{x|q}(x|q) = \frac{1}{f_V\left(F_V^{-1}(q)\right)} f_\varepsilon\left(F_V^{-1}(q) - h(x)\right)$ and $f_V(v) > 0$, $L_{x|q}^\dagger y(q) = 0$ for all $q \in [0,1]$, implies that $\int_0^1 f_\varepsilon(v - h(x))\, y(x)\, dx = 0$ for all $v \in \mathbb{R}$. By Lemma C.2.3, $y(x) = 0$ for almost all x. ■

Define the operator $L_{x_1|x_2} : L_1([0,1]) \to L_1([0,1])$ as $L_{x_1|x_2} m(x_1) = \int f_{x_1|x_2}(x_1|x_2)\, m(x_2)\, dx_2$ for any function $m \in L_1([0,1])$.

Lemma C.2.5 *$L_{x_1|x_2}$ is injective if (i) f_ε has a non-vanishing characteristic function (ii) F_V is continuous and strictly increasing and (iii) $h(x)$ has full support on $\mathbb{R}$. Further, $L_{x_1|x_2}$ is bounded, and $L^{-1}_{x_1|x_2}$ exists and is densely defined.*

Proof. Note that

$$\begin{aligned}
\int_0^1 f_{x_1|x_2}(x_1|x_2)\, m(x_2)\, dx_2 &= \int_0^1 \left(\int_0^1 f(x_1, q|x_2)\, dq\right) m(x_2)\, dx_2 \\
&= \int_0^1 \left(\int_0^1 f_{x_1|q}(x_1|q)\, f_{q|x_2}(q|x_2)\, dq\right) m(x_2)\, dx_2 \\
&= \int_0^1 \left(\int f_{x_1|v}(x_1|v)\, f_{v|x_2}(v|x_2)\, dv\right) m(x_2)\, dx_2 \\
&= \int\int_0^1 \frac{f_\varepsilon(v - h(x_1))}{f_V(v)} f_{v|x_2}(v|x_2)\, m(x_2)\, dx_2 dv \\
&= \int \frac{f_\varepsilon(v - h(x_1))}{f_V(v)} \left(\int_0^1 f_\varepsilon(v - h(x_2))\, m(x_2)\, dx_2\right) dv
\end{aligned}$$

where we use (i) $f(x_1, q|x_2) = f(x_1|q, x_2)\, f_{q|x_2}(q|x_2) = f_{x_1|q}(x_1|q)\, f_{q|x_2}(q|x_2)$, (ii) a change of variables $F_V(v) = q$, (iii) $f_{x_1|v}(x_1|v) = \frac{f_\varepsilon(v-h(x_1))}{f_V(v)}$, $f_{q|x_2}(F_V(v)|x_2) = \frac{1}{f_V(v)} f_{v|x_2}(v|x_2)$ and (iv) a change in the order of integration by Fubini's theorem.

By Lemma C.2.3, $f_\varepsilon(v - h(x_1))$ is complete, and consequently, $\int_0^1 f_{x_1|x_2}(x_1|x_2)\, m(x_2)\, dx_2 = 0$ implies $\frac{1}{f_V(v)} \int_0^1 f_\varepsilon(v - h(x_2))\, m(x_2)\, dx_2 = 0$ for almost all v. Since $f_V(v) > 0$, we have that $\int_0^1 f_\varepsilon(v - h(x_2))\, m(x_2)\, dx_2 = 0$ for almost all v. A second application of Lemma C.2.3 implies that $m(x) = 0$ for all x (Lemma C.2.3). Hence, $f_{x_1|x_2}(x_1|x_2)$ is complete, and as noted in Hu and Schennach (2008), completeness implies injectivity.

Note that the (operator) norm of $L_{x_1|x_2}$ is

$$\begin{aligned}
\sup_{m\in\mathcal{L}_1([0,1])} \frac{\left\|L_{x_1|x_2} m\right\|_1}{\|m\|_1} &\le \sup_{m\in\mathcal{L}_1([0,1])} \frac{1}{\|m\|_1} \int_0^1 \int f_\varepsilon(v - h(x_1)) \frac{1}{f_V(v)} \left(\int_0^1 f_\varepsilon(v - h(x_2))\, |m(x_2)|\, dx_2\right) dv dx_1 \\
&= \sup_{m\in\mathcal{L}_1([0,1])} \frac{1}{\|m\|_1} \int_0^1 \int \left(\frac{1}{f_V(v)} \int_0^1 f_\varepsilon(v - h(x_1))\, dx_1\right) f_\varepsilon(v - h(x_2))\, dv\, |m(x_2)|\, dx_2 \\
&= \sup_{m\in\mathcal{L}_1([0,1])} \frac{1}{\|m\|_1} \int_0^1 \int f_\varepsilon(v - h(x_2))\, dv\, |m(x_2)|\, dx_2 = 1,
\end{aligned}$$

where we use (i) change in the order of integration by Fubini's theorem, and (ii) the identity $f_V(v) = \int_0^1 f_\varepsilon(v - h(x_1))\, dx_1$.

As argued in the proof of HS, Lemma 1, to show that $L^{-1}_{x_1|x_2}$ is densely defined, it is sufficient to show that the adjoint $L^\dagger_{x_1|x_2}$ is injective. Note that for any y in the space of bounded functions (dual of the domain of $L_{x|q}$), it must be that

$$\begin{aligned}
\left\langle m, L^\dagger_{x_1|x_2} y\right\rangle = \left\langle L_{x_1|x_2} m, y\right\rangle &= \int_0^1 \left(\int_0^1 \left(\int_0^1 f_{x_1|q}(x_1|q)\, f_{q|x_2}(q|x_2)\, dq\right) m(x_2)\, dx_2\right) y(x_1)\, dx_1 \\
&= \int_0^1 \left(\int_0^1 \left(\int_0^1 f_{x_1|q}(x_1|q)\, f_{q|x_2}(q|x_2)\, dq\right) y(x_1)\, dx_1\right) m(x_2)\, dx_2
\end{aligned}$$

where we changed the order of integration using Fubini's theorem. Hence,

$$L^\dagger_{x_1|x_2} y(x_2) = \int_0^1 \left(\int_0^1 f_{x_1|q}(x_1|q)\, f_{q|x_2}(q|x_2)\, dq\right) y(x_1)\, dx_1.$$

The arguments made above for $L_{x_1|x_2}$ also imply that $L^{\dagger}_{x_1|x_2}$ is injective. ■

For a function m, define the operator $L_{z;x_1|x_2}$ as

$$L_{z;x_1|x_2} m(x_1) = \int_0^1 \int_0^1 f_{z|q}(z|q) f_{x_1|q}(x_1|q) f_{q|x_2}(q|x_2) m(x_2) dq dx_2$$

where $f_{z|q}(z|q)$, $f_{x_1|q}(x_1|q)$, and $f_{q|x_2}(q|x_2)$ are the conditional densities of the respective random variables

Lemma C.2.6 *For any Borel set $\Lambda \subseteq \mathbb{R}$, let $P(\Lambda) = L_{x_1|q} I_\Lambda L^{-1}_{x_1|q}$ and $I_\Lambda m(q) = 1\left(f_{z|q}(z|q) \in \Lambda\right) m(q)$. If (i) $L_{x_1|q}$ is a bounded invertible operator (ii) $L^{-1}_{x_1|q}$ is densely defined and (iii) $L^{-1}_{x_1|x_2}$ exists and is densely defined, then P is the unique projection-valued measure for which $T = L_{z;x_1|x_2} L^{-1}_{x_1|x_2} = \int \lambda P(d\lambda)$.*

Proof. Hu and Schennach (2008) (henceforth HS) show that $L_{z;x_1|x_2} = L_{x_1|q} \Delta_{z;q} L^{-1}_{x_1|q} L_{x_1|x_2}$, and therefore $T = L_{z;x_1|x_2} L^{-1}_{x_1|x_2} = L_{x_1|q} \Delta_{z;q} L^{-1}_{x_1|q}$ where $\Delta_{z;q} m(q) = f_{z|q}(z|q) m(q)$. This allows them to re-write $T = \int \lambda P(d\lambda)$.

HS show that $T = L_{z;x_1|x_2} L^{-1}_{x_1|x_2}$ has a unique resolution of the identity by appealing to Theorem XV.4.5 in Dunford and Schwartz (1971) since, under Assumption HS.1 $f_{z|q}(z|q)$, is bounded, and therefore T is a bounded operator. However, $f_{z|q}(z|q)$ may not be bounded in our case. We therefore appeal to Corollary XVIII.14 in Dunford and Schwartz (1971), which shows an analogous result for unbounded scalar type operators. The result applies to our model if the projection-valued measure P is strongly countably additive, thereby satisfying Definition XVIII.10 in Dunford and Schwartz (1971)

To complete the proof, we need to show that if Λ_i is a countable sequence of disjoint Borel sets, then

$$P\left(\cup_{i=1}^n \Lambda_i\right) = L_{x_1|q} I_{\cup_{i=1}^n \Lambda_i} L^{-1}_{x_1|q} = L_{x_1|q} \left(\sum_{i=1}^n I_{\Lambda_i}\right) L^{-1}_{x_1|q}$$

converges to $P\left(\cup_{i=1}^\infty \Lambda_i\right) = L_{x_1|q} I_{\cup_{i=1}^\infty \Lambda_i} L^{-1}_{x_1|q}$ in the strong operator topology. Equivalently, we need to show that for any integrable function m, $\left\| L_{x_1|q} I_{\cup_{i=1}^n \Lambda_i} L^{-1}_{x_1|q} m - L_{x_1|q} I_{\cup_{i=1}^\infty \Lambda_i} L^{-1}_{x_1|q} m \right\| \to 0$. Since $L^{-1}_{x_1|q}$ is densely defined, it is enough to show this for any m in the domain of $L^{-1}_{x_1|q}$.

Let $\tilde{m} = L^{-1}_{x_1|q} m$, and note that

$$\left(I_{\cup_{i=1}^n \Lambda_i} - I_{\cup_{i=1}^\infty \Lambda_i}\right) \tilde{m}(q) = \sum_{i=n+1}^{\infty} 1\left(f_{z|q}(z|q) \in \Lambda_i\right) \tilde{m}(q)$$

converges pointwise to zero since $f_{z|q}(z|q)$ can live in only one set Λ_i. Since $\tilde{m}$ is in the domain of $L_{x|q}$, $\tilde{m}$ is integrable. By the Dominated Convergence Theorem, we have that$\left\| \left(I_{\cup_{i=1}^n \Lambda_i} - I_{\cup_{i=1}^\infty \Lambda_i}\right) \tilde{m} \right\| \to 0$. Now, note that

$$\begin{aligned} \left\| L_{x_1|q} I_{\cup_{i=1}^n \Lambda_i} L^{-1}_{x_1|q} m - L_{x_1|q} I_{\cup_{i=1}^\infty \Lambda_i} L^{-1}_{x_1|q} m \right\| &= \left\| L_{x_1|q} \left(I_{\cup_{i=1}^n \Lambda_i} - I_{\cup_{i=1}^\infty \Lambda_i}\right) L^{-1}_{x_1|q} m \right\| \\ &\leq \left\| L_{x_1|q} \right\| \left\| \left(I_{\cup_{i=1}^n \Lambda_i} - I_{\cup_{i=1}^\infty \Lambda_i}\right) L^{-1}_{x_1|q} m \right\| \end{aligned}$$

where $\left\| L_{x_1|q} \right\|$ is finite since $L_{x_1|q}$ is bounded. Further, since $\tilde{m} = L^{-1}_{x_1|q} m$, the right-hand side converges to zero. ■

Our final preliminary result will use the following condition on $f_{x|q}(x|q)$:

Condition C.2.2 *(i) For all $q_1^* \neq q_2^* \in [0,1]$, the set $\left\{x : f_{x|q}(x|q_1^*) \neq f_{x|q}(x|q_2^*)\right\}$ has positive probability under f_x. (ii) $f_{x|q}(x|q)$ continuously differentiable in q. (iii) for all $q \in (0,1)$, there exists an x, such that*

$\frac{\partial f_{x|q}(x|q)}{\partial q} \neq 0$. *(iv)* $f_q(q) = 1\{q \in [0,1]\}$. *(v)* $\int_0^1 f_{x|q}(x|q)\, m(q)\, dq = 0$ *for all* x*, implies that* $m(q) = 0$. *(vi)* $f(x) = \int_0^1 f_{x|q}(x|q)\, dq$.

Lemma C.2.7 *Consider two conditional densities* $\tilde{f}_{\tilde{q}}(x|\tilde{q})$ *and* $f_q(x|q)$ *satisfying Condition C.2.2. If there exists a bijection* $Q : [0,1] \to [0,1]$ *such that* $\tilde{f}_{\tilde{q}}(x|Q(q)) = f_q(x|q)$ *then* Q *is the identity.*

Proof. We will show that if there exists a reindexing $\tilde{f}_{x|\tilde{q}}(x|\tilde{q})$ of q via a bijection $Q : [0,1] \to [0,1]$ such that $\tilde{f}_{x|\tilde{q}}(x|Q(q)) = f_{x|q}(x|q)$ (Assumption 4 in Hu and Schennach (2008) requires $Q(q)$ to be injective on $[0,1]$ and the support assumption in the hypothesis implies surjectivity). If $f_q(q) = 1$ and $f_{\tilde{q}}(\tilde{q}) = 1$, then $Q(\cdot)$ is the identity.

By the assumptions of the theorem,

$$f(x) = \int_0^1 f_{x|q}(x|q)\, dq = \int_0^1 \tilde{f}_{x|\tilde{q}}(x|q)\, dq = \int_0^1 f_{x|q}(x|Q(q))\, dq.$$

A change of variables, $q' = Q(q)$ yields that

$$\int_0^1 f_{x|q}(x|Q(q))\, dq = \int_0^1 f_{x|q}(x|q')\, dQ^{-1}(q') = \int_0^1 f_{x|q}(x|q') \frac{1}{Q'(Q^{-1}(q'))} dq'.$$

The second inequality follows from the inverse function theorem. Differentiability of Q follows from the implicit function theorem: $Q(q)$ is defined implicitly from $\tilde{f}_{x|\tilde{q}}(x|Q(q)) - f_{x|q}(x|q) = 0$, where for every Q there exists an x such that $\tilde{f}_{x|\tilde{q}}(x|Q(q))$ has a non-zero derivative under the hypotheses of the theorem. Hence,

$$\int_0^1 f_{x|q}(x|q)\, dq - \int_0^1 f_{x|q}(x|Q(q))\, dq = 0 \Rightarrow \int_0^1 f_{x|q}(x|q)\left(1 - \frac{1}{Q'(Q^{-1}(q))}\right) dq = 0.$$

By Lemma C.2.4, for all $q \in [0,1]$, $\left(1 - \frac{1}{Q'(Q^{-1}(q))}\right) = 0$. Therefore $Q(\cdot)$ is the identity since $Q'(q) = 1$. ■

D Detailed Proofs: Estimation

D.1 Lemmata Used in Proposition 3

We will use the following lemmata for the result, which are stated, for each dimension of ψ. We omit the dimension index for notational simplicity. Define $q_{N,V}(q) = F_V\left(F_{V_N}^{-1}(q)\right)$ and $q_{N,U}(q) = F_U\left(F_{U_N}^{-1}(q)\right)$.

Lemma D.1.8 *Suppose that* f_V *and* f_U *are continuous, and* Γ_X *and* Γ_Z *are respectively* μ_X *and* μ_Z *Donsker. The stochastic process defined by*

$$\begin{bmatrix} \sqrt{N}\left(q_{N,V}(q_X) - q_X\right) \\ \sqrt{N/2}\left(q_{N,U}(q_Z) - q_Z\right) \\ \sqrt{N}\left(\mu_{X_N} - \mu_X\right)(\gamma_X) \\ \sqrt{N/2}\left(\mu_{Z_N} - \mu_Z\right)(\gamma_Z) \end{bmatrix}$$

indexed by $q_X, q_Z \in [0,1]$, $\gamma_X \in \Gamma_X$ *and* $\gamma_Z \in \Gamma_Z$ *is asymptotically equivalent to the empirical process*

$$\begin{bmatrix} \sqrt{N}\left(\mu_{(X,\varepsilon)_N} - \mu_{X,\varepsilon}\right)\left(1\left\{h(x;\theta_0) + \varepsilon \leq F_V^{-1}(q_X)\right\}\right) \\ \sqrt{N/2}\left(\mu_{(Z,\eta)_N} - \mu_{Z,\eta}\right)\left(1\left\{g(z;\theta_0) + \eta \leq F_U^{-1}(q_Z)\right\}\right) \\ \sqrt{N}\left(\mu_{X_N} - \mu_X\right)(\gamma_X) \\ \sqrt{N/2}\left(\mu_{Z_N} - \mu_Z\right)(\gamma_Z) \end{bmatrix},$$

which converges weakly to the mean-zero Gaussian process with covariance kernel given by

$$
\begin{aligned}
\Omega(q_X,q_Z) &= \Omega(q_Z,\gamma_X)=\Omega(q_X,\gamma_Z)=\Omega(\gamma_X,\gamma_Z)=0 \\
\Omega(q_Z,\gamma_Z) &= \mu_{Z,\eta}\left(\gamma_Z 1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q_Z)\right\}\right)-\mu_Z(\gamma_Z)\,\mu_{Z,\eta}\left(1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q_Z)\right\}\right) \\
\Omega(q_X,\gamma_X) &= \mu_{X,\varepsilon}\left(\gamma_X 1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q_X)\right\}\right)-\mu_X(\gamma_X)\,\mu_{X,\varepsilon}\left(1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q_X)\right\}\right) \\
\Omega(\gamma_X,\gamma_X') &= \mu_X(\gamma_X\gamma_X')-\mu_X(\gamma_X)\,\mu_X(\gamma_X') \\
\Omega(\gamma_Z,\gamma_Z') &= \mu_Z(\gamma_Z\gamma_Z')-\mu_Z(\gamma_Z)\,\mu_Z(\gamma_Z') \\
\Omega(q_X,q_X') &= \mu_{X,\varepsilon}\left(1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q_X)\right\}1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q_X')\right\}\right) \\
&\quad -\mu_{X,\varepsilon}\left(1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q_X)\right\}\right)\mu_{X,\varepsilon}\left(1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q_X')\right\}\right) \\
\Omega(q_Z,q_Z') &= \mu_{Z,\eta}\left(1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q_Z)\right\}1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q_Z')\right\}\right) \\
&\quad -\mu_{Z,\eta}\left(1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q_Z)\right\}\right)\mu_{Z,\eta}\left(1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q_Z')\right\}\right),
\end{aligned}
$$

where $q_X,q_X'\in[0,1]$, $q_Z,q_Z'\in[0,1]$, $\gamma_X,\gamma_X'\in\Gamma_X$ *and* $\gamma_Z,\gamma_Z'\in\Gamma_Z$.

Proof. Note that $F_{V_N}\left(F_V^{-1}(.)\right)$ is the empirical cumulative distribution function of a uniformly distributed random variable, and $q_{N,V}:[0,1]\to[0,1]$ is its associated quantile function. Therefore, $F_{V_N}\left(F_V^{-1}(.)\right)$ satisfied the assumptions for theorem 4 of Csorgo and Revesz (1978). This theorem implies that

$$\sqrt{N}\sup_{q\in[0,1]}\left|\left[F_{V_N}\left(F_V^{-1}(q)\right)-F_V\left(F_V^{-1}(q)\right)\right]-\left[F_V\left(F_{V_N}^{-1}(q)\right)-F_V\left(F_V^{-1}(q)\right)\right]\right|=o_p(1).$$

Since $q_{N,V}(q)=F_V\left(F_{V_N}^{-1}(q)\right)$ and $F_V\left(F_V^{-1}(q)\right)=q$, we therefore have that

$$\sqrt{N}\sup_{q\epsilon[0,1]}\left|\left[F_{V_N}\left(F_V^{-1}(q)\right)-F_V\left(F_V^{-1}(q)\right)\right]-\left[q_{N,V}(q)-q\right]\right|=o_p(1).$$

By an identical argument,

$$\sqrt{\frac{N}{2}}\sup_{q\epsilon[0,1]}\left|\left[F_{U_N}\left(F_U^{-1}(q)\right)-F_U\left(F_U^{-1}(q)\right)\right]-\left[q_{N,U}(q)-q\right]\right|=o_p(1).$$

Note that

$$F_{V_N}\left(F_V^{-1}(q)\right)-F_V\left(F_V^{-1}(q)\right)=\left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q)\right\}\right)$$

and

$$F_{U_N}\left(F_U^{-1}(q)\right)-F_U\left(F_U^{-1}(q)\right)=\left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)\left(1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q)\right\}\right).$$

The result therefore follows from the functional central limit theorem, since the first and last two components of

$$\begin{bmatrix}\sqrt{N}\left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(1\left\{h(x;\theta_0)+\varepsilon\leq F_V^{-1}(q_X)\right\}\right)\\ \sqrt{N}\left(\mu_{X_N}-\mu_X\right)(\gamma_X)\\ \sqrt{N/2}\left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)\left(1\left\{g(z;\theta_0)+\eta\leq F_U^{-1}(q_Z)\right\}\right)\\ \sqrt{N/2}\left(\mu_{Z_N}-\mu_Z\right)(\gamma_Z)\end{bmatrix}$$

are two independent empirical processes index by $\mu_{X,\varepsilon}$ and $\mu_{Z,\eta}$ Donsker classes. ■

Lemma D.1.9 *(i) If Assumption 4(i) is satisfied,* $E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)-\psi$ *converges in probability to* 0 *as* $N\to\infty$.

(ii) If Assumption 4(ii) is satisfied, then for any bounded μ_X-Donsker class Γ_X and bounded μ_Z-Donsker class Γ_Z,

$$\begin{bmatrix} \sqrt{N}\left(E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)-\psi\right) \\ \sqrt{N}\left(\mu_{X_N}-\mu_X\right)(\gamma_X) \\ \sqrt{N/2}\left(\mu_{Z_N}-\mu_Z\right)(\gamma_Z) \end{bmatrix}$$

is asymptotically equivalent to the process

$$\begin{bmatrix} \sqrt{N}\int_0^1 \nabla\tilde{\psi}_q(q,q,q)\cdot\begin{bmatrix} \left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(1\left\{h(x;\theta_0)+\varepsilon\le F_V^{-1}(q_X)\right\}\right) \\ \left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(1\left\{h(x;\theta_0)+\varepsilon\le F_V^{-1}(q_X)\right\}\right) \\ \left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)\left(1\left\{g(z;\theta_0)+\eta\le F_U^{-1}(q_Z)\right\}\right) \end{bmatrix} dq \\ \sqrt{N}\left(\mu_{X_N}-\mu_X\right)(\gamma_X) \\ \sqrt{N/2}\left(\mu_{Z_N}-\mu_Z\right)(\gamma_Z) \end{bmatrix},$$

which converges weakly to a mean-zero Gaussian process with covariance kernel given by

$$\begin{aligned}
V'(\gamma_X,\gamma_Z) &= 0 \\
V'(\gamma_\Psi,\gamma_Z) &= \sqrt{2}\int_0^1 \tilde{\psi}_{q,3}(q_Z,q_Z,q_Z)\,\Omega(q_Z,\gamma_Z)\,dq_Z \\
V'(\gamma_\Psi,\gamma_X) &= \int_0^1 \left(\tilde{\psi}_{q,1}(q_X,q_X,q_X)+\tilde{\psi}_{q,2}(q_X,q_X,q_X)\right)\Omega(q_X,\gamma_X)\,dq_X \\
V'(\gamma_\Psi,\gamma_\Psi) &= \int_0^1\int_0^1 \left(\tilde{\psi}_{q,1}(q_X,q_X,q_X)+\tilde{\psi}_{q,2}(q_X,q_X,q_X)\right) \\
&\quad \left(\tilde{\psi}_{q,1}(q'_X,q'_X,q'_X)+\tilde{\psi}_{q,2}(q'_X,q'_X,q'_X)\right)\Omega(q_X,q'_X)\,dq_X dq'_X \\
&\quad +2\int_0^1\int_0^1 \tilde{\psi}_{q,3}(q_Z,q_Z,q_Z)\,\tilde{\psi}_{q,3}(q'_Z,q'_Z,q'_Z)\,\Omega(q_Z,q'_Z)\,dq_Z dq'_Z \\
V'(\gamma_X,\gamma'_X) &= \Omega(\gamma_X,\gamma'_X) \\
V'(\gamma_Z,\gamma'_Z) &= \Omega(\gamma_Z,\gamma'_Z),
\end{aligned}$$

where $\gamma_X,\gamma'_X\in\Gamma_X$, $\gamma_Z,\gamma'_Z\in\Gamma_Z$ and γ_Ψ indexes $\sqrt{N}\left(E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)-\psi\right)$.

Proof. The quantity $E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)$ can be computed by using the fact that for all $1\le k\le J$, the k'th most desirable firm is occupied by the $2k$-th and the $(2k-1)$-th most desirable workers. By definition, the conditional expectation of $\Psi(x_1,x_2,z)$ given μ_{V_N}, μ_{U_N} for the k'th desirable job is $\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right),F_{V_N}^{-1}\left(\frac{2k}{N}\right),F_{U_N}^{-1}\left(\frac{k}{N/2}\right)\right)$ where F_{V_N} and F_{U_N} are the cdfs representing the empirical measures μ_{V_N} and μ_{U_N} respectively. Therefore,

$$\begin{aligned}
E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right) &= \frac{1}{N/2}\sum_{k=1}^{N/2}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right),F_{V_N}^{-1}\left(\frac{2k}{N}\right),F_{U_N}^{-1}\left(\frac{k}{N/2}\right)\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{i}{N}\right),F_{V_N}^{-1}\left(\frac{i}{N}\right),F_{U_N}^{-1}\left(\frac{i}{N}\right)\right)+R. \qquad \text{(D.1.23)}
\end{aligned}$$

where

$$R = \frac{1}{N/2}\sum_{k=1}^{N/2}\left[\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{k}{N/2}\right)\right)\right.$$
$$-\frac{1}{2}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{U_N}^{-1}\left(\frac{2k-1}{N}\right)\right)$$
$$\left.-\frac{1}{2}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right]. \tag{D.1.24}$$

Our proof of part (i) proceeds by showing that $R \to 0$ and that

$$\frac{1}{N}\sum_{i=1}^{N}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{U_N}^{-1}\left(\frac{i}{N}\right)\right) - \psi \to 0. \tag{D.1.25}$$

The proof of part (ii) is analogous. It characterizes the limit distribution of

$$\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{U_N}^{-1}\left(\frac{i}{N}\right)\right) - \psi\right)$$

under stronger assumptions.

Proof of Part (i): We begin by bounding the absolute value of R in equation (D.1.24) using the triangle inequality as:

$$|R| \leq \frac{1}{N}\sum_{k=1}^{N/2}\left|\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right.$$
$$\left.-\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right|$$
$$+\frac{1}{N}\sum_{k=1}^{N/2}\left|\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right.$$
$$\left.-\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right|.$$

For any $\delta \in \left(0, \frac{1}{2}\right)$, we have that:

$$|R| \leq \frac{1}{N}\sum_{\lceil \delta N/2\rceil < k < \lfloor (1-\delta)N/2\rfloor}\left|\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right.$$
$$\left.-\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right|$$
$$+\frac{1}{N}\sum_{\lceil \delta N/2\rceil < k < \lfloor (1-\delta)N/2\rfloor}\left|\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right.$$
$$\left.-\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{V_N}^{-1}\left(\frac{2k-1}{N}\right), F_{U_N}^{-1}\left(\frac{2k}{N}\right)\right)\right|$$
$$+4\delta\left\|\Psi\right\|_\infty$$
$$= \tilde{R} + 4\delta\left\|\Psi\right\|_\infty. \tag{D.1.26}$$

Since $\tilde{\psi}$ is Lipschitz continuous

$$\tilde{R} \leq \sup_{\lceil J\delta \rceil < k < \lfloor J(1-\delta) \rfloor} \left|\tilde{\psi}\right|_{LC} \left[2 \left| F_{V_N}^{-1}\left(\frac{2k-1}{N}\right) - F_{V_N}^{-1}\left(\frac{2k}{N}\right) \right| \right], \tag{D.1.27}$$

where $\left|\tilde{\psi}\right|_{LC}$ denotes the Lipschitz constant. By Example 3.9.21 in van der Vaart and Wellner (2000), for all $\lceil \delta N/2 \rceil < k < \lfloor (1-\delta) N/2 \rfloor$ $\left|F_{V_N}^{-1}\left(\frac{2k-1}{N}\right) - F_{V_N}^{-1}\left(\frac{2k}{N}\right)\right|$ converges in probability to 0 uniformly in k (Assumption 4(i)b. implies that f_V is continuous with full support). Therefore, since $\tilde{R} \geq 0$, it converges in probability to 0.

Now, we show that the difference in equation (D.1.25) converges in probability to 0. Note that F_{U_N} is constant on each interval $[\frac{k-1}{N/2}, \frac{k}{N/2})$ and F_{V_N} is constant on $[\frac{i-1}{N}, \frac{i}{N})$. Hence,

$$\begin{aligned}
& \frac{1}{N}\sum_{i=1}^{N} \tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{U_N}^{-1}\left(\frac{i}{N}\right)\right) - \psi \\
= & \int_0^1 \tilde{\psi}\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) dq - \int_0^1 \tilde{\psi}\left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right) dq \\
= & \int_\delta^{1-\delta} \left[\tilde{\psi}\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) - \tilde{\psi}\left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right)\right] dq \\
& + \left(\int_0^\delta + \int_{1-\delta}^1\right) \left[\tilde{\psi}\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) - \tilde{\psi}\left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right)\right] dq \\
= & T_1 + T_2
\end{aligned} \tag{D.1.28}$$

where $\delta \in \left(0, \frac{1}{2}\right)$.

We now bound T_1 and T_2 in terms of δ. Since $\|\Psi\|_\infty < \infty$, $|T_2| \leq 4\delta \|\Psi\|_\infty$. To bound T_1, note that

$$\begin{aligned}
|T_1| = & \left| \int_\delta^{1-\delta} \left[\tilde{\psi}\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) - \tilde{\psi}\left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right)\right] dq \right| \\
\leq & \int_\delta^{1-\delta} \left|\tilde{\psi}\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) - \tilde{\psi}\left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right)\right| dq \\
\leq & \sup_{q\in[\delta,1-\delta]} \left|\tilde{\psi}\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) - \tilde{\psi}\left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right)\right| \\
\leq & \left|\tilde{\psi}\right|_{LC} \sup_{q\in[\delta,1-\delta]} \left|\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) - \left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right)\right|.
\end{aligned} \tag{D.1.29}$$

Combining equations (D.1.23) - (D.1.29) and the bound on T_2, we have that

$$\begin{aligned}
& \left|E\left(\psi_N | \mu_{V_N}, \mu_{U_N}\right) - \psi\right| \\
\leq & \left|\frac{1}{N}\sum_{i=1}^{N} \tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{V_N}^{-1}\left(\frac{i}{N}\right), F_{U_N}^{-1}\left(\frac{i}{N}\right)\right) - \psi\right| + |R| \\
\leq & |T_1| + |T_2| + \left|\tilde{R}\right| + 4\delta \|\Psi\|_\infty \\
\leq & \left|\tilde{\psi}\right|_{LC} \sup_{q\in[\delta,1-\delta]} \left|\left(F_{V_N}^{-1}(q), F_{V_N}^{-1}(q), F_{U_N}^{-1}(q)\right) - \left(F_V^{-1}(q), F_V^{-1}(q), F_U^{-1}(q)\right)\right| + 8\delta \|\Psi\|_\infty + o_p(1)
\end{aligned}$$

since $|T_2| \leq 4\delta \|\Psi\|_\infty$ and $\left|\tilde{R}\right| = o_p(1)$.

We now show that $\left|E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)-\psi\right|\to 0$ in probability as $N\to\infty$. Fix $\varepsilon>0$ and choose $\delta=\frac{\varepsilon}{16\|\Psi\|_\infty}$. By Example 3.9.21 in van der Vaart and Wellner (2000),

$$\sup_{q\in[\delta,1-\delta]}\left|\left(F_{V_N}^{-1}(q),F_{V_N}^{-1}(q),F_{U_N}^{-1}(q)\right)-\left(F_V^{-1}(q),F_V^{-1}(q),F_U^{-1}(q)\right)\right|$$

converges in probability to 0 (Assumption 4(i)b. implies that f_V and f_U are continuous with full support). Hence, for sufficiently large N we have

$$P\left(\left|\tilde{\psi}\right|_{LC}\sup_{q\in[\delta,1-\delta]}\left|\left(F_{V_N}^{-1}(q),F_{V_N}^{-1}(q),F_{U_N}^{-1}(q)\right)-\left(F_V^{-1}(q),F_V^{-1}(q),F_U^{-1}(q)\right)\right|>\frac{\varepsilon}{2}\right)<\varepsilon.$$

This implies $P\left(\left|E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)-\psi\right|>\varepsilon\right)<\varepsilon$, proving the desired convergence in probability to 0.

Proof of Part (ii): Let $q_{N,V}(q)=F_V\left(F_{V_N}^{-1}(q)\right)$, $q_{N,U}(q)=F_U\left(F_{U_N}^{-1}(q)\right)$, and

$$\tilde{\psi}_q(q_1,q_2,q_3)=\tilde{\psi}\left(F_V^{-1}(q_1),F_V^{-1}(q_2),F_U^{-1}(q_3)\right).$$

Equation (D.1.24) can be rewritten by using this notation as

$$\begin{aligned}R \;=\; & \frac{1}{N/2}\sum_{k=1}^{N/2}\left[\tilde{\psi}_q\left(q_{N,V}\left(\frac{2k-1}{N}\right),q_{N,V}\left(\frac{2k}{N}\right),q_{N,U}\left(\frac{k}{N/2}\right)\right)\right.\\ & -\frac{1}{2}\tilde{\psi}_q\left(q_{N,V}\left(\frac{2k-1}{N}\right),q_{N,V}\left(\frac{2k-1}{N}\right),q_{N,U}\left(\frac{k}{N/2}\right)\right)\\ & \left.-\frac{1}{2}\tilde{\psi}_q\left(q_{N,V}\left(\frac{2k}{N}\right),q_{N,V}\left(\frac{2k}{N}\right),q_{N,U}\left(\frac{k}{N/2}\right)\right)\right].\end{aligned}$$

By the triangle inequality and the assumption that $\tilde{\psi}_q$ has a bounded derivative,

$$|R|\le\frac{1}{N}\sum_{k=1}^{N/2}\left\|\nabla\tilde{\psi}_q\right\|_\infty 2\left|q_{N,V}\left(\frac{2k-1}{N}\right)-q_{N,V}\left(\frac{2k}{N}\right)\right|.$$

Since $q_{N,V}(q)$ is monotonic in q and has range $[0,1]$, we have that

$$\begin{aligned}&\sum_{k=1}^{N/2}\left|q_{N,V}\left(\frac{2k-1}{N}\right)-q_{N,V}\left(\frac{2k}{N}\right)\right|\\ =\;&\left|\sum_{k=1}^{N/2}q_{N,V}\left(\frac{2k-1}{N}\right)-q_{N,V}\left(\frac{2k}{N}\right)\right|\\ \le\;& 1.\end{aligned}$$

Therefore, since $\left\|\nabla\tilde{\psi}_q\right\|_\infty<\infty$,

$$\sqrt{N}\,|R|\le\frac{1}{\sqrt{N}}\left\|\nabla\tilde{\psi}_q\right\|_\infty\to 0. \tag{D.1.30}$$

Now, we compute the limit distribution of

$$\begin{aligned}
& \frac{1}{N}\sum_{i=1}^{N}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{i}{N}\right),F_{V_N}^{-1}\left(\frac{i}{N}\right),F_{U_N}^{-1}\left(\frac{i}{N}\right)\right)-\psi \\
= & \frac{1}{N}\sum_{i=1}^{N}\tilde{\psi}_q\left(q_{N,V}\left(\frac{i}{N}\right),q_{N,V}\left(\frac{i}{N}\right),q_{N,U}\left(\frac{i}{N}\right)\right)-\psi \\
= & \int_0^1\tilde{\psi}_q\left(q_{N,V}\left(q\right),q_{N,V}\left(q\right),q_{N,U}\left(q\right)\right)dq-\int_0^1\tilde{\psi}_q\left(q,q,q\right)dq.
\end{aligned}$$

By Taylor's theorem,

$$\begin{aligned}
& \tilde{\psi}_q\left(q_{N,V}\left(q\right),q_{N,V}\left(q\right),q_{N,U}\left(q\right)\right)-\tilde{\psi}_q\left(q,q,q\right) \\
= & \tilde{\psi}_{q,1}\left(q,q,q\right)\left(q_{N,V}\left(q\right)-q\right)+\tilde{\psi}_{q,2}\left(q,q,q\right)\left(q_{N,V}\left(q\right)-q\right)+\tilde{\psi}_{q,3}\left(q,q,q\right)\left(q_{N,U}\left(q\right)-q\right)+R_q.
\end{aligned}$$

Since,

$$\sup_q\left|R_q\right|=o\left(\sup_q\left\|q_{N,V}\left(q\right)-q\right\|+\sup_q\left\|q_{N,V}\left(q\right)-q\right\|+\sup_q\left\|q_{N,U}\left(q\right)-q\right\|\right),$$

we have that $\sqrt{N}\sup_q\left|R_q\right|\rightarrow_p 0$. Therefore,

$$\begin{aligned}
& \sqrt{N}\left(E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)-\psi\right) \\
= & \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{\psi}\left(F_{V_N}^{-1}\left(\frac{i}{N}\right),F_{V_N}^{-1}\left(\frac{i}{N}\right),F_{U_N}^{-1}\left(\frac{i}{N}\right)\right)-\psi\right)+o_p\left(1\right) \\
= & \sqrt{N}\int_0^1\nabla\tilde{\psi}_q\left(q,q,q\right)\cdot\begin{bmatrix} q_{N,V}\left(q\right)-q \\ q_{N,V}\left(q\right)-q \\ q_{N,U}\left(q\right)-q \end{bmatrix}dq+o_p\left(1\right),
\end{aligned}$$

showing the required asymptotic equivalence. Lemma D.1.8 characterizes the limit distribution of

$$\begin{bmatrix} \sqrt{N}\left(q_{N,V}\left(q_X\right)-q_X\right) \\ \sqrt{N/2}\left(q_{N,U}\left(q_Z\right)-q_Z\right) \\ \sqrt{N}\left(\mu_{X_N}-\mu_X\right)\left(\gamma_X\right) \\ \sqrt{N/2}\left(\mu_{Z_N}-\mu_Z\right)\left(\gamma_Z\right) \end{bmatrix}$$

indexed by $q_X,q_Z\in[0,1]$, $\gamma_X\in\Gamma_X$ and $\gamma_Z\in\Gamma_Z$. Therefore,

$$\begin{bmatrix} \sqrt{N}\left(E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)-\psi\right) \\ \sqrt{N}\left(\mu_{X_N}-\mu_X\right)\left(\gamma_X\right) \\ \sqrt{N/2}\left(\mu_{Z_N}-\mu_Z\right)\left(\gamma_Z\right) \end{bmatrix}$$

converges to a mean-zero Gaussian process with covariance kernel V'. ■

Lemma D.1.10 *(i) $\psi_N-E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)$ converges in probability to 0 if $\left\|\Psi\right\|_\infty<\infty$.*

(ii) Suppose Assumption 4(ii)b is satisfied. For any bounded functions $\gamma_{X,\varepsilon}$ on the domain of (X,ε) and $\gamma_{Z,\eta}$ on the domain of (Z,η),

$$\sqrt{N/2}\left[\psi_N-E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right),\left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(\gamma_Z\right)\ ,\left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)\left(\gamma_Z\right)\right]$$

converges to a multivariate normal distribution with mean 0 *and covariance kernel*

$$V''(\gamma_X,\gamma_Z)=\begin{bmatrix}\int_0^1 var_{q,\Psi}(q,q,q)\,dq & \int_0^1 cov_q(\Psi,\gamma_{X,\varepsilon}|q,q,q)\,dq & \int_0^1 cov_q(\Psi,\gamma_{Z,\eta}|q,q,q)\,dq\\ \int_0^1 cov_q(\Psi,\gamma_{X,\varepsilon}|q,q,q)\,dq & \frac{1}{2}Var(\gamma_{X,\varepsilon}) & 0\\ \int_0^1 cov_q(\Psi,\gamma_{Z,\eta}|q,q,q)\,dq & 0 & Var(\gamma_{Z,\eta})\end{bmatrix}.$$

Proof. Let $v^{(k)}$ and $u^{(k)}$ be k'th order statistics of worker and firm desirability and let $\left(X^{(k)},\varepsilon^{(k)}\right)$ and $\left(Z^{(k)},\eta^{(k)}\right)$ be the corresponding observations drawn from $\mu_{X,\varepsilon|v^{(k)}}$ and $\mu_{Z,\eta|u^{(k)}}$ respectively. We will use $J=N/2$ in this proof. Rewrite:

$$\psi_N-E(\psi_N|\mu_{U_N},\mu_{V_N})=\frac{1}{J}\left(\sum_{k=1}^{J}\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(i)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right).$$

Proof of Part (i): The conditional variance of $\psi_N-E(\psi_N|\mu_{U_N},\mu_{V_N})$ given (μ_{V_N},μ_{U_N}) is

$$\begin{aligned}&\frac{1}{J^2}E\left(\left(\sum_{i=1}^{J}\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)^2\middle|\mu_{V_N},\mu_{U_N}\right)\\ =&\frac{1}{J^2}\sum_{i=1}^{J}E\left(\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)^2\middle|\mu_{V_N},\mu_{U_N}\right)\\ \leq&\frac{1}{J}4\|\Psi\|_\infty^2,\end{aligned}$$

where the first equality follows from conditional independence.

However, since $\psi_N-E(\psi_N|\mu_{V_N},\mu_{U_N})$ is by definition mean zero, it follows that the unconditional variance of $\psi_N-E(\psi_N|\mu_{V_N},\mu_{U_N})$ is bounded above by $\frac{1}{J}4\|\Psi\|_\infty^2$, by the law of total variance. By Chebychev's inequality, $\sqrt{J}\left(\psi_N-E(\psi_N|\mu_{V_N},\mu_{U_N})\right)=O_p(1)$ and thus $\psi_N-E(\psi_N|\mu_{V_N},\mu_{U_N})=o_p(1)$.

Proof of Part (ii): We will show that the random variables

$$\begin{aligned}\psi_N-E(\psi_N|\mu_{V_N},\mu_{U_N})&=\frac{1}{J}\left(\sum_{k=1}^{J}\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right),\\ \left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)(\gamma_{X,\varepsilon})&=\frac{1}{2J}\sum_{k=1}^{2J}\gamma_{X,\varepsilon}\left(X^{(k)},\varepsilon^{(k)}\right)-E(\gamma_{X,\varepsilon}),\ \text{and}\\ \left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)(\gamma_Z,\eta)&=\frac{1}{J}\sum_{k=1}^{J}\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E(\gamma_Z,\eta),\end{aligned}\tag{D.1.31}$$

are jointly asymptotically normal. The latter two random variables are jointly asymptotically normal by the standard CLT. We will characterize the joint limiting distribution of these three random variables by calculating their joint moment generating function and comparing it with the moment generating function of a normal random variable. We do this by computing the limiting variance-covariance matrices of the first random variable with each of the other two (note that the second and third random variables are independent), and then using a Taylor expansion of the moment generating function to show that the leading terms match the moment generating function of a normal random variable and that higher order terms are asymptotically negligible.

The sample variances of $\gamma_{X,\varepsilon}$ and $\gamma_{Z,\eta}$ and their covariance converge in probability to $Var(\gamma_{X,\varepsilon})$, $Var(\gamma_{Z,\eta})$ and 0 by the standard law of large numbers.

To show that the sample variances converge, we show that the second moment of the sample variances (of the random variables above) converge to 0. If these variance of the sample variances converge to 0, then the relevant sample variances will converge in probability (by Chebychev's inequality). To bound the variance of the first sample variance, by the law of total variance, rewrite

$$\begin{aligned}
& Var\left(\frac{1}{J}\sum_{k=1}^{J}\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right]^2\right) \\
= & \frac{1}{J^2}E\left(\sum_{k=1}^{J}Var\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\Big|\mu_{V_N},\mu_{U_N}\right]^2\right) \\
& +Var\left(\frac{1}{J}\sum_{k=1}^{J}E\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\Big|\mu_{V_N},\mu_{U_N}\right]^2\right) \\
= & T_1+T_2.
\end{aligned}$$

To bound the variance of the sample covariance of the first and second random variables, rewrite

$$\begin{aligned}
& Var\left(\frac{1}{J}\sum_{k=1}^{J}\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right]\right. \\
& \left.\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]\right) \\
= & \frac{1}{J^2}E\left(\sum_{k=1}^{J}Var\left\{\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right]\right.\right. \\
& \left.\left.\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]\Big|\mu_{V_N},\mu_{U_N}\right\}\right) \\
& +Var\left(\frac{1}{J}\sum_{k=1}^{J}E\left\{\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right]\right.\right. \\
& \left.\left.\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]\Big|\mu_{V_N},\mu_{U_N}\right\}\right) \\
= & R_1+R_2.
\end{aligned}$$

To bound the variance of the sample covariance of the first and third random variables, rewrite

$$\begin{aligned}
& Var\left(\frac{1}{J}\sum_{k=1}^{J}\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right]\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\gamma_{Z,\eta}\right]\right) \\
= & \frac{1}{J^2}E\left(\sum_{k=1}^{J}Var\left\{\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right]\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\gamma_{Z,\eta}\right]\Big|\mu_{V_N},\mu_{U_N}\right\}\right) \\
& +Var\left(\frac{1}{J}\sum_{k=1}^{J}E\left\{\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right]\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\gamma_{Z,\eta}\right]\Big|\mu_{V_N},\mu_{U_N}\right\}\right) \\
= & V_1+V_2.
\end{aligned}$$

Note that T_1, R_1 and V_1 are the sum of J bounded terms divided by J^2 and hence converge in probability to 0. To show that T_2, R_2 and V_2 converge in probability to 0, we compute the relevant conditional expectations.

For T_2, we have that

$$\frac{1}{J}\sum_{k=1}^{J} E\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\Big| \mu_{V_N}, \mu_{U_N}\right)^2 = 0$$

since

$$\begin{aligned}
& E\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\Big| \mu_{V_N}, \mu_{U_N}\right) \\
= & E\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) |\mu_{V_N}, \mu_{U_N}\right) - E\left[\tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right) |v^{(2k-1)}, v^{(2k)}, u^{(k)}\right] \\
= & E\left[\tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right) |v^{(2k-1)}, v^{(2k)}, u^{(k)}\right] - E\left[\tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right) |v^{(2k-1)}, v^{(2k)}, u^{(k)}\right] = 0
\end{aligned}$$

by definition of $\tilde{\psi}$.

For later calculations, it will be useful to compute the variance of $\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right)$ conditional on μ_{V_N}, μ_{U_N}.

$$\begin{aligned}
& \frac{1}{J}\sum_{k=1}^{J}\left[E\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right)^2\Big| \mu_{V_N}, \mu_{U_N}\right) - E\left(\tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\Big| \mu_{V_N}, \mu_{U_N}\right)^2\right] \\
= & \frac{1}{J}\sum_{k=1}^{J} var\left(\Psi|v^{(2k-1)}, v^{(2k)}, u^{(k)}\right) \\
= & \frac{1}{2J}\sum_{k=1}^{J}\left(var\left(\Psi|v^{(2k)}, v^{(2k)}, u^{(k)}\right) + var\left(\Psi|v^{(2k-1)}, v^{(2k-1)}, u^{(k)}\right)\right) \\
& -\frac{1}{2J}\sum_{k=1}^{J}\left(var\left(\Psi|v^{(2k)}, v^{(2k)}, u^{(k)}\right) - var\left(\Psi|v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right) \\
& -\frac{1}{2J}\sum_{k=1}^{J}\left(var\left(\Psi|v^{(2k-1)}, v^{(2k-1)}, u^{(k)}\right) - var\left(\Psi|v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right) \qquad \text{(D.1.32)}
\end{aligned}$$

The first term in the summation is

$$\begin{aligned}
& \frac{1}{2J}\left(\sum_{k=1}^{J} var\left(\Psi|v^{(2k)}, v^{(2k)}, u^{(k)}\right) + \sum_{k=1}^{J} var\left(\Psi|v^{(2k-1)}, v^{(2k-1)}, u^{(k)}\right)\right) \\
= & \int_0^1 var_q\left(\Psi|q_{N,V}\left(q\right), q_{N,V}\left(q\right), q_{N,U}\left(q\right)\right) dq.
\end{aligned}$$

Since $\|\Psi\|_\infty < \infty$, by the dominated convergence theorem,

$$\begin{aligned}
& \frac{1}{2J}\left(\sum_{k=1}^{J} var\left(\Psi|v^{(2k)}, v^{(2k)}, u^{(k)}\right) + \sum_{k=1}^{J} var\left(\Psi|v^{(2k-1)}, v^{(2k-1)}, u^{(k)}\right)\right) \\
\to & \int_0^1 var_q\left(\Psi|q, q, q\right) dq \qquad \text{(D.1.33)}
\end{aligned}$$

almost surely. Note that

$$
\begin{aligned}
& \left|\frac{1}{2J}\sum_{k=1}^{J} var\left(\Psi|v^{(2k)},v^{(2k)},u^{(k)}\right)-var\left(\Psi|v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right| \\
\leq\ & \left|\frac{1}{2J}\sum_{k=1}^{J} E\left(\Psi^2|v^{(2k)},v^{(2k)},u^{(k)}\right)-E\left(\Psi^2|v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right| \\
& +\left|\frac{1}{2J}\sum_{k=1}^{J} E\left(\Psi|v^{(2k)},v^{(2k)},u^{(k)}\right)^2-E\left(\Psi|v^{(2k-1)},v^{(2k)},u^{(k)}\right)^2\right| \\
\leq\ & \left|\frac{1}{2J}\sum_{k=1}^{J}\int_0^{\|\Psi\|_\infty^2}\left(P\left(\Psi^2\geq c|v^{(2k)},v^{(2k)},u^{(k)}\right)-P\left(\Psi^2\geq c|v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)dc\right| \\
& +\left|\frac{\|\Psi\|_\infty}{J}\sum_{k=1}^{J}\int_0^{\|\Psi\|_\infty}\left(P\left(\Psi\geq c|v^{(2k)},v^{(2k)},u^{(k)}\right)-P\left(\Psi\geq c|v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)dc\right| \\
\to\ & 0 \qquad \text{(D.1.34)}
\end{aligned}
$$

since

$$
\begin{aligned}
P\left(\Psi^2\geq c|v_1,v_2,u\right) &= \int 1\left\{\Psi\left(x_1,x_2,z\right)^2\geq c\right\}d\mu_{X_1|v}d\mu_{X_2|v_2}d\mu_{Z|u} \\
&= \frac{\int 1\left\{\Psi\left(x_1,x_2,z\right)^2\geq c\right\}f_\varepsilon\left(v_1-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v_2-h\left(x_2;\theta\right)\right)f_\eta\left(u-g\left(z;\theta\right)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(v_2-h\left(x_1;\theta\right)\right)d\mu_X\int f_\varepsilon\left(v_1-h\left(x_2;\theta\right)\right)d\mu_X\int f_\varepsilon\left(u-g\left(z;\theta\right)\right)d\mu_X}
\end{aligned}
$$

is continuous in v_1, v_2 and u (implied by Assumption 4(ii)b), and $\|\Psi\|_\infty<\infty$.

Therefore, by equations (D.1.32), (D.1.33) and (D.1.34),

$$
\frac{1}{J}\sum_{k=1}^{J}E\left(\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)^2\middle|\mu_{V_N},\mu_{U_N}\right)\to\int_0^1 var\left(\psi|q,q,q\right)dq \quad \text{(D.1.35)}
$$

almost surely.

Similarly, for R_2,

$$\begin{aligned}
& \frac{1}{J}\sum_{k=1}^{J} E\left\{\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right)\right. \\
& \left.\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)}, \varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)}, \varepsilon^{(2k)}\right) - E\left(\gamma_{X,\varepsilon}\right)\right]\right| \mu_{V_N}, \mu_{U_N}\Big\} \\
= & \frac{1}{J}\sum_{k=1}^{J} E\left\{\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right)\right. \\
& \left(\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)}, \varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)}, \varepsilon^{(2k)}\right)\right. \\
& \left.\left. -\frac{1}{2}E\left[\gamma_{X,\varepsilon}\left(X^{(2k-1)}, \varepsilon^{(2k-1)}\right) + \gamma_{X,\varepsilon}\left(X^{(2k)}, \varepsilon^{(2k)}\right)\Big| \mu_{V_N}, \mu_{U_N}\right]\right)\right| \mu_{V_N}, \mu_{U_N}\Big\} \\
& +\frac{1}{J}\sum_{k=1}^{J} E\left\{\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right)\Big| \mu_{V_N}, \mu_{U_N}\right\} \times \\
& \left(\frac{1}{2}E\left[\gamma_{X,\varepsilon}\left(X^{(2k-1)}, \varepsilon^{(2k-1)}\right) + \gamma_{X,\varepsilon}\left(X^{(2k)}, \varepsilon^{(2k)}\right)\Big| \mu_{V_N}, \mu_{U_N}\right] - E\left(\gamma_{X,\varepsilon}\right)\right) \\
= & \frac{1}{J}\sum_{k=1}^{J} cov\left(\Psi, \gamma_{X,\varepsilon}|v^{(2k-1)}, v^{(2k)}, u^{(k)}\right) \\
& +\frac{1}{J}\sum_{k=1}^{J} 0 \times \left(\frac{1}{2}E\left[\gamma_{X,\varepsilon}\left(X^{(2k-1)}, \varepsilon^{(2k-1)}\right) + \gamma_{X,\varepsilon}\left(X^{(2k)}, \varepsilon^{(2k)}\right)\Big| \mu_{V_N}, \mu_{U_N}\right] - E\left(\gamma_{X,\varepsilon}\right)\right) \\
= & \int_0^1 cov_q\left(\Psi, \gamma_{X,\varepsilon}|q, q, q\right) dq + o(1)
\end{aligned} \tag{D.1.36}$$

where the last equality follows from arguments identical to showing equation (D.1.35) and

$$\begin{aligned}
& cov\left(\Psi, \gamma_{X,\varepsilon}|v_1, v_2, u\right) \\
= & \int \Psi\left(x_1, x_2, z\right)\left(\frac{1}{2}\gamma_{X,\varepsilon}\left(x_1, \varepsilon_1\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(x_2, \varepsilon_2\right)\right) d\mu_{(X_1,\varepsilon_1)|v_1} d\mu_{(X_2,\varepsilon_2)|v_2} d\mu_{Z|u} \\
& - \int \Psi\left(x_1, x_2, z\right) d\mu_{X_1|v_1} d\mu_{X_2|v_2} d\mu_{Z|u} \int \left(\frac{1}{2}\gamma_{X,\varepsilon}\left(x_1, \varepsilon_1\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(x_2, \varepsilon_2\right)\right) d\mu_{(X_1,\varepsilon_1)|v_1} d\mu_{(X_2,\varepsilon_2)|v_2}.
\end{aligned}$$

Similarly, for V_2, we have that

$$\begin{aligned}
& \frac{1}{J}\sum_{k=1}^{J} E\left\{\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right)\left[\gamma_{Z,\eta}\left(Z^{(k)}, \eta^{(k)}\right) - E\gamma_{Z,\eta}\right] |\mu_{V_N}, \mu_{U_N}\right\} \\
= & \frac{1}{J}\sum_{k=1}^{J} E\left\{\left(\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right) \times \left(\gamma_{Z,\eta}\left(Z^{(k)}, \eta^{(k)}\right) - E\left\{\gamma_{Z,\eta}\left(Z^{(k)}, \eta^{(k)}\right) |\mu_{V_N}, \mu_{U_N}\right\}\right)\right. \\
& +\frac{1}{J}\sum_{k=1}^{J} E\left\{\Psi\left(X^{(2k-1)}, X^{(2k)}, Z^{(k)}\right) - \tilde{\psi}\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right) |\mu_{V_N}, \mu_{U_N}\right\}\left(E\left\{\gamma_{Z,\eta}\left(Z^{(k)}, \eta^{(k)}\right) |\mu_{V_N}, \mu_{U_N}\right\} - E\gamma_{Z,\eta}\right) \\
= & \frac{1}{J}\sum_{k=1}^{J} cov\left(\Psi, \gamma_{Z,\eta}|\left(v^{(2k-1)}, v^{(2k)}, u^{(k)}\right)\right) + \frac{1}{J}\sum_{k=1}^{J} 0 \times \left(E\left\{\gamma_{Z,\eta}\left(Z^{(k)}, \eta^{(k)}\right) |\mu_{V_N}, \mu_{U_N}\right\} - E\gamma_{Z,\eta}\right) \\
= & \int_0^1 cov_q\left(\Psi, \gamma_{Z,\eta}|q, q, q\right) dq + o(1).
\end{aligned}$$

Note that these three calculations imply T_2, R_2, and V_2 are variances of bounded random variables which converge in probability, and hence converge to 0. It follows that the sample variances converge in probability to their mean, which we now compute.

By the law of iterated expectations and arguments identical to showing equation (D.1.35),

$$\begin{aligned}
& E\frac{1}{J}\sum_{k=1}^{J}\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)^2 \\
= \; & E\frac{1}{J}\sum_{k=1}^{J}E\left\{\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)^2\Big|\mu_{V_N},\mu_{U_N}\right\} \\
= \; & \int_0^1 var\left(\Psi|q,q,q\right)dq+o\left(1\right) \qquad \text{(D.1.38)}
\end{aligned}$$

and

$$\begin{aligned}
& E\frac{1}{J}\sum_{k=1}^{J}\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right] \\
= \; & E\frac{1}{J}\sum_{k=1}^{J}E\left\{\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]\Big|\mu_{V_N},\mu_{U_N}\right\} \\
= \; & \int_0^1 cov_q\left(\Psi,\gamma_{X,\varepsilon}|\left(q,q,q\right)\right)dq+o\left(1\right) \qquad \text{(D.1.39)}
\end{aligned}$$

and

$$\begin{aligned}
& E\frac{1}{J}\sum_{k=1}^{J}\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\gamma_{Z,\eta}\right] \\
= \; & E\frac{1}{J}\sum_{k=1}^{J}E\left\{\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\gamma_{Z,\eta}\right]\Big|\mu_{V_N},\mu_{U_N}\right\} \\
= \; & \int_0^1 cov_q\left(\Psi,\gamma_{Z,\eta}|q,q,q\right)dq+o\left(1\right) \qquad \text{(D.1.40)}
\end{aligned}$$

This characterizes the asymptotic variance of the random variables in equation (D.1.31).

We now characterize the limiting distribution by computing the limit of the moment generating function. For arbitrary $C_1, C_2, C_3 > 0$ we must compute

$$E\left(\exp C_1\left[\psi_N-E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)\right]+C_2\left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right)+C_3\left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right)\right).$$

$$\begin{aligned}
& E\exp\left(\left[C_1\sqrt{J}\left[\psi_N-E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)\right]+C_2\sqrt{J}\left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right)+C_3\sqrt{J}\left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right)\right]\right) \\
= \; & E\exp\frac{1}{\sqrt{J}}\left(\sum_{k=1}^{J}C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right. \\
& \left.+\sum_{k=1}^{J}C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\right) \\
= \; & E\Pi_{k=1}^{J}\exp\frac{1}{\sqrt{J}}\left(C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right. \\
& \left.+C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\right)
\end{aligned}$$

By the Law of Iterated Expectations, this equals

$$
\begin{aligned}
& E\left[E\left[\Pi_{k=1}^{J}\exp\frac{1}{\sqrt{J}}\left(C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right.\right.\right.\\
& \left.\left.\left.+C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\right)\right|\mu_{V_N},\mu_{U_N}\right]\right]\\
= \; & E\Pi_{k=1}^{J}E\left[\exp\frac{1}{\sqrt{J}}\left(C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right.\right.\\
& \left.\left.+C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\right)\right|\mu_{V_N},\mu_{U_N}\right]\\
= \; & E\exp\sum_{k=1}^{J}\log E\left[\exp\frac{1}{\sqrt{J}}\left(C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right.\right.\\
& \left.\left.+C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\right)\right|\mu_{V_N},\mu_{U_N}\right]
\end{aligned}
$$

where the first equality follows from conditional independence of the terms k and $l \neq k$. Replacing the inner $\exp(x)$ by its Taylor expansion $\exp(x) = 1 + x + \frac{1}{2}x^2 + R(x)$ yields the expression

$$
\begin{aligned}
& E\exp\left(\left[C_1\sqrt{J}\left[\psi_N - E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)\right]+C_2\sqrt{J}\left(\mu_{(X,\varepsilon)_N}-\mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right)+C_3\sqrt{J}\left(\mu_{(Z,\eta)_N}-\mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right)\right]\right)\\
= \; & E\exp\sum_{k=1}^{J}\log E\left[1+\frac{1}{\sqrt{J}}\left(C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right.\right.\\
& +C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\Big)\\
& +\frac{1}{2J}\left(C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right.\\
& +C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]\\
& \left.\left.+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\right)^2+\frac{R_k}{J^{\frac{3}{2}}}\right|\mu_{V_N},\mu_{U_N}\Bigg]\\
= \; & E\exp\sum_{k=1}^{J}\log E\,[1+\\
& +\frac{C_2}{\sqrt{J}}\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)}\right)-E\left(\gamma_{X\varepsilon}\right)\right]+\frac{C_3}{\sqrt{J}}\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\\
& +\frac{1}{2J}\left(C_1\left(\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\right)\right.\\
& +C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right)+\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},,\varepsilon^{(2k)}\right)-E\left(\gamma_{X,\varepsilon}\right)\right]\\
& \left.\left.+C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right)-E\left(\gamma_{Z,\eta}\right)\right]\right)^2+\frac{R_k}{J^{\frac{3}{2}}}\right|\mu_{V_N},\mu_{U_N}\Bigg]
\end{aligned}
$$

where the first term $E\left[\Psi\left(X^{(2k-1)},X^{(2k)},Z^{(k)}\right)-\tilde{\psi}\left(v^{(2k-1)},v^{(2k)},u^{(k)}\right)\Big|\mu_{V_N},\mu_{U_N}\right]=0$ by the definition of $\tilde{\psi}$.

Since γ_X, γ_Z and Ψ are bounded, we approximate $\log(1+x)$ by its Taylor expansion $\log(1+x) = x - \frac{1}{2}x^2 + r(x)$ and keep track only of terms J^{-1} and lower (note that R_k is bounded as well). The above equation simplifies to

$$
\begin{aligned}
&E \exp \sum_{k=1}^{J} E \left\{ \frac{1}{\sqrt{J}} C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] + \frac{1}{\sqrt{J}} C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right. \\
&+ \frac{1}{2J} \left[C_1 \left(\Psi \left(X^{(2k-1)}, X^{(2k)}, Z^{(k)} \right) - \tilde{\psi} \left(v^{(2k-1)}, v^{(2k)}, u^{(k)} \right) \right) \right. \\
&\left. + C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] + C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right]^2 \\
&- \frac{1}{2J} \left(C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] \right)^2 - \frac{1}{2J} \left(C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right)^2 \\
&\left. - \frac{1}{2J} 2 C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right| \mu_{V_N}, \mu_{U_N} \right\} \\
&+ o\left(J^{-1} \right)
\end{aligned}
$$

Since $\frac{1}{J} \sum_{k=1}^{J} \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right]$ converges in probability to 0, we can rewrite this as

$$
\begin{aligned}
&E \exp \sum_{k=1}^{J} E \left\{ \frac{1}{\sqrt{J}} C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] + \frac{1}{\sqrt{J}} C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right. \\
&+ \frac{1}{2J} \left[C_1 \left(\Psi \left(X^{(2k-1)}, X^{(2k)}, Z^{(k)} \right) - \tilde{\psi} \left(v^{(2k-1)}, v^{(2k)}, u^{(k)} \right) \right) \right. \\
&\left. + C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] + C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right]^2 \\
&- \frac{1}{2J} \left(C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] \right)^2 \\
&\left. - \frac{1}{2J} \left(C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right)^2 \right| \mu_{V_N}, \mu_{U_N} \right\} + o(1)
\end{aligned}
$$

By the variance computations in equations (D.1.38), (D.1.39) and (D.1.40),

$$
\begin{aligned}
&\sum_{k=1}^{J} \frac{1}{2J} \left[C_1 \left(\Psi \left(X^{(2k-1)}, X^{(2k)}, Z^{(k)} \right) - \tilde{\psi} \left(v^{(2k-1)}, v^{(2k)}, u^{(k)} \right) \right) \right. \\
&+ C_2 \left[\frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k-1)}, \varepsilon^{(2k-1)} \right) + \frac{1}{2} \gamma_{X,\varepsilon} \left(X^{(2k)}, \varepsilon^{(2k)} \right) - E\left(\gamma_{X,\varepsilon} \right) \right] \\
&\left. + C_3 \left[\gamma_{Z,\eta} \left(Z^{(k)}, \eta^{(k)} \right) - E \gamma_{Z,\eta} \right] \right]^2
\end{aligned}
$$

converges in probability to

$$
\begin{aligned}
V_1 \quad = \quad & \frac{C_1^2}{2} \int_0^1 var_q \left(q, q, q \right) dq + C_1 C_2 \int_0^1 cov_q \left(\Psi, f | q, q, q \right) dq \\
& + C_1 C_3 \int_0^1 cov_q \left(\Psi, g | q, q, q \right) dq \\
& + \frac{C_2^2}{2} \frac{1}{2} Var \left(\gamma_{X,\varepsilon} \right) + \frac{C_3^2}{2} Var \left(\gamma_{Z,\eta} \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E\exp\left(\left[C_1\sqrt{J}\left[\psi_N - E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)\right] + C_2\sqrt{J}\left(\mu_{(X,\varepsilon)_N} - \mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right) + C_3\sqrt{J}\left(\mu_{(Z,\eta)_N} - \mu_{(Z,\eta)}\right)\left(\gamma_{Z,\eta}\right)\right]\right) \\
= \; & \exp\left(V_1\right) E\exp\left\{\frac{1}{\sqrt{J}}\sum_{k=1}^{J} E\left[C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right) - E\left(\gamma_{X,\varepsilon}\right)\right]\right.\right. \\
& \left. +C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right) - E\gamma_{Z,\eta}\right]\right| \mu_{V_N},\mu_{U_N}\Big] \\
& -\frac{1}{2J}\sum_{k=1}^{J} E\left[C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right) - E\left(\gamma_{X,\varepsilon}\right)\right]\right. \\
& \left.\left. + C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right) - E\gamma_{Z,\eta}\right]\right| \mu_{V_N},\mu_{U_N}\right]^2\Bigg\} + o(1).
\end{aligned}
$$

Since convergence in distribution implies convergence of moment generating functions and

$$
\frac{1}{2J}\sum_{k=1}^{J} E\left[C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right) - E\left(\gamma_{X,\varepsilon}\right)\right] + C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right) - E\gamma_{Z,\eta}\right]\right| \mu_{V_N},\mu_{U_N}\Bigg]^2
$$

converges in probability to

$$
\frac{1}{2}C_2^2\frac{1}{2}Var\left(\gamma_{X,\varepsilon}\right) + \frac{1}{2}C_3^2 Var\left(\gamma_{Z,\eta}\right),
$$

we can rewrite,

$$
\begin{aligned}
& E\exp\left(\left[C_1\sqrt{J}\left[\psi_N - E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)\right] + C_2\sqrt{J}\left(\mu_{(X,\varepsilon)_N} - \mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right) + C_3\sqrt{J}\left(\mu_{(Z,\eta)_N} - \mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right)\right]\right) \\
= \; & \exp\left(V_1\right) E\exp\left(\sum_{k=1}^{J} E\left[\frac{1}{\sqrt{J}}\left(C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right) - E\left(\gamma_{X,\varepsilon}\right)\right]\right.\right.\right. \\
& \left.\left. + C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right) - E\gamma_{Z,\eta}\right]\right)\right| \mu_{V_N},\mu_{U_N}\Bigg] \left. -\frac{1}{2}C_2^2\frac{1}{2}Var\left(\gamma_{X,\varepsilon}\right) - \frac{1}{2}C_3^2 Var\left(\gamma_{Z,\eta}\right)\right) + o(1) \\
= \; & \exp\left(V_1\right)\exp\left(-\frac{1}{2}C_2^2\frac{1}{2}Var\left(\gamma_{X,\varepsilon}\right) - \frac{1}{2}C_3^2 Var\left(\gamma_{Z,\eta}\right)\right) \times \\
& E\exp\sum_{k=1}^{J} E\left[\frac{1}{\sqrt{J}}\left(C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right) - E\left(\gamma_{X,\varepsilon}\right)\right]\right.\right. \\
& \left.\left. + C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right) - E\gamma_{Z,\eta}\right]\right)\right| \mu_{V_N},\mu_{U_N}\Bigg] + o(1).
\end{aligned}
$$

By the Levy continuity theorem and the equality $E\exp(tX) = \exp\left(E[X]' t + \frac{1}{2}t'V(X)^{-1}t\right)$ for normally distributed random variables, the product of the second and third terms,

$$
\begin{aligned}
& \exp\left(-\frac{1}{2}C_2^2\frac{1}{2}Var\left(\gamma_{X,\varepsilon}\right) - \frac{1}{2}C_3^2 Var\left(\gamma_{Z,\eta}\right)\right) \times \\
& E\exp\sum_{k=1}^{J} E\left[\frac{1}{\sqrt{J}}\left(C_2\left[\frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k-1)},\varepsilon^{(2k-1)}\right) + \frac{1}{2}\gamma_{X,\varepsilon}\left(X^{(2k)},\varepsilon^{(2k)}\right) - E\left(\gamma_{X,\varepsilon}\right)\right]\right.\right. \\
& \left.\left. +C_3\left[\gamma_{Z,\eta}\left(Z^{(k)},\eta^{(k)}\right) - E\gamma_{Z,\eta}\right]\right)\right| \mu_{V_N},\mu_{U_N}\Bigg]
\end{aligned}
$$

converges to 1. Hence,

$$E\exp\left(\left[C_1\sqrt{J}\left[\psi_N - E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)\right] + C_2\sqrt{J}\left(\mu_{(X,\varepsilon)_N} - \mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right) + C_3\sqrt{J}\left(\mu_{(Z,\eta)_N} - \mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right)\right]\right)$$

converges in probability to $\exp\left(V_1\right)$. Therefore, by Levy continuity,

$$\begin{bmatrix} \sqrt{J}\left(\psi_N - E\left(\psi_N|\mu_{V_N},\mu_{U_N}\right)\right) \\ \sqrt{J}\left(\mu_{(X,\varepsilon)_N} - \mu_{X,\eta}\right)\left(\gamma_{X,\varepsilon}\right) \\ \sqrt{J}\left(\mu_{(Z,\eta)_N} - \mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right) \end{bmatrix}$$

converges in distribution to a mean-zero normal with covariance

$$V''\left(\gamma_X,\gamma_Z\right) = \begin{bmatrix} \int_0^1 \sigma^2_{q,\Psi}\left(q,q,q\right)dq & \int_0^1 cov_q\left(\Psi,\gamma_{X,\varepsilon}|q,q,q\right)dq & \int_0^1 cov_q\left(\Psi,\gamma_{Z,\eta}|q,q,q\right)dq \\ \int_0^1 cov_q\left(\Psi,\gamma_{X,\varepsilon}|q,q,q\right)dq & \frac{1}{2}Var\left(\gamma_{X,\varepsilon}\right) & 0 \\ \int_0^1 cov_q\left(\Psi,\gamma_{Z,\eta}|q,q,q\right)dq & 0 & Var\left(\gamma_{Z,\eta}\right) \end{bmatrix}.$$

■

Lemma D.1.11 *Suppose Assumption 4(ii) is satisfied. For any $\mu_{X,\varepsilon}-$ Donsker class $\Gamma_{X,\varepsilon}$ of bounded functions on (X,ε) and $\mu_{Z,\eta}$ Donsker class $\Gamma_{Z,\eta}$ of bounded functions on (Z,η),*

$$\begin{bmatrix} \sqrt{N/2}\left(\psi_N - E\left(\psi_N|\mu_{U_N},\mu_{V_N}\right)\right) \\ \sqrt{N/2}\left(\mu_{(X,\varepsilon)_N} - \mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right) \\ \sqrt{N/2}\left(\mu_{(Z,\eta)_N} - \mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right) \end{bmatrix}$$

indexed by $\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}$ and $\gamma_{Z,\eta}\in\Gamma_{Z,\eta}$ converges to a Gaussian process whose covariance kernel characterized by V''.

Proof. Let $\gamma_{X,\varepsilon}$ be a linear combination of a finite number of elements of $\Gamma_{X,\varepsilon}$ and $\gamma_{Z,\eta}$ be a linear combination of a finite number of elements of $\Gamma_{Z,\eta}$. By Lemma D.1.10,

$$\sqrt{N/2}\left(\psi_N - E\left(\psi_N|\mu_{U_N},\mu_{V_N}\right), \left(\mu_{(X,\varepsilon)_N} - \mu_{X,\varepsilon}\right)\left(\gamma_{X,\varepsilon}\right), \left(\mu_{(Z,\eta)_N} - \mu_{Z,\eta}\right)\left(\gamma_{Z,\eta}\right)\right)$$

converges in distribution to $N\left(0,V''\left(\gamma_{X,\varepsilon},\gamma_{Z,\eta}\right)\right)$. Let H_N be the stochastic process $\sqrt{N/2}\left(\psi_N - E\left(\psi_N|\mu_{U_N},\mu_{V_N}\right)\right)$ jointly with the empirical processes on $\Gamma_{X,\varepsilon}$ and $\Gamma_{Z,\eta}$. We index these stochastic processes with $\gamma\in\Gamma$ where we endow Γ with the L_2 metric. By the Cramer-Wold device, the finite dimensional distributions of H_N converge to a Gaussian process whose covariance kernel is defined as follows:

For two elements of $\Gamma_{X,\varepsilon}$ and $\Gamma_{Z,\eta}$, the covariance kernel is that of the associated empirical processes. The covariance of an element of $\Gamma_{X,\varepsilon}$ and an element of $\Gamma_{Z,\eta}$ is 0. The covariance of $\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}$ with $\sqrt{N/2}\left(\psi_N - E\left(\psi_N|\mu_{U_N},\mu_{V_N}\right)\right)$ is $\int_0^1 cov_q\left(\Psi,\gamma_{X,\varepsilon}|q,q,q\right)dq$. The covariance of $\gamma_{Z,\eta}\in\Gamma_{Z,\eta}$ with the term $\sqrt{N}\left(\psi_N - E\left(\psi_N|\mu_{U_N},\mu_{V_N}\right)\right)$ is $\int_0^1 cov_q\left(\Psi,\gamma_{Z,\eta}|q,q,q\right)dq$. The variance of $\sqrt{N/2}\left(\psi_N - E\left(\psi_N|\mu_{U_N},\mu_{V_N}\right)\right)$ is $\int_0^1 \sigma^2_{q,\Psi}\left(q,q,q\right)dq$.

We now verify equicontinuity to show weak convergence of H_N. We prove this directly using equicontinuity properties of the empirical processes on $\Gamma_{X,\varepsilon}$ and $\Gamma_{Z,\varepsilon}$. Denote

$$\begin{aligned} Var_{V,Z}(m_V, m_Z) &= \int Var(\Psi(X_1, X_2, Z) | v, Z = z) \, dm_Z dm_V \\ Var_{U,X}(m_U, m_X) &= \int Var(\Psi(X_1, X_2, Z) | u, X_1 = x_1, X_2 = x_2) \, dm_X dm_X dm_U. \end{aligned}$$

Let $Var(\Psi(X_1, X_2, Z) | v_1, v_2, Z = z) = Var(v_1, v_2, z)$. Consider the quantity $\frac{1}{N/2} \sum_{i=1}^{N/2} Var\left(v^{(2i-1)}, v^{(2i)}, z^{(i)}\right)$, and note that since Var is bounded and uniformly continuous, it is equal to

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^{N/2} \left[Var\left(v^{(2i-1)}, v^{(2i-1)}, z^{(i)}\right) + Var\left(v^{(2i)}, v^{(2i)}, z^{(i)}\right) \right] + o(1) \\ = \; & Var_{V,Z}\left(\mu_{V_N}, \mu_{Z_N}\right) + o(1) \\ = \; & Var_{V,Z}(\mu_V, \mu_Z) + o(1) \end{aligned}$$

An identical argument implies

$$\frac{1}{N/2} \sum_{i=1}^{N/2} Var\left(x^{(2i-1)}, x^{(2i)}, u^{(i)}\right) = Var_{U,X}(\mu_U, \mu_X) + o(1).$$

Since $\mu_{X|v}$ and $\mu_{Z|u}$ are not degenerate, $Var_{V,Z}(\mu_V, \mu_Z)$ and $Var_{U,X}(\mu_U, \mu_X)$ are strictly positive. Hence, $\lim_N \sup_{\mu_{V_N}, \mu_{Z_N}} Var_{V,Z}\left(\mu_{V_N}, \mu_{Z_N}\right)$ and $\lim_N \sup_{\mu_{U_N}, \mu_{X_N}} Var_{U,X}\left(\mu_{U_N}, \mu_{X_N}\right)$ are strictly positive. Hence, for large enough N, there is a $\delta > 0$ such that a $\delta-$ball around $H_N(\gamma) = \sqrt{N/2}\left(\psi_N - E\left(\psi_N | \mu_{V_N}, \mu_{U_N}\right)\right)$ contains no other element $H_N(\gamma')$ for $\gamma' \neq \gamma$. Pick $\delta > 0$ such that the δ ball around $\sqrt{N/2}\left(\psi_N - E\left(\psi_N | \mu_{V_N}, \mu_{U_N}\right)\right)$ is a singleton. Hence, if $B_{\Gamma_{X,\varepsilon}}(\gamma, \delta) = B(\gamma, \delta) \cap \Gamma_{X,\varepsilon}$, and $B_{\Gamma_{Z,\eta}}(\gamma, \delta) = B(\gamma, \delta) \cap \Gamma_{Z,\eta}$,

$$\begin{aligned} & \sup_{\gamma \in \Gamma} \sup_{\gamma' \in B(\gamma, \delta)} |H_N(\gamma) - H_N(\gamma')| \\ \leq \; & \sup_{\gamma_{X,\varepsilon} \in \Gamma_{X,\varepsilon}} \sup_{\gamma'_{X,\varepsilon} \in B_{\Gamma_{X,\varepsilon}}(\gamma_{X,\varepsilon}, \delta)} \left| H_N(\gamma_{X,\varepsilon}) - H_N(\gamma'_{X,\varepsilon}) \right| + \sup_{\gamma_{Z,\eta} \in \Gamma_{Z,\eta}} \sup_{\gamma'_{X,\varepsilon} \in B_{\Gamma_{X,\varepsilon}}(\gamma_{Z,\eta}, \delta)} \left| H_N(\gamma_{Z,\eta}) - H_N(\gamma'_{X,\varepsilon}) \right| \\ & + \sup_{\gamma_{X,\varepsilon} \in \Gamma_{X,\varepsilon}} \sup_{\gamma'_{Z,\eta} \in B_{\Gamma_{Z,\eta}}(\gamma_{X,\varepsilon}, \delta)} \left| H_N(\gamma_{X,\varepsilon}) - H_N(\gamma'_{Z,\eta}) \right| + \sup_{\gamma_{Z,\eta} \in \Gamma_{Z,\eta}} \sup_{\gamma'_{Z,\eta} \in B_{\Gamma_{Z,\eta}}(\gamma_{Z,\eta}, \delta)} \left| H_N(\gamma_{Z,\eta}) - H_N(\gamma'_{Z,\eta}) \right| \end{aligned}$$

since $B(\gamma, \delta) = \{\gamma\}$ if $H_N(\gamma) = \sqrt{N/2}\left(\psi_N - E\left(\psi_N | \mu_{U_N}, \mu_{V_N}\right)\right)$.

For a fixed $\varepsilon, \eta > 0$, there is (by definition of stochastic equicontinuity) there exists $\delta > 0$ such that

$$\lim_{N \to \infty} \sup P\left(\sup_{\gamma_{X,\varepsilon} \in \Gamma_{X,\varepsilon}} \sup_{\gamma'_{X,\varepsilon} \in B_{\Gamma_{X,\varepsilon}}(\gamma_{X,\varepsilon}, \delta)} \left| H_N(\gamma_{X,\varepsilon}) - H_N(\gamma'_{X,\varepsilon}) \right| > \frac{\varepsilon}{6} \right) < \frac{\eta}{6}$$

and

$$\lim_{N \to \infty} \sup P\left(\sup_{\gamma_{Z,\eta} \in \Gamma_{Z,\eta}} \sup_{\gamma'_{Z,\eta} \in B_{\Gamma_{Z,\eta}}(\gamma_{Z,\eta}, \delta)} \left| H_N(\gamma_{Z,\eta}) - H_N(\gamma'_{Z,\eta}) \right| > \frac{\varepsilon}{6} \right) < \frac{\eta}{6}.$$

Now we show that

$$\limsup_{N\to\infty} P\left(\sup_{\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}}\ \sup_{\gamma'_{Z,\eta}\in B_{\Gamma_{Z,\eta}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{X,\varepsilon}\right)-H_N\left(\gamma'_{Z,\eta}\right)\right|>\frac{\varepsilon}{3}\right)<\frac{\eta}{6}.$$

Note that independence of empirical processes on $\Gamma_{X,\varepsilon}$ and $\Gamma_{Z,\eta}$ implies that $B_{\Gamma_{Z,\eta}}\left(\gamma_{X,\varepsilon},\delta\right)$ is nonempty only if $\gamma_{X,\varepsilon}$ has L^2 norm less than δ. If this is the case, every element of $B_{\Gamma_{Z,\eta}}\left(\gamma_{X,\varepsilon},\delta\right)$ also has L^2 norm less than δ. Therefore,

$$\begin{aligned}
& P\left(\sup_{\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}}\ \sup_{\gamma'_{Z,\eta}\in B_{\Gamma_{Z,\eta}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{X,\varepsilon}\right)-H_N\left(\gamma'_{Z,\eta}\right)\right|>\frac{\varepsilon}{3}\right)\\
\leq\ & P\left(\sup_{\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}}\ \sup_{\gamma'_{Z,\eta}\in B_{\Gamma_{Z,\eta}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{X,\varepsilon}\right)\right|+\left|H_N\left(\gamma'_{Z,\eta}\right)\right|>\frac{\varepsilon}{3}\right)\\
\leq\ & P\left(\sup_{\gamma_{Z,\eta}\in\Gamma_{Z,\eta}}\ \sup_{\gamma'_{Z,\eta}\in B_{\Gamma_{Z,\eta}}\left(\gamma_{Z,\eta},\delta\right)}\left|H_N\left(\gamma_{Z,\eta}\right)-H_N\left(\gamma'_{Z,\eta}\right)\right|+\sup_{\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}}\ \sup_{\gamma'_{X,\varepsilon}\in B_{\Gamma_{X,\varepsilon}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{X,\varepsilon}\right)-H_N\left(\gamma'_{X,\varepsilon}\right)\right|>\frac{\varepsilon}{3}\right)\\
\leq\ & P\left(\sup_{\gamma_{Z,\eta}\in\Gamma_{Z,\eta}}\ \sup_{\gamma'_{Z,\eta}\in B_{\Gamma_{Z,\eta}}\left(\gamma_{Z,\eta},\delta\right)}\left|H_N\left(\gamma_{Z,\eta}\right)-H_N\left(\gamma'_{Z,\eta}\right)\right|>\frac{\varepsilon}{6}\right)\\
& +P\left(\sup_{\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}}\ \sup_{\gamma'_{X,\varepsilon}\in B_{\Gamma_{X,\varepsilon}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{X,\varepsilon}\right)-H_N\left(\gamma'_{X,\varepsilon}\right)\right|>\frac{\varepsilon}{6}\right)
\end{aligned}$$

where the second inequality follows from the triangle inequality since a constant 0 function is an element of both $\Gamma_{X,\varepsilon}$ and $\Gamma_{Z,\eta}$. By the same argument

$$\begin{aligned}
& P\left(\sup_{\gamma_{Z,\eta}\in\Gamma_{Z,\eta}}\ \sup_{\gamma'_{X,\varepsilon}\in B_{\Gamma_{X,\varepsilon}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{Z,\eta}\right)-H_N\left(\gamma'_{X,\varepsilon}\right)\right|>\frac{\varepsilon}{3}\right)\\
\leq\ & P\left(\sup_{\gamma_{Z,\eta}\in\Gamma_{Z,\eta}}\ \sup_{\gamma'_{Z,\eta}\in B_{\Gamma_{Z,\eta}}\left(\gamma_{Z,\eta},\delta\right)}\left|H_N\left(\gamma_{Z,\eta}\right)-H_N\left(\gamma'_{Z,\eta}\right)\right|>\frac{\varepsilon}{6}\right)\\
& +P\left(\sup_{\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}}\ \sup_{\gamma'_{X,\varepsilon}\in B_{\Gamma_{X,\varepsilon}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{X,\varepsilon}\right)-H_N\left(\gamma'_{X,\varepsilon}\right)\right|>\frac{\varepsilon}{6}\right)
\end{aligned}$$

and thus

$$\begin{aligned}
& \limsup_{n\to\infty} P\left(\sup_{\gamma\in\Gamma}\ \sup_{\gamma'\in B(\gamma,\delta)}\left|H_N\left(\gamma\right)-H_N\left(\gamma'\right)\right|>\varepsilon\right)\\
\leq\ & 3\limsup_{n\to\infty} P\left(\sup_{\gamma_{X,\varepsilon}\in\Gamma_{X,\varepsilon}}\ \sup_{\gamma'_{X,\varepsilon}\in B_{\Gamma_{X,\varepsilon}}\left(\gamma_{X,\varepsilon},\delta\right)}\left|H_N\left(\gamma_{X,\varepsilon}\right)-H_N\left(\gamma'_{X,\varepsilon}\right)\right|>\frac{\varepsilon}{6}\right)\\
& +3\limsup_{n\to\infty} P\left(\sup_{\gamma_{Z,\eta}\in\Gamma_{Z,\eta}}\ \sup_{\gamma'_{Z,\eta}\in B_{\Gamma_{Z,\eta}}\left(\gamma_{Z,\eta},\delta\right)}\left|H_N\left(\gamma_{Z,\eta}\right)-H_N\left(\gamma'_{Z,\eta}\right)\right|>\frac{\varepsilon}{6}\right)\\
<\ & \eta.
\end{aligned}$$

This proves stochastic equicontinuity of $H_N(\gamma)$ and hence weak convergence to the Gaussian process defined above. ■

D.2 Proof of Proposition 4

We will show that the Hadamard derivative of $\psi^{\delta} : L_{\infty}^{\Gamma} \to L_{\infty}^{\Theta}$ evaluated at (μ_X, μ_Z) in the direction (G_X, G_Z) is

$$
\begin{aligned}
& \nabla_{(G_X,G_Z)} \tilde{\psi}^{\delta} [\mu_X, \mu_Z](\theta) \\
= & \int_{\delta}^{1-\delta} G_U^q(\theta) \frac{\int \Psi(x_1, x_2, z)\, \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, f_{\eta}'\left(F_{U;\theta}^{-1}(q) - h(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} G_V^q(\theta) \frac{\int \Psi(x_1, x_2, z)\, \phi_{\varepsilon}(q, x_1; \theta)\, f_{\varepsilon}'\left(F_{V;\theta}^{-1}(q) - h(x_2;\theta)\right) \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} G_V^q(\theta) \frac{\int \Psi(x_1, x_2, z)\, f_{\varepsilon}'\left(F_{V;\theta}^{-1}(q) - h(x_1;\theta)\right) \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1, x_2, z)\, \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, dG_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1, x_2, z)\, \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_X dG_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1, x_2, z)\, \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} dG_Z}{\int \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1, x_2, z)\, \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1; \theta)\, \phi_{\varepsilon}(q, x_2; \theta)\, \phi_{\eta}(q, z; \theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} L_G[\mu_X, \mu_Z](\theta, q)\, dq.
\end{aligned}
$$

where

$$
\begin{aligned}
G_V^q(\theta) &= \frac{1}{f_{V;\theta}\left(F_{V,\theta}^{-1}(q)\right)} \int G_X\left(1\left\{h(x;\theta) + \varepsilon \le F_{V;\theta}^{-1}(q)\right\}\right) dF_{\varepsilon}, \\
G_U^q(\theta) &= \frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)} \int G_Z\left(1\left\{g(z;\theta) + \eta \le F_{U;\theta}^{-1}(q)\right\}\right) dF_{\eta},
\end{aligned}
$$

and $L_G[\mu_X, \mu_Z](\theta, q)$ is the negative of

$$\begin{aligned}
&G_U^q(\theta) \frac{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) f'_\eta\left(F_{U;\theta}^{-1}(q) - g(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
&+G_V^q(\theta) \frac{\int \phi_\varepsilon(q, x_1; \theta) f'_\varepsilon\left(F_{V;\theta}^{-1}(q) - h(x_2;\theta)\right) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
&+G_V^q(\theta) \frac{\int f'_\varepsilon\left(F_{V;\theta}^{-1}(q) - h(x_1;\theta)\right) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
&+\frac{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) dG_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
&+\frac{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_X dG_{X_2} d\mu_Z}{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
&+\frac{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} dG_{X_3}}{\int \phi_\varepsilon(q, x_1; \theta) \phi_\varepsilon(q, x_2; \theta) \phi_\eta(q, z; \theta) d\mu_{X_1} d\mu_{X_2} d\mu_Z}
\end{aligned}$$

and

$$\begin{aligned}
\phi_\eta(q, z; \theta) &= f_\eta\left(F_{u;\theta,\mu_Z}^{-1}(q) - g(z;\theta)\right) \\
\phi_\varepsilon(q, x; \theta) &= f_\varepsilon\left(F_{v;\theta,\mu_X}^{-1}(q) - h(x;\theta)\right).
\end{aligned}$$

Proof. Let

$$\begin{aligned}
\phi_{\eta,N}(q, z; \theta) &= f_\eta\left(F_{N,U;\theta,\mu_{Z_N}}^{-1}(q) - g(z;\theta)\right) \\
\phi_{\varepsilon,N}(q, x; \theta) &= f_\varepsilon\left(F_{N,V;\theta,\mu_{X_N}}^{-1}(q) - h(x;\theta)\right)
\end{aligned}$$

where $F_{N,U;\theta}(u) = \int F_\eta(u - g(Z;\theta)) d\mu_{Z_N}$ and $F_{N,V;\theta}(v) = \int F_\varepsilon(v - h(X;\theta)) d\mu_{X_N}$.

Consider a sequence of measures $\left(\mu_{X_N}, \mu_{Z_N}\right)$ and a sequence of scalars $h_N \to 0$ such that $\frac{1}{h_N}\left(\mu_{X_N} - \mu_X, \mu_{Z_N} - \mu_Z\right)$

converges to $G = (G_X, G_Z)$ uniformly in L_∞^Γ, where G is bounded and uniformly continuous. We can rewrite

$$\begin{aligned}
& \int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\
& -\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}dq \\
= & \int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\left(d\mu_{X_1}d\mu_{X_2}d\mu_Z - d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}\right)}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\
& +\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\
& -\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \times \\
& \frac{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}dq \\
= & \int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\left(d\mu_{X_1}d\mu_{X_2}d\mu_Z - d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}\right)}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\
& +\left[\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right. \\
& \left.-\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right] \\
& +\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \times \\
& \left(1 - \frac{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}\right)dq \\
= & T_1 + T_2 + T_3 \\
= & \int_\delta^{1-\delta} T_1(q)\,dq + \int_\delta^{1-\delta} T_2(q)\,dq + \int_\delta^{1-\delta} T_3(q)\,dq
\end{aligned} \tag{D.2.41}$$

To compute $T_1(q)$ note that

$$\begin{aligned}
& d\mu_{X_1}d\mu_{X_2}d\mu_Z - d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N} \\
= & \left(d\mu_{X_1} - d\mu_{X_{N,1}}\right)d\mu_{X_2}d\mu_Z + d\mu_{X_{N,1}}d\mu_{X_2}d\mu_Z - d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N} \\
= & \left(d\mu_{X_1} - d\mu_{X_{N,1}}\right)d\mu_{X_2}d\mu_Z + d\mu_{X_{N,1}}\left(d\mu_{X_2} - d\mu_{X_{N,2}}\right)d\mu_Z \\
& + d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_Z - d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N} \\
= & \left(d\mu_{X_1} - d\mu_{X_{N,1}}\right)d\mu_{X_2}d\mu_Z + d\mu_{X_{N,1}}\left(d\mu_{X_2} - d\mu_{X_{N,2}}\right)d\mu_Z + d\mu_{X_{N,1}}d\mu_{X_{N,2}}\left(d\mu_Z - d\mu_{Z_N}\right) \\
= & \left(d\mu_{X_1} - d\mu_{X_{N,1}}\right)d\mu_{X_2}d\mu_Z + d\mu_X\left(d\mu_{X_2} - d\mu_{X_{N,2}}\right)d\mu_Z \\
& + \left(d\mu_{X_{N,1}} - d\mu_X\right)\left(d\mu_{X_2} - d\mu_{X_{N,2}}\right)d\mu_Z + d\mu_{X_1}d\mu_{X_2}\left(d\mu_Z - d\mu_{Z_N}\right) \\
& + \left(d\mu_{X_{N,1}} - d\mu_{X_1}\right)d\mu_{X_2}\left(d\mu_Z - d\mu_{Z_N}\right) + d\mu_{X_{N,1}}\left(d\mu_{X_{N,2}} - d\mu_{X_2}\right)\left(d\mu_Z - d\mu_{Z_N}\right).
\end{aligned}$$

Hence,

$$\begin{aligned} T_1(q) &= \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\left(d\mu_{X_1}-d\mu_{X_{N,1}}\right)d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \\ &+\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_X\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \\ &+\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}\left(d\mu_Z-d\mu_{Z_N}\right)}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \\ &+\frac{R(q)}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \end{aligned} \tag{D.2.42}$$

where

$$\begin{aligned} R(q) &= \int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\left(d\mu_{X_{N,1}}-d\mu_X\right)\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)d\mu_Z \\ &\quad+\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\left(d\mu_{X_{N,1}}-d\mu_{X_1}\right)d\mu_{X_2}\left(d\mu_Z-d\mu_{Z_N}\right) \\ &\quad+\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_{N,1}}\left(d\mu_{X_{N,2}}-d\mu_{X_2}\right)\left(d\mu_Z-d\mu_{Z_N}\right) \\ &= R_1(q)+R_2(q)+R_3(q). \end{aligned}$$

Now we show that each of $\frac{1}{h_N}R_1$, $\frac{1}{h_N}R_2$ and $\frac{1}{h_N}R_3$ are negligible. To show that $\frac{1}{h_N}R_1(q)$ is negligible, we rewrite it as

$$\begin{aligned} &\frac{1}{h_N}R_1(q) \\ &= \int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)d\mu_Z + \\ &\quad\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\left(\frac{1}{h_N}\left(d\mu_{X_{N,1}}-d\mu_X\right)-dG_{X_1}\right)\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)d\mu_Z \\ &= S_1(q)+S_2(q) \end{aligned}$$

and show that S_1 and S_2 are negligible. Note that

$$\begin{aligned} &\sup_{q\in(\delta,1-\delta),\theta}|S_2(q)| \\ &\leq \int \sup_{q\in(\delta,1-\delta),\theta}\left|\Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\left(\frac{1}{h_N}\left(d\mu_{X_{N,1}}-d\mu_X\right)-dG_{X_1}\right)\right|\left(d\mu_{X_2}+d\mu_{X_{N,2}}\right)d\mu_Z \\ &\leq 2\,\|\Psi\|_\infty\,\|f_\varepsilon\|_\infty\,\|f_\eta\|_\infty \sup_{q\in(\delta,1-\delta),\theta}\left|\phi_\varepsilon(q,x_1;\theta)\left(\frac{1}{h_N}\left(d\mu_{X_{N,1}}-d\mu_X\right)-dG_{X_1}\right)\right| \\ &= o(1). \end{aligned}$$

since

$$\phi_\varepsilon(q,x_2;\theta)=f_\varepsilon\left(F_V^{-1}(q)-h(x;\theta)\right)$$

indexed by q, θ is a sub-class of Γ. Turning to S_1, note that

$$S_1(q)=\int \tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)d\mu_Z$$

where $\tilde{\Psi}(x_1,x_2,z,q,\theta)=\Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)$ and $\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}$ is a bounded uniformly continuous function of (x_2,z,q,θ) for $q\in(\delta,1-\delta)$. For any $\varepsilon>0$, fix a compact set $\bar{\chi}=1\{x:c_1\le x\le c_2\}$ for $c_1,c_2\in\mathbb{R}^{k_X}$, such that $\mu_X(\chi\backslash\bar{\chi})\le\varepsilon$. By the triangle inequality,

$$\begin{aligned}&\left|\int\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)\right|\\ \le\;&\|G\|_\infty\left(\mu_X(\chi\backslash\bar{\chi})+\mu_{X_N}(\chi\backslash\bar{\chi})\right)+\left|\int_{\bar{\chi}}\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)\right|.\end{aligned}$$

Since G is uniformly continuous, there exists a collection $\chi^1,\ldots,\chi^M$ of subsets $\chi^i=\left\{x:c_1^i\le x\le c_2^i\right\}$ containing points $x^1,\ldots,x^M$ that cover $\bar{\chi}$ such that

$$\left|\int_{\bar{\chi}}\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)-\sum_{i=1}^{M}\int\tilde{\Psi}\left(x_1,x^i,z,q,\theta\right)dG_{X_1}\left(\mu_{X_2}-\mu_{X_{N,2}}\right)(\chi_i)\right|<\varepsilon.$$

Note that $1\left\{x\in\chi^i\right\}\in\Gamma_X$. By the triangle inequality,

$$\begin{aligned}&\left|\int_{\bar{\chi}}\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)\right|\\ <\;&\varepsilon+\left|\sum_{i=1}^{M}\int\tilde{\Psi}\left(x_1,x^i,z,q,\theta\right)dG_{X_1}\left(\mu_{X_2}(\chi_i)-\mu_{X_{N,2}}(\chi_i)\right)\right|\\ \le\;&\varepsilon+M\|G\|_\infty\left\|\mu_{X_2}-\mu_{X_{N,2}}\right\|_\infty,\end{aligned}$$

where $\left\|\mu_{X_2}-\mu_{X_{N,2}}\right\|_\infty=\sup_{\gamma_X\in\Gamma_X}\left|\left(\mu_{X_2}-\mu_{X_{N,2}}\right)(\gamma_X)\right|$. Thus,

$$\begin{aligned}&\left|\int\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)\right|\\ \le\;&\|G\|_\infty\left(\mu_X(\chi\backslash\bar{\chi})+\mu_{X_N}(\chi\backslash\bar{\chi})\right)+\varepsilon+M\|G\|_\infty\left\|d\mu_{X_2}-d\mu_{X_{N,2}}\right\|_\infty.\end{aligned}$$

Since $\limsup_{N\to\infty}\left\|d\mu_{X_2}-d\mu_{X_{N,2}}\right\|_\infty=0$, we have that

$$\begin{aligned}&\limsup_{N\to\infty}\sup_{z,q}\left|\int\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)\right|\\ \le\;&2\|G\|_\infty\mu_X(\chi\backslash\bar{\chi})+\varepsilon\\ \le\;&(2\|G\|_\infty+1)\,\varepsilon\end{aligned}$$

Since this inequality holds for all $\varepsilon>0$,

$$\lim_{N\to\infty}\sup_{z,q}\left|\int\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)\right|=0$$

Thus,

$$\begin{aligned}\sup_{q\in(\delta,1-\delta),\theta}|S_1(q,N,\theta)|\;\le\;&\sup_{q\in(\delta,1-\delta),\theta}\left|\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)d\mu_Z\right|\\ \le\;&\sup_{z,q}\left|\int\int\tilde{\Psi}(x_1,x_2,z,q,\theta)\,dG_{X_1}\left(d\mu_{X_2}-d\mu_{X_{N,2}}\right)\right|\to 0.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{q\in(\delta,1-\delta),\theta}\left|\frac{1}{h_N}R_1\left(q,N,\theta\right)\right| &\leq \sup_{q\in(\delta,1-\delta),\theta}\left|S_1\left(q,N,\theta\right)\right|+\sup_{q\in(\delta,1-\delta),\theta}\left|S_2\left(q,N,\theta\right)\right| \\ &\rightarrow 0\end{aligned}$$

Identical arguments show that $R_2\rightarrow 0$ and $R_3\rightarrow 0$. Lemma D.2.12 implies that

$$\inf_{q\in(\delta,1-\delta),\theta\in\Theta}\int\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z>0.$$

Therefore,

$$\frac{R\left(q\right)}{\int\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\rightarrow 0.$$

It follows that equation (D.2.42) can be re-written as

$$\begin{aligned}\frac{1}{h_N}T_1 &= \int_{\delta}^{1-\delta}\frac{1}{h_N}T_1\left(q\right)dq \\ &= \int_{\delta}^{1-\delta}\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)dG_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\ &\quad+\int_{\delta}^{1-\delta}\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_X dG_{X_2}d\mu_Z}{\int\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\ &\quad+\int_{\delta}^{1-\delta}\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}dG_Z}{\int\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\ &\quad+o\left(1\right).\end{aligned} \tag{D.2.43}$$

To compute the limit of T_2, rewrite

$$\begin{aligned}T_2\left(q\right) &= \frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z} \\ &\quad-\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\end{aligned}$$

by observing that

$$\begin{aligned}&\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta}\left(q,z;\theta\right)-\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right) \\ =\;&\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\left[\phi_{\eta}\left(q,z;\theta\right)-\phi_{\eta,N}\left(q,z;\theta\right)\right]+\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right) \\ &-\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right) \\ =\;&\phi_{\varepsilon}\left(q,x_1;\theta\right)\phi_{\varepsilon}\left(q,x_2;\theta\right)\left[\phi_{\eta}\left(q,z;\theta\right)-\phi_{\eta,N}\left(q,z;\theta\right)\right] \\ &+\phi_{\varepsilon}\left(q,x_1;\theta\right)\left[\phi_{\varepsilon}\left(q,x_2;\theta\right)-\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\right]\phi_{\eta,N}\left(q,z;\theta\right) \\ &+\left[\phi_{\varepsilon}\left(q,x_1;\theta\right)-\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\right]\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right).\end{aligned}$$

Hence,

$$
\begin{aligned}
& \frac{1}{h_N} T_2(q) \\
= \; & \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\frac{1}{h_N}\left[\phi_\eta(q,z;\theta)-\phi_{\eta,N}(q,z;\theta)\right] d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& + \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\frac{1}{h_N}\left[\phi_\varepsilon(q,x_2;\theta)-\phi_{\varepsilon,N}(q,x_2;\theta)\right]\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& + \frac{\int \Psi(x_1,x_2,z)\,\frac{1}{h_N}\left[\phi_\varepsilon(q,x_1;\theta)-\phi_{\varepsilon,N}(q,x_1;\theta)\right]\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
= \; & \frac{1}{h_N}\left(F_{U;\theta}^{-1}(q)-F_{N,U;\theta}^{-1}(q)\right)\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,f_\eta'\left(F_{U;\theta}^{-1}(q)-g(z;\theta)\right) d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& + \frac{1}{h_N}\left(F_{V;\theta}^{-1}(q)-F_{N,V;\theta}^{-1}(q)\right)\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,f_\varepsilon'\left(F_{V;\theta}^{-1}(q)-h(x_2;\theta)\right)\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& + \frac{1}{h_N}\left(F_{V;\theta}^{-1}(q)-F_{N,V;\theta}^{-1}(q)\right)\frac{\int \Psi(x_1,x_2,z)\,f_\varepsilon'\left(F_{V;\theta}^{-1}(q)-h(x_1;\theta)\right)\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& + o(1) \\
= \; & K_1(q)+K_2(q)+K_3(q)+o(1), \qquad \text{(D.2.44)}
\end{aligned}
$$

where the equality follows from a Taylor expansion and dominated convergence theorem (since f_ε' and f_η' are bounded).

Rewrite $K_1(q)$ as

$$
\begin{aligned}
& \frac{1}{h_N}\left(F_{U;\theta}^{-1}(q)-F_{N,U;\theta}^{-1}(q)\right)\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,f_\eta'\left(F_{U;\theta}^{-1}(q)-g(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& + \frac{1}{h_N}\left(F_{U;\theta}^{-1}(q)-F_{N,U;\theta}^{-1}(q)\right)\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,f_\eta'\left(F_{U;\theta}^{-1}(q)-g(z;\theta)\right) d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& - \frac{1}{h_N}\left(F_{U;\theta}^{-1}(q)-F_{N,U;\theta}^{-1}(q)\right)\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,f_\eta'\left(F_{U;\theta}^{-1}(q)-g(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
= \; & C_1(q)+C_2(q)-C_3(q)
\end{aligned}
$$

where $C_2(q)-C_3(q)$ is not greater in absolute value than

$$
\begin{aligned}
& \frac{\sup_{\theta,q\in(\delta,1-\delta)}\left|\frac{1}{h_N}\left(F_{U;\theta}^{-1}(q)-F_{N,U;\theta}^{-1}(q)\right)\right|}{\inf_\theta \int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1} d\mu_{X_2} d\mu_Z} \times \\
& \left|\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,f_\eta'\left(F_{U;\theta}^{-1}(q)-g(z;\theta)\right)\left(d\mu_{X_1} d\mu_{X_2} d\mu_Z - d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{N,Z}\right)\right|
\end{aligned}
$$

which goes to 0 uniformly in $q\in(\delta,1-\delta)$ by the same argument used to compute the limit of $T_1(q)$. To compute the limit of $C_1(q)$, note

$$\begin{aligned}
& F_{U;\theta}^{-1}(q) - F_{N,U;\theta}^{-1}(q) \\
= & \frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)}\left(F_{U;\theta}\left(F_{U;\theta}^{-1}(q)\right) - F_{N,U;\theta}\left(F_{U;\theta}^{-1}(q)\right)\right) + o(1) \\
= & \frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)}\int F_{\eta}\left(F_{U;\theta}^{-1}(q) - g(z;\theta)\right)\left(d\mu_Z - d\mu_{Z_N}\right) + o(1) \\
= & \frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)}\int \left(\mu_Z - \mu_{Z_N}\right)\left(1\left\{g(z;\theta) + \eta \leq F_{U;\theta}^{-1}(q)\right\}\right) dF_{\eta} + o(1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{h_N}\left(F_{U;\theta}^{-1}(q) - F_{N,U;\theta}^{-1}(q)\right) & \to \frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)}\int G_Z\left(1\left\{g(z;\theta) + \eta \leq F_{U;\theta}^{-1}(q)\right\}\right) dF_{\eta} \\
& = G_U^q(\theta)
\end{aligned}$$

uniformly in $q \in (\delta, 1-\delta)$. Hence, $K_1(q)$ converges to

$$G_U^q(\theta)\frac{\int \Psi(x_1, x_2, z)\,\phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,f_{\eta}'\left(F_{U;\theta}^{-1}(q) - g(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}$$

Similar arguments show that $K_2(q)$ and $K_3(q)$ respectively converge to

$$G_V^q(\theta)\frac{\int \Psi(x_1, x_2, z)\,\phi_{\varepsilon}(q, x_1;\theta)\,f_{\varepsilon}'\left(F_{V;\theta}^{-1}(q) - h(x_2;\theta)\right)\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}$$

and

$$G_V^q(\theta)\frac{\int \Psi(x_1, x_2, z)\,f_{\varepsilon}'\left(F_{V;\theta}^{-1}(q) - h(x_1;\theta)\right)\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}.$$

Consequently, equation (D.2.44) can be written as

$$\begin{aligned}
\frac{1}{h_n}T_2 = & \int_{\delta}^{1-\delta}\frac{1}{h_n}T_2(q)\,dq \\
= & \int_{\delta}^{1-\delta} G_U^q(\theta)\frac{\int \Psi(x_1, x_2, z)\,\phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,f_{\eta}'\left(F_{U;\theta}^{-1}(q) - g(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} G_V^q(\theta)\frac{\int \Psi(x_1, x_2, z)\,\phi_{\varepsilon}(q, x_1;\theta)\,f_{\varepsilon}'\left(F_{V;\theta}^{-1}(q) - h(x_2;\theta)\right)\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + \int_{\delta}^{1-\delta} G_V^q(\theta)\frac{\int \Psi(x_1, x_2, z)\,f_{\varepsilon}'\left(F_{V;\theta}^{-1}(q) - h(x_1;\theta)\right)\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1;\theta)\,\phi_{\varepsilon}(q, x_2;\theta)\,\phi_{\eta}(q, z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& + o(1). \qquad \text{(D.2.45)}
\end{aligned}$$

Finally, we rewrite

$$\begin{aligned}
& T_3(q) = \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}} \times \\
& \left(1-\frac{\int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}\right) \\
= \; & \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}} \times \\
& \left(\frac{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N} - \int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}{\int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}\right) \\
= \; & \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}} \times \left(-\tilde{T}_1(q)-\tilde{T}_2(q)\right) \\
= \; & \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}{\int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}} \times \left(-\tilde{T}_1(q)-\tilde{T}_2(q)\right) \\
& + \left(\frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}\right. \\
& \left. -\frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}{\int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}\right) \times \left(-\tilde{T}_1(q)-\tilde{T}_2(q)\right)
\end{aligned} \tag{D.2.46}$$

where $\tilde{T}_1(q) = T_1(q)$ and $\tilde{T}_2(q) = T_2(q)$ evaluated at $\Psi = 1$. Since $\sup_{\theta,q\in(\delta,1-\delta),N}\left|\frac{1}{h_N}\tilde{T}_1(q)\right|$ and $\sup_{\theta,q\in(\delta,1-\delta),N}\left|\frac{1}{h_N}\tilde{T}_2(q)\right|$ are finite, and

$$\begin{aligned}
\sup_{\theta,q\in(\delta,1-\delta),N} & \left|\frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}\right. \\
& \left. -\frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}{\int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}\right| \to 0,
\end{aligned}$$

we have that

$$\frac{1}{h_N}T_3(q) = \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}}{\int \phi_{\varepsilon}(q,x_1;\theta)\,\phi_{\varepsilon}(q,x_2;\theta)\,\phi_{\eta}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z}} \times \frac{1}{h_N}\left(-\tilde{T}_1(q)-\tilde{T}_2(q)\right) + o(1).$$

Equations (D.2.43) and (D.2.45), along with $\tilde{T}_1(q) = T_1(q)$ and $\tilde{T}_2(q) = T_2(q)$, imply that

$$\begin{aligned}
& \frac{1}{h_N}\tilde{T}_1(q) + \frac{1}{h_N}\tilde{T}_2(q) \to \\
& G_U^q(\theta)\frac{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\, f'_\eta\left(F_{U;\theta}^{-1}(q) - g(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& +G_V^q(\theta)\frac{\int \phi_\varepsilon(q,x_1;\theta)\, f'_\varepsilon\left(F_{V;\theta}^{-1}(q) - h(x_2;\theta)\right)\phi_\eta(q,z;\theta)\, d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& +G_V^q(\theta)\frac{\int f'_\varepsilon\left(F_{V;\theta}^{-1}(q) - h(x_1;\theta)\right)\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& +\frac{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, dG_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& +\frac{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_X dG_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
& +\frac{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} dG_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\
= \; & -L_G(\theta,q) \qquad \text{(D.2.47)}
\end{aligned}$$

uniformly in θ and $q \in (\delta, 1-\delta)$. Equations (D.2.46) and (D.2.47) imply that

$$\frac{1}{h_N}T_3 = \int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} L_G(\theta,q)\, dq + o(1) \qquad \text{(D.2.48)}$$

uniformly in θ.

Together, equations (D.2.43), (D.2.45), (D.2.47) and (D.2.48) imply that

$$\begin{aligned}
& \frac{1}{h_N}\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
& -\frac{1}{h_N}\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\, d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\, d\mu_{X_{N,1}} d\mu_{X_{N,2}} d\mu_{Z_N}} dq
\end{aligned}$$

converges to

$$
\begin{aligned}
Lim_{G;\delta}(\theta) &= \int_{\delta}^{1-\delta} G_U^q(\theta) \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\, f_\eta'\left(F_{U;\theta}^{-1}(q) - g(z;\theta)\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
&+ \int_{\delta}^{1-\delta} G_V^q(\theta) \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\, f_\varepsilon'\left(F_{V;\theta}^{-1}(q) - h(x_2;\theta)\right) \phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
&+ \int_{\delta}^{1-\delta} G_V^q(\theta) \frac{\int \Psi(x_1,x_2,z)\, f_\varepsilon'\left(F_{V;\theta}^{-1}(q) - h(x_1;\theta)\right) \phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
&+ \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, dG_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
&+ \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_X dG_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
&+ \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} dG_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\
&+ \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} L_G(\theta,q)\, dq, \qquad \text{(D.2.49)}
\end{aligned}
$$

where $L_G(\theta, q)$ is defined in equation (D.2.47). This expression is therefore the Hadamard derivative of interest. ■

Lemma D.2.12 *Suppose that f_ε is bounded away from zero on every compact interval, and $h(x;\theta)$ is uniformly μ_X−integrable over $\theta \in \Theta$, then for every $q \in (0,1)$, $\inf_{\theta\in\Theta} \int f_\varepsilon\left(F_{V;\theta}^{-1}(q) - h(x;\theta)\right) d\mu_X > 0$.*

Proof. First, we show that there exists $M < \infty$, such that $\inf_{\theta\in\Theta} F_{V;\theta}^{-1}(q) > -M$ and $\sup_{\theta\in\Theta} F_{V;\theta}^{-1}(q) < M$. To do so, it is enough to show that for any $\delta > 0$, there exists M such that $\sup_\theta \mathbb{P}(|h(x;\theta) + \varepsilon| > M) < \delta$. The triangle inequality implies that $\sup_\theta \mathbb{P}(|h(x;\theta) + \varepsilon| > M) \le \sup_{\theta\in\Theta} \mathbb{P}\left(|h(x;\theta)| > \frac{M}{2}\right) + \mathbb{P}\left(|\varepsilon| > \frac{M}{2}\right)$. For large enough M, the second term is less than $\frac{\delta}{2}$ by definition and the first term is less than $\frac{\delta}{2}$ since $h(x;\theta)$ is uniformly integrable.

Since for each q the map from θ to $F_{V;\theta}^{-1}(q)$ lives in a compact interval, $F_{V;\theta}^{-1}(q) - h(x;\theta)$ is a uniformly integrable family. Therefore, $\inf_{\theta\in\Theta} \int f_\varepsilon\left(F_{V;\theta}^{-1}(q) - h(x;\theta)\right) d\mu_X > 0$ since f_ε is bounded away from zero on any compact interval. ■

D.3 Proof of Proposition 5

Proof of Part (i): We need to show that $\sup_\theta |(\psi_N - \psi_N(\theta)) - (\psi - \psi(\theta))|$ converges in probability to zero. By the triangle inequality,

$$\sup_\theta |(\psi_N - \psi_N(\theta)) - (\psi - \psi(\theta))| \le |\psi_N - \psi| + \sup_\theta |\psi_N(\theta) - \psi(\theta)|.$$

Proposition 3(i) shows that $|\psi_N - \psi|$ converges in probability to 0. We now show that the second term also converges in probability to zero.

By definition of $\psi^\delta[m_X, m_Z]$,

$$\psi_N(\theta) - \psi(\theta) = \psi^0\left[\mu_{X_N}, \mu_{Z_N}\right](\theta) - \psi^0\left[\mu_X, \mu_Z\right](\theta).$$

Further, for any $\delta \in \left(0, \frac{1}{2}\right)$, we have that

$$\left|\psi^0\left[\mu_{X_N}, \mu_{Z_N}\right](\theta) - \psi^0\left[\mu_X, \mu_Z\right](\theta)\right| \leq \left|\psi^\delta\left[\mu_{X_N}, \mu_{Z_N}\right](\theta) - \psi^\delta\left[\mu_X, \mu_Z\right](\theta)\right| + 2\left\|\Psi\right\|_\infty \delta.$$

Proposition 4 implies that $\psi^\delta\left[\mu_X, \mu_Z\right] : L_\infty^\Gamma \to L_\infty^\Theta$ is uniformly continuous in μ_X, μ_Z. Since Γ_X is μ_X-Glivenko Cantelli, and Γ_Z is μ_Z-Glivenko Cantelli, $\sup_\theta \left|\psi^\delta\left[\mu_{X_N}, \mu_{Z_N}\right](\theta) - \psi^\delta\left[\mu_X, \mu_Z\right](\theta)\right|$ converges in probability to zero for any $\delta \in \left(0, \frac{1}{2}\right)$ by the continuous mapping theorem. Hence, $\sup_\theta \left|\psi_N(\theta) - \psi(\theta)\right|$ converges in probability to 0.

Proof of Part (ii): Consider the process

$$\begin{bmatrix} \sqrt{N}\left(\psi_N - \psi(\theta_0)\right) \\ \sqrt{N}\left(\mu_{X_N} - \mu_X\right) \\ \sqrt{N/2}\left(\mu_{Z_N} - \mu_Z\right) \end{bmatrix},$$

where $\sqrt{N}\left(\mu_{X_N} - \mu_X\right)$ is the empirical process indexed by Γ_X and $\sqrt{N/2}\left(\mu_{Z_N} - \mu_Z\right)$ is the empirical process indexed by Γ_Z. Proposition 3(ii) shows that this process converges weakly to the Gaussian process, $\tilde{G} = (G_\Psi, G_X, G_Z)$, which a mean zero Gaussian process with covariance kernel V.

By the functional delta method and the Hadamard derivative derived in Proposition 4, we have that

$$m_N^\delta(\theta) = \sqrt{N}\left(\psi_N - \psi(\theta_0)\right) - \sqrt{N}\left(\psi_N^\delta(\theta) - \psi^\delta(\theta)\right)$$

converges weakly to a mean zero Gaussian process

$$G_\Psi - \nabla_{(G_X, G_Z)}\psi^\delta\left[\mu_X, \mu_Z\right](\theta).$$

Therefore, there exists a sequence δ_N of positive numbers decreasing to 0 such that

$$d\left(m_N^{\delta_N}(\cdot), G_\Psi - \nabla_{(G_X, G_Z)}\psi^{\delta_N}\left[\mu_X, \mu_Z\right](\cdot)\right) \to 0,$$

where d is a metric for weak convergence, and (by Assumption 6(ii)c.)

$$\sup_{\|\theta - \theta_0\| \leq b_N}\left|\nabla_{(G_X, G_Z)}\psi^{\delta_N}\left[\mu_X, \mu_Z\right](\theta) - \nabla_{(G_X, G_Z)}\psi^{\delta_N}\left[\mu_X, \mu_Z\right](\theta_0)\right| = o_p(1).$$

In what follows, we fix such a sequence of δ_N.

We derive the limit distribution of $m_N^0(\theta_0) = \sqrt{N}\left(\psi_N - \psi_N(\theta_0)\right)$ to show Condition 1(ii) a. By the triangle inequality,

$$\begin{aligned} & d\left(m_N^0(\theta_0), G_\Psi - \nabla_{(G_X, G_Z)}\psi^0\left[\mu_X, \mu_Z\right](\theta_0)\right) \\ \leq \quad & d\left(m_N^0(\theta_0), m_N^{\delta_N}(\theta_0)\right) + d\left(m_N^{\delta_N}(\theta_0), G_\Psi - \nabla_{(G_X, G_Z)}\psi^{\delta_N}\left[\mu_X, \mu_Z\right](\theta_0)\right) \\ & + d\left(G_\Psi - \nabla_{(G_X, G_Z)}\psi^{\delta_N}\left[\mu_X, \mu_Z\right](\theta_0), G_\Psi - \nabla_{(G_X, G_Z)}\psi^0\left[\mu_X, \mu_Z\right](\theta_0)\right) \end{aligned}$$

The first term converges to zero as $N \to \infty$ by Assumption 6(ii)b. The second term converges to zero by the choice of δ_N. The third term goes to zero since

$$\left(G_\Psi - \nabla_{(G_X, G_Z)}\psi^{\delta_N}\left[\mu_X, \mu_Z\right](\theta_0)\right) - \left(G_\Psi - \nabla_{(G_X, G_Z)}\psi^0\left[\mu_X, \mu_Z\right](\theta_0)\right)$$

converges in probability and therefore in distribution to 0 (by Assumption 6(ii)d). Hence, $m_N^0(\theta_0)$ converges in distribution to $G_\Psi - \nabla_{(G_X,G_Z)}\psi^0[\mu_X,\mu_Z](\theta_0)$. Note that this limiting random variable is distributed $N\left(0,\lim_{\delta\to 0}V^\delta\right)$ where V^δ is the variance of $G_\Psi - \nabla_{(G_X,G_Z)}\psi^\delta[\mu_X,\mu_Z](\theta_0)$.

Now, we verify Condition 1(ii) b. By the triangle inequality, for any sequence $\{b_N\}$ of positive numbers converging to zero,

$$\sup_{\|\theta-\theta_0\|\le b_N}\left|m_N^0(\theta_0)-m_N^0(\theta)\right| \le \sup_{\|\theta-\theta_0\|\le b_N}\left|m_N^{\delta_N}(\theta_0)-m_N^{\delta_N}(\theta)\right| + 2\sup_{\|\theta-\theta_0\|\le b_N}\left|m_N^0(\theta)-m_N^{\delta_N}(\theta)\right|.$$

Note that, by the triangle inequality,

$$\begin{aligned} d\left(m_N^{\delta_N}(\theta_0),m_N^{\delta_N}(\theta)\right) &\le 2d\left(m_N^{\delta_N}(\cdot),G_\Psi-\nabla_{(G_X,G_Z)}\psi^{\delta_N}[\mu_X,\mu_Z](\cdot)\right) \\ &\quad + d\left(G_\Psi-\nabla_{(G_X,G_Z)}\psi^{\delta_N}[\mu_X,\mu_Z](\theta),G_\Psi-\nabla_{(G_X,G_Z)}\psi^{\delta_N}[\mu_X,\mu_Z](\theta_0)\right)\end{aligned}$$

converges to 0 since $\sup_{\|\theta-\theta_0\|\le b_N}\left|\nabla_{(G_X,G_Z)}\psi^{\delta_N}[\mu_X,\mu_Z](\theta)-\nabla_{(G_X,G_Z)}\psi^{\delta_N}[\mu_X,\mu_Z](\theta_0)\right| = o_p(1)$. Assumption 6(ii)b. implies that $2E\sup_{\|\theta-\theta_0\|\le b_N}\left|m_N^0(\theta)-m_N^{\delta_N}(\theta)\right|$ converges to zero as $N\to\infty$. Therefore,

$$2\sup_{\|\theta-\theta_0\|\le b_N}\left|m_N^0(\theta)-m_N^{\delta_N}(\theta)\right|$$

converges in probability to zero. Hence,

$$\begin{aligned} \sqrt{N}\left((\psi(\theta_0)-\psi_N(\theta_0))-(\psi(\theta)-\psi_N(\theta))\right) &= m_N^0(\theta_0)-m_N^0(\theta) \\ \Rightarrow \sup_{\|\theta-\theta_0\|\le b_N}\left|\sqrt{N}\left((\psi(\theta_0)-\psi_N(\theta_0))-(\psi(\theta)-\psi_N(\theta))\right)\right| &= \sup_{\|\theta-\theta_0\|\le b_N}\left|m_N^0(\theta_0)-m_N^0(\theta)\right| = o_p(1).\end{aligned}$$

E Auxiliary Results on Estimation

E.1 Primitive conditions for Assumption 4(i)

Assumption E.1.8 *(i)* $\Psi(x_1,x_2,z)$ *is bounded and symmetric in* x_1 *and* x_2

(ii) The quantities $\int\frac{\left|f_\varepsilon'(v-h(x;\theta_0))\right|}{\int f_\varepsilon(v-h(X;\theta_0))d\mu_X}d\mu_X$ *and* $\int\frac{\left|f_\eta'(u-g(z;\theta_0))\right|}{\int f_\eta(u-g(Z;\theta_0))d\mu_Z}d\mu_Z$ *are uniformly bounded in* v *and* u *respectively*

Lemma E.1.13 *If Assumption E.1.8 is satisfied, then* $\left\|\nabla\tilde{\psi}\right\|_\infty<\infty$. *Hence,* $\tilde{\psi}(v_1,v_2,u;\theta_0)$ *is Lipschitz continuous in* v_1, v_2 *and* u.

Proof. Note that

$$\begin{aligned} &\tilde{\psi}(v_1,v_2,u) \\ =& \int\Psi(X_1,X_2,Z)\,d\mu_{X|v_1}d\mu_{X|v_2}d\mu_{Z|u} \\ =& \int\Psi(X_1,X_2,Z)\;\tilde{f}_{v,x}(v_1,X_1)\;\tilde{f}_{v,x}(v_2,X_2)\;\tilde{f}_{u,z}(u,Z)\;d\mu_{X_1}d\mu_{X_2}d\mu_Z \\ &\text{where } \tilde{f}_{v,x}(v,x)=\frac{f_\varepsilon(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X} \text{ and } \tilde{f}_{u,z}(u,z)=\frac{f_\eta(u-g(z,\theta_0))}{\int f_\eta(u-g(Z,\theta_0))\,d\mu_Z}\end{aligned}$$

We will only show $\tilde{\psi}(v_1, v_2, u)$ has a bounded derivative with respect to v_1 as the proof for the other two arguments are identical. Note that

$$\begin{aligned}
&\frac{\partial}{\partial v}\frac{f_\varepsilon(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X} \\
=\;& \frac{f_\varepsilon'(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X} - \frac{\frac{f_\varepsilon'(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))d\mu_X}\int f_\varepsilon'(v-h(X;\theta_0))\,d\mu_X}{\left(\int f_\varepsilon(u-h(X;\theta_0))\,d\mu_X\right)^2}
\end{aligned} \tag{E.1.50}$$

If the expression in equation (E.1.50) is μ_X integrable in X, then the Dominated Convergence Theorem implies that the derivative $\frac{\partial}{\partial v_1}\tilde{\psi}(v_1, v_2, u)$ exists and is given by

$$\int \Psi(X_1, X_2, Z)\frac{\partial}{\partial v_1}\tilde{f}_{v,x}(v_1, X_1)\;\tilde{f}_{v,x}(v_2, X_2)\;\tilde{f}_{u,z}(u, Z)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z.$$

To proceed, we will show that

$$\begin{aligned}
\sup_v\left|\int\left(\frac{f_\varepsilon'(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X} - \frac{f_\varepsilon(v-h(x;\theta_0))\int f_\varepsilon'(v-h(X;\theta_0))\,d\mu_X}{\left(\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X\right)^2}\right)d\mu_X\right| &< \infty \\
\sup_u\left|\int\left(\frac{f_\eta'(u-g(z;\theta_0))}{\int f_\eta(u-g(Z,\theta_0))\,d\mu_Z} - \frac{f_\eta(u-g(z,\theta_0))\int f_\eta'(u-g(Z,\theta_0))\,d\mu_Z}{\left(\int f_\eta(u-g(Z,\theta_0))\,d\mu_Z\right)^2}\right)d\mu_Z\right| &< \infty
\end{aligned}$$

for the first expression since the proof of the other expression is identical. Note that

$$\begin{aligned}
& \sup_v\left|\int\left(\frac{f_\varepsilon'(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X} - \frac{f_\varepsilon(v-h(x;\theta_0))\int f_\varepsilon'(v-h(X;\theta_0))\,d\mu_X}{\left(\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X\right)^2}\right)d\mu_X\right| \\
\leq\;& \sup_v\int\left|\frac{f_\varepsilon'(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X} - \frac{f_\varepsilon(v-h(x;\theta_0))\int f_\varepsilon'(v-h(X;\theta_0))\,d\mu_X}{\left(\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X\right)^2}\right|d\mu_X \\
\leq\;& \sup_v\int\left|\frac{f_\varepsilon'(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X}\right| + \left|\frac{f_\varepsilon(v-h(x;\theta_0))\int f_\varepsilon'(v-h(X;\theta_0))\,d\mu_X}{\left(\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X\right)^2}\right|d\mu_X \\
\leq\;& \sup_v\int\frac{|f_\varepsilon'(v-h(x;\theta_0))|}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X}d\mu_X + \sup_v\int\frac{f_\varepsilon(v-h(x;\theta_0))\sup_v\int|f_\varepsilon'(v-h(X;\theta_0))|\,d\mu_X}{\left(\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X\right)^2}d\mu_X \\
\leq\;& \sup_v\int\frac{|f_\varepsilon'(v-h(x;\theta_0))|}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X}d\mu_X \\
& +\sup_v\int\frac{f_\varepsilon(v-h(x;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X}d\mu_X\sup_v\int\frac{|f_\varepsilon'(v-h(X;\theta_0))|}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X}d\mu_X \\
\leq\;& \sup_v\int\frac{|f_\varepsilon'(v-h(x;\theta_0))|}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X}d\mu_X\left(1+\sup_v\int\frac{f_\varepsilon(v-h(X;\theta_0))}{\int f_\varepsilon(v-h(X;\theta_0))\,d\mu_X}d\mu_X\right) < \infty
\end{aligned}$$

by Assumption E.1.8 (ii).

Since $\|\Psi\|_\infty < \infty$ (Assumption E.1.8 (i)) and

$$\begin{aligned}
&\int\tilde{f}_{v,x}(v_2, X_2)\,\tilde{f}_{u,z}(u, Z)\,d\mu_{X_1}d\mu_Z \\
=\;& \int\frac{f_\varepsilon(v_1-h(X_1;\theta_0))}{\int f_\varepsilon(v_1-h(X_1;\theta_0))\,d\mu_{X_1}}d\mu_{X_1}\int\frac{f_\eta(u-g(z,\theta_0))}{\int f_\eta(u-g(Z,\theta_0))\,d\mu_Z}d\mu_Z \leq 1,
\end{aligned}$$

we have that

$$
\begin{aligned}
& \frac{\partial}{\partial v_1}\tilde{\psi}\left(v_1,v_2,u\right) \\
\leq\quad & \left\|\Psi\right\|_\infty \left|\int\left(\frac{f_\varepsilon'\left(v_1-h\left(x;\theta_0\right)\right)}{\int f_\varepsilon\left(v_1-h\left(X;\theta_0\right)\right)d\mu_X}-\frac{f_\varepsilon\left(v_1-h\left(x;\theta_0\right)\right)\int f_\varepsilon'\left(v_1-h\left(X;\theta_0\right)\right)d\mu_X}{\left(\int f_\varepsilon\left(v_1-h\left(X;\theta_0\right)\right)d\mu_X\right)^2}\right)d\mu_X\right|\times \\
& \int\frac{f_\varepsilon\left(v_1-h\left(X_1;\theta_0\right)\right)}{\int f_\varepsilon\left(v_1-h\left(X_1;\theta_0\right)\right)d\mu_{X_1}}d\mu_{X_1}\int\frac{f_\eta\left(u-g\left(z,\theta_0\right)\right)}{\int f_\eta\left(u-g\left(Z,\theta_0\right)\right)d\mu_Z}d\mu_Z \\
<\quad & \infty.
\end{aligned}
$$

■

E.2 Primitive conditions for Assumption 6(ii)

For each x and z, define the Lipschitz constants $h_{LC}\left(x\right)=\sup_{\theta\in\Theta}\frac{\left|h(x;\theta)-h\left(x;\theta'\right)\right|}{\|\theta-\theta'\|}$, and $g_{LC}\ \left(z\right)=\sup_{\theta\in\Theta}\frac{\left|g(z;\theta)-g\left(z;\theta'\right)\right|}{\|\theta-\theta'\|}$.

Assumption E.2.9 *(i)* $\Psi\left(x_1,x_2,z\right)$ *indexed by* x_2 *and* z *is* μ_X*-Donsker and* $\Psi\left(x_1,x_2,z\right)$ *indexed by* x_1 *and* x_2 *is* μ_Z*-Donsker*

(ii) f_ε *and* f_η *are bounded away from zero on any compact interval of* $\mathbb{R}$*, and have continuous first derivatives*

(iii) there exist constants C_1*,* $C_2>0$ *such that*

$$\max\left\{f_\varepsilon\left(v\right),\ f_\eta\left(v\right),\ \left|f_\varepsilon'\left(v\right)\right|,\ \left|f_\eta'\left(v\right)\right|,\sup_{\theta\in\Theta}P\left(\left|h\left(x;\theta\right)\right|>v\right),\sup_{\theta\in\Theta}P\left(\left|g\left(z;\theta\right)\right|>v\right)\right\}\leq C_1\exp\left(-C_2\left|v\right|\right)$$

(iv) $\int h_{LC}\left(X\right)^4d\mu_X$*,* $\int g_{LC}\left(Z\right)^4d\mu_Z$*, and* $\left\|\nabla\tilde{\psi}_q\right\|_\infty$ *are finite*

(v) $\Psi\left(x_1,x_2,z\right)=\sum_{k=1}^K a_k\Psi_1^k\left(x_1\right)\Psi_2^k\left(x_2\right)\Psi_z\left(z\right)$ *with* $\left\|\Psi_\cdot^k\right\|_\infty<\infty$ *for some constants* $a_1,\ldots,a_K$

(vi) $\left\|f_\varepsilon''\right\|_\infty$*,* $\int_{-\infty}^{\infty}\left|f_\varepsilon''\left(v\right)\right|dv$*,* $\left\|f_\eta''\right\|_\infty$*, and* $\int_{-\infty}^{\infty}\left|f_\eta''\left(v\right)\right|dv$ *are finite*

(vii) ε *and* η *have full support on* $\mathbb{R}$

Theorem E.2.5 *If Assumption E.2.9 is satisfied, then Assumption 6(ii) is satisfied.*

Proof. Assumption 6(ii) a. is verified by Proposition E.3.7.

Assumption 6(ii) b. is verified by Proposition E.4.8.

Assumption 6(ii) c. is verified by Proposition E.5.9.

Assumption 6(ii) d. is verified by Proposition E.6.10. ■

E.3 Donsker Properties for Γ_X and Γ_Z

For each x, define the Lipschitz constant $h_{LC}\left(x\right)=\sup_{\theta\in\Theta}\frac{\left|h(x;\theta)-h\left(x;\theta'\right)\right|}{\|\theta-\theta'\|}$.

Claim E.3.1 *Suppose*

1. $\left(\int h_{LC}\left(x\right)^2d\mu_X\right)^{1/2}$, $\left\|f_\varepsilon\right\|_\infty$ and $\left\|\Psi\right\|_\infty$ are finite
2. $\Psi\left(x_1,x_2,z\right)$ indexed by x_2 and z is μ_X-Donsker

Then, we have that

1. $F_{\varepsilon}(c-h(x;\theta))$ indexed by c and θ is a μ_X-Donsker class.
2. If $\int_{-\infty}^{\infty}|f_{\varepsilon}'(v)|\,dv<\infty$, then $f_{\varepsilon}(c-h(x;\theta))$ indexed by c and θ is a μ_X-Donsker class
3. If $\int_{-\infty}^{\infty}|f_{\varepsilon}''(v)|\,dv<\infty$, then $f_{\varepsilon}'(c-h(x;\theta))$ indexed by c and θ is a μ_X-Donsker class.

Proof. We only spell out the argument for the second statement since the other two are analogous, as $\int_{-\infty}^{\infty}|f_{\varepsilon}(v)|\,dv=1$ by definition. Consider the class

$$f_{\varepsilon}(c-h(x;\theta))$$

indexed by $c\in\mathbb{R}$ and $\theta\in\Theta$. We will show that this class is Donsker by bounding its L_2-bracketing number.

Fix a partition $-\infty=c_0<c_1<c_2<\ldots<c_N=\infty$. Lets compute

$$\begin{aligned}
& \sup_{\theta\in\Theta}\int\left[f_{\varepsilon}(c_n-h(x;\theta))-f_{\varepsilon}(c_{n+1}-h(x;\theta))\right]^2 d\mu_X \\
\leq\;& 2\|f_{\varepsilon}\|_{\infty}\sup_{\theta\in\Theta}\int\left|f_{\varepsilon}(c_n-h(x;\theta))-f_{\varepsilon}(c_{n+1}-h(x;\theta))\right| d\mu_X \\
\leq\;& 2\|f_{\varepsilon}\|_{\infty}\sup_{\theta\in\Theta}\int\int_{c_n}^{c_{n+1}}\left|f_{\varepsilon}'(c-h(x;\theta))\right| dc\,d\mu_X
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_n\sup_{\theta\in\Theta}\int\left[f_{\varepsilon}(c_n-h(x;\theta))-f_{\varepsilon}(c_{n+1}-h(x;\theta))\right]^2 d\mu_X \\
\leq\;& 2\|f_{\varepsilon}\|_{\infty}\sup_{\theta\in\Theta}\int\sum_n\int_{c_n}^{c_{n+1}}\left|f_{\varepsilon}'(c-h(x;\theta))\right| dc\,d\mu_X \\
=\;& 2\|f_{\varepsilon}\|_{\infty}\sup_{\theta\in\Theta}\int\int_{-\infty}^{\infty}\left|f_{\varepsilon}'(c-h(x;\theta))\right| dc\,d\mu_X \\
=\;& 2\|f_{\varepsilon}\|_{\infty}\int_{-\infty}^{\infty}\left|f_{\varepsilon}'(c)\right| dc \\
=\;& \tilde{K}<\infty
\end{aligned} \tag{E.3.51}$$

where $\tilde{K}$ does not depend on the choice of $c_0<c_1<c_2<\ldots<c_N$. Now, consider the function

$$\tilde{f}(a)=2\|f_{\varepsilon}\|_{\infty}\int_{-\infty}^{a}\left|f_{\varepsilon}'(c)\right| dc.$$

Note that $\tilde{f}(a)$ is continuous, non-decreasing and has image $\left[0,\tilde{f}(\infty)\right]$. For any N and $n\in\{0,\ldots,N\}$ define

$$c_i=\tilde{f}^{-1}\left(\frac{n}{N}\right).$$

Then, for each n inequality in equation (E.3.51),

$$\sup_{\theta\in\Theta}\int\left[f_{\varepsilon}(c_n-h(x;\theta))-f_{\varepsilon}(c_{n+1}-h(x;\theta))\right]^2 d\mu_X\leq\frac{\tilde{K}}{N}.$$

Consider an $1/\sqrt{N}$-net $\Theta \subseteq \mathbb{R}^d$, Θ_i for $i \in \{1,\ldots,D\}$. Note that $D = \left(\sqrt{N}\mathrm{diam}\left(\Theta\right)\right)^d$. For each Θ_i and each n, define the bracket

$$\left[\inf_{\theta\in\Theta_i}\inf_{c\in[c_n,c_{n+1}]} f_\varepsilon\left(c-h\left(x;\theta\right)\right), \sup_{\theta\in\Theta_i}\sup_{c\in[c_n,c_{n+1}]} f_\varepsilon\left(c-h\left(x;\theta\right)\right)\right].$$

The volume of these brackets are

$$\begin{aligned}
&\left(\int\left[\sup_{\theta\in\Theta_i}\sup_{c\in[c_n,c_{n+1}]} f_\varepsilon\left(c-h\left(x;\theta\right)\right)-\inf_{\theta\in\Theta_i}\inf_{c\in[c_n,c_{n+1}]} f_\varepsilon\left(c-h\left(x;\theta\right)\right)\right]^2 d\mu_X\right)^{1/2}\\
=\;&\left(\int\left[f_\varepsilon\left(c^+-h\left(x;\theta^+\right)\right)-f_\varepsilon\left(c^--h\left(x;\theta^-\right)\right)\right]^2 d\mu_X\right)^{1/2}\\
\leq\;&\left(\int\left[f_\varepsilon\left(c^+-h\left(x;\theta^+\right)\right)-f_\varepsilon\left(c^--h\left(x;\theta^+\right)\right)\right]^2 d\mu_X\right)^{1/2}\\
&+\left(\int\left[f_\varepsilon\left(c^--h\left(x;\theta^+\right)\right)-f_\varepsilon\left(c^--h\left(x;\theta^-\right)\right)\right]^2 d\mu_X\right)^{1/2}\\
\leq\;&\left(\frac{\tilde{K}}{N}\right)^{1/2}+\|f_\varepsilon'\|_\infty\left(\int h_{LC}\left(x\right)^2 d\mu_X\right)^{1/2}\sup_{\theta,\theta'\in\Theta_i}\left\|\theta-\theta'\right\|\\
=\;&\left(\frac{\tilde{K}}{N}\right)^{1/2}+\frac{\|f_\varepsilon'\|_\infty}{\sqrt{N}}\left(\int h_{LC}\left(x\right)^2 d\mu_X\right)^{1/2}=KN^{-1/2}.
\end{aligned}$$

Therefore, the ε-bracketing number is bounded by a polynomial in $1/\varepsilon$. Therefore, $\int_0^\infty\sqrt{\log\mathcal{N}\left(\varepsilon\right)}d\varepsilon$ is finite, where $\mathcal{N}\left(\varepsilon\right)$ be the ε bracketing number of this class. By van der Vaart (2000) Theorem 2.5.6, it follows that $f_\varepsilon\left(c-h\left(x;\theta\right)\right)$ indexed by $c\in\mathbb{R}$ and $\theta\in\Theta$ is a μ_X-Donsker class. ■

Proposition E.3.7 *Suppose that the conditions for Claim E.3.1 hold and* $\|f_\eta\|_\infty$*, then* Γ_X *is a* μ_X-*Donsker class. Analogous conditions imply that* Γ_Z *is a* μ_Z-*Donsker class.*

Proof. We only need to show that the terms

$$\Psi\left(x_1,x_2,z\right) f_\varepsilon\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x_1;\theta\right)\right) f_\varepsilon\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x_2;\theta\right)\right) f_\eta\left(F_{U;\theta}^{-1}\left(q\right)-g\left(z;\theta\right)\right)$$

and

$$\Psi\left(x_1,x_2,z\right) f_\varepsilon'\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x_1;\theta\right)\right) f_\varepsilon\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x_2;\theta\right)\right) f_\eta\left(F_{U;\theta}^{-1}\left(q\right)-g\left(z;\theta\right)\right)$$

indexed by (x_1,z,q,θ) are μ_X-Donsker classes. This is because the terms $1\left\{c_1\leq x\leq c_2\right\}$ are μ_X-Donsker since they are intersections of half-spaces, and therefore suitably measurable VC-classes. The remaining terms are μ_X-Donsker by Claim E.3.1.

Note that $f_\varepsilon\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x_2;\theta\right)\right)$ indexed by (q,θ) is a sub-class of the μ_X-Donsker class $f_\varepsilon\left(c-h\left(x_2;\theta\right)\right)$ indexed by (c,θ), and is therefore μ_X-Donsker. Further, the quantities

$$\Psi\left(x_1,x_2,z\right) f_\varepsilon\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x_1;\theta\right)\right) f_\eta\left(F_{U;\theta}^{-1}\left(q\right)-g\left(z;\theta\right)\right)$$

are uniformly bounded and measurable since $\|\Psi\|_{\infty}$, $\|f_{\varepsilon}\|_{\infty}$ and $\|f_{\eta}\|_{\infty}$ are finite. Since the product of two bounded Donsker classes is Donsker (van der Vaart (2000), example 2.10.8), we have that

$$\Psi(x_1, x_2, z) f_{\varepsilon}\left(F_{V;\theta}^{-1}(q) - h(x_1; \theta)\right) f_{\varepsilon}\left(F_{V;\theta}^{-1}(q) - h(x_2; \theta)\right) f_{\eta}\left(F_{U;\theta}^{-1}(q) - g(z; \theta)\right)$$

and

$$\Psi(x_1, x_2, z) f_{\varepsilon}'\left(F_{V;\theta}^{-1}(q) - h(x_1; \theta)\right) f_{\varepsilon}\left(F_{V;\theta}^{-1}(q) - h(x_2; \theta)\right) f_{\eta}\left(F_{U;\theta}^{-1}(q) - g(z; \theta)\right)$$

indexed by (x_2, z, q, θ) are μ_X-Donsker classes. ■

E.4 Primitive Conditions for Assumption 6(ii) b.

Our result verifying Assumption 6(ii) b. is stated in Proposition E.4.8 below. The main technical difficulty is solved in the following lemma. This result requires preliminaries proved below in Appendix E.4.1.

For each x and z, define the Lipschitz constants $h_{LC}(x) = \sup_{\theta \in \Theta} \frac{\left|h(x;\theta) - h\left(x;\theta'\right)\right|}{\|\theta - \theta'\|}$, and $g_{LC}(z) = \sup_{\theta \in \Theta} \frac{\left|g(z;\theta) - g\left(z;\theta'\right)\right|}{\|\theta - \theta'\|}$.

Lemma E.4.14 *Suppose that $\int h_{LC}(X)^4 d\mu_X$ is finite, and there exist constants C_1, $C_2 > 0$ such that*

$$\max\left\{\left|f_{\varepsilon}'(v)\right|, \sup_{\theta \in \Theta} P\left(|h(x;\theta)| > v\right)\right\} \leq C_1 \exp\left(-C_2 |v|\right).$$

Then, for any function $\Psi(x)$ with $\|\Psi\|_{\infty} < \infty$, we have that (i)

$$E \sup_{\theta} \int \left| \sqrt{N}\left(\mu_{X_N} - \mu_X\right)\left(\Psi(X) f_{\varepsilon}(v - h(X;\theta))\right)\right| dv$$

is bounded and

(ii) for any sequence of positive numbers $\{r_N\}$ which decrease to 0 as $N \to \infty$,

$$E \sup_{\|\theta_1 - \theta_2\| \leq r_N} \int \left| \sqrt{N}\left(\mu_{X_N} - \mu_X\right)\left(\Psi(X)\left[f_{\varepsilon}(v - h(X;\theta_1)) - f_{\varepsilon}(v - h(X;\theta_2))\right]\right)\right| dv \to 0.$$

Proof. The argument combines ideas from Pollard (2002) recursive proof of Ossiander's bracketing functional central limit theorem and an application of Boucheron et al. (2003) (Theorem 2) concentration inequality.

Let D be the diameter of the parameter space Θ and for nonnegative integer i, let $\delta_i = D2^{-i}$. Fix a natural number i^*. Fix a δ_{i^*} net of Θ of size $N(\delta_{i^*})$ and for each $\theta \in \Theta$ let $B(\theta; i^*)$ be the center of a ball in this δ_{i^*} net which contains θ. For any nonnegative integer $i < i^*$, fix a δ_i net of Θ of size $N(\delta_i)$ and recursively define $B(\theta; i)$ to be the center of a ball in this δ_i net which contains $B(\theta; i+1)$. Note that this definition implies $d(\theta; B(\theta; i^*)) \leq \delta_{i^*}$, $d(B(\theta, i), B(\theta; i+1)) \leq \delta_i$, and that $B(\theta; i)$ takes on at most $N(\delta_i)$ distinct values. By repeated application of the triangle inequality, $d(B(\theta; i), \theta) \leq 2\delta_i$ for all θ. Let $C_{\varepsilon} = \int_{-\infty}^{\infty} |f_{\varepsilon}'(v)| dv$. Note that $C_{\varepsilon} < \infty$ by our exponential tail bound on f_{ε}'.

For each $i \leq i^*$, let $V_i = \sqrt{N} \frac{\delta_i}{\sqrt{\log N(\delta_i)}}$, let

$$R_i(\theta, v) = \sqrt{N}\left(\mu_{X_N} - \mu_X\right)\Psi(X)\left[f_{\varepsilon}(v - h(X;\theta)) - f_{\varepsilon}(v - h(X; B(\theta; i)))\right]$$

and $T_i(\theta) = \left\{x : h_{LC}(x) \leq \frac{V_i}{2\delta_i}\right\}$.

To prove part (i), we separately bound $E \sup_{\theta} \int |R_0(\theta, v) T_0(\theta)| dv$ and $E \sup_{\theta} \int |R_0(\theta, v) T_0^c(\theta)| dv$. To prove part (ii), we must similarly show that $E \sup_{\theta} \int |R_i(\theta, v) T_i(\theta)| dv$ and $E \sup_{\theta} \int |R_i(\theta, v) T_i^c(\theta)| dv$ go to 0 as $i \to \infty$.

As noted by Pollard (2002),

$$R_i T_i = R_{i+1} T_{i+1} - R_{i+1} T_i^c T_{i+1} + (R_i - R_{i+1}) T_i T_{i+1} + R_i T_i T_{i+1}^c.$$

It follows that

$$\begin{aligned}
& E \sup_\theta \int |R_i(\theta, v) T_i(\theta)| \, dv \\
\leq \quad & E \sup_\theta \int |R_{i+1}(\theta, v)| T_{i+1}(\theta) \, dv + E \sup_\theta \int |R_{i+1}(\theta, v)| T_i^c(\theta) T_{i+1}(\theta) \, dv && \text{(E.4.52)} \\
& + E \sup_\theta \int |(R_i(\theta, v) - R_{i+1}(\theta, v))| T_i(\theta) T_{i+1}(\theta) \, dv \\
& + E \sup_\theta \int |(R_i(\theta, v))| T_i(\theta) T_{i+1}^c(\theta) \, dv \\
\Rightarrow \quad & E \sup_\theta \int |R_0(\theta, v) T_0(\theta)| \, dv \\
\leq \quad & E \sup_\theta \int |R_{i^*}(\theta, v) T_{i^*}(\theta)| \, dv && \text{(E.4.53)} \\
& + \sum_{i=0}^{i^*-1} \left\{ E \sup_\theta \int |R_{i+1}(\theta, v)| T_i^c(\theta) T_{i+1}(\theta) \, dv \right. \\
& + E \sup_\theta \int |R_i(\theta, v)| T_i(\theta) T_{i+1}^c(\theta) \, dv \\
& \left. + E \sup_\theta \int |(R_i(\theta, v) - R_{i+1}(\theta, v))| T_i(\theta) T_{i+1}(\theta) \, dv \right\}. && \text{(E.4.54)}
\end{aligned}$$

We need to show that each of the terms above is bounded. First, we show that summation is bounded. Lemmas E.4.15 and E.4.16 (below) imply that there exists a constant K such that each of the terms in the summation is no greater than $K\sqrt{\log N(\delta_i)}\delta_i$. Therefore, equation (E.4.54) implies

$$E \sup_\theta \int |R_i(\theta, v) T_i(\theta)| \, dv \leq E \sup_\theta \int |R_{i^*}(\theta, v) T_{i^*}(\theta)| \, dv + K \sum_{j=i}^{i^*-1} \sqrt{\log N(\delta_j)}\delta_j. \qquad \text{(E.4.55)}$$

We now show that as $i^* \to \infty$,

$$E \sup_\theta \int |R_{i^*}(\theta, v) T_{i^*}(\theta)| \, dv \to 0.$$

For any i, we have that,

$$\begin{aligned}
& E\sup_{\theta}\int\left|R_i\left(\theta,v\right)T_i\left(\theta\right)\right|dv \\
= \;& E\sup_{\theta}\int\left|\sqrt{N}\left(\mu_{X_N}-\mu_X\right)\Psi\left(X\right)\left[f_\varepsilon\left(v-h\left(X;\theta\right)\right)-f_\varepsilon\left(v-h\left(X;B\left(\theta;i\right)\right)\right)\right]\right|dv \\
\leq \;& \sqrt{N}E\sup_{\theta}\int\left(\mu_{X_N}\left\|\Psi\right\|_\infty\left|f_\varepsilon\left(v-h\left(X;\theta\right)\right)-f_\varepsilon\left(v-h\left(X;B\left(\theta;i\right)\right)\right)\right|\right. \\
& \left.+\mu_X\left\|\Psi\right\|_\infty\left|f_\varepsilon\left(v-h\left(X;\theta\right)\right)-f_\varepsilon\left(v-h\left(X;B\left(\theta;i\right)\right)\right)\right|\right)dv \\
\leq \;& \sqrt{N}\left\|\Psi\right\|_\infty E\sup_{\theta}\frac{1}{N}\sum_{j=1}^{N}\int\left(\left|f_\varepsilon\left(v-h\left(X_j;\theta\right)\right)-f_\varepsilon\left(v-h\left(X_j;B\left(\theta;i\right)\right)\right)\right|\right. \\
& \left.+\mu_X\left\|\Psi\right\|_\infty\left|f_\varepsilon\left(v-h\left(X;\theta\right)\right)-f_\varepsilon\left(v-h\left(X;B\left(\theta;i\right)\right)\right)\right|\right)dv \\
\leq \;& \sqrt{N}\left\|\Psi\right\|_\infty E\sup_{\theta}\frac{1}{N}\sum_{j=1}^{N}C_\varepsilon\left(\left|h\left(X_j;\theta\right)-h\left(X_j;B\left(\theta;i\right)\right)\right|+\mu_X\left|h\left(X;\theta\right)-h\left(X;B\left(\theta;i\right)\right)\right|\right) \\
\leq \;& \sqrt{N}\left\|\Psi\right\|_\infty E\left(\frac{1}{N}\sum_{j=1}^{N}C_\varepsilon\left(2\delta_i\left|h_{LC}\left(X_j\right)\right|+\mu_X 2\delta_i\left|h_{LC}\left(X\right)\right|\right)\right) \\
= \;& 4\delta_i\left\|\Psi\right\|_\infty C_\varepsilon\sqrt{N}\mu_X h_{LC}\left(X\right).
\end{aligned} \tag{E.4.56}$$

Hence, equation (E.4.55) implies that for any $i^* > i$,

$$E\sup_{\theta}\int\left|R_i\left(\theta,v\right)T_i\left(\theta\right)\right|dv\leq 4\delta_{i^*}\left\|\Psi\right\|_\infty C_\varepsilon\sqrt{N}\mu_X h_{LC}\left(X\right)+K\sum_{j=i}^{\infty}\delta_j\sqrt{\log N\left(\delta_j\right)}.$$

Therefore, for a universal constant K',

$$\begin{aligned}
E\sup_{\theta}\int\left|R_i\left(\theta,v\right)T_i\left(\theta\right)\right|dv &\leq K'\int_0^{\delta_i}\sqrt{\log N\left(\delta\right)}d\delta \\
\text{and } E\sup_{\theta}\int\left|R_0\left(\theta,v\right)T_0\left(\theta\right)\right|dv &\leq K'\int_0^{\infty}\sqrt{\log N\left(\delta\right)}d\delta<\infty.
\end{aligned}$$

Note that

$$\begin{aligned}
& E\sup_{\theta}\int\left|R_i\left(\theta,v\right)T_i^c\left(\theta\right)\right|dv \\
= \;& E\sup_{\theta}\int_{-\infty}^{\infty}\left|\sqrt{N}\left(\mu_{X_N}-\mu_X\right)\left[f_\varepsilon\left(v-h\left(X;\theta\right)\right)-f_\varepsilon\left(v-h\left(X;B\left(\theta;i\right)\right)\right)\right]\left\{X:h_{LC}\left(X\right)>\frac{V_i}{2\delta_i}\right\}\right|dv \\
\leq \;& E\sup_{\theta}\int_{-\infty}^{\infty}\left|\sqrt{N}\left(\mu_{X_N}-\mu_X\right)\left[f_\varepsilon\left(v-h\left(X;\theta\right)\right)-f_\varepsilon\left(v-h\left(X;B\left(\theta;i\right)\right)\right)\right]\left\{X:h_{LC}\left(X\right)>\frac{\sqrt{N}}{2\sqrt{\log N\left(\delta_i\right)}}\right\}\right|dv \\
\leq \;& 2\delta_i E\sup_{\theta}\sqrt{N}\left(\mu_{X_N}+\mu_X\right)h_{LC}\left(X\right)\left\{X:h_{LC}\left(X\right)>\frac{\sqrt{N}}{2\sqrt{\log N\left(\delta_i\right)}}\right\}\int_{-\infty}^{\infty}\left|f_\varepsilon'\left(v\right)\right|dv \\
\leq \;& 4\delta_i\sqrt{N}\int_{-\infty}^{\infty}\left|f_\varepsilon'\left(v\right)\right|dv\mu_X h_{LC}\left(X\right)^4\left[\frac{2\sqrt{\log N\left(\delta_i\right)}}{\sqrt{N}}\right]^3 \\
= \;& 32\frac{1}{N}\int_{-\infty}^{\infty}\left|f_\varepsilon'\left(v\right)\right|dv\mu_X h_{LC}\left(X\right)^4\delta_i\left(\log N\left(\delta_i\right)\right)^{\frac{3}{2}}
\end{aligned}$$

Since $N(\delta_i)$ is not greater than some polynomial in $\frac{1}{\delta_i}$, $\sup_i \delta_i (\log N(\delta_i))^{\frac{3}{2}} < \infty$, we have that

$$\begin{aligned} & \sup_N E \sup_\theta \int |R_0(\theta,v)|\,dv \\ \leq\ & \sup_N E \sup_\theta \int |R_0(\theta,v) T_0(\theta)|\,dv + \sup_N E \sup_\theta \int |R_0(\theta,v) T_0^c(\theta)|\,dv \\ \leq\ & K' \int_0^\infty \sqrt{\log N(\delta)} d\delta + 32 \frac{1}{N} \int_{-\infty}^\infty |f_\varepsilon'(v)|\,dv \mu_X h_{LC}(X)^4 \delta_0 (\log N(\delta_0))^{\frac{3}{2}} \\ <\ & \infty. \end{aligned}$$

This completes the proof for Part (i). Similarly, for any sequence of $i_N \to \infty$, as $N \to \infty$,

$$\begin{aligned} & E \sup_\theta \int |R_{i_N}(\theta,v)|\,dv \\ \leq\ & E \sup_\theta \int |R_{i_N}(\theta,v) T_{i_N}(\theta)|\,dv + E \sup_\theta \int \left|R_{i_N}(\theta,v) T_{i_N}^c(\theta)\right| dv \to 0 \end{aligned}$$

■

We are now ready to show the main result:

Proposition E.4.8 *If the following assumptions are satisfied*

(i) Γ_X *and* Γ_Z *are respectively* μ_X*- and* μ_Z*- Donsker*

(ii) f_ε *and* f_η *are bounded away from zero on any compact interval of* $\mathbb{R}$*, and have continuous first derivatives*

(iii) there exist constants C_1*,* $C_2 > 0$ *such that*

$$\max\left\{ f_\varepsilon(v),\ f_\eta(v),\ |f_\varepsilon'(v)|,\ \left|f_\eta'(v)\right|, \sup_{\theta\in\Theta} P(|h(x;\theta)| > v), \sup_{\theta\in\Theta} P(|g(z;\theta)| > v) \right\} \leq C_1 \exp(-C_2 |v|)$$

(iv) $\int h_{LC}(X)^4 d\mu_X$*,* $\int g_{LC}(Z)^4 d\mu_Z$*, and* $\left\|\nabla\tilde{\psi}_q\right\|_\infty$ *are finite*

(v) $\Psi(x_1,x_2,z) = \sum_{k=1}^K a_k \Psi_1^k(x_1) \Psi_2^k(x_2) \Psi_z(z)$ *with* $\left\|\Psi_\cdot^k\right\|_\infty < \infty$ *for some constants* $a_1,\ldots,a_K$

then for any sequence of positive δ_N *and* r_N *decreasing to 0*

$$\sqrt{N} E \sup_{\|\theta-\theta_0\|\leq r_N} \left| \left(\psi[\mu_X,\mu_Z](\theta) - \psi\left[\mu_{X_N},\mu_{Z_N}\right](\theta)\right) - \left(\psi^{\delta_N}[\mu_X,\mu_Z](\theta) - \psi^{\delta_N}\left[\mu_{X_N},\mu_{Z_N}\right](\theta)\right)\right| = o(1)$$

as $N \to \infty$*.*

Proof. The proof proceeds by first manipulating this expression into a sum of similar terms which can all be handed by Lemma E.4.14. To ease notation, define

$$\begin{aligned} \phi_\eta(q,z;\theta) &= f_\eta\left(F_{U;\theta}^{-1}(q) - g(z;\theta)\right) \\ \phi_{\eta,N}(q,z;\theta) &= f_\eta\left(F_{N,U;\theta}^{-1}(q) - g(z;\theta)\right) \\ \phi_\varepsilon(q,x;\theta) &= f_\varepsilon\left(F_{V;\theta}^{-1}(q) - h(x;\theta)\right) \\ \phi_{\varepsilon,N}(q,x;\theta) &= f_\varepsilon\left(F_{N,V;\theta}^{-1}(q) - h(x;\theta)\right). \end{aligned}$$

First, note that

$$\begin{aligned}
& \left(\psi\left[\mu_X,\mu_Z\right](\theta)-\psi\left[\mu_{X_N},\mu_{Z_N}\right](\theta)\right)-\left(\psi^{\delta}\left[\mu_X,\mu_Z\right](\theta)-\psi^{\delta}\left[\mu_{X_N},\mu_{Z_N}\right](\theta)\right)\\
= & \left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\\
& -\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}dq\\
= & \left(\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right.\\
& \left.-\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right)\\
& +\left(\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right.\\
& \left.-\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}dq\right)\\
= & A_1+A_2
\end{aligned}$$

First, we bound the absolute value of A_1. Since $q_{N,V;\theta}(q)=F_{V;\theta}\left(F^{-1}_{N,V;\theta}(q)\right)$ and $\phi_\varepsilon\left(q_{N,V;\theta}(q),x;\theta\right)=\phi_{\varepsilon,N}\left(q,x;\theta\right)$, we have that

$$\begin{aligned}
A_1 = & \left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\\
& -\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_{\varepsilon,N}\left(q,x_1;\theta\right)\phi_{\varepsilon,N}\left(q,x_2;\theta\right)\phi_{\eta,N}\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\\
= & \left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon\left(q,x_1;\theta\right)\phi_\varepsilon\left(q,x_2;\theta\right)\phi_\eta\left(q,z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\\
& -\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi\left(x_1,x_2,z\right)\phi_\varepsilon\left(q_{N,V;\theta}(q),x_1;\theta\right)\phi_\varepsilon\left(q_{N,V;\theta}(q),x_2;\theta\right)\phi_\eta\left(q_{N,U;\theta}(q),z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon\left(q_{N,V;\theta}(q),x_1;\theta\right)\phi_\varepsilon\left(q_{N,V;\theta}(q),x_2;\theta\right)\phi_\eta\left(q_{N,U;\theta}(q),z;\theta\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq.
\end{aligned}$$

By a first order Taylor expansion,

$$\begin{aligned}
\sqrt{N}E\sup_\theta\left|A_1\right| & \leq \sqrt{N}\left[\int_0^1-\int_\delta^{1-\delta}\right]\left\|\nabla\tilde{\psi}_q\right\|_\infty\left(2E\sup_\theta\left|q_{N,V;\theta}(q)-q\right|+E\sup_\theta\left|q_{N,U;\theta}(q)-q\right|\right)dq\\
& \leq 4\delta\left\|\nabla\tilde{\psi}_q\right\|_\infty\left(E\sup_\theta\left|q_{N,V;\theta}(q)-q\right|+E\sup_\theta\left|q_{N,U;\theta}(q)-q\right|\right).
\end{aligned}$$

Since $\left\|\nabla\tilde{\psi}_q\right\|_\infty<\infty$, we only need to show that $\sqrt{N}E\sup_{q,\theta}\left|q_{N,V;\theta}(q)-q\right|$ and $\sqrt{N}E\sup_{q,\theta}\left|q_{N,U;\theta}(q)-q\right|$ are finite. Note that

$$
\begin{aligned}
q_{N,V;\theta}(q)-q &= F_{V;\theta}\left(F_{N,V;\theta}^{-1}(q)\right)-F_{N,V;\theta}\left(F_{N,V;\theta}^{-1}(q)\right)\\
&= \left(\mu_X-\mu_{X_N}\right)\left(F_\varepsilon\left(F_{N,V;\theta}^{-1}(q)-h(X;\theta)\right)\right)\\
\Rightarrow \sqrt{N}E\sup_{q,\theta}\left|q_{N,V;\theta}(q)-q\right| &\leq \sqrt{N}E\sup_{v,\theta}\left|\left(\mu_X-\mu_{X_N}\right)\left(F_\varepsilon\left(v-h(X;\theta)\right)\right)\right|\\
&< \infty
\end{aligned}
$$

since $F_\varepsilon(v-h(X;\theta))$ indexed by v and θ is μ_X-Donsker. An identical argument implies that $\sqrt{N}E\sup_{q,\theta}\left|q_{N,U;\theta}(q)-q\right|$ is finite.

To bound the absolute value of A_2, let

$$
\rho_{\eta,N;\theta}(v,z)=f_\eta\left(F_{N,U;\theta}^{-1}\left(F_{N,V;\theta}(v)\right)-g(z;\theta)\right)
$$

$$
\begin{aligned}
& A_2\\
=\; & \left(\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}\,dq\right.\\
& \left.-\left[\int_0^1-\int_\delta^{1-\delta}\right]\frac{\int\Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}\,dq\right)\\
=\; & \left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}\int f_\varepsilon(v-h(x_1;\theta))\,d\mu_{X_N}\,dv\\
& -\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int f_\varepsilon(v-h(x_1;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_{N,1}}d\mu_{Z_N}}\,dv\\
=\; & \left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}\int f_\varepsilon(v-h(x_1;\theta))\left(d\mu_{X_N}-d\mu_X\right)dv\\
& +\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon(v-h(x_1;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_1}d\mu_Z}\frac{\int f_\varepsilon(v-h(x_1;\theta))\,d\mu_X}{\int f_\varepsilon(v-h(x_1;\theta))\,d\mu_X}\,dv\\
& -\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-u_\theta(x_1))\,f_\varepsilon(v-h(x_2;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int f_\varepsilon(v-h(x_1;\theta))\,\rho_{\eta,N;\theta}(v,z)\,d\mu_{X_{N,1}}d\mu_{Z_N}}\,dv\\
=\; & T_1+T_2-T_3
\end{aligned}
$$

where the first equality follows from the change of variable $v=F_{N,V;\theta}^{-1}(q)$.

Note that

$$
\begin{aligned}
\sqrt{N}\left|T_1\right| &\leq \sqrt{N}\left\|\Psi\right\|_\infty\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left|\int f_\varepsilon(v-h(x_1;\theta))\left(d\mu_{X_N}-d\mu_X\right)\right|dv\\
&\leq \sqrt{N}\left\|\Psi\right\|_\infty\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left|\int f_\varepsilon(v-h(x_1;\theta_0))\left(d\mu_{X_N}-d\mu_X\right)\right|dv\\
&\quad +\sqrt{N}\left\|\Psi\right\|_\infty\int_{-\infty}^{\infty}\left|\int\left[f_\varepsilon(v-h(x_1;\theta))-f_\varepsilon(v-h(x_1;\theta_0))\right]\left(d\mu_{X_N}-d\mu_X\right)\right|dv.
\end{aligned}
$$

Hence, $E\sqrt{N}\sup_{\|\theta-\theta_0\|\leq r_N}\left(\left|T_1\right|\right)|_{\delta=\delta_N}\to 0$ for any sequence of positive δ_N and r_N decreasing to 0 by Lemmas E.4.14 and E.4.19.

Now we bound $T_2 - T_3$ by splitting it into three terms, and bounding them,

$$
\begin{aligned}
& T_2 - T_3 \\
= \; & \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{X_2} d\mu_{Z}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{Z}} dv \\
& - \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{Z_N}} dv \\
& - \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\left(\mu_{X_{N,2}} - \mu_{X_2}\right) d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{Z_N}} dv \\
= \; & R_1 + R_2 - R_3,
\end{aligned}
$$

where

$$
\begin{aligned}
R_3 \; = \; & \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\left(\mu_{X_{N,2}} - \mu_{X_2}\right) d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{Z_N}} dv \\
= \; & \sum a_k \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \frac{\int \Psi_1^k\left(x_1\right) \Psi_z^k\left(z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{Z_N}} \times \\
& \int \Psi_2^k\left(x_2\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) d\left(\mu_{X_{N,2}} - \mu_{X_2}\right) d\mu_{X_2} dv \\
\Rightarrow \; & \sqrt{N}\left|R_3\right| \leq \sum_{k=1}^{K} a_k \left\|\Psi_1^k\right\|_{\infty} \left\|\Psi_z^k\right\|_{\infty} \sqrt{N} \left[\begin{array}{c} \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \left|\int \Psi_2^k\left(x\right) f_{\varepsilon}\left(v - h\left(x;\theta_0\right)\right)\left(d\mu_{X_{N,2}} - d\mu_{X_2}\right)\right| dv \\ + \int_{-\infty}^{\infty} \left|\int \Psi_2^k\left(x\right)\left[f_{\varepsilon}\left(v - h\left(x;\theta\right)\right) - f_{\varepsilon}\left(v - h\left(x;\theta_0\right)\right)\right]\left(d\mu_{X_{N,2}} - d\mu_{X_2}\right)\right| dv \end{array}\right]
\end{aligned}
$$

Hence, $E\sqrt{N}\sup_{\|\theta-\theta_0\|\leq r_N}\left(\left|R_3\right|\right)|_{\delta=\delta_N} \to 0$ for any sequence of positive δ_N and r_N decreasing to 0 by Lemmas E.4.14 and E.4.19.

We will now break $R_1 + R_2$ into three terms

$$
\begin{aligned}
& \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{X_2} d\mu_{Z}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{Z}} dv \\
& - \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{Z_N}} dv \\
= \; & \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \int f_{\varepsilon}\left(v - h\left(x;\theta\right)\right) d\mu_{X} \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{X_2} d\mu_{Z}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{X_2} d\mu_{Z}} dv \\
& - \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \int f_{\varepsilon}\left(v - h\left(x;\theta\right)\right) d\mu_{X_2} \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}} dv \\
= \; & \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \int f_{\varepsilon}\left(v - h\left(x;\theta\right)\right) d\mu_{X} \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{X_2} d\mu_{Z}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_1} d\mu_{X_2} d\mu_{Z}} dv \\
& - \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \int f_{\varepsilon}\left(v - h\left(x;\theta\right)\right) d\mu_{X_N} \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}} dv \\
& + \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right) \int f_{\varepsilon}\left(v - h\left(x;\theta\right)\right) \left(d\mu_{X_N} - d\mu_{X}\right) \frac{\int \Psi\left(x_1, x_2, z\right) f_{\varepsilon}\left(u - u_{\theta}\left(x_1\right)\right) f_{\varepsilon}\left(u - u_{\theta}\left(x_2\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}}{\int f_{\varepsilon}\left(v - h\left(x_1;\theta\right)\right) f_{\varepsilon}\left(v - h\left(x_2;\theta\right)\right) \rho_{\eta,N;\theta}\left(v, z\right) d\mu_{X_{N,1}} d\mu_{X_2} d\mu_{Z_N}} dv \\
= \; & M_1 - M_2 + M_3,
\end{aligned}
$$

where

$$\begin{aligned}
\sqrt{N}\left|M_3\right| &\leq \sqrt{N}\left\|\Psi\right\|_\infty\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left|\int f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\left(d\mu_{X_N}-d\mu_X\right)\right|dv \\
&\quad +\sqrt{N}\left\|\Psi\right\|_\infty\int_{-\infty}^{\infty}\left|\int\left[f_\varepsilon\left(v-h\left(x;\theta\right)\right)-f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\right]\left(d\mu_{X_N}-d\mu_X\right)\right|dv,
\end{aligned}$$

so $E\sup_{\|\theta-\theta_0\|\leq r_N}\sqrt{N}\left(\left|M_3\right|\right)|_{\delta=\delta_N}\to 0$ for our sequences r_N,δ_N by the same argument applied to T_1.

We rewrite M_1-M_2 as

$$\begin{aligned}
&= \left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left(\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-u_\theta\left(x_2\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
&\quad \left.-\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_{N,1}}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)\int f_\varepsilon\left(v-h\left(x;\theta\right)\right)d\mu_X dv \\
&= \left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left(\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_2}d\mu_Z}\right. \\
&\quad \left.-\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(u-u_\theta\left(x_1\right)\right)f_\varepsilon\left(u-u_\theta\left(x_2\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(u-u_\theta\left(x_2\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_2}d\mu_{Z_N}}\right)dv \\
&\quad +\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\left(\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)\left(d\mu_{X_{N,1}}-d\mu_{X_1}\right)d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_2}d\mu_{Z_N}}\right)dv \\
&= N_1+N_2,
\end{aligned}$$

where

$$\sqrt{N}\left|N_2\right|\leq\sqrt{N}\sum_{k=1}^{K}a_k\left\|\Psi_2^k\right\|_\infty\left\|\Psi_z^k\right\|_\infty\left[\begin{array}{c}\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left|\int\Psi_1^k\left(x\right)f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\left(d\mu_{X_{N,1}}-d\mu_{X_1}\right)\right|dv \\ \int_{-\infty}^{\infty}\left|\int\Psi_1^k\left(x\right)\left[f_\varepsilon\left(v-h\left(x;\theta\right)\right)-f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\right]\left(d\mu_{X_{N,1}}-d\mu_{X_1}\right)\right|dv\end{array}\right]$$

so $E\sup_{\|\theta-\theta_0\|\leq r_N}\sqrt{N}\left(\left|N_2\right|\right)|_{\delta=\delta_N}\to 0$ by the same argument bounding T_1. We now split N_1 into three pieces

$$\begin{aligned}
N_1 &= \left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left(\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
&\quad \left.-\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_{N,1}}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)\int f_\varepsilon\left(v-h\left(x;\theta\right)\right)d\mu_X dv \\
&= \left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\int f_\varepsilon\left(v-h\left(x;\theta\right)\right)d\mu_X dv \\
&\quad -\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\int f_\varepsilon\left(v-h\left(x;\theta\right)\right)d\mu_X dv \\
&\quad +\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\frac{\int\Psi\left(x_1,x_2,z\right)f_\varepsilon\left(v-h\left(x_1;\theta\right)\right)f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)\left(d\mu_{X_1}-d\mu_{X_{N,1}}\right)d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(v-h\left(x_2;\theta\right)\right)\rho_{\eta,N;\theta}\left(v,z\right)d\mu_{X_2}d\mu_{Z_N}}dv \\
&= O_1+O_2+O_3
\end{aligned}$$

where

$$\sqrt{N}\left|O_3\right|\leq\sqrt{N}\sum_{k=1}^{K}a_k\left\|\Psi_2^k\right\|_\infty\left\|\Psi_z^k\right\|_\infty\left[\begin{array}{c}\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left|\int\Psi_1\left(x\right)f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\left(d\mu_{X_1}-d\mu_{X_{N,1}}\right)\right|dv \\ +\int_{-\infty}^{\infty}\left|\int\Psi_1\left(x\right)\left(f_\varepsilon\left(v-h\left(x;\theta\right)\right)-f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\right)\left(d\mu_{X_1}-d\mu_{X_{N,1}}\right)\right|dv\end{array}\right],$$

$E\sup_{\|\theta-\theta_0\|\leq r_N}\sqrt{N}\left(|O_3|\right)|_{\delta=\delta_N}\to 0$ by the same argument bounding T_1.

Now we rewrite O_1+O_2 by substituting $\rho_{\eta,N;\theta}(v,z)=f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)$.

$$
\begin{aligned}
& O_1+O_2 \\
= & \left(\int_{-\infty}^{\infty}-\int_{F^{-1}_{N,V;\theta}(\delta)}^{F^{-1}_{N,V;\theta}(1-\delta)}\right)\left(\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
& \left.-\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)\int f_\varepsilon(v-h(x;\theta))\,d\mu_X\,dv \\
= & \left(\int_{-\infty}^{\infty}-\int_{F^{-1}_{N,V;\theta}(\delta)}^{F^{-1}_{N,V;\theta}(1-\delta)}\right)\left(\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
& \left.-\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)\int f_\varepsilon(v-h(x;\theta))\,d\mu_{X_N}\,dv \\
& +\left(\int_{-\infty}^{\infty}-\int_{F^{-1}_{N,V;\theta}(\delta)}^{F^{-1}_{N,V;\theta}(1-\delta)}\right)\left(\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
& \left.-\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon(v-h(x_1;\theta))\,f_\varepsilon(v-h(x_2;\theta))\,f_\eta\left(F^{-1}_{N,U;\theta}\left(F^{-1}_{N,V;\theta}(v)\right)-g(z;\theta)\right)d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)\int f_\varepsilon(v-h(x;\theta))\left(d\mu_X-d\mu_{X_N}\right)dv \\
= & P_1+P_2,
\end{aligned}
$$

where

$$
\sqrt{N}\,|P_2|\leq\sqrt{N}2\,\|\Psi\|_\infty\left[\begin{array}{c}\left(\int_{-\infty}^{\infty}-\int_{F^{-1}_{N,V;\theta}(\delta)}^{F^{-1}_{N,V;\theta}(1-\delta)}\right)\left|\int f_\varepsilon(v-h(x;\theta_0))\left(d\mu_X-d\mu_{X_N}\right)\right|dv \\ +\int_{-\infty}^{\infty}\left|\int\left[f_\varepsilon(v-h(x;\theta))-f_\varepsilon(v-h(x;\theta_0))\right]\left(d\mu_X-d\mu_{X_N}\right)\right|dv\end{array}\right],
$$

so $E\sup_{\|\theta-\theta_0\|\leq r_N}\sqrt{N}\left(|P_2|\right)|_{\delta=\delta_N}\to 0$ by the same argument bounding T_1.

By change of variables,

$$
\begin{aligned}
& q=F^{-1}_{N,V;\theta}(v) \\
\Rightarrow\quad & dq=\int f_\varepsilon(v-h(x;\theta))\,d\mu_{X_N}dv\text{ and }F^{-1}_{N,V;\theta}(q)=v
\end{aligned}
$$

followed by a change of variables

$$
q=\int F_\eta(u-g(z;\theta))\,d\mu_{Z_N}
$$

we rewrite P_1 as

$$\begin{aligned}
& \left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta)}^{F_{N,V;\theta}^{-1}(1-\delta)}\right)\left(\frac{\int \Psi(x_1,x_2,z) f_\varepsilon(v-h(x_1;\theta)) f_\varepsilon(v-h(x_2;\theta)) f_\eta\left(F_{N,U;\theta}^{-1}\left(F_{N,V;\theta}^{-1}(v)\right)-g(z;\theta)\right) d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon(v-h(x_1;\theta)) f_\varepsilon(v-h(x_2;\theta)) f_\eta\left(F_{N,U;\theta}^{-1}\left(F_{N,V;\theta}^{-1}(v)\right)-g(z;\theta)\right) d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
& \left.-\frac{\int \Psi(x_1,x_2,z) f_\varepsilon(v-h(x_1;\theta)) f_\varepsilon(v-u_\theta(x_2)) f_\eta\left(F_{N,U;\theta}^{-1}\left(F_{N,V;\theta}^{-1}(v)\right)-g(z;\theta)\right) d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon(v-h(x_1;\theta)) f_\varepsilon(v-h(x_2;\theta)) f_\eta\left(F_{N,U;\theta}^{-1}\left(F_{N,V;\theta}^{-1}(v)\right)-g(z;\theta)\right) d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)\int f_\varepsilon(v-h(x;\theta))\,d\mu_{X_N}dv \\
= & \left(\int_0^1-\int_\delta^{1-\delta}\right)\left(\frac{\int \Psi(x_1,x_2,z)\phi_{\varepsilon,N}(q,x_1;\theta)\phi_{\varepsilon,N}(q,x_2;\theta)\phi_{\eta,N}(q,x_1;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\phi_{\varepsilon,N}(q,x_2;\theta)\phi_{\eta,N}(q,x_1;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
& \left.-\frac{\int \Psi(x_1,x_2,z)\phi_{\varepsilon,N}(q,x_1;\theta)\phi_{\varepsilon,N}(q,x_2;\theta)\phi_{\eta,N}(q,x_1;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\phi_{\varepsilon,N}(q,x_2;\theta)\phi_{\eta,N}(q,x_1;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)dq \\
= & \left(\int_{-\infty}^{\infty}-\int_{F_{N,U;\theta}^{-1}(\delta)}^{F_{N,U;\theta}^{-1}(1-\delta)}\right)\left(\frac{\int \Psi(x_1,x_2,z) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}\right. \\
& \left.-\frac{\int \Psi(x_1,x_2,z) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}\right)\int f_\eta(u-g(z;\theta))\,d\mu_{Z_N}du \\
= & \left(\int_{-\infty}^{\infty}-\int_{F_{N,U;\theta}^{-1}(\delta)}^{F_{N,U;\theta}^{-1}(1-\delta)}\right)\frac{\int \Psi(x_1,x_2,z) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) d\mu_{X_1}d\mu_{X_2}}du \\
& +\left(\int_{-\infty}^{\infty}-\int_{F_{N,U;\theta}^{-1}(\delta)}^{F_{N,U;\theta}^{-1}(1-\delta)}\right)\frac{\int \Psi(x_1,x_2,z) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \\
& \times\int f_\eta(u-g(z;\theta))\left(d\mu_{Z_N}-d\mu_Z\right)du \\
& -\left(\int_{-\infty}^{\infty}-\int_{F_{N,U;\theta}^{-1}(\delta)}^{F_{N,U;\theta}^{-1}(1-\delta)}\right)\frac{\int \Psi(x_1,x_2,z) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) d\mu_{X_1}d\mu_{X_2}}du \\
= & Q_1+Q_2+Q_3
\end{aligned}$$

where $\sqrt{N}\,|Q_2|\leq\sqrt{N}\,\|\Psi\|_\infty\left[\begin{array}{c}\left(\int_{-\infty}^{\infty}-\int_{F_{N,U;\theta}^{-1}(\delta)}^{F_{N,U;\theta}^{-1}(1-\delta)}\right)\left|\int f_\eta(u-g(z;\theta_0))\left(d\mu_{Z_N}-d\mu_Z\right)\right|du \\ \int_{-\infty}^{\infty}\left|\int\left[f_\eta(u-g(z;\theta))-f_\eta(u-g(z;\theta_0))\right]\left(d\mu_{Z_N}-d\mu_Z\right)\right|du\end{array}\right]$, and so

$$E\sup_{\|\theta-\theta_0\|\leq r_N}\sqrt{N}\left(|Q_2|\right)|_{\delta=\delta_N}\rightarrow 0$$

by the same argument bounding T_1.. Finally, to bound Q_1+Q_3, note that

$$\begin{aligned}
& \sqrt{N}\,|Q_1+Q_3| \\
\leq & \sqrt{N}\left(\int_{-\infty}^{\infty}-\int_{F_{N,U;\theta}^{-1}(\delta)}^{F_{N,U;\theta}^{-1}(1-\delta)}\right)\left|\frac{\int \Psi(x_1,x_2,z) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) d\mu_{X_1}d\mu_{X_2}}\right. \\
& \left.-\frac{\int \Psi(x_1,x_2,z) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) f_\eta(u-g(z;\theta))\,d\mu_{X_1}d\mu_{X_2}d\mu_{Z_N}}{\int f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_1;\theta)\right) f_\varepsilon\left(F_{N,V;\theta}^{-1}\left(F_{N,U;\theta}^{-1}(u)\right)-h(x_2;\theta)\right) d\mu_{X_1}d\mu_{X_2}}\right|du \\
\leq & \sqrt{N}\sum a_k\|\Psi_1\|_\infty\|\Psi_2\|_\infty\left[\begin{array}{c}\left(\int_{-\infty}^{\infty}-\int_{F_{N,U;\theta}^{-1}(\delta)}^{F_{N,U;\theta}^{-1}(1-\delta)}\right)\int_{-\infty}^{\infty}\left|\int\Psi_z(z)\,f_\eta(u-g(z;\theta_0))\left(d\mu_Z-d\mu_{Z_N}\right)\right|du \\ +\int_{-\infty}^{\infty}\left|\int\Psi_z(z)\left[f_\eta(u-g(z;\theta))-f_\eta(u-g(z;\theta_0))\right]\left(d\mu_Z-d\mu_{Z_N}\right)\right|du\end{array}\right],
\end{aligned}$$

and so $E\sup_{\|\theta-\theta_0\|\leq r_N}\sqrt{N}\left(|Q_1+Q_3|\right)|_{\delta=\delta_N}\rightarrow 0$ by the same argument bounding T_1.By the triangle inequality, the expression

$$\begin{aligned}
& \sqrt{N}\left|\left(\int_0^1-\int_\delta^{1-\delta}\right)\frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta)\,\phi_\varepsilon(q,x_2;\theta)\,\phi_\eta(q,z;\theta)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right. \\
& \left.-\left(\int_0^1-\int_\delta^{1-\delta}\right)\frac{\int \Psi(x_1,x_2,z)\,\phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}{\int \phi_{\varepsilon,N}(q,x_1;\theta)\,\phi_{\varepsilon,N}(q,x_2;\theta)\,\phi_{\eta,N}(q,z;\theta)\,d\mu_{X_{N,1}}d\mu_{X_{N,2}}d\mu_{Z_N}}dq\right|
\end{aligned}$$

is bounded by the sum of

$$\begin{array}{l}\sqrt{N}\left|A_1\right|+\sqrt{N}\left|T_1\right|+\sqrt{N}\left|R_3\right|+\sqrt{N}\left|M_3\right| \\ +\sqrt{N}\left|N_2\right|+\sqrt{N}\left|O_3\right|+\sqrt{N}\left|P_2\right|+\sqrt{N}\left|Q_1+Q_3\right|+\sqrt{N}\left|Q_2\right|\end{array}$$

so $\sqrt{N}E\sup_{\|\theta-\theta_0\|\leq r_N}\left|\left(\psi\left[\mu_X,\mu_Z\right](\theta)-\psi\left[\mu_{X_N},\mu_{Z_N}\right](\theta)\right)-\left(\psi^{\delta_N}\left[\mu_X,\mu_Z\right](\theta)-\psi^{\delta_N}\left[\mu_{X_N},\mu_{Z_N}\right](\theta)\right)\right|=o(1)$ as desired. ■

E.4.1 Preliminaries for Proposition E.4.8

Lemma E.4.15 *If $C_\varepsilon=\int_{-\infty}^{\infty}\left|f_\varepsilon'(v)\right|dv$, $\mu_X h_{LC}(X)^2$ and $\Psi(X)$ are bounded, then*

$$E\sup_\theta\int\left|R_{i+1}(\theta,v)\right|T_i^c(\theta)T_{i+1}(\theta)\,dv\leq\sqrt{N}\left\|\Psi\right\|_\infty 6\delta_{i+1}\frac{\delta_i}{V_i}C_\varepsilon dv\mu_X h_{LC}(X)^2$$

and

$$E\sup_\theta\int\left|\left(R_i(\theta,v)\right)T_i(\theta)T_{i+1}^c(\theta)\right|dv\leq 6\left\|\Psi\right\|_\infty\delta_i\frac{\delta_{i+1}}{V_{i+1}}C_\varepsilon\mu_X h_{LC}(X)^2.$$

Proof. We first show that

$$E\sup_\theta\int\left|R_{i+1}(\theta,v)\right|T_i^c(\theta)T_{i+1}(\theta)\,dv\leq\sqrt{N}6\left\|\Psi\right\|_\infty\delta_{i+1}\frac{\delta_i}{V_i}C_\varepsilon dv\mu_X h_{LC}(X)^2.$$

Note that

$$\begin{aligned}&E\sup_\theta\int\left|R_{i+1}(\theta,v)\right|T_i^c(\theta)T_{i+1}(\theta)\,dv\\ \leq\;&E\sup_\theta\int\left|R_{i+1}(\theta,v)\right|T_i^c(\theta)\,dv\\ =\;&E\sup_\theta\int\left|\sqrt{N}\left(\mu_{X_N}-\mu_X\right)\left(\Psi(X)\left[f_\varepsilon(v-h(X;\theta))-f_\varepsilon(v-h(X;B(\theta;i+1)))\right]\left\{h_{LC}(X)>\frac{V_i}{2\delta_i}\right\}\right)\right|dv\\ \leq\;&E\sup_\theta\frac{1}{\sqrt{N}}\int\sum_{i=1}^{n}\left|\Psi(X)\left\{f_\varepsilon(v-h(X_j;\theta))-f_\varepsilon(v-h(X_j;B(\theta;i+1)))\right\}\right|\left\{h_{LC}(X_j)>\frac{V_i}{2\delta_i}\right\}dv\\ &+\sqrt{N}\mu_X\int\left(\left|\Psi(X)f_\varepsilon(v-h(X_j;\theta))-f_\varepsilon(v-h(X_j;B(\theta;i+1)))\right|\left\{h_{LC}(X)>\frac{V_i}{2\delta_i}\right\}\right)dv\end{aligned}$$

where the last inequality is a consequence of the triangle inequality. The second term is not greater than

$$\begin{aligned}&\sqrt{N}\left\|\Psi\right\|_\infty\delta_{i+1}\int_{-\infty}^{\infty}\left|f_\varepsilon'(v)\right|dv\mu_X h_{LC}(X)\left\{h_{LC}(X)>\frac{V_i}{2\delta_i}\right\}\\ \leq\;&2\left\|\Psi\right\|_\infty\sqrt{N}\frac{\delta_i\delta_{i+1}}{V_i}C_\varepsilon\mu_X h_{LC}(X)^2\end{aligned}$$

and the first term is not greater than

$$
\begin{aligned}
& E\sup_{\theta}\frac{1}{\sqrt{N}}\sum_{i=1}^{n}\left\{h_{LC}\left(X_j\right)>\frac{V_i}{2\delta_i}\right\}\int\left\|\Psi\right\|_{\infty}\left|\left\{f_{\varepsilon}\left(v-h\left(X_j;\theta\right)\right)-f_{\varepsilon}\left(v-h\left(X_j;B\left(\theta;i+1\right)\right)\right)\right\}\right|dv \\
\leq\ & \left\|\Psi\right\|_{\infty}E\sup_{\theta}\frac{1}{\sqrt{N}}C_{\varepsilon}\sum_{i=1}^{n}\left\{h_{LC}\left(X_j\right)>\frac{V_i}{2\delta_i}\right\}\left|h\left(X_j;\theta\right)-h\left(X_j;B\left(\theta;i+1\right)\right)\right| \\
\leq\ & \left\|\Psi\right\|_{\infty}E\sup_{\theta}2\delta_{i+1}\sqrt{N}C_{\varepsilon}\mu_{X_N}\left|h_{LC}\left(X\right)\left\{h_{LC}\left(X\right)>\frac{V_i}{2\delta_i}\right\}\right| \\
\leq\ & 4\left\|\Psi\right\|_{\infty}\delta_i\frac{\delta_{i+1}}{V_i}\sqrt{N}C_{\varepsilon}\mu_X\left|h_{LC}\left(X\right)^2\right|.
\end{aligned}
$$

By an identical argument,

$$
\begin{aligned}
& E\sup_{\theta}\int\left|\left(R_i\left(\theta,v\right)\right)T_i\left(\theta\right)T_{i+1}^{c}\left(\theta\right)\right|dv \\
\leq\ & E\sup_{\theta}\int\left|\sqrt{N}\left(\mu_{X_N}-\mu_X\right)\left(\Psi\left(X\right)\left[f_{\varepsilon}\left(v-h\left(X;\theta\right)\right)-f_{\varepsilon}\left(v-h\left(X;B\left(\theta;i\right)\right)\right)\right]\left\{h_{LC}\left(X\right)>\frac{V_{i+1}}{2\delta_{i+1}}\right\}\right)\right|dv \\
\leq\ & 6\left\|\Psi\right\|_{\infty}\delta_i\frac{\delta_{i+1}}{V_{i+1}}C_{\varepsilon}\mu_X h_{LC}\left(X\right)^2.
\end{aligned}
$$

■

Lemma E.4.16 *Let* $\mathcal{E}\left(x\right)=2\frac{\exp(x)-1-x}{x^2}$, *and let* $N\left(\delta_{i+1}\right)$ *be the* δ_{i+1} *covering number of* Θ *in the Euclidean metric. If Assumption E.2.9 is satisfied, then*

$$
\begin{aligned}
& E\sup_{\theta}\int\left|R_i\left(\theta,v\right)-R_{i+1}\left(\theta,v\right)\right|T_i\left(\theta\right)T_{i+1}\left(\theta\right)dv \\
\leq\ & \delta_i\sqrt{\log\left(2N\left(\delta_{i+1}\right)\right)}\left(1+12C_{\varepsilon}^2\left\|\Psi\right\|_{\infty}^2\mu_X\left(h_{LC}\left(X\right)^2\right)+18C_{\varepsilon}^4\left\|\Psi\right\|_{\infty}^4\mu_X\left(h_{LC}\left(X\right)^2\right)\mathcal{E}\left(6\right)\right) \\
& +2\left\|\Psi\right\|_{\infty}K\delta_i.
\end{aligned}
$$

for some constant K. *Hence, if* $N\left(\delta_{i+1}\right)$ *is finite, there is a* $K_1<\infty$ *such that*

$$
E\sup_{\theta}\int\left|R_i\left(\theta,v\right)-R_{i+1}\left(\theta,v\right)\right|T_i\left(\theta\right)T_{i+1}\left(\theta\right)dv<K_1\delta_i\sqrt{\log\left(N\left(\delta_i\right)\right)}.
$$

Proof. To simplify notation, let $\Delta_i^f\left(X;\theta,v\right)=f_{\varepsilon}\left(v-h\left(X;B\left(\theta;i+1\right)\right)\right)-f_{\varepsilon}\left(v-h\left(X;B\left(\theta;i\right)\right)\right)$ and $\Delta_i^h\left(X;\theta\right)=h\left(X;B\left(\theta;i+1\right)\right)-h\left(X;B\left(\theta;i\right)\right)$. By the triangle inequality,

$$
\begin{aligned}
& E\sup_{\theta}\int\left|\left(R_i\left(\theta,v\right)-R_{i+1}\left(\theta,v\right)\right)\right|T_i\left(\theta\right)T_{i+1}\left(\theta\right)dv \\
\leq\ & E\sup_{\theta}\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{h_{LC}\left(X\right)\leq\frac{V_i}{2\delta_i}\right\}dv \\
\leq\ & E\sup_{\theta}\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv \\
\leq\ & E\sup_{\theta}\left[\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right. \\
& \left.-E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right] \\
& +\sup_{\theta}E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv \qquad \text{(E.4.57)}
\end{aligned}
$$

We now bound the two terms individually.

The first term in equation (E.4.57) is bounded by using the bound on its moment generating function for a fixed θ (derived in Lemma E.4.17) and the concentration inequality of Theorem 2 in Boucheron et al. (2003).

By Jensen's inequality, note that for any λ_i,

$$\begin{aligned}
&\exp\left(\lambda_i\left(E\sup_\theta\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right.\right.\\
&\left.\left.-E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right)\right)\\
\leq\;&E\exp\left(\lambda_i\left(\sup_\theta\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right.\right.\\
&\left.\left.-E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right)\right).
\end{aligned}$$

Note that $B\left(\theta;i+1\right)$ takes on at most $N\left(\delta_{i+1}\right)$. Since the expectation of a maximum of finitely many nonnegative random variables is less than the sum of their expectations, the expression above is no greater than

$$\begin{aligned}
&\sum_{\theta\in\operatorname{Im}B(\theta;i+1)}E\exp\left(\lambda_i\left|\int\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right.\right.\\
&\left.\left.-E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right|\right)\\
\leq\;&\sum_{\theta\in\operatorname{Im}B(\theta;i+1)}E\exp\left(\lambda_i\left(\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right.\right.\\
&\left.\left.-E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right)\right)\\
&+\sum_{\theta\in\operatorname{Im}B(\theta;i+1)}E\exp\left(-\lambda_i\left(\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right.\right.\\
&\left.\left.-E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\right)\right)
\end{aligned}$$

Lemma E.4.17 implies this is not greater than

$$\begin{aligned}
&2N\left(\delta_{i+1}\right)\max_{\theta\in\Theta}\exp\left(C_\varepsilon^2\left\|\Psi\right\|_\infty^2\lambda_i^2 12\mu_X\left(\Delta_i^h\left(X;\theta\right)\right)^2+\frac{18}{n}C_\varepsilon^4\left\|\Psi\right\|_\infty^4\lambda_i^4V_i^2E\left(\left|\Delta_i^h\left(X_j;\theta\right)\right|^2\right)\mathcal{E}\left(\frac{6\lambda_i^2}{n}V_i^2\right)\right)\\
\leq\;&2N\left(\delta_{i+1}\right)\max_{\theta\in\Theta}\exp\left(C_\varepsilon^2\left\|\Psi\right\|_\infty^2\lambda_i^2\delta_i^2 12\mu_X\left(h_{LC}\left(X\right)^2\right)+\frac{18}{n}C_\varepsilon^4\left\|\Psi\right\|_\infty^4\lambda_i^4\delta_i^2V_i^2\mu_X\left(h_{LC}\left(X\right)^2\right)\mathcal{E}\left(\frac{6\lambda_i^2}{n}V_i^2\right)\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
&E\sup_\theta\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\\
&-E\int\left|\sqrt{N}\left(\left(\mu_X-\mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right|\leq V_i\right\}dv\\
\leq\;&\frac{\log\left(2N\left(\delta_{i+1}\right)\right)}{\lambda_i}+\frac{1}{\lambda_i}\left(C_\varepsilon^2\left\|\Psi\right\|_\infty^2\delta_i^2\lambda_i^2 12\mu_X\left(h_{LC}\left(X\right)\right)^2+\frac{18}{n}C_\varepsilon^4\left\|\Psi\right\|_\infty^4\lambda_i^4\delta_i^2V_i^2\mu_X\left(h_{LC}\left(X\right)^2\right)\mathcal{E}\left(\frac{6\lambda_i^2}{n}V_i^2\right)\right)
\end{aligned}$$

Recall that $V_i = \frac{\sqrt{N}}{\lambda_i}$ and choose $\lambda_i = \frac{\sqrt{N}}{V_i} = \frac{\sqrt{\log(2N(\delta_{i+1}))}}{\delta_i}$ which yields the upper bound

$$\delta_i \sqrt{\log\left(2N\left(\delta_{i+1}\right)\right)} \left(1 + 12C_\varepsilon^2 \left\|\Psi\right\|_\infty^2 \mu_X \left(h_{LC}\left(X\right)^2\right) + 18C_\varepsilon^4 \left\|\Psi\right\|_\infty^4 \mu_X \left(h_{LC}\left(X\right)^2\right) \mathcal{E}\left(6\right)\right)$$

for the first term in equation (E.4.57).

We bound the second term in equation (E.4.57) using Lemma E.4.18. Note that

$$\begin{aligned}
& \sup_\theta E \int \left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right| \left\{\left|\Delta_i^h\left(X;\theta\right)\right| \le V_i\right\} dv \\
\le \; & \sup_\theta \int E\left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right| dv
\end{aligned}$$

By Jensen's inequality this is not greater than

$$\begin{aligned}
& \sup_\theta \int \sqrt{E\left(\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right)^2} dv \\
= \; & \sup_\theta \int \sqrt{\mu_X \Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)^2} dv \\
\le \; & \left\|\Psi\right\|_\infty \sup_\theta \int \sqrt{\mu_X\left(f_\varepsilon\left(v - h\left(X;B\left(\theta;i+1\right)\right)\right) - f_\varepsilon\left(v - h\left(X;B\left(\theta;i\right)\right)\right)\right)^2} dv \\
\le \; & \left\|\Psi\right\|_\infty K \sup_{\theta_i \in B(\theta;i)} \left\|\theta_{i+1} - \theta_i\right\| \\
\le \; & 2\left\|\Psi\right\|_\infty \delta_i K
\end{aligned}$$

for some constant $K \in (0,\infty)$. The second to last inequality follows from Lemma E.4.18, and the last inequality follows from the definitions of $B\left(\theta,i\right)$ and δ_i. ■

Lemma E.4.17 *For each $\theta \in \Theta$, and any $\lambda_i > 0$,*

$$\begin{aligned}
& E\exp\left(\pm\lambda_i\left(\int\left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right| \le V_i\right\} dv\right.\right. \\
& \left.\left. -E\int\left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Psi\left(X\right)\left\{h\left(X;B\left(\theta;i\right)\right) \le v\right\} - \left\{h\left(X;B\left(\theta;i+1\right)\right) \le v\right\}\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right| \le V_i\right\} dv\right)\right) \\
\le \; & \exp\left(\lambda_i^2 C_\varepsilon^2 \left\|\Psi\right\|_\infty^2 12\mu_X \Delta_i^h\left(X;\theta\right)^2 + \frac{18}{n} C_\varepsilon^4 \left\|\Psi\right\|_\infty^4 \lambda_i^4 V_i^2 E\left(\left|\Delta_i^h\left(X;\theta\right)\right|^2\right)\mathcal{E}\left(\frac{6\lambda_i^2}{n}V_i^2\right)\right)
\end{aligned}$$

where $\Delta_i^f\left(X;\theta,v\right) = f_\varepsilon\left(v - h\left(X;B\left(\theta;i+1\right)\right)\right) - f_\varepsilon\left(v - h\left(X;B\left(\theta;i\right)\right)\right)$ and $\Delta_i^h\left(X;\theta\right) = h\left(X;B\left(\theta;i+1\right)\right) - h\left(X;B\left(\theta;i\right)\right)$.

Proof. Let $\left(X_{(1)}, X_{(2)}, ..., X_{(n)}\right)$ be an independently drawn copy of $\left(X_1, X_2, ...X_n\right)$, and let $\mu_X^{n,(j)}$ be the empirical measure induced by replacing X_j by $X_{(j)}$. Let

$$Z = \int\left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right| \le V_i\right\} dv$$

and

$$Z^{(j)} = \int\left|\sqrt{N}\left(\left(\mu_X - \mu_X^{n,(j)}\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right| \le V_i\right\} dv.$$

By Theorem 2 in Boucheron et al. (2003), for any $0 < \theta < \frac{1}{|\lambda_i|}$

$$\log E \exp\left(\pm\lambda_i\left(Z - E\left[Z\right]\right)\right) \leq \frac{\lambda_i \theta}{1-\lambda_i\theta} \log E \exp\left(\frac{\lambda_i}{\theta} E\left[\sum_{j=1}^{n}\left(Z - Z^{(j)}\right)^2 |\mu_{X_N}\right]\right)$$

so it is enough to bound the moment generating function of $E\left(\sum_{j=1}^{n}\left(Z - Z^{(j)}\right)^2 |\mu_{X_N}\right)$ to prove the lemma. Note that

$$\begin{aligned}
& \left|\int\left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right| \leq V_i\right\}dv\right. \\
& \left.-\int\left|\sqrt{N}\left(\left(\mu_X - \mu_X^{n,(j)}\right)\Psi\left(X\right)\Delta_i^f\left(X;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X;\theta\right)\right| \leq V_i\right\}dv\right| \\
\leq\ & \frac{1}{\sqrt{N}}\left|\int\left|\Psi\left(X_j\right)\left(\Delta_i^f\left(X_j;\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X_j;\theta\right)\right| \leq V_i\right\}dv\right| \\
& +\frac{1}{\sqrt{N}}\left|\int\left|\Psi\left(X_{(j)}\right)\left(\Delta_i^f\left(X_{(j)};\theta,v\right)\right)\right|\left\{\left|\Delta_i^h\left(X_{(j)};\theta\right)\right| \leq V_i\right\}dv\right| \\
\leq\ & \frac{1}{\sqrt{N}}\left\|\Psi\right\|_\infty \int_{-\infty}^{\infty}\left|f_\varepsilon'\left(v\right)\right|dv \min\left(\left|\Delta_i^h\left(X_j;\theta\right)\right|, V_i\right) + \frac{1}{\sqrt{N}}\left\|\Psi\right\|_\infty \int_{-\infty}^{\infty}\left|f_\varepsilon'\left(v\right)\right|dv\left|\Delta_i^h\left(X_{(j)};\theta\right)\right|.
\end{aligned}$$

Since $\left(a+b\right)^2 \leq 3a^2 + 3b^2$, it follows that

$$\sum_{j=1}^{n}\left(Z - Z^{(j)}\right)^2 \leq \frac{3}{n}C_\varepsilon^2\left\|\Psi\right\|_\infty^2 \sum_{j=1}^{n}\min\left(\left|\Delta_i^h\left(X_j;\theta\right)\right|^2, V_i^2\right) + \left(\Delta_i^h\left(X_{(j)};\theta\right)\right)^2,$$

and this upper bound has conditional expectation given μ_{X_N} of

$$\begin{aligned}
& 3C_\varepsilon^2\left\|\Psi\right\|_\infty^2 \mu_X\left(\Delta_i^h\left(X;\theta\right)\right)^2 + 3C_\varepsilon^2\left\|\Psi\right\|_\infty^2 \mu_{X_N}\min\left(\left|\Delta_i^h\left(X;\theta\right)\right|^2, V_i^2\right) \\
\leq\ & 6C_\varepsilon^2\left\|\Psi\right\|_\infty^2 \mu_X\left(\Delta_i^h\left(X;\theta\right)\right)^2 + 3C_\varepsilon^2\left\|\Psi\right\|_\infty^2\left(\mu_{X_N} - \mu_X\right)\min\left(\left|\Delta_i^h\left(X;\theta\right)\right|^2, V_i^2\right)
\end{aligned}$$

Hence, the moment generating function of this conditional expectation is not greater than

$$\begin{aligned}
& \exp\left(\lambda_i C_\varepsilon^2\left\|\Psi\right\|_\infty^2 6\mu_X\left(\Delta_i^h\left(X;\theta\right)\right)^2\right) \\
& \times E\exp\left(\lambda_i C_\varepsilon^2\left\|\Psi\right\|_\infty^2 3\left(\mu_{X_N} - \mu_X\right)\min\left(\left(\Delta_i^h\left(X;\theta\right)\right)^2, V_i^2\right)\right).
\end{aligned}$$

Since μ_{X_N} is a sum of i.i.d. random variables,

$$\begin{aligned}
& E\exp\left(\lambda_i C_\varepsilon^2\left\|\Psi\right\|_\infty^2 3\left(\mu_{X_N} - \mu_X\right)\min\left(\left(\Delta_i^h\left(X;\theta\right)\right)^2, V_i^2\right)\right) \\
=\ & \prod_{j=1}^{n} E\exp\left(C_\varepsilon^2\left\|\Psi\right\|_\infty^2 \frac{3\lambda_i}{n}\left(\min\left(\left|\Delta_i^h\left(X_j;\theta\right)\right|^2, V_i^2\right) - \mu_X\min\left(\left|\Delta_i^h\left(X;\theta\right)\right|^2, V_i^2\right)\right)\right).
\end{aligned}$$

To bound this note that

$$\exp\left(x\right) = 1 + x + \frac{1}{2}x^2\mathcal{E}\left(x\right)$$

where $\mathcal{E}\left(x\right) = 2\frac{\exp(x)-1-x}{x^2}$ is strictly increasing. This implies that if V is a mean zero random variable bounded by a constant K,

$$E\exp\left(\lambda V\right) \leq 1 + \frac{1}{2}\lambda^2\mathcal{E}\left(\lambda K\right)E\left(V^2\right) \leq \exp\left(\frac{1}{2}\lambda^2\mathcal{E}\left(\lambda K\right)E\left(V^2\right)\right).$$

Hence,

$$
\begin{aligned}
& E\exp\left(C_\varepsilon^2 \|\Psi\|_\infty^2 \frac{3\lambda_i}{n}\left(\min\left(\left|\Delta_i^h (X_j;\theta)\right|^2, V_i^2\right)\right.\right. \\
& \left.\left. -\mu_X \min\left(\left|\Delta_i^h (X;\theta)\right|^2, V_i^2\right)\right)\right) \\
\leq\ & \exp\left(\frac{9}{2}C_\varepsilon^4 \|\Psi\|_\infty^4 \frac{\lambda_i^2}{n^2} Var\left(\min\left(\left|\Delta_i^h (X_j;\theta)\right|^2, V_i^2\right)^2\right)\mathcal{E}\left(C_\varepsilon^2 \|\Psi\|_\infty^2 \frac{3\lambda_i}{n} V_i^2\right)\right) \\
\leq\ & \exp\left(\frac{9}{2}C_\varepsilon^4 \|\Psi\|_\infty^4 \frac{\lambda_i^2}{n^2} V_i^2 E\left(\Delta_i^h (X_j;\theta)^2\right)\mathcal{E}\left(C_\varepsilon^2 \|\Psi\|_\infty^2 \frac{3\lambda_i}{n} V_i^2\right)\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& E\exp\left(\lambda_i E\left[\sum_{j=1}^n \left(Z - Z^{(j)}\right)^2 |\mu_{X_N}\right]\right) \\
\leq\ & \exp\left(\lambda_i C_\varepsilon^2 \|\Psi\|_\infty^2 6\mu_X \left(\Delta_i^h (X;\theta)^2\right) + \frac{9}{2}C_\varepsilon^4 \|\Psi\|_\infty^4 \frac{\lambda_i^2}{n} V_i^2 E\left(\Delta_i^h (X_j;\theta)\right)^2 \mathcal{E}\left(C_\varepsilon^2 \|\Psi\|_\infty^2 \frac{3\lambda_i}{n} V_i^2\right)\right)
\end{aligned}
$$

By Theorem 2 of Boucheron et al. (2003), this implies for all $\gamma_i > 0$ and $\lambda_i \in \left(0, \frac{1}{\gamma_i}\right)$

$$
\begin{aligned}
& \log E\exp\left(\pm\lambda_i\left(\int\left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Delta_i^f (X;\theta,v)\right)\right|\left\{\left|\Delta_i^h (X;\theta)\right| \leq V_i\right\} dv\right.\right. \\
& \left.\left. -E\int\left|\sqrt{N}\left(\left(\mu_X - \mu_{X_N}\right)\Delta_i^f (X;\theta,v)\right)\right|\left\{\left|\Delta_i^h (X;\theta)\right| \leq V_i\right\} dv\right)\right) \\
\leq\ & \frac{\lambda_i\gamma_i}{1-\lambda_i\gamma_i}\log E\exp\left(\frac{\lambda_i}{\gamma_i}E\left[\sum_{j=1}^n \left(Z - Z^{(j)}\right)^2 |\mu_{X_N}\right]\right) \\
\leq\ & \frac{\lambda_i\gamma_i}{1-\lambda_i\gamma_i}\frac{\lambda_i}{\gamma_i}C_\varepsilon^2 \|\Psi\|_\infty^2 6\mu_X\left(\left|\Delta_i^h (X_j;\theta)\right|^2\right) \\
& +\frac{\lambda_i\gamma_i}{1-\lambda_i\gamma_i}\frac{9}{2n}C_\varepsilon^4 \|\Psi\|_\infty^4 \left(\frac{\lambda_i}{\gamma_i}\right)^2 V_i^2 E\left(\left|\Delta_i^h (X_j;\theta)\right|^2\right)\mathcal{E}\left(\frac{3\lambda_i}{n\gamma_i}V_i^2\right)
\end{aligned}
$$

If we pick γ_i so that $\lambda_i\gamma_i = \frac{1}{2}$ we get the upper bound

$$
\lambda_i^2 C_\varepsilon^2 \|\Psi\|_\infty^2 12\mu_X\left(\Delta_i^h (X;\theta)^2\right) + \frac{18}{n}C_\varepsilon^4 \lambda_i^4 \|\Psi\|_\infty^4 V_i^2 E\left(\left|\Delta_i^h (X_j;\theta)\right|^2\right)\mathcal{E}\left(\frac{6\lambda_i^2}{n}V_i^2\right)
$$

as desired. ■

Lemma E.4.18 *Suppose that*

(i) for some constants C_1, $C_2 > 0$, *we have that* $\max\left\{\left|f_\varepsilon'(v)\right|, \sup_{\theta\in\Theta} P\left(\left|h(x;\theta)\right| > v\right)\right\} \leq C_1 \exp\left(-C_2|v|\right)$

(ii) $\int h_{LC}(X)^4 d\mu_X$ *is finite*

then there exists a constant such that

$$
\left|\int\sqrt{\mu_X\left(f_\varepsilon\left(v - h(X;\theta_1)\right) - f_\varepsilon\left(v - h(X;\theta_2)\right)\right)^2}dv\right| \leq K\|\theta_1 - \theta_2\|.
$$

Proof. It is enough to show that the following term

$$\sup_{\theta\in\Theta}\int\sqrt{\mu_X\left(\nabla_\theta f_\varepsilon\left(v-h\left(X;\theta\right)\right)\right)^2}dv\leq\sup_{\theta\in\Theta}\int_{-\infty}^{\infty}\left(\int f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^2h_{LC}^2\left(x\right)d\mu_X\right)^{\frac{1}{2}}dv$$

is finite. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int\left(\int f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^2h_{LC}^2\left(x\right)d\mu_X\right)^{\frac{1}{2}}dv\\ \leq\;&\int\left(\int f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^4d\mu_X\int h_{LC}^4\left(x\right)d\mu_X\right)^{\frac{1}{4}}dv\\ =\;&\left(\int h_{LC}^4\left(x\right)d\mu_X\right)^{\frac{1}{4}}\int\left(\int f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^4d\mu_X\right)^{\frac{1}{4}}dv.\end{aligned}$$

The first term is bounded by assumption. The second term is finite if, for all $\theta\in\Theta$, the integrand

$$\int f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^4d\mu_X\leq K_1\exp\left(-K_2\left|v\right|\right)$$

for some constants K_1 and K_2. Note that

$$\begin{aligned}&\int f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^4d\mu_X\\ =\;&\int\left\{\left|h\left(x;\theta\right)\right|\geq\frac{v}{2}\right\}f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^4d\mu_X+\int\left\{\left|h\left(x;\theta\right)\right|<\frac{v}{2}\right\}f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^4d\mu_X\\ \leq\;&C_1\left\|f_\varepsilon'\right\|_\infty^4\exp\left(-C_2\left|\frac{v}{2}\right|\right)+\int\left\{\left|h\left(x;\theta\right)\right|<\frac{v}{2}\right\}f_\varepsilon'\left(v-h\left(x;\theta\right)\right)^4d\mu_X\\ \leq\;&C_1\left\|f_\varepsilon'\right\|_\infty^2\exp\left(-C_2\left|\frac{v}{2}\right|\right)+\int C_1\exp\left(-4C_2\left|\frac{v}{2}\right|\right)d\mu_X\\ =\;&K_1\exp\left(-K_2\left|v\right|\right)\end{aligned}$$

since $\left\|f_\varepsilon'\right\|_\infty<C_1$ by our bound. ■

Lemma E.4.19 *If the Assumptions in Proposition E.4.8 are satisfied, then for any sequence of positive numbers δ_N and r_N decreasing to 0, as $N\to\infty$,*

$$\sqrt{N}E\sup_{\left\|\theta-\theta_0\right\|\leq r_N}\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}\left(\delta_N\right)}^{F_{N,V;\theta}^{-1}\left(1-\delta_N\right)}\right)\left|\int\Psi\left(x\right)f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\left(d\mu_{X_N}-d\mu_X\right)\right|dv\to0.$$

Proof. We bound this term as follows:

$$
\begin{aligned}
& \sqrt{N} E \sup_{\|\theta-\theta_0\|\leq r_N} \left(\int_{-\infty}^{\infty} - \int_{F_{N,V;\theta}^{-1}(\delta_N)}^{F_{N,V;\theta}^{-1}(1-\delta_N)} \right) \left| \int \Psi(x) f_\varepsilon (v - h(x;\theta_0)) \left(d\mu_{X_N} - d\mu_X \right) \right| dv \\
\leq \; & \sqrt{N} E \left(\int_{-\infty}^{\infty} - \int_{V_1}^{V_2} \right) \left| \int \Psi(x) f_\varepsilon (v - h(x;\theta_0)) \left(d\mu_{X_N} - d\mu_X \right) \right| dv \\
& + \sqrt{N} E \sup_{\|\theta-\theta_0\|\leq r_N} \left[\left\{ F_{N,V;\theta}^{-1}(\delta_N) \geq V_1 \right\} + \left\{ F_{N,V;\theta}^{-1}(1-\delta_N) \leq V_2 \right\} \right] \int_{-\infty}^{\infty} \left| \int \Psi(x) f_\varepsilon (v - h(x;\theta_0)) \left(d\mu_{X_N} - d\mu_X \right) \right| dv \\
\leq \; & \left(\int_{-\infty}^{\infty} - \int_{V_1}^{V_2} \right) \left(\int \Psi(x)^2 f_\varepsilon^2 (v - h(x;\theta_0)) \, d\mu_X \right)^{\frac{1}{2}} dv \\
& + \left[E \sup_{\|\theta-\theta_0\|\leq r_N} \left[\left\{ F_{N,V;\theta}^{-1}(\delta_N) \geq V_1 \right\} + \left\{ F_{N,V;\theta}^{-1}(1-\delta_N) \leq V_2 \right\} \right] \right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(\int \Psi(x)^2 f_\varepsilon^2 (v - h(x;\theta_0)) \, d\mu_X \right)^{\frac{1}{2}} dv \\
\leq \; & \|\Psi\|_\infty \left(\int_{-\infty}^{\infty} - \int_{V_1}^{V_2} \right) \left(\int f_\varepsilon^2 (v - h(x;\theta_0)) \, d\mu_X \right)^{\frac{1}{2}} dv \\
& + \|\Psi\|_\infty \left[E \sup_{\|\theta-\theta_0\|\leq r_N} \left[\left\{ F_{N,V;\theta}^{-1}(\delta_N) \geq V_1 \right\} + \left\{ F_{N,V;\theta}^{-1}(1-\delta_N) \leq V_2 \right\} \right] \right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(\int f_\varepsilon^2 (v - h(x;\theta_0)) \, d\mu_X \right)^{\frac{1}{2}} dv
\end{aligned}
$$

We now show that $E \sup_{\|\theta-\theta_0\|\leq r_N} \left[\left\{ F_{N,V;\theta}^{-1}(\delta_N) \geq V_1 \right\} \right]$ converges to zero. Note that for any $\epsilon > 0$

$$
\begin{aligned}
& E \sup_{\|\theta-\theta_0\|\leq r_N} \left\{ F_{N,V;\theta}^{-1}(\delta) \geq V_1 \right\} \\
\leq \; & \left\{ F_{V;\theta_0}^{-1}(\delta) \geq V_1 - 2\epsilon \right\} + E \sup_{\|\theta-\theta_0\|\leq r_N} \left\{ \left| F_{V;\theta}^{-1}(\delta) - F_{N,V;\theta}^{-1}(\delta) \right| \geq \epsilon \right\} + \sup_{\|\theta-\theta_0\|\leq r_N} \left\{ \left| F_{V;\theta_0}^{-1}(\delta) - F_{V;\theta}^{-1}(\delta) \right| \geq \epsilon \right\}
\end{aligned}
$$

We first bound these terms for a fixed δ. The first term equals 0 for $\delta < F_{V;\theta_0}(V_1 - 2\epsilon)$. By definition,

$$
\begin{aligned}
\delta = \mu_X \left(F_\varepsilon \left(F_{V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right) &= \mu_{X_N} \left(F_\varepsilon \left(F_{N,V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right) \\
\Rightarrow \mu_X \left(F_\varepsilon \left(F_{V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right) - \mu_X \left(F_\varepsilon \left(F_{N,V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right) &= \left(\mu_{X_N} - \mu_X \right) \left(F_\varepsilon \left(F_{N,V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right)
\end{aligned}
$$

Note that $\sqrt{N} E \sup_\theta \left(\mu_{X_N} - \mu_X \right) \left(F_\varepsilon \left(F_{N,V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right) \leq \sqrt{N} E \sup_{\theta,v} \left(\mu_{X_N} - \mu_X \right) \left(F_\varepsilon (v - h(x;\theta)) \right) < \infty$. Thus, $E \sup_\theta \left| \mu_X \left(F_\varepsilon \left(F_{V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right) - \mu_X \left(F_\varepsilon \left(F_{N,V;\theta}^{-1}(\delta) - h(x;\theta) \right) \right) \right| = O\left(1/\sqrt{N}\right)$. Lemma D.2.12 implies that $\frac{d}{dv} \mu_X \left(F_\varepsilon (v - h(x;\theta)) \right) = \int f_\varepsilon (v - h(x;\theta)) \, d\mu_X$ is bounded away from 0 over all θ and all v in a compact intervals. Therefore, we have that $E \sup_\theta \left| F_{V;\theta}^{-1}(\delta) - F_{N,V;\theta}^{-1}(\delta) \right| \to 0$. Finally, for any $\delta > 0$ and $q \in (\delta, 1-\delta)$

$$
\nabla_\theta F_{V;\theta}^{-1}(q) = \frac{\int \nabla_\theta h(X;\theta) f_\varepsilon \left(F_{V;\theta}^{-1}(q) - h(X;\theta) \right) d\mu_X}{\int f_\varepsilon \left(F_{V;\theta}^{-1}(q) - h(X;\theta) \right) d\mu_X}
$$

is bounded over all $\theta \in \Theta$ since $\nabla_\theta h(X;\theta) \leq h_{LC}(X)$ and $\int h_{LC}(X)^2 \, d\mu_X < \infty$ and $\int f_\varepsilon \left(F_{V;\theta}^{-1}(q) - h(X;\theta) \right) d\mu_X$ is bounded away from zero. Hence, for r_N sufficiently small, the third term is

$$
\sup_{\|\theta-\theta_0\|\leq r_N} \left\{ \left| F_{V;\theta_0}^{-1}(\delta) - F_{V;\theta}^{-1}(\delta) \right| \geq \epsilon \right\} = 0.
$$

Therefore, there exists a sequence of $\tilde{\delta}_N$ decreasing to 0, such that

$$\sup_{\delta\in\left(\tilde{\delta}_N,1-\tilde{\delta}_N\right)}\left[E\sup_{\|\theta-\theta_0\|\le r_N}\left\{\left|F_{V;\theta}^{-1}(\delta)-F_{N,V;\theta}^{-1}(\delta)\right|\ge\epsilon\right\}+\sup_{\|\theta-\theta_0\|\le r_N}\left\{\left|F_{V;\theta_0}^{-1}(\delta)-F_{V;\theta}^{-1}(\delta)\right|\ge\epsilon\right\}\right]\to 0.$$

Since $\left\{F_{V;\theta_0}^{-1}\left(\tilde{\delta}_N\right)\ge V_1-2\epsilon\right\}\to 0$, we have that

$$E\sup_{\|\theta-\theta_0\|\le r_N}\left\{F_{N,V;\theta}^{-1}(\delta_N)\ge V_1\right\}\le E\sup_{\|\theta-\theta_0\|\le r_N}\left\{F_{N,V;\theta}^{-1}\left(\max\left(\delta_N,\tilde{\delta}_N\right)\right)\ge V_1\right\}\to 0.$$

Similar arguments show that $E\sup_{\|\theta-\theta_0\|\le r_N}\left\{F_{N,V;\theta}^{-1}(1-\delta_N)\le V_2\right\}\to 0$. It follows that there exist a sequence of $V_{1,N}\to-\infty,\ V_{2,N}\to\infty$ such that

$$\left[E\sup_{\|\theta-\theta_0\|\le r_N}\left[\left\{F_{N,V;\theta}^{-1}(\delta_N)\ge V_{1,N}\right\}+\left\{F_{N,V;\theta}^{-1}(1-\delta_N)\le V_{2,N}\right\}\right]\right]^{\frac{1}{2}}\to 0.$$

Therefore,

$$\sqrt{N}E\sup_{\|\theta-\theta_0\|\le r_N}\left(\int_{-\infty}^{\infty}-\int_{F_{N,V;\theta}^{-1}(\delta_N)}^{F_{N,V;\theta}^{-1}(1-\delta_N)}\right)\left|\int\Psi(x)\,f_\varepsilon\left(v-h\left(x;\theta_0\right)\right)\left(d\mu_{X_N}-d\mu_X\right)\right|dv\to 0.$$

■

E.5 Primitives for Assumption 6(ii) c.

Proposition E.5.9 *If Assumption E.2.9 is satisfied,then, for any sequence of positive numbers b_N decreasing to 0, and for any $\delta>0$,*

$$\sup_{\|\theta-\theta_0\|\le b_N}\left|\nabla_{(G_X,G_Z)}\psi^\delta(\theta)-\nabla_{(G_X,G_Z)}\psi^\delta(\theta_0)\right|=o_p(1).$$

Proof. For a fixed $\delta>0$, consider the Gaussian process $\nabla_{(G_X,G_Z)}\psi^\delta(\theta)$, indexed by Θ. The expression for this term (given in Appendix D.2) is a sum and product of finitely many terms of the form

$$\int_\delta^{1-\delta}\frac{\int\Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1,\theta_1)\,\phi_\varepsilon(q,x_2,\theta_1)\,\phi_\eta(q,z,\theta_1)\,dG_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon(q,x_1,\theta_1)\,\phi_\varepsilon(q,x_2,\theta_1)\,\phi_\eta(q,z,\theta_1)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq,$$

$$\text{and}\int_\delta^{1-\delta}G_V^q(\theta_1)\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\,\phi_\eta(q,z;\theta_1)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon(q,x_1;\theta_1)\,\phi_\varepsilon(q,x_2;\theta_1)\,\phi_\eta(q,z;\theta_1)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq$$

and analogous terms with G_Z and G_U^q instead of G_{X_1} and G_V^q. We will show that for any sequence of positive numbers b_N decreasing to 0,

$$\sup_{\|\theta-\theta_0\|\le b_N}\left|\nabla_{(G_X,G_Z)}\psi^\delta(\theta)-\nabla_{(G_X,G_Z)}\psi^\delta(\theta_0)\right|=o_p(1)$$

by individually analyzing these terms.

First consider the Gaussian process $\tilde{G}(\theta)$ indexed by Θ, given by

$$\tilde{G}(\theta) = \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1, x_2, z)\, \phi_{\varepsilon}(q, x_1, \theta_1)\, \phi_{\varepsilon}(q, x_2, \theta_1)\, \phi_{\eta}(q, z, \theta_1)\, dG_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q, x_1, \theta_1)\, \phi_{\varepsilon}(q, x_2, \theta_1)\, \phi_{\eta}(q, z, \theta_1)\, d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq.$$

We show that for any sequence of positive numbers b_N decreasing to 0, we have that

$$\sup_{\|\theta-\theta_0\| \leq b_N} \left| \tilde{G}(\theta) - \tilde{G}(\theta_0) \right| = o_p(1).$$

To do so, it is enough to show that $\tilde{G}$ has almost surely uniformly continuous sample paths in θ. By Dudley's Theorem (e.g. Theorem 2.6.1 of Dudley (2014)), $\tilde{G}(\theta)$ has almost surely uniformly continuous sample paths if $\int_0^{\infty} \sqrt{\log N_{\tilde{G}}(\epsilon)} d\epsilon$ is finite, where $N_{\tilde{G}}(\epsilon)$ is the $\epsilon - L_2$ covering number for $\tilde{G}$. Note that if $N_{\tilde{G}}(\epsilon) \leq C_0 \epsilon^d$ for some constant C_0 and natural number d, this integral is finite. A sufficient condition is that $\left(E \left(\tilde{G}(\theta_1) - \tilde{G}(\theta_2) \right)^2 \right)^{\frac{1}{2}} < K \|\theta_1 - \theta_2\|$ since Θ is finite dimensional.

Hence, we must bound

$$\begin{aligned}
& \left[E \left(\begin{array}{c} \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\phi_{\varepsilon}(q,x_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq dG_{X_1} \\ - \int_{\delta}^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq dG_{X_1} \end{array} \right)^2 \right]^{\frac{1}{2}} \\
= \; & Var \left(\begin{array}{c} \int_{\delta}^{1-\delta} \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\ - \int_{\delta}^{1-\delta} \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \end{array} \right)^{\frac{1}{2}} \\
\leq \; & \left(E \left(\int_{\delta_1}^{1-\delta_1} \left(\begin{array}{c} \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1)}{\int \phi_{\varepsilon}(q,x_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\ - \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \end{array} \right) dq d\mu_{X_2} d\mu_Z \right)^2 \right)^{\frac{1}{2}} \\
\leq \; & \left(E \left(\int_0^1 \left(\begin{array}{c} \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1)}{\int \phi_{\varepsilon}(q,x_1,\theta_1)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\ - \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \end{array} \right) dq d\mu_{X_2} d\mu_Z \right)^2 \right)^{\frac{1}{2}} \\
& + \left(E \left(\int_0^1 \left(\begin{array}{c} \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_1)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\ - \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_1) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \end{array} \right) dq d\mu_{X_2} d\mu_Z \right)^2 \right)^{\frac{1}{2}} \\
& + \left(E \left(\int_0^1 \left(\begin{array}{c} \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_1) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \\ - \frac{\int \Psi(X_1,x_2,z)\phi_{\varepsilon}(q,X_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_2} d\mu_Z}{\int \phi_{\varepsilon}(q,x_1,\theta_2)\phi_{\varepsilon}(q,x_2,\theta_2)\phi_{\eta}(q,z,\theta_2) d\mu_{X_1} d\mu_{X_2} d\mu_Z} \end{array} \right) dq d\mu_{X_2} d\mu_Z \right)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

By a change of variables, $v = F_{V;\theta_1}^{-1}(q)$, the first of these 3 terms equals

$$\left[E\left(\int_{-\infty}^{\infty}\frac{\int\Psi(X_1,x_2,z)\left(f_\varepsilon(v-h(X_1;\theta_1))-f_\varepsilon(v-h(X_1;\theta_2))\right)\phi_\varepsilon(q,x_2,\theta_2)\,\phi_\eta(q,z,\theta_1)\,d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon(q,x_2,\theta_2)\,\phi_\eta(q,z,\theta_1)\,d\mu_{X_2}d\mu_Z}dv\right)^2\right]^{\frac{1}{2}}$$

$$\begin{aligned}
&\le \|\Psi\|_\infty\left(E\left(\int_{-\infty}^{\infty}\left|\left(f_\varepsilon(v-h(X_1;\theta_1))-f_\varepsilon(v-h(X_1;\theta_2))\right)\right|dv\right)^2\right)^{\frac{1}{2}} \\
&= \|\Psi\|_\infty\left(E\left(h(X_1;\theta_1)-h(X_1;\theta_2)\right)^2\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\left|f_\varepsilon'(v)\right|dv \\
&\le \|\theta_1-\theta_2\|\left(\int h_{LC}(X)^2\,d\mu_X\right)^{\frac{1}{2}}\|\Psi\|_\infty\int_{-\infty}^{\infty}\left|f_\varepsilon'(v)\right|dv < K\,\|\theta_1-\theta_2\|
\end{aligned}$$

for a finite constant K. The next two terms are handled similarly. Hence, $\tilde{G}(\theta)$ has almost surely uniformly continuous sample paths.

By a similar argument, a bound on

$$\left[E\left[\begin{array}{c}\int_\delta^{1-\delta}G_V^q(\theta_1)\frac{\int\Psi(x_1,x_2,z)f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq \\ -\int_\delta^{1-\delta}G_V^q(\theta_2)\frac{\int\Psi(x_1,x_2,z)f_\varepsilon'\left(F_{V;\theta_2}^{-1}(q)-h(x_1;\theta_2)\right)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon(q,x_1;\theta_2)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\end{array}\right]^2\right]^{\frac{1}{2}}$$

implies that

$$\int_\delta^{1-\delta}G_V^q(\theta_1)\frac{\int\Psi(x_1,x_2,z)\,f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\,\phi_\eta(q,z;\theta_1)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int\phi_\varepsilon(q,x_1;\theta_1)\,\phi_\varepsilon(q,x_2;\theta_1)\,\phi_\eta(q,z;\theta_1)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq$$

has almost surely uniformly continuous sample paths. Note that

$$
\begin{aligned}
& \left[E\left[\begin{array}{l}
\int_\delta^{1-\delta} G_V^q\left(\theta_1\right) \frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z} dq \\
-\int_\delta^{1-\delta} G_V^q\left(\theta_2\right) \frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_2}^{-1}(q)-h(x_1;\theta_2)\right)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_2)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z} dq
\end{array}\right]^2\right]^{\frac{1}{2}} \\
= & \left[E\left[\begin{array}{l}
\int_\delta^{1-\delta} \frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)} \int G_X\left(1\left\{h(x;\theta_1)+\varepsilon \leq F_{V;\theta_1}^{-1}(q)\right\}\right) dF_\varepsilon \frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z} dq \\
-\int_\delta^{1-\delta} \frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)} \int G_X\left(1\left\{h(x;\theta_2)+\varepsilon \leq F_{V;\theta_2}^{-1}(q)\right\}\right) dF_\varepsilon \frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_2}^{-1}(q)-h(x_1;\theta_2)\right)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_2)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z} dq
\end{array}\right]^2\right]^{\frac{1}{2}} \\
\leq & \left[E\left[\begin{array}{c}
\int_\delta^{1-\delta}\left(\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)} \int G_X\left(1\left\{h(x;\theta_1)+\varepsilon \leq F_{V;\theta_1}^{-1}(q)\right\}\right) dF_\varepsilon-\frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)} \int G_X\left(1\left\{h(x;\theta_2)+\varepsilon \leq F_{V;\theta_2}^{-1}(q)\right\}\right) dF_\varepsilon\right) \times \\
\frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z} dq
\end{array}\right]^2\right]^{\frac{1}{2}} \\
& +\left[E\left[\begin{array}{c}
\int_\delta^{1-\delta} \frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)} \int G_X\left(1\left\{h(x;\theta_2)+\varepsilon \leq F_{V;\theta_2}^{-1}(q)\right\}\right) dF_\varepsilon \times \\
\left(\begin{array}{l}
\frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_2}^{-1}(q)-h(x_1;\theta_2)\right)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_2)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z} \\
-\frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}
\end{array}\right) dq
\end{array}\right]^2\right]^{\frac{1}{2}} \\
= & \left[E\left[\begin{array}{c}
\int_\delta^{1-\delta}\left(\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)} \int 1\left\{h(X;\theta_1)+\varepsilon \leq F_{V;\theta_1}^{-1}(q)\right\} dF_\varepsilon-\frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)} \int 1\left\{h(X;\theta_2)+\varepsilon \leq F_{V;\theta_2}^{-1}(q)\right\} dF_\varepsilon\right) \times \\
\frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z} dq
\end{array}\right]^2\right]^{\frac{1}{2}} \\
& +\left[E\left[\begin{array}{c}
\int_\delta^{1-\delta} \frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)} \int 1\left\{h(X;\theta_2)+\varepsilon \leq F_{V;\theta_2}^{-1}(q)\right\} dF_\varepsilon \times \\
\left(\begin{array}{l}
\frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_2}^{-1}(q)-h(x_1;\theta_2)\right)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_2)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2)d\mu_{X_1}d\mu_{X_2}d\mu_Z} \\
-\frac{\int \Psi(x_1,x_2,z) f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h(x_1;\theta_1)\right)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1)d\mu_{X_1}d\mu_{X_2}d\mu_Z}
\end{array}\right) dq
\end{array}\right]^2\right]^{\frac{1}{2}} \\
= & T_1+T_2
\end{aligned}
$$

where the last equality follows from the definition of G_X's covariance kernel. To bound T_1, note that for any $\delta>0$ and all $q \in (\delta, 1-\delta)$, we have that

$$
\left|\frac{\int \Psi\left(x_1, x_2, z\right) f_\varepsilon'\left(F_{V;\theta_1}^{-1}(q)-h\left(x_1;\theta_1\right)\right) \phi_\varepsilon\left(q, x_2;\theta_1\right) \phi_\eta\left(q, z;\theta_1\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon\left(q, x_1;\theta_1\right) \phi_\varepsilon\left(q, x_2;\theta_1\right) \phi_\eta\left(q, z;\theta_1\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z}\right|<M<\infty
$$

since $\inf_{\theta, q\in(\delta,1-\delta)} \int \phi_\varepsilon\left(q, x_1;\theta_1\right) \phi_\varepsilon\left(q, x_2;\theta_1\right) \phi_\eta\left(q, z;\theta_1\right) d\mu_{X_1} d\mu_{X_2} d\mu_Z>0$ (Lemma D.2.12) and the numerator is uniformly bounded. Hence, the T_1 no greater than

$$
\begin{aligned}
& M\left[E\left[\int_{\delta}^{1-\delta}\left|\begin{array}{l}\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_1\right)+\varepsilon\leq F_{V;\theta_1}^{-1}\left(q\right)\right\}dF_{\varepsilon}\\ -\frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_2\right)+\varepsilon\leq F_{V;\theta_2}^{-1}\left(q\right)\right\}dF_{\varepsilon}\end{array}\right|dq\right]^2\right]^{\frac{1}{2}} \\
\leq\ & M\left[E\left[\int_{\delta}^{1-\delta}\left|\begin{array}{l}\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_1\right)+\varepsilon\leq F_{V;\theta_1}^{-1}\left(q\right)\right\}dF_{\varepsilon}\\ -\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_2\right)+\varepsilon\leq F_{V;\theta_2}^{-1}\left(q\right)\right\}dF_{\varepsilon}\end{array}\right|dq\right]^2\right]^{\frac{1}{2}} \\
& +M\left[E\left[\int_{\delta}^{1-\delta}\left|\begin{array}{l}\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_2\right)+\varepsilon\leq F_{V;\theta_2}^{-1}\left(q\right)\right\}dF_{\varepsilon}\\ -\frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_2\right)+\varepsilon\leq F_{V;\theta_2}^{-1}\left(q\right)\right\}dF_{\varepsilon}\end{array}\right|dq\right]^2\right]^{\frac{1}{2}} \\
\leq\ & M\left[E\left[\int_{\delta}^{1-\delta}\left|\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}\left(q\right)\right)}\left(F_{\varepsilon}\left(F_{V;\theta_1}^{-1}\left(q\right)-h\left(X;\theta_1\right)\right)-F_{\varepsilon}\left(F_{V;\theta_2}^{-1}\left(q\right)+h\left(X;\theta_2\right)\right)\right)\right|dq\right]^2\right]^{\frac{1}{2}} \\
& +M\left[E\left[\int_{\delta}^{1-\delta}\left|\begin{array}{l}\frac{1}{f_{V;\theta_1}\left(F_{V;\theta_1}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_2\right)+\varepsilon\leq F_{V;\theta_2}^{-1}\left(q\right)\right\}dF_{\varepsilon}\\ -\frac{1}{f_{V;\theta_2}\left(F_{V;\theta_2}^{-1}(q)\right)}\int 1\left\{h\left(X;\theta_2\right)+\varepsilon\leq F_{V;\theta_2}^{-1}\left(q\right)\right\}dF_{\varepsilon}\end{array}\right|dq\right]^2\right]^{\frac{1}{2}} \\
=\ & R_1+R_2
\end{aligned}
$$

Note that

$$\nabla_{\theta}F_{V;\theta}^{-1}\left(q\right)=\frac{\int f_{\varepsilon}\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x;\theta\right)\right)\nabla_{\theta}h\left(x;\theta\right)d\mu_X}{f_{V;\theta}\left(F_{V;\theta}^{-1}\left(q\right)\right)}.$$

and $\inf_{\theta,q\in(\delta,1-\delta)}f_{V;\theta}\left(F_{V;\theta}^{-1}\left(q\right)\right)>0$ (Lemma D.2.12). Hence, R_1 is no greater than

$$
\begin{aligned}
& M\left(\frac{1}{\inf_{\theta,q\in(\delta,1-\delta)}f_{V;\theta}\left(F_{V;\theta}^{-1}\left(q\right)\right)}\right)\times \\
& \left[E\left[\int_{\delta}^{1-\delta}\left\|f_{\varepsilon}\right\|_{\infty}\left|\frac{\int f_{\varepsilon}\left(F_{V;\theta}^{-1}\left(q\right)-h\left(x;\theta\right)\right)\nabla_{\theta}h\left(x;\theta\right)d\mu_X}{f_{V;\theta}\left(F_{V;\theta}^{-1}\left(q\right)\right)}-\nabla_{\theta}h\left(X;\theta\right)\right|\left\|\theta_1-\theta_2\right\|dq\right]^2\right]^{\frac{1}{2}} \\
\leq\ & M\frac{\left\|f_{\varepsilon}\right\|_{\infty}\left\|\theta_1-\theta_2\right\|}{\inf_{\theta,q\in(\delta,1-\delta)}f_{V;\theta}\left(F_{V;\theta}^{-1}\left(q\right)\right)}\left(\left|\frac{\left\|f_{\varepsilon}\right\|_{\infty}}{\inf_{\theta,q\in(\delta,1-\delta)}f_{V;\theta}\left(F_{V;\theta}^{-1}\left(q\right)\right)}\int h_{LC}\left(x\right)d\mu_X\right|+\left(\int h_{LC}\left(x\right)^2d\mu_X\right)^{\frac{1}{2}}\right) \\
<\ & K_1M\left\|\theta_1-\theta_2\right\|
\end{aligned}
$$

since $\frac{1}{\inf_{\theta,q\in(\delta,1-\delta)}f_{V;\theta}\left(F_{V;\theta}^{-1}(q)\right)}$, $\left\|f_{\varepsilon}\right\|_{\infty}$ and $\left(\int h_{LC}\left(x\right)^2d\mu_X\right)^{\frac{1}{2}}$ are finite.

Similarly, to bound R_2, note that $\nabla_\theta \left(\frac{1}{f_{V;\theta}\left(F^{-1}_{V;\theta}(q)\right)} \right)$ is given by

$$
\begin{aligned}
& -\frac{\int \left(\nabla_\theta F^{-1}_{V;\theta}(q) - \nabla_\theta h(x;\theta)\right) f'_\varepsilon \left(F^{-1}_{V;\theta}(q) - h(x;\theta)\right) d\mu_X}{\left(f_{V;\theta}\left(F^{-1}_{V;\theta}(q)\right)\right)^2} \\
= \; & -\frac{1}{\left(f_{V;\theta}\left(F^{-1}_{V;\theta}(q)\right)\right)^2} \int \left(\frac{\int f_\varepsilon \left(F^{-1}_{V;\theta}(q) - h(x;\theta)\right) \nabla_\theta h(x;\theta)\, d\mu_X}{f_{V;\theta}\left(F^{-1}_{V;\theta}(q)\right)} - \nabla_\theta h(x;\theta) \right) f'_\varepsilon \left(F^{-1}_{V;\theta}(q) - h(x;\theta)\right) d\mu_X .
\end{aligned}
$$

Hence, $\sup_{\theta, q \in (\delta, 1-\delta)} \left| \nabla_\theta \left(\frac{1}{f_{V;\theta}\left(F^{-1}_{V;\theta}(q)\right)} \right) \right|$ is at most

$$
\left(\frac{1}{\inf_{\theta,q\in(\delta,1-\delta)} f_{V;\theta}\left(F^{-1}_{V;\theta}(q)\right)} \right)^2 \left\| f'_\varepsilon \right\|_\infty \left(\frac{1}{\inf_{\theta,q\in(\delta,1-\delta)} f_{V;\theta}\left(F^{-1}_{V;\theta}(q)\right)} \left\| f_\varepsilon \right\|_\infty + 1 \right) \int \left| h_{LC}(x) \right| d\mu_X < K_2 < \infty
$$

for each $q \in (\delta, 1-\delta)$. Therefore, the R_2 is at most $MK_2 \left\| \theta_1 - \theta_2 \right\|$. Similarly, since $\left| \frac{1}{f_{V;\theta_2}\left(F^{-1}_{V;\theta_2}(q)\right)} \int 1 \left\{ h(X;\theta_2) + \varepsilon \le F^{-1}_{V;\theta_2} (q \right.\right.$ is bounded, a uniform bound on the derivative of

$$
\frac{\int \Psi(x_1, x_2, z) f'_\varepsilon \left(F^{-1}_{V;\theta_1}(q) - h(x_1;\theta_1)\right) \phi_\varepsilon(q, x_2; \theta_1) \phi_\eta(q, z; \theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q, x_1; \theta_1) \phi_\varepsilon(q, x_2; \theta_1) \phi_\eta(q, z; \theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z}
$$

with respect to θ_1 implies that $T_2 \le K_3 \left\| \theta_1 - \theta_2 \right\|$ for some constant K_3. This follows from identical arguments as the ones above.

Hence,

$$
\left[E \left[\begin{array}{c} \int_\delta^{1-\delta} G^q_V(\theta_1) \frac{\int \Psi(x_1,x_2,z) f'_\varepsilon\left(F^{-1}_{V;\theta_1}(q) - h(x_1;\theta_1)\right) \phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_1)\phi_\varepsilon(q,x_2;\theta_1)\phi_\eta(q,z;\theta_1) d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \\ - \int_\delta^{1-\delta} G^q_V(\theta_2) \frac{\int \Psi(x_1,x_2,z) f'_\varepsilon\left(F^{-1}_{V;\theta_2}(q) - h(x_1;\theta_2)\right) \phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2) d\mu_{X_1} d\mu_{X_2} d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_2)\phi_\varepsilon(q,x_2;\theta_2)\phi_\eta(q,z;\theta_2) d\mu_{X_1} d\mu_{X_2} d\mu_Z} dq \end{array} \right]^2 \right]^{\frac{1}{2}} \le T_1 + T_2 \le K \left\| \theta_1 - \theta_2 \right\|
$$

for some constant $K \in (0, \infty)$.

The proof for the remaining terms in $\nabla_{(G_X, G_Z)} \psi^\delta(\theta)$ is analogous. Therefore,

$$
\left(E \left(\nabla_{(G_X,G_Z)} \psi^\delta(\theta_1) - \nabla_{(G_X,G_Z)} \psi^\delta(\theta_2) \right)^2 \right)^{\frac{1}{2}} < \tilde{K} \left\| \theta_1 - \theta_2 \right\|
$$

for some constant $\tilde{K}$, implying that the $\epsilon-$ L^2 covering numbers are bounded above by a polynomial in $\frac{1}{\epsilon}$, completing the proof. ■

E.6 Primitives for Assumption 6(ii) d.

Proposition E.6.10 *If $\|\Psi\|_\infty < \infty$, $\left\| \nabla \tilde{\psi}_q \right\|^2_\infty < \infty$, $F_{U;\theta_0}$ and $F_{V;\theta_0}$ have full support on $\mathbb{R}$, and and $g(Z;\theta_0)$ and $h(X;\theta_0)$ have finite second moments, then*

$$
\left| \nabla_{\tilde{G}} \psi^\delta \left[\mu_X, \mu_Z\right](\theta_0) - \nabla_{\tilde{G}} \psi^0 \left[\mu_X, \mu_Z\right](\theta_0) \right|
$$

converges in probability to 0 *as* $\delta \to 0$.

Proof. The expression for $\nabla_{\tilde{G}}\psi^{\delta}[\mu_X,\mu_Z]$ is given in equation (D.2.49). We show convergence of each of the terms in $Lim_{G;\delta}(\theta_0)$ as $\delta \to 0$. First, we show that

$$\left(\int_0^1 - \int_\delta^{1-\delta}\right) \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,dG_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq$$

converges weakly as $\delta \to 0$.

This term has mean zero and variance not greater than

$$\begin{aligned}
&\int \left[\left(\int_0^1 - \int_\delta^{1-\delta}\right) \frac{\int \Psi(X_1,x_2,z)\,\phi_\varepsilon(q,X_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right]^2 d\mu_{X_1}\\
\leq\;& \|\Psi\|_\infty^2 \int \left[\left(\int_0^1 - \int_\delta^{1-\delta}\right) \frac{\int \phi_\varepsilon(q,X_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\right]^2 d\mu_{X_1}\\
=\;& \|\Psi\|_\infty^2 \int \left[\left(\int_0^1 - \int_\delta^{1-\delta}\right) \frac{f_\varepsilon\left(F_{V;\theta_0}^{-1}(q) - h(X_1;\theta_0)\right)}{\int f_\varepsilon\left(F_{V;\theta_0}^{-1}(q) - h(X_1;\theta_0)\right)d\mu_{X_1}}dq\right]^2 d\mu_{X_1}\\
=\;& \|\Psi\|_\infty^2 \int \left[\left(\int_{-\infty}^{\infty} - \int_{F_{V;\theta_0}^{-1}(\delta)}^{F_{V;\theta_0}^{-1}(1-\delta)}\right) f_\varepsilon(v - h(X_1;\theta_0))\,dv\right]^2 d\mu_{X_1}.
\end{aligned}$$

where the last equality follows from a change of variables, $v = F_{V;\theta_0}^{-1}(q)$. Since $\int_{-\infty}^{\infty} f_\varepsilon(v - h(X_1;\theta_0))\,dv = 1$ for all X_1, and $F_{V;\theta_0}^{-1}(\delta) \to -\infty$ and $F_{V;\theta_0}^{-1}(1-\delta) \to \infty$ as $\delta \to 0$, the bound above converges to 0 as $\delta \to 0$ by the dominated convergence theorem. This proves that the term

$$\left(\int_0^1 - \int_\delta^{1-\delta}\right) \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,dG_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq$$

converges to 0 in probability as $\delta \to 0$.

Next, recall that

$$\frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)} \int G_Z\left(1\left\{g(z;\theta) + \eta \leq F_{U;\theta}^{-1}(q)\right\}\right) dF_\eta = G_U^q(\theta).$$

Consider the terms that include $G_U^q(\theta_0)$ in the expression for $\nabla_{(G_X,G_Z)}\psi^{\delta}[\mu_X,\mu_Z](\theta_0)$. The sum of these are given by

$$\begin{aligned}
&\int_\delta^{1-\delta} G_U^q(\theta_0) \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,f_\eta'\left(F_{U;\theta}^{-1}(q) - g(z;\theta_0)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq\\
&-\int_\delta^{1-\delta} \frac{\int \Psi(x_1,x_2,z)\,\phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z} \times\\
&G_U^q(\theta) \frac{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,f_\eta'\left(F_{U;\theta}^{-1}(q) - g(z;\theta_0)\right)d\mu_{X_1}d\mu_{X_2}d\mu_Z}{\int \phi_\varepsilon(q,x_1;\theta_0)\,\phi_\varepsilon(q,x_2;\theta_0)\,\phi_\eta(q,z;\theta_0)\,d\mu_{X_1}d\mu_{X_2}d\mu_Z}dq.
\end{aligned}$$

Note that this term is equal to

$$
\begin{aligned}
& \int_{\delta}^{1-\delta} G_U^q(\theta) \frac{\partial}{\partial q_3} \tilde{\psi}_q dq \\
= & \int_{\delta}^{1-\delta} \frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)} \int G_Z\left(1\left\{g(z;\theta)+\eta \leq F_{U;\theta}^{-1}(q)\right\}\right) dF_\eta \frac{\partial}{\partial q_3} \tilde{\psi}_q dq .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \nabla_{(G_X,G_Z)} \psi^{\delta}[\mu_X,\mu_Z](\theta_0) - \nabla_{(G_X,G_Z)} \psi^{0}[\mu_X,\mu_Z](\theta_0) \\
= & \left(\int_0^1 - \int_{\delta}^{1-\delta}\right) \frac{1}{f_{U;\theta}\left(F_{U,\theta}^{-1}(q)\right)} \int G_Z\left(1\left\{g(z;\theta)+\eta \leq F_{U;\theta}^{-1}(q)\right\}\right) dF_\eta \frac{\partial}{\partial q_3} \tilde{\psi}_q dq \\
= & \left(\int_{-\infty}^{\infty} - \int_{F_{U;\theta_0}^{-1}(\delta)}^{F_{U;\theta_0}^{-1}(1-\delta)}\right) \int G_Z\left(1\{g(z;\theta)+\eta \leq u\}\right) dF_\eta \left.\frac{\partial}{\partial q_3} \tilde{\psi}_q\right|_{q_3=F_{U;\theta}(u)} du
\end{aligned}
$$

has mean zero and variance not greater than

$$
\begin{aligned}
& \left\|\nabla \tilde{\psi}_q\right\|_\infty^2 \int \left[\left(\int_{-\infty}^{\infty} - \int_{F_{U;\theta_0}^{-1}(\delta)}^{F_{U;\theta_0}^{-1}(1-\delta)}\right)\left(\int \left(1\{g(Z;\theta_0)+\eta \leq u\}\right) dF_\eta - E\int \left(1\{g(Z;\theta_0)+\eta \leq u\}\right) dF_\eta\right) du\right]^2 d\mu_Z \\
= & \left\|\nabla \tilde{\psi}_q\right\|_\infty^2 \int \left[\left(\int_{-\infty}^{\infty} - \int_{F_{U;\theta_0}^{-1}(\delta)}^{F_{U;\theta_0}^{-1}(1-\delta)}\right)\left[F_\eta(u-g(Z;\theta_0)) - EF_\eta(u-g(Z;\theta_0))\right] du\right]^2 d\mu_Z .
\end{aligned}
$$

By the Efron-Stein inequality, let $Z^{(i)}$ have the same distribution as Z, and note that

$$
\begin{aligned}
& \int \left[\int_{-\infty}^{\infty} \left[F_\eta(u-g(Z;\theta_0)) - EF_\eta(u-g(Z;\theta_0))\right] du\right]^2 d\mu_Z \\
\leq & \frac{1}{2} \int\int \left[\int_{-\infty}^{\infty} \left[F_\eta(u-g(Z;\theta_0)) - F_\eta\left(u-g\left(Z^{(i)};\theta_0\right)\right)\right] du\right]^2 d\mu_Z d\mu_{Z^{(i)}} \\
= & \frac{1}{2} \int\int \left[\int_{-\infty}^{\infty} \left[\int_{g(Z;\theta_0)}^{g\left(Z^{(i)};\theta_0\right)} f_\eta(u-g)\, dg\right] du\right]^2 d\mu_Z d\mu_{Z^{(i)}} \\
= & \frac{1}{2} \int\int \left[\int_{g(Z;\theta_0)}^{g\left(Z^{(i)};\theta_0\right)} \int_{-\infty}^{\infty} f_\eta(u-g)\, du dg\right]^2 d\mu_Z d\mu_{Z^{(i)}} \\
= & \frac{1}{2} \int\int \left[g(Z;\theta_0) - g\left(Z^{(i)};\theta_0\right)\right]^2 d\mu_Z d\mu_{Z^{(i)}} \\
= & Var\left(g(Z;\theta_0)\right) < \infty
\end{aligned}
$$

where the second-last equality follows from the fact that $\int_{-\infty}^{\infty} f_\eta(u-g)\, du = 1$. Since $F_{V;\theta_0}^{-1}(\delta) \to -\infty$ and $F_{V;\theta_0}^{-1}(1-\delta) \to \infty$ as $\delta \to 0$,

$$
\int \left[\int_{-\infty}^{\infty} \left[F_\eta(u-g(Z;\theta_0)) - EF_\eta(u-g(Z;\theta_0))\right] du\right]^2 d\mu_Z
$$

converges to 0 as $\delta \to 0$ by the dominated convergence theorem.

The other terms in the expression for $Lim_{G;\delta}(\theta_0)$ converge to 0 in probability by analogous arguments. ■

F Parametric Bootstrap

Let $\{z_j\}_{j=1}^{J}$ be a sample of firm characteristics and $\{x_i\}_{i=1}^{N}$ denote a sample of worker characteristics. The parametric bootstrap for the estimate $\hat{\theta} = \arg\min_{\theta\in\Theta} \hat{Q}_N(\theta)$ is constructed by the following procedure for $b = \{1, \dots, 500\}$

1. Sample J firms with replacement from the empirical sample $\{z_j\}_{j=1}^{J}$. Denote this sample with $\left\{z_j^b\right\}_{j=1}^{J}$.
2. Draw N^b workers with replacement from the empirical sample $\{x_i\}_{i=1}^{N}$, where $N^b = \sum c_j^b$ and c_j^b is capacity of the j-th sampled firm in the bootstrap sample.
3. Simulate the unobservables ε_j^b and η_i^b.
4. Compute the quantities v_i^b and u_j^b at $\hat{\theta}$ from equations (17) and (18).
5. Compute a pairwise stable match for the bootstrap sample.
6. Compute $\hat{\theta}_b = \arg\min_{\theta\in\Theta} \hat{Q}_N^b(\theta)$ using the bootstrap pairwise stable match and an independent set of simulations for $\hat{Q}_N^b(\theta)$.