Crowding Out in Ricardian Economies

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\textsuperscript{†}I thank Joshua Abel and Michael Roberts for helpful discussions and Anna Cororaton and Amora Elsaify for excellent research assistance.

Abstract

The crowding-out coefficient is the ratio of the reduction in privately-issued bonds to the increase in government bonds that are issued to finance a tax cut. If (1) Ricardian equivalence holds, and (2) households do not simultaneously borrow risklessly and have positive gross positions in other riskless assets, the crowding-out coefficient equals the fraction of the aggregate tax cut that accrues to households that borrow. In the conventional case in which all households receive equal tax cuts, the crowding-out coefficient equals the fraction of households that borrow. In the United States, about 75\% of households borrow, so the crowding-out coefficient is predicted to be 0.75, which differs from econometric estimates that are around 0.5. I explore extensions of the model, such as a departure from Ricardian Equivalence or the introduction of cross-sectional variation in taxes, that might account for this difference.

Keywords: Crowding-Out Coefficient; Ricardian Equivalence

JEL classification: E62, H6
In the absence of transactions costs, any individual household could increase its holdings of riskless government bonds and reduce its holdings of privately-issued riskless bonds by equal amounts. Such a reallocation of riskless assets in the individual’s portfolio might be described as government bonds crowding out an equal value of privately-issued bonds. For an individual household, the crowding-out coefficient, which is the ratio of the decrease in privately-issued bonds to the increase in government bonds, equals one. In a closed economy, however, the household sector as a whole cannot simply increase its aggregate holdings of government bonds. The aggregate supply of government bonds is determined by the fiscal actions of the government—in particular, tax cuts or increases in government purchases will increase the amount of government bonds outstanding, and tax increases or cuts in government purchases will decrease the amount of government bonds outstanding. I will focus on Ricardian tax changes, which are lump-sum changes in current taxes accompanied by offsetting lump-sum changes in future taxes of equal present value. If Ricardian Equivalence holds, then a Ricardian tax cut has no effect on consumption, investment, output, or asset prices. Households will simply use the tax cut to purchase newly-issued government bonds and then will use these bonds, with accrued interest, to pay for increased future taxes levied to enable the government to pay the interest and principal on these bonds. No other net asset positions are affected and hence it would appear that the crowding-out coefficient is zero.

In a closed economy that consists entirely of identical households, there are no privately-issued bonds. However, in an open economy with identical households, domestic households can issue bonds that will be held by foreigners, and domestic tax cuts can crowd out domestic privately-issued bonds. If all domestic households choose to issue bonds, then a sufficiently small bond-financed tax cut will reduce the amount of bonds issued by domestic households by the amount of the tax cut; the crowding-out coefficient will be 1.0. If the tax cut

1 There are no financial intermediaries or firms that issue debt.
is as large as the amount of privately-issued bonds initially outstanding, it will completely
eliminate privately-issued bonds and the crowding-out coefficient will also be 1.0. Larger tax
cuts will also eliminate privately-issued bonds, and the amount of newly-issued government
bonds will exceed the reduction in privately-issued bonds. Thus, the crowding-out coefficient
will be less than 1.0 for sufficiently large tax cuts. As I show, the crowding-out coefficient
in an open economy with identical households can be anywhere between zero and one.²

To examine bonds issued by households in a closed economy requires relaxing the as-
sumption of identical households so that some households borrow from other households.
Barro and Mollerus (2014) analyze privately-issued bonds in a closed economy in which half
of the households are borrowers and the other half are lenders. They present an illuminating
numerical example in which Ricardian Equivalence holds and yet each additional dollar of
government bonds associated with a Ricardian tax cut will crowd out private bonds by 50
cents, so that the crowding-out coefficient is 0.5. This example clearly refutes, in a closed
economy, the argument above that Ricardian Equivalence implies zero crowding out when
the additional government bonds are issued to finance a lump-sum tax cut.

In this paper, I analyze the crowding out of bonds issued by households in an economy
in which Ricardian Equivalence holds. I adopt the assumption in Barro and Mollerus
(2014) that any given household cannot simultaneously issue riskless bonds and hold positive
amounts of riskless assets. I extend and generalize the analysis of crowding out in six
ways: (1) I allow an arbitrary cross-sectional distribution of riskless asset positions, including
borrowers who have various negative positions in riskless assets and lenders who have various
positive positions in riskless assets; (2) I allow for open as well as closed economies; (3) I
analyze tax increases as well as tax cuts and show that the crowding-out coefficient for tax
increases is greater than or equal to the crowding-out coefficient for tax cuts; (4) I allow tax
cuts that are large enough to induce some borrowers to become lenders and tax increases

²More precisely, it can be any rational number between zero and one.
that are large enough to induce some lenders to become borrowers; (5) I generalize the
notion of Ricardian tax changes to allow for cross-household heterogeneity in the size of the
current lump-sum tax change; and (6) I do not restrict the analysis to particular parametric
specifications of utility functions or the evolution of output. I show that the crowding-out coefficient equals the fraction of the aggregate tax cut that accrues to households that borrow; if all households receive identical lump-sum tax cuts, the crowding-out coefficient is simply the fraction of households that borrow.

To obtain an empirical measure of the crowding-out coefficient in the model, both with identical tax changes across households and with cross-household heterogeneity in tax changes, I use data from the Survey of Consumer Finances and from the Congressional Budget Office. I find that in the conventional case in which all households receive identical small tax cuts, the crowding-out coefficient, which simply equals the fraction of households that borrow, is about 0.75. Allowing for realistic heterogeneity in tax changes increases the crowding-out coefficient to about 0.85. Krishnamurthy and Vissing-Jorgenson (2013, p. 1) and Gorton, Lewellen, and Metrick (2012, Table 1) empirically estimate the crowding-out coefficient to be around 0.5. To address this discrepancy from the prediction of the model, I introduce a particular departure from Ricardian Equivalence, parametrized by $0 \leq \gamma \leq 1$, where $\gamma = 0$ represents Ricardian Equivalence and $\gamma = 1$ represents the situation in which households do not change their total holdings of riskless assets at all in response to a bond-financed lump-sum tax cut. It turns out that $\gamma = 1/3$ reconciles the value of the crowding-out coefficient in the model under the conventional assumption of identical tax changes across households with empirical estimates of the crowding-out coefficient. With cross-household heterogeneity in tax changes, a value of $\gamma = 0.412$ reconciles the model with empirical estimates.

I describe the Ricardian economy and the menu of safe assets and safe liabilities in Section 1. In that section, I also analyze the implications of the restriction that individuals will not
simultaneously borrow risklessly and hold positive amounts of riskless assets. In Section 2., I define Ricardian tax changes, which lead each household to reduce its net position in riskless assets by the amount of the tax increase it faces, or equivalently, to increase its net position in riskless assets by the amount of the cut it receives. I define the crowding-out coefficient in Section 3., and show that it is weakly increasing in the size of a Ricardian tax change—that is, weakly increasing in the size of a Ricardian tax increase and weakly decreasing in the size of a Ricardian tax cut. Section 4. focuses on the conventional case in which all households receive identical tax cuts. This section also presents a closed-form solution for the crowding-out coefficient in open economies populated by identical households who borrow from foreign lenders. In Section 5., I introduce a particular class of non-Ricardian economies parametrized by a scalar that measures the departure from Ricardian Equivalence. In Section 6., I use data from the Survey of Consumer Finances to measure the model’s crowding-out coefficient if all households receive identical tax cuts. To analyse the crowding-out coefficient when there is cross-sectional variation in tax cuts, I also use data from the Congressional Budget Office on the share of total Federal taxes paid by each quintile of income. Section 7. presents concluding remarks.

1. **A Ricardian Economy**

The conditions under which Ricardian Equivalence holds have been widely studied in the four decades since Barro’s (1974) seminal article. Ricardian Equivalence does not depend on the particular specification of utility functions or technologies, the presence or absence of risk, or whether the economy is open or closed. Because Ricardian Equivalence holds in such a wide array of situations, I will not specify a particular utility function or technology. I assume that all economic agents, including the government, have the opportunity to borrow and lend risklessly at the same riskless interest rate. I assume that households’ positions in riskless assets are determined by a standard intertemporal first-order condition, often called
an Euler equation. That is, there are no operative constraints on borrowing or lending. I also assume that any agent who receives a lump-sum tax cut at the current time will, with certainty, face an increase in future lump-sum taxes with present value equal to the current tax cut. The optimal response to such a lump-sum tax cut is to use the tax cut to purchase riskless bonds and to use those bonds, with accrued interest, to pay the increased future lump-sum taxes. With this response, there is no change in optimal consumption, investment, output, or equilibrium asset prices. For simplicity, I assume that all economic activity is conducted directly by households; there are no firms or financial intermediaries. I do not model any activity at future times. All that is relevant about the future is that any household that receives a lump-sum tax cut at the current time will, with certainty, pay increased future lump-sum taxes with present value equal to the current tax cut. It does not matter when future taxes are increased, as long as the household will have to pay the increased taxes (or the household’s heirs, linked to the household by operative altruistic bequest motives, have to pay the increased future taxes). Nor does the maturity of the bonds matter, provided that the sequence of additional government bonds can, if held by households, be used to pay the increased future lump-sum taxes. The focus of the analysis will be the optimal holdings of riskless assets by households at the current time, which will increase by the amount of any lump-sum cut in current taxes matched by an increase in future lump-sum taxes of equal present value.

1.1. Riskless Assets Held by Households

Consider an open economy populated by a unit measure of domestic households indexed by $i \in [0, 1]$. All households of a given type $i$ are identical in all respects. They have identical preferences and have identical opportunities to earn income and to hold assets and issue liabilities. Each household optimally chooses consumption and saving and a portfolio of assets that may include both riskless and risky assets and liabilities. In particular, households can hold government bonds, which are riskless, and can issue riskless liabilities.
in the form of bonds. There are no binding constraints on the intertemporal allocation of consumption, so that optimal positions in riskless assets are characterized by a standard Euler equation, and hence Ricardian Equivalence holds. That is, a bond-financed lump-sum cut in current taxes that is offset by a riskless future lump-sum tax increase of equal present value will have no effect on consumption or asset prices; it will, however, increase households’ net positions in riskless assets by an amount equal to the tax cut received.

Let \( A(i) \) be the net holding of riskless assets (described below) by households of type \( i \). Households with \( A(i) < 0 \) are borrowers. Define the measure \( F(i) \) of households so that \( \int_{A(i)<0} dF(i) \) is the fraction of households that are borrowers, that is, issuers of riskless private bonds.

Domestic households of type \( i \) may hold four types of riskless assets in the following amounts: (1) \( b^G(i) \geq 0 \) of riskless domestic government bonds, where the non-negativity constraint reflects the fact that households cannot issue government bonds and the assumption that they cannot short government bonds; (2) \( b^P(i) \) of riskless bonds issued by domestic households. For domestic households that hold these bonds as assets, their net holdings are \( b^P(i) > 0 \), and for domestic households that issue these bonds as liabilities, their net holdings are \( b^P(i) < 0 \); (3) \( a^H(i) \geq 0 \) of other domestic riskless assets (the superscript \( H \) denotes assets in the home country), such as riskless physical assets, where the non-negativity constraint reflects the assumption that households cannot short physical assets; and (4) \( a^F(i) \geq 0 \) of foreign riskless assets, including foreign-issued riskless bonds, where the non-negativity constraint reflects the assumption that domestic households cannot short foreign assets. Of course, households in an open economy can issue their own riskless bonds, which could be held by foreign agents; such bonds would appear as negative values of \( b^P(i) \) for domestic households.

\( ^3 I \) will not specify the utility function other than that utility is an increasing concave function of the household’s consumption at every point of time and does not depend directly on taxes *per se* or asset holdings *per se*. 
Household $i$’s overall holding of riskless assets, $A(i)$, is

$$A(i) = a^H(i) + a^F(i) + b^G(i) + b^P(i),$$

where

$$a^H(i) \geq 0, a^F(i) \geq 0, b^G(i) \geq 0.$$  \hfill (2)

All four riskless assets pay the same interest rate. Therefore, a household’s allocation of a given positive total amount of riskless assets, $A(i) > 0$, among non-negative holdings of the four types of riskless assets is indeterminate. However, following Barro and Mollerus (2014), I assume that individual households will never borrow and lend simultaneously.\(^4\)\(^5\) Therefore, $b^P(i)$ cannot be negative if any of the other holdings of riskless assets, $a^H(i)$, $a^F(i)$, and $b^G(i)$, are positive. That is,

$$b^P(i) a^H(i) \geq 0$$  \hfill (3)

$$b^P(i) a^F(i) \geq 0$$  \hfill (4)

and

$$b^P(i) b^G(i) \geq 0.$$  \hfill (5)

**Lemma 1** Assume that equations (2), (3), (4), and (5) hold.

1. If $b^P(i) < 0$, then $A(i) = b^P(i) < 0$ and
2. If $A(i) \leq 0$, then $b^P(i) = A(i) \leq 0$.

\(^4\)Of course, this assumption rules out financial intermediaries as economic agents.

\(^5\)Barro and Mollerus (2014, p. 26) assume that "there is an infinitesimal amount of transaction costs for bond issuance or collection of interest and principal" that prevents individual households from simultaneously borrowing and lending. Under such costs, households may avoid vanishingly small positive or negative positions in riskless assets and their positions in riskless assets may be insensitive to vanishingly small tax changes. To avoid that potential insensitivity, one could assume that households incur a fixed cost if they hold both positive and negative gross positions in riskless assets but do not incur these costs if they hold only one of the these gross positions. That assumption is tantamount to simply assuming that no households simultaneously borrow and lend riskless assets, and I will simply adopt that assumption here.
Lemma 1 implies that a household is a borrower, that is, an issuer of private bonds, if and only if its net holding of riskless assets is negative. Therefore, the aggregate amount of riskless bonds issued by domestic households is

\[ B^P \equiv -\int_{A(i)<0} b^P(i) dF(i) \geq 0. \]  

(6)

Statement 2 of Lemma 1 directly implies the following corollary, which states that the aggregate amount of riskless bonds issued by domestic households, \( B^P \), equals the negative of the aggregate value of the net positions in riskless assets of households with \( A(i) < 0 \).

**Corollary 2 to Lemma 1** \( B^P \equiv -\int_{A(i)<0} b^P(i) dF(i) = -\int_{A(i)<0} A(i) dF(i). \)

In a closed economy, every outstanding privately-issued bond is the liability of some household and an asset of other some other household. Therefore, \( \int_0^1 b^P(i) dF(i) = 0 \), so \( B^P \equiv -\int_{A(i)<0} b^P(i) dF(i) = \int_{A(i)\geq0} b^P(i) dF(i) \). That is, in a closed economy, the amount of privately-issued riskless bonds can be measured either as the aggregate amount of these bonds issued by domestic borrowers or as the aggregate amount of these bonds held by domestic lenders. In an open economy, some domestic privately-issued riskless bonds may be held by foreign holders, so that \( \int_0^1 b^P(i) dF(i) \leq 0 \). Therefore, in an open economy, the aggregate amount of riskless bonds issued by domestic private borrowers, \( B^P \equiv -\int_{A(i)<0} b^P(i) dF(i) \), will be greater than or equal to the aggregate holding of these bonds by domestic lenders, \( \int_{A(i)\geq0} b^P(i) dF(i) \), because foreigners may hold some of these privately-issued domestic bonds.\(^6\)

2. Ricardian Tax Changes

Consider a tax change that increases current lump-sum taxes paid by each household of type \( i \) by \( \tau z(i) \) and reduces that household’s future lump-sum taxes by an amount with

\(^6\)Any lending by domestic households to foreign borrowers is part of \( a^F(i) > 0 \).
present value known to equal \( \tau z(i) \), where \( z(i) > 0 \). The factor \( z(i) \) allows for cross-sectional variation in the size of the change in current taxes for households of different types. I will normalize \( z(i) \) so that \( \int z(i) \, dF(i) = 1 \), which implies that the aggregate increase in current taxes is \( \tau \). If \( \tau > 0 \), households pay increased current taxes and receive a cut in future taxes. Alternatively, if \( \tau < 0 \), households receive a cut in current taxes and face an increase in future taxes. In either case, the present value of the current and future taxes paid by each household is unchanged by the tax change. I will call this tax change a *Ricardian tax change*.

Assume that households do not face any binding constraints on the intertemporal allocation of consumption and that Ricardian Equivalence holds. That is, in response to a Ricardian tax change, households of type \( i \) do not change consumption or the holdings of any assets, except that they reduce their net holdings of riskless assets, \( A(i) \), by \( \tau z(i) \) to pay for the increased current taxes. If \( \tau < 0 \), households receive a cut in current taxes in the amount \(-\tau z(i)\) and increase their holdings of riskless assets by \(-\tau z(i)\) to pay their increased future taxes. These additional riskless assets are made available by the government, which finances the aggregate tax cut by issuing additional bonds in the amount \(-\tau > 0\). The following definition formalizes the notion of a Ricardian tax change in an economy in which Ricardian Equivalence holds.

**Definition 3** Under Ricardian Equivalence, a Ricardian tax increase of aggregate size \( \tau \) increases current lump-sum taxes of type \( i \) households by \( \tau z(i) \), reduces their future taxes by an equal present value, and induces them to reduce their current holdings of riskless assets by \( \tau z(i) \).

Note that Definition 3 allows \( \tau \) to be negative as well as positive. When \( \tau > 0 \), households pay increased taxes in the current period and receive a tax cut in the future; when \( \tau < 0 \), households receive a tax cut in the current period and pay increased taxes in the future.

Consider an initial situation, before a tax change, and use the subscript 0 to denote the values of variables in this situation. From Corollary 2, the aggregate amount of privately-
issued domestic bonds in this initial situation is

\[ B_0^P = - \int_{A_0(i)<0} A_0(i) dF(i). \]  \hspace{1cm} (7)

In response to a Ricardian tax change of \( \tau z(i) \), regardless of whether \( \tau \) is positive or negative, households of type \( i \) change their holdings of riskless assets, \( A(i) \), by \(-\tau z(i)\) to \( A_1(i) = A_0(i) - \tau z(i) \), so the total amount of privately-issued domestic bonds after the tax change is

\[ B_1^P \equiv - \int_{A_0(i)<\tau z(i)} (A_0(i) - \tau z(i)) dF(i). \]  \hspace{1cm} (8)

The change in the amount of privately-issued domestic bonds associated with the tax change can be calculated by subtracting equation (7) from equation (8) to obtain

\[ B_1^P - B_0^P = \int_{A_0(i)<0} A_0(i) dF(i) - \int_{A_0(i)<\tau z(i)} (A_0(i) - \tau z(i)) dF(i). \]  \hspace{1cm} (9)

Equation (9) holds for reductions in current taxes \( (\tau < 0) \) as well as for increases in current taxes \( (\tau > 0) \).

2.1. Ricardian Tax Cuts: Incumbent Borrowers, Former Borrowers, and Non-Borrowers

Consider the case with \( \tau < 0 \) so that households receive a cut in their current taxes. Households of type \( i \) receive a tax cut of \(-\tau z(i) > 0\) and increase their net holdings of riskless assets to \( A_0(i) - \tau z(i) \). It is convenient to put each household into one of three categories depending on whether the household borrows before and after the tax cut (incumbent borrowers), borrows before but not after the tax cut (former borrowers), or does not borrow either before or after the tax cut (non-borrowers). Households with \( A_0(i) < \tau z(i) < 0 \) initially borrow more than \(-\tau z(i)\) and have \( A_1(i) = A_0 - \tau z(i) < 0 \) after the tax cut, so they will continue to borrow after the tax cut. Therefore, these households are incumbent borrowers. Households with \( \tau z(i) \leq A_0(i) < 0 \) borrow an amount less than or equal to \(-\tau z(i)\) before the tax cut, but since \( A_1(i) = A_0(i) - \tau z(i) \geq 0 \), they will not borrow after
the tax cut. Therefore, these households are *former borrowers*. Finally, households with $A_0(i) \geq 0$ will have $A_1(i) > 0$ after the tax cut and will be called *non-borrowers* because they do not borrow either before or after the tax cut.

In the case of tax cuts ($\tau < 0$), the change in the aggregate amount of privately-issued bonds in equation (9) can be rewritten as

$$B_1^P - B_0^P = \int_{\tau z(i) \leq A_0(i) < 0} A_0(i) \, dF(i) + \tau \int_{A_0(i) < \tau z(i)} z(i) \, dF(i) \leq 0, \quad \text{for } \tau < 0. \quad (10)$$

The left-hand side of equation (10) is the (negative of the) amount of privately-issued domestic bonds that are crowded out by the Ricardian tax cut. The first term on the right hand side of equation (10) is the (negative of the) amount of bonds initially issued by former borrowers; these former borrowers pay off these bonds completely when they receive the tax cut. The second term is the (negative of the) reduction in bonds issued by incumbent borrowers; these incumbent borrowers each reduce their bonds outstanding by $-\tau z(i)$.

### 2.2. Ricardian Tax Increases: Incumbent Borrowers, New Borrowers, and Non-Borrowers

Now consider the case in which $\tau > 0$ so that households of type $i$ face an increase of $\tau z(i) > 0$ in current lump-sum taxes and receive a future lump-sum tax cut with known present value equal to $\tau z(i)$. After the tax increase, there will be incumbent borrowers, new borrowers, and non-borrowers. Households with $A_0(i) < 0$ are incumbent borrowers because they have negative positions in riskless assets before and after the tax increase since $A_1(i) = A_0(i) - \tau z(i) < A_0(i) < 0$. Households with $0 \leq A_0(i) < \tau z(i)$ are new borrowers because they have non-negative positions in riskless assets before the tax increase and negative positions $A_1(i) = A_0(i) - \tau z(i) < 0$ after the tax increase. Households with $A_0(i) \geq \tau z(i) > 0$ have $A_1(i) = A_0(i) - \tau z(i) \geq 0$ and thus have non-negative positions in riskless assets before and after the tax increase; they are non-borrowers.

In the case of a tax increase ($\tau > 0$), the change in the aggregate amount of privately-
issued bonds in equation (9) can be written as

\[ B_1^P - B_0^P = \tau \int_{A_0(i) < \tau z(i)} z(i) \, dF(i) - \int_{0 \leq A_0(i) < \tau z(i)} A_0(i) \, dF(i), \quad \text{for } \tau > 0. \] (11)

The first term on the right hand side of equation (11) is the aggregate increase in taxes paid by incumbent borrowers and new borrowers. The second term is the amount of positive riskless assets held by new borrowers before the tax increase. They sell all of these riskless assets to pay (at least part of) their increased current taxes. Thus, the right hand side of equation (11) is the increase in borrowing by incumbent borrowers and new borrowers as a result of the current tax increase.

2.3. Shares of the Aggregate Tax Change

The impact of Ricardian tax changes depends on the shares of the aggregate tax change paid by, or accruing to, borrowers of various types. To simplify notation, I will define the function \( G(a; Z) \) to be the share of the aggregate tax change that accrues to households for whom \( \frac{A_0(i)}{z(i)} < a \), which implies that initial riskless assets, \( A_0(i) \), are less than \( az(i) \). Specifically, let \( Z \) represent the cross-sectional distribution of \( z(i) > 0 \) and define

\[ G(a; Z) \equiv \int_{\frac{A_0(i)}{z(i)} < a} z(i) \, dF(i) \] (12)

as the share of an aggregate tax change \( a \) that is paid (or, if \( a < 0 \), is received) by households with initial positions in riskless assets, \( A_0(i) \), less than \( az(i) \). Observe that \( G(a; Z) \) is (weakly) increasing in \( a \). \( G(a; Z) \) can be discontinuous at \( a \) if the distribution \( F(i) \) has positive mass at \( A_0(i) = az(i) \). In that case, \( G(a^+; Z) \equiv \lim_{a \searrow a} G(x; Z) > G(a; Z) \).

Recall from Lemma 1 that if \( A(i) \leq 0 \), then \( b^P(i) = A(i) \). Therefore, if \( a \leq 0 \), then

\[ G(a; Z) \equiv \int_{\frac{A_0(i)}{z(i)} < a} z(i) \, dF(i) = \int_{\frac{A_0(i)}{z(i)} < a} z(i) \, dF(i), \] which is the fraction of the aggregate tax change \( \tau \) that is paid (or received, if \( \tau < 0 \)) by households with initial net holdings of domestic privately-issued bonds less than \( az(i) \leq 0 \). Therefore, \( G(0; Z) \) is the fraction of
the aggregate tax change that accrues to or from households that are borrowers before the
tax change. After the tax change, the set of borrowers (consisting of incumbent borrowers
and, in the case of a tax increase, new borrowers) is the set for which \( A_1 (i) = A_0 - \tau z (i) < 0, \)
or equivalently, \( \frac{A_0}{z(i)} < \tau. \) Therefore, \( G (\tau; Z) \) is the fraction of the aggregate tax change that
accrues to households that are borrowers after the tax change.

If, as is standard in analyses of Ricardian Equivalence, there is no cross-sectional variation
in changes in current taxes or in future taxes, then \( z (i) \equiv 1. \) I will use the notation \( G (a; 1) \)
to denote the value of \( G (a; Z) \) in this case. It follows immediately from the definition of
\( G (a; Z) \) that \( G (a; 1) = \int_{A_0 (i) < a} dF (i), \) so that \( G (0; 1) \) is the fraction of households that are
borrowers before the tax change and \( G (\tau; 1) \) is the fraction of households that are borrowers
after the tax change.

The function \( G (a; 1) \) provides a simple characterization of the fractions of households
that comprise the various categories of borrowers after a tax change. First consider a tax
increase so that \( \tau > 0 \) and \( G (\tau; 1) \geq G (0; 1) \) since \( G (a; 1) \) is (weakly) increasing in \( a. \)
With \( \tau > 0, \) there are \( G (0; 1) \) incumbent borrowers; \( G (\tau; 1) - G (0; 1) \) new borrowers; and
\( 1 - G (\tau; 1) \) non-borrowers. Now consider a tax cut so that \( \tau < 0 \) and \( G (0; 1) \geq G (\tau; 1). \)
With \( \tau < 0, \) there are \( G (\tau; 1) \) incumbent borrowers; \( G (0; 1) - G (\tau; 1) \) former borrowers;
and \( 1 - G (0; 1) \) non-borrowers. These population shares of various borrower categories are
summarized in Table 1.

3. The Crowding-Out Coefficient

Define \( \theta (\tau) \) to be the crowding-out coefficient associated with a Ricardian tax change of
size \( \tau \neq 0. \) If \( \tau > 0, \) so the tax change is an increase in current taxes, then \( \theta (\tau) \) is the
increase in privately-issued domestic bonds divided by the decrease in government bonds
outstanding, \( \tau. \) Alternatively, if \( \tau < 0, \) so that the tax change is a cut in current taxes,
then \( \theta (\tau) \) is the reduction in the amount of privately-issued domestic bonds divided by the
amount of additional domestic government bonds outstanding, \(-\tau\). Formally,

\textbf{Definition 4} The crowding-out coefficient associated with a Ricardian tax change of size \(\tau \neq 0\) is \(\theta (\tau) \equiv \frac{1}{\tau} (B_1^\tau - B_0^\tau)\). For vanishingly small tax changes, define \(\theta (0^+) \equiv \lim_{\tau \searrow 0} \theta (\tau)\) and \(\theta (0^-) \equiv \lim_{\tau \nearrow 0} \theta (\tau)\).

The following lemma presents an expression for the crowding-out coefficient that follows directly from substituting equation (9) into Definition 4.

\textbf{Lemma 5} The crowding-out coefficient associated with a Ricardian tax change of size \(\tau \neq 0\) is \(\theta (\tau) = \frac{1}{\tau} \int_{A_0(i) < 0} A_0 (i) dF (i) - \frac{1}{\tau} \int_{A_0(i) < \tau z(i)} A_0 (i) dF (i) + G (\tau; 1)\).

Lemma 5 provides an exact expression for the crowding-out coefficient as a function of \(\tau\). In this section, I analyze properties of this function.

\textbf{Proposition 6} If \(\tau > 0\), then \(G (0^+; Z) \leq \theta (\tau) \leq G (\tau; Z)\). If \(\tau < 0\), then \(G (\tau; Z) \leq \theta (\tau) \leq G (0; Z)\).

In the case of a tax increase \((\tau > 0)\), there are \(G (0; Z)\) incumbent borrowers, each of whom increases borrowing by the amount of their tax increase, and there are \(G (\tau; Z) - G (0; Z)\) new borrowers, who increase their borrowing by an amount less than or equal to the tax increase. Therefore, in the aggregate, incumbent borrowers increase their borrowing by \(\tau G (0; Z)\) and new borrowers increase their borrowing by an amount no greater than \(\tau (G (\tau; Z) - G (0; Z))\), so the total increase in private borrowing is no more than \(\tau G (\tau; Z)\). Households with \(A_0 (i) = 0\) are new borrowers and increase their borrowing by the full amount of the tax increase. Therefore, the aggregate increase in borrowing by all new borrowers is at least \(\tau (G (0^+; Z) - G (0; Z))\) so the aggregate increase in borrowing by incumbent borrowers and new borrowers is at least \(\tau G (0^+, Z)\). Therefore, \(G (0^+; Z) \leq \theta (\tau) \leq G (\tau; Z)\).

In the case of a tax cut \((\tau < 0)\), there are \(G (\tau; Z)\) incumbent borrowers, each of whom reduces borrowing by the amount of their tax cut, and there are \(G (0; Z) - G (\tau; Z)\) former
borrowers, who reduce their borrowing by an amount less than or equal to the tax cut. Therefore, in the aggregate, incumbent borrowers reduce their borrowing by \(-\tau G(\tau; Z)\) and former borrowers reduce their borrowing by an amount no greater than \(-\tau (G(0; Z) - G(\tau; Z))\).

Hence the total reduction in private borrowing is at least \(-\tau G(\tau; Z)\) but no more than \(-\tau G(0; Z)\), so \(G(\tau; Z) \leq \theta (\tau) \leq G(0; Z)\).

The following corollary presents expressions for the crowding-out coefficients for vanishingly small tax increases, \(\theta (0^+) \equiv \lim_{\tau \searrow 0} \theta (\tau)\), and vanishingly small tax cuts, \(\theta (0^-) \equiv \lim_{\tau \nearrow 0} \theta (\tau)\).

**Corollary 7** \(\theta (0^-) = G(0; Z) \leq G(0^+; Z) = \theta (0^+), \) where \(G(0^+; Z) \equiv \lim_{\tau \searrow 0} G(\tau; Z)\).

The crowding-out coefficient for a vanishingly small tax increase, \(\theta (0^+)\), will exceed the crowding-out coefficient for a vanishingly small tax cut, \(\theta (0^-)\), if \(G(0^+; Z) > G(0; Z)\), that is, if a non-infinitesimal share of the aggregate tax increase is paid by households with initial riskless assets precisely equal to zero. In this situation, this positive mass of households are non-borrowers in the event of a tax cut, but are new borrowers in the case of a tax increase. As new borrowers, they further increase the amount by which privately-issued domestic bonds increase in addition to the increase in bonds issued by incumbent borrowers. Therefore, if there is a positive mass of households with precisely zero riskless assets in the initial situation, the crowding-out coefficient for a tax increase exceeds the crowding-out coefficient for a tax cut, even if the tax changes are small.

**Proposition 8** If \(\tau_2 > \tau_1\) for nonzero \(\tau_1\) and \(\tau_2\), then \(\theta (\tau_2) \geq \theta (\tau_1)\).

Proposition 8 states that the crowding-out coefficient, \(\theta (\tau)\), is (weakly) increasing in \(\tau\). Therefore, \(\theta (\tau)\) is (weakly) increasing in the aggregate size of a tax increase, \(\tau > 0\), but is (weakly) decreasing in the aggregate size of a tax cut \(-\tau > 0\). First consider a tax increase. A tax increase, \(\tau > 0\), induces some households that were not borrowers before the tax increase to become new borrowers after the tax increase. These new borrowers do not reduce
the measure of incumbent borrowers, all of whom increase their borrowing by the amount of
their increased taxes. Therefore, these new borrowers increase the amount of privately-issued
domestic bonds outstanding beyond the increase in borrowing from incumbent borrowers.
Since larger tax increases induce more households to become new borrowers, without reducing
the measure of incumbent borrowers, the crowding-out coefficient \( \theta (\tau) \) is larger for larger
tax increases.

Now consider a tax cut. A tax cut, \( -\tau > 0 \), induces some initial borrowers to pay off
all of their outstanding bonds and to become former borrowers. Further increases in the
tax cut cannot further reduce the debt issued by these households since they have already
paid off their debt. Former borrowers reduce their borrowing by an amount less than or
equal to the amount that incumbent borrowers reduce their borrowing. Since larger tax cuts
induce more households to become former borrowers and reduce the measure of incumbent
borrowers, the crowding-out coefficient, \( \theta (\tau) \), is smaller for larger tax cuts.

Proposition 8 immediately implies the following corollary.

**Corollary 9** Starting from a given initial situation, the crowding-out coefficient for tax
increases is at least as large as the crowding-out coefficient for tax cuts, with strict inequality
for large tax changes.

4. **Identical Tax Changes for All Households**

I have broadened the concept of Ricardian tax changes (Definition 3) to allow for cross-
sectional variation in tax changes. This more expansive definition includes the conventional
case in which all households receive identical tax changes, which is represented by \( Z = 1 \).
In this section, I examine two cases in which \( Z = 1 \).

4.1. **Closed Economies with Identical Small Tax Cuts for All Households**

Consider a closed economy in which all households receive identical tax cuts \( -\tau > 0 \), and
assume that \( \tau \) is vanishingly small so that \( \theta (\tau) = \theta (0^-) = G(0; 1) \). In principle, the value
of $G(0;1)$ can be calculated in two different ways. Perhaps the more obvious calculation is simply to identify whether or not each household borrows, and then to compute $G(0;1)$ as the fraction of households that borrow. An alternative calculation is based on the ratio of the average amount borrowed by borrowing households to the average amount of lending by lending households. I develop that calculation in this subsection.

Define

$$\bar{b}^- \equiv \frac{-\int_{b^P(i)<0} b^P(\cdot) \, dF(i)}{\int_{b^P(i)<0} dF(i)}$$

(13)

as the average amount of bonds issued per household in the set of households that borrow and

$$\bar{b}^+ \equiv \frac{\int_{b^P(i)>0} b^P(\cdot) \, dF(i)}{\int_{b^P(i)>0} dF(i)}$$

(14)

as the average amount of privately-issued domestic bonds held per household in the set of domestic households that lend. In a closed economy, the total amount of bonds issued by households that borrow, $G(0;1)\bar{b}^-$, equals the total amount of privately-issued domestic bonds held by households that lend, $(1-G(0^+;1))\bar{b}^+$, that is,

$$G(0;1)\bar{b}^- = (1 - G(0^+;1))\bar{b}^+. \quad (15)$$

Equation (15) leads to the following proposition.

**Proposition 10**  Consider a closed economy with nonzero household borrowing. If all households receive identical tax cuts, i.e., if $Z = 1$, then

$$\theta(0^-) = \frac{1 - \omega_0}{1 + \lambda} \quad \text{and} \quad \theta(0^+) = \frac{1 + \omega_0 \lambda}{1 + \lambda},$$

where $\lambda \equiv \frac{\bar{b}^-}{\bar{b}^+}$ is the ratio of the average amount of bonds issued by borrowing households to the average amount of privately-issued domestic bonds held by lending households before the tax cut, and $\omega_0 \equiv G(0^+;1) - G(0;1)$ is the measure of households that have zero positions in riskless assets before the tax cut.

Proposition 10 applies to any distribution of bond holdings in a closed economy. Re-
markably, for any distribution of bond holdings, the crowding-out coefficients $\theta (0^+)$ and $\theta (0^-)$ in a closed economy are simple functions of two parameters: $\omega_0$, which is the measure of households that neither issue nor hold privately-issued bonds before the tax change; and $\lambda$, which indicates the degree of asymmetry of the distribution of holdings of riskless bonds.

If $\omega_0 = 0$, so that there is a zero measure of households that neither issue nor hold privately-issued bonds before the tax change, then $\theta (0^+) = \theta (0^-)$ is simply $\frac{1}{1 - \lambda}$. If, in addition, the distribution is symmetric, then $\lambda = 1$ and the crowding-out coefficient is simply $\theta (0^+) = \theta (0^-) = 0.5$. However, if $\omega_0 > 0$, then $\theta (0^-) = \frac{1 - \omega_0}{2}$ and $\theta (0^+) = \frac{1 + \omega_0}{2}$ for any symmetric distribution of bond holdings. Therefore, symmetry alone is not sufficient for either $\theta (0^+)$ or $\theta (0^-)$ to equal 0.5.

4.2. Open Economies with Identical Households

In this subsection I derive a simple expression for the crowding-out coefficient in open economies in which all domestic households have identical preferences, face identical taxes, and have identical opportunities to hold and issue assets. All households will have the same amount of riskless assets, $A(i) = A$. If $A \geq 0$, then no households issue domestic bonds. If $A < 0$, then (Lemma 1) $b^P(i) = A(i) = A < 0$, so the amount of bonds issued by households is $-A$, and these bonds are held by foreign investors. Therefore, the aggregate amount of domestic privately-issued bonds in an open economy with identical households is

$$B^P = -\min |A, 0| \geq 0. \quad (16)$$

That is, if $A \geq 0$, there are no privately-issued domestic bonds, and if $A < 0$, then $B^P = -A > 0$, since $b^P(i) = A(i) = A$. In the initial situation, before the tax change, the

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7 Households may choose different combinations of $a^H(i), a^F(i), b^G(i)$, and $b^P(i)$ provided that they sum to $A$ and satisfy equations (2), (3), (4), and (5). Therefore, if $A \geq 0$, then $b^P(i)$ cannot be negative because that would require at least one of $a^H(i), a^F(i)$ or $b^G(i)$ to be positive, which would violate one of equations (3), (4) or (5).
aggregate amount of bonds issued by domestic households is

$$B^P_0 = -\min [A_0, 0] \geq 0.$$  \hspace{1cm} (17)

A Ricardian tax change $\tau$ changes $A$ to $A_0 - \tau$, so the aggregate amount of domestic privately-issued bonds after the tax change is

$$B^P_1 = -\min [A_0 - \tau, 0] \geq 0.$$  \hspace{1cm} (18)

Subtract equation (17) from equation (18) to obtain

$$B^P_1 - B^P_0 = \min [A_0, 0] - \min [A_0 - \tau, 0].$$  \hspace{1cm} (19)

The crowding-out coefficient, $\theta(\tau) \equiv \frac{1}{\tau} (B^P_1 - B^P_0)$, is calculated by dividing equation (19) by $\tau$. For clarity, I will examine the crowding-out coefficients for tax increases ($\tau > 0$) and tax cuts ($\tau < 0$) separately. For tax increases, equation (19) implies

$$\begin{align*}
0, & \quad \text{if } A_0 \geq \tau > 0 \\
\text{for } \tau > 0, & \quad \theta(\tau) = 1 - \frac{A_0}{\tau} < 1, \quad \text{if } 0 < A_0 < \tau \\
1, & \quad \text{if } A_0 \leq 0
\end{align*}$$  \hspace{1cm} (20)

If $A_0 \geq \tau > 0$, households hold non-negative positions in riskless assets both before the tax increase ($A_0 > 0$) and after the tax increase ($A_0 - \tau \geq 0$). Since they have no bonds outstanding either before the tax increase or after the tax increase, the crowding-out coefficient is zero. If $0 < A_0 < \tau$, households have positive positions in riskless assets before the tax increase but negative positions in riskless assets ($A_0 - \tau < 0$) after the tax increase. Therefore, the amount of privately-issued domestic bonds increases by $-(A_0 - \tau)$ while government bonds decrease by $\tau$, so the crowding-out coefficient is $1 - \frac{A_0}{\tau} < 1$. If $A_0 \leq 0$, then households issue additional bonds to pay for the entire tax increase, so the crowding-out coefficient equals one.
For tax cuts, equation (19) implies

\[
0, \quad \text{if } A_0 \geq 0 \\
0, \quad \text{if } \tau < 0, \quad \theta(\tau) = \frac{A_0}{\tau} < 1, \quad \text{if } \tau < A_0 < 0, \\
1, \quad \text{if } A_0 \leq \tau < 0 
\]  

(21)

If \( A_0 \geq 0 \), then there are no privately-issued domestic riskless bonds before or after the tax cut. Therefore, the crowding-out coefficient for \( \tau < 0 \) is zero if \( A_0 \geq 0 \). If \( \tau < A_0 < 0 \), then \( A_1 = A_0 - \tau > 0 \), so households borrow (from foreign lenders) before the tax cut but lend after the tax cut because the tax cut is larger than the amount of privately-issued domestic bonds outstanding before the tax cut. Therefore, the crowding-out coefficient is positive but smaller than one in this case. More precisely, the tax cut eliminates all privately-issued domestic bonds so that the reduction in these bonds is \(-A_0\), while the increase in government bonds is \(-\tau > 0\). Therefore, the crowding-out coefficient is \( \theta(\tau) < 1 \).

If \( A_0 \leq \tau < 0 \), each household receives a tax cut \(-\tau > 0\) that is smaller than the amount of bonds it has outstanding before the tax cut. Therefore, these households reduce their borrowing by the full amount of the tax cut and the crowding-out coefficient equals one.

5. The Crowding-Out Coefficient in the Case of a Particular Departure from Ricardian Equivalence

In a Ricardian economy, when households of type \( i \) face an increase in current taxes of \( \tau z(i) \) accompanied by a decrease in future taxes with present value \( \tau z(i) \), they reduce their current holdings of riskless assets by the amount of the tax increase, \( \tau z(i) \). Now consider a departure from this Ricardian framework and suppose that all households reduce their holdings of riskless assets by a fraction \( 1 - \gamma \) of the increase in their current taxes. That is, households of type \( i \) reduce \( A(i) \) by \( (1 - \gamma) \tau z(i) \), where \( 0 \leq \gamma \leq 1 \). For instance, households may (for unspecified reasons) reduce their current consumption by a fraction \( \gamma \) of the tax increase and pay for the remainder of the tax increase by reducing their current
holding of riskless assets. When $\gamma = 0$ Ricardian Equivalence holds, but when $\gamma > 0$, 
Ricardian Equivalence fails to hold. The parameter $\gamma$ indicates the degree of the departure 
from Ricardian Equivalence.

**Definition 11** Define $\Theta_N(\tau, \gamma) \equiv \frac{1}{\tau} (B_1^P - B_0^P)$ as the non-Ricardian 
crowding-out coefficient associated with a tax change of aggregate size $\tau \neq 0$, in which households of type 
i pay a current increase in taxes of $\tau z(i)$ and reduce their current holdings of riskless 
assets by $(1 - \gamma) \tau z(i)$, where $0 \leq \gamma \leq 1$. For vanishingly small tax changes, define 
$\Theta_N(0^+, \gamma) \equiv \lim_{\tau \searrow 0} \Theta_N(\tau, \gamma)$ and $\Theta_N(0^-, \gamma) \equiv \lim_{\tau \nearrow 0} \Theta_N(\tau, \gamma)$.

The amount of riskless assets held by households of type $i$ after reducing their holdings 
of riskless assets by $(1 - \gamma) \tau z(i)$ in response to a tax increase of $\tau z(i)$ is 
$A_1(i) = A_0(i) - (1 - \gamma) \tau z(i)$. Therefore, the total amount of domestic privately-issued bonds outstanding 
after the tax change is

$$B_1^P = -\int_{A_1(i)<0} b_1^P(i) dF(i) = -\int_{A_0(i)-(1-\gamma)\tau z(i)<0} [A_0(i) - (1 - \gamma) \tau z(i)] dF(i). \quad (22)$$

First consider the case in which $\tau > 0$, so that current taxes increase. Subtracting $B_0^P$ 
in equation (7) from $B_1^P$ in equation (22), and rearranging, yields the change in the amount 
of privately-issued domestic bonds outstanding following the tax increase

$$B_1^P - B_0^P = \frac{(1 - \gamma) \tau \int_{A_0(i)<0} z(i) dF(i)}{-\int_{0\leq A_0(i)<(1-\gamma)\tau z(i)} [A_0(i) - (1 - \gamma) \tau z(i)] dF(i)}, \quad \text{if } \tau > 0. \quad (23)$$

Households with $A_0(i) < 0$ are incumbent borrowers because they had negative positions 
in riskless assets before the tax increase and increase their borrowing by $(1 - \gamma) \tau > 0$ after 
the tax increase. The first term on the right hand side of equation (23) is the increase in 
outstanding bonds issued by incumbent borrowers, who increase their borrowing to pay the 
increased tax in the current period. Households with $0 \leq A_0(i) < (1 - \gamma) \tau z(i)$ are new 
borrowers because $0 \leq A_0(i) < (1 - \gamma) \tau z(i)$ implies that these households did not borrow before the tax 
increase; however, $A_0(i) < (1 - \gamma) \tau z(i)$ implies that these households have negative net
holdings of riskless assets, \( A_1(i) = A_0(i) - (1 - \gamma) \tau z(i) < 0 \), after the tax increase, and hence become borrowers after the tax increase. The second term on the right hand side of equation (23) is the aggregate amount of bonds issued by new borrowers after the tax increase.

Now consider the case in which \( \tau < 0 \) so that current taxes are reduced. In this case, subtracting \( B_0^P \) in equation (7) from \( B_1^P \) in equation (22), and rearranging, yields the change in the amount of private bonds outstanding

\[
B_1^P - B_0^P = -(1 - \gamma) \tau \int_{A_0(i) < (1 - \gamma) \tau z(i)} z(i) dF(i) - \int_{(1 - \gamma) \tau z(i) \leq A_0(i) < 0} A_0(i) dF(i), \quad \text{if } \tau < 0. \tag{24}
\]

Households with \( A_0(i) < (1 - \gamma) \tau z(i) \leq 0 \) are incumbent borrowers because they had negative positions in riskless assets before the tax cut and continue to have negative positions in riskless assets, \( A_1(i) = A_0(i) - (1 - \gamma) \tau z(i) < 0 \), after the tax cut. The first term on the right hand side of equation (24) is the reduction in outstanding bonds issued by incumbent borrowers because each of these households reduces its outstanding bonds by \(- (1 - \gamma) \tau z(i) > 0 \). Households with \( (1 - \gamma) \tau z(i) \leq A_0(i) < 0 \) are former borrowers because they had negative positions in riskless assets before the tax cut and non-negative positions in riskless assets, \( A_1(i) = A_0(i) - (1 - \gamma) \tau z(i) \geq 0 \), after the tax cut. The second term on the right hand side of equation (24) is the reduction in outstanding bonds issued by former borrowers because each of these households completely pays off its initially outstanding bonds of \(-A_0(i)\).

**Proposition 12** \( \Theta_N(\tau, \gamma) = (1 - \gamma) \theta ((1 - \gamma) \tau) \) for \( \tau \neq 0 \). For vanishingly small tax changes, \( \Theta_N(0^+, \gamma) \equiv \lim_{\tau \to 0^+} \Theta_N(\tau, \gamma) = (1 - \gamma) \theta (0^+) \) and \( \Theta_N(0^-, \gamma) \equiv \lim_{\tau \to 0^-} \Theta_N(\tau, \gamma) = (1 - \gamma) \theta (0^-) \).

Proposition 12 applies to tax increases \((\tau > 0)\) as well as to tax cuts \((\tau < 0)\). For vanishingly small tax changes, the non-Ricardian crowding-out coefficient, \( \Theta_N \), is simply the Ricardian crowding-out coefficient, \( \theta \), multiplied by \((1 - \gamma)\).
6. The Crowding-Out Coefficient in the United States

In this section I present empirical measures of the crowding-out coefficient in the model developed here, and I compare these measures to existing econometric estimates of the crowding-out coefficient. Let \( \hat{\theta} \) denote an econometric estimate of the crowding-out coefficient. Empirical estimates by Krishnamurthy and Vissing-Jorgenson (2013) and Gorton, Lewellen, and Metrick (2012) can be summarized approximately as \( \hat{\theta} = 0.5 \). I will analyze the extent to which the framework in this paper can account for that value of the crowding-out coefficient. I focus on tax cuts that are small enough that no households become former borrowers after the tax cut. Formally, this assumption implies that the crowding-out coefficient of interest is \( \theta (0^-) = G (0; Z) \), which is the fraction of the aggregate tax cut that is received by households that borrow.

6.1. Identical Tax Cuts for All Households

In the conventional case in which all households receive identical tax cuts, \( z (i) \equiv 1 \) for all \( i \) and hence \( \theta (0^-) = G (0; 1) \), which is simply the fraction of households that are borrowers. Table 2 presents, at three-year intervals from 1989 to 2013, the percentage of households that are borrowers.\(^8\) The values of \( G (0; 1) \) for these years are shown in the final column of Table 2 and range from 72.3% to 77.0% over this 24-year period. That is, the values of \( G (0; 1) \) are fairly tightly clustered around 0.75, which is substantially different than \( \hat{\theta} = 0.5 \).

Proposition 12 implies that if \( z (i) \equiv 1 \) and the economy deviates from Ricardian Equivalence as modeled in Section 5., the crowding-out coefficient for small tax cuts is \( \Theta (0^-; \gamma) = (1 - \gamma) \theta (0^-) = (1 - \gamma) G (0; 1) \), where \( \gamma \) indicates the degree of departure from Ricardian Equivalence. The crowding-out coefficient \( \Theta (0^-; \gamma) \) will match the econometrically estimated value \( \hat{\theta} \) if \( 1 - \gamma = \frac{\hat{\theta}}{G (0; 1)} \). With \( \hat{\theta} = 0.5 \) and \( G (0; 1) = 0.75 \), the implied value of

\(^8\) The data are taken from the Federal Reserve’s triennial Survey of Consumer Finances as reported in its 2013 Chartbook (unnumbered page 834 of the pdf file downloaded from http://www.federalreserve.gov/econresdata/scf/files/BulletinCharts.pdf.)
6.2. Cross-Sectional Heterogeneity in Tax Cuts

To derive an empirical measure of the crowding-out coefficient in the case of heterogeneous tax cuts, I partition the households in the Survey of Consumer Finances into 10 sets by partitioning each quintile of the income distribution into a set consisting of households that borrow and a set consisting of households that do not borrow. Let $\beta_j$ denote the fraction of households in the $j$–th income quintile that borrow; the measure of these households is $\beta_j/5$, and the measure of households in the $j$–th income quintile that do not borrow is $(1 - \beta_j)/5$. Let $T_j$ be the average amount of Federal taxes paid by each household in the $j$–th income quintile, so that total taxes paid by households in the $j$–th income quintile is $T_j/5$. Therefore, the aggregate amount of Federal taxes paid by all households, $T$, is $T = \sum_{j=1}^{5} T_j/5$. Since there is a unit measure of households in the economy, the economy-wide average amount of taxes per household also equals $T$. Finally, define $\sigma_j \equiv \frac{T_j/5}{T} = T_j/\sum_{i=1}^{5} T_i$ as the fraction of aggregate taxes that are paid by households in the $j$–th income quintile.

Consider a lump-sum tax change of aggregate size $\tau$. For now, assume that all households $i$ in the $j$–th income quintile face identical tax changes $\tau z(i) = 5\sigma_j \tau$. Consistent with the assumption that the economy-wide average value of $z(i)$ equals one, it is straightforward to verify that the aggregate size of the tax change is $\int \tau z(i) dF(i) = \sum_{j=1}^{5} \frac{1}{5} (5\sigma_j \tau) = \tau$. Since the measure of households that are both in the $j$–th income quintile and are borrowers is $\beta_j/5$, the definition of $G(a; Z) \equiv \int_{(a)} z(i) dF(i)$ implies that the fraction of the aggregate tax change paid by, or received by, households that borrow is $G(0; Z) = \int_{(a)} z(i) dF(i) = \frac{1}{\tau} \sum_{j=1}^{5} (\beta_j/5) (5\sigma_j \tau) = \sum_{j=1}^{5} \beta_j \sigma_j$. Table 2 reports the values of $\beta_j$, the percentage of the $j$–th income quintile that borrows, for $j = 1, 2, 3, 4, 5$, for each year of the triennial Survey of Consumer Finances from 1989 to 2013. Table 3 reports the values of $\sigma_j$, the percentage of total Federal tax liabilities paid by the $j$–th income quintile. These data are from
the Congressional Budget Office and they do not include the year 2013. Therefore, Table 3 presents data for each year of the Survey of Consumer Finances from 1989 to 2010. The final column of Table 3 reports the value of $G(0; Z) = \sum_{j=1}^{5} \beta_j \sigma_j$, which is the value of $\theta(0^-)$, in each of the displayed years. The values of $G(0; Z)$ are tightly clustered around 0.85, ranging from 0.836 to 0.871. Allowing for cross-sectional heterogeneity in tax cuts increases the value of the crowding-out coefficient $\theta(0^-) = G(0; Z)$ by about 0.10 from its value of about 0.75 when all households face identical tax changes. Taking account of heterogeneity in tax changes increases the crowding-out coefficient because the propensity for a household to borrow and the share of the aggregate tax cut received by a household both monotonically increase in income across income quintiles.\footnote{Let $T_{j,B}$ be the average tax cut received by each borrowing household in the $j$-th income quintile and let $T_{j,N}$ be the average tax cut received by each non-borrowing household in the $j$-th income quintile. Since a fraction $\beta_j$ of the households in the $j$-th income quintile and $\sigma_j$ are positively correlated across quintiles so $\text{Cov}(\beta_j, \sigma_j) = \sum_{j=1}^{5} \frac{1}{5} \beta_j \sigma_j - \left(\sum_{j=1}^{5} \frac{1}{5} \beta_j\right) \left(\sum_{j=1}^{5} \frac{1}{5} \sigma_j\right) > 0$, which implies $G(0; Z) = \sum_{j=1}^{5} \beta_j \sigma_j > \left(\sum_{j=1}^{5} \frac{1}{5} \beta_j\right) \left(\sum_{j=1}^{5} \sigma_j\right) = \frac{1}{5} \sum_{j=1}^{5} \beta_j = G(0; 1).$}

The value of 0.85 for $G(0; Z)$ exceeds the econometrically estimated value $\hat{\theta}$ by 0.35. In the presence of heterogeneous tax cuts, the crowding-out coefficient $\Theta_N(0^-; \gamma)$ will match the econometrically estimated value $\hat{\theta}$ if $1 - \gamma = \frac{\hat{\theta}}{G(0; Z)}$. With $\hat{\theta} = 0.5$ and $G(0; Z) = 0.85$, the implied value of $\gamma$ is $\gamma = 0.412$, again a substantial departure from Ricardian Equivalence.

6.2.1. Different Tax Cuts for Borrowers and Non-Borrowers within Income Quintiles

I have assumed so far that all households in a given income quintile face identical tax changes. To the extent that there is a systematic difference in tax changes facing borrowers and those facing non-borrowers within an income quintile, the value of $G(0; Z)$ will differ from $\sum_{j=1}^{5} \beta_j \sigma_j$. 
are borrowers and a fraction \(1 - \beta_j\) are non-borrowers,

\[
\beta_j T_{j,B} + (1 - \beta_j) T_{j,N} = T_j, \quad \text{for } j = 1, 2, 3, 4, 5. \tag{25}
\]

Instead of assuming that \(T_{j,N} = T_{j,B}\), as above, now assume that

\[
T_{j,N} = \eta T_{j,B}, \quad \text{for } j = 1, 2, 3, 4, 5, \tag{26}
\]

with \(\eta > 0\). Substitute equation (26) into equation (25) and rearrange to obtain

\[
T_{j,B} = \frac{T_j}{\beta_j + (1 - \beta_j) \eta}, \quad \text{for } j = 1, 2, 3, 4, 5. \tag{27}
\]

The tax cut received by each borrowing household \(i\) in the \(j\)-th income quintile is

\[
\frac{5\sigma_j \tau}{\beta_j + (1 - \beta_j) \eta}. \tag{28}
\]

Since the measure of households that are in the \(j\)-th income quintile and are borrowers is \(\beta_j / 5\), the definition of \(G(a; Z) \equiv \int_{\frac{a}{1-\beta}}^{\infty} z(i) dF(i)\) implies that the fraction of the aggregate tax cut accruing to households that borrow is

\[
G(0; Z) = \int_{\frac{a}{1-\beta}}^{\infty} z(i) dF(i) = \sum_{j=1}^{5} \left(\frac{\beta_j / 5}{\beta_j + (1 - \beta_j) \eta}\right) \frac{5\sigma_j \tau}{\beta_j + (1 - \beta_j) \eta} = \sum_{j=1}^{5} \frac{\beta_j \sigma_j}{\beta_j + (1 - \beta_j) \eta}. \tag{29}
\]

Since \(\sum_{j=1}^{5} \frac{\beta_j \sigma_j}{\beta_j + (1 - \beta_j) \eta}\) is strictly decreasing in \(\eta > 0\) for \(0 < \beta_j < 1\) and \(\sigma_j > 0\), an increase in \(\eta\) decreases the calculated value of \(G(0; Z)\).

Table 4 reports the value of \(G(0; Z)\) for 2010 using the values of \(\beta_j\) and \(\sigma_j\) from Tables 2 and 3, respectively, and various values of \(\eta\). The values of \(G(0; Z)\) in Table 4 fall monotonically as \(\eta\) increases, as noted above. Of course, when \(\eta = 1\), all households in a given quintile receive identical tax cuts, and the value of \(G(0; Z)\) is identical to its value in the final row and final column of Table 3. Note that for \(\eta = 6\), the value of \(G(0; Z)\) equals 0.500. That is, in order for the calculated value of the crowding-out coefficient to equal \(\hat{\theta} = 0.5\), the average tax cut received by non-borrowing households in a given income quintile would have been 6 times as large as the average tax cut received by borrowing households in that same income quintile. Such dramatic within-income-quintile variation in tax changes is far from the spirit of the sort of lump-sum taxes typically associated with Ricardian tax changes.
6.3. Discrepancies between Model and Empirical Estimates of the Crowding-Out Coefficient

I have pointed out that the value of the crowding-out coefficient predicted by the model (around 0.75 if households face homogeneous tax changes and around 0.85 if they face heterogeneous tax changes) far exceeds the values (about 0.5) estimated by Gorton, Lewellen, and Metrick (2012) and Krishnamurthy and Vissing-Jorgenson (2013). I have also shown that introducing a particular ad hoc departure from Ricardian Equivalence can reduce the value of the crowding-out coefficient predicted by the model so that it will equal 0.5. But there are other reasons why the model’s predicted crowding-out coefficient may differ from empirical estimates cited here. First, the privately-issued debt in the model is entirely household debt, whereas the privately-issued debt in empirical studies is, in general, corporate debt, much of it issued by financial intermediaries. In the model, the amount of debt issued by households is determinate because of the assumption that no household—or any economic entity—will simultaneously borrow risklessly and hold riskless assets. This assumption rules out the existence of financial intermediaries. The next challenge for using the model to account for empirically-estimated crowding-out coefficients is to find a way to incorporate financial intermediaries and to allow households to simultaneously borrow and lend risklessly.

Another major discrepancy between the model and the data used to estimate the crowding-out coefficient is that the model focuses on Ricardian tax changes as the only source of changes in government debt. The important feature of Ricardian tax changes in a Ricardian economy is that optimal consumption, investment, and output, and equilibrium asset prices remain unchanged in the presence of such changes, while each household’s optimal holding of riskless assets increases by the amount of its current Ricardian tax cut. In the aggregate, households increase their holding of riskless assets by the amount of the tax cut, which equals the increase in government bonds outstanding. Empirically, however, tax cuts generally
are not Ricardian. The most obvious non-Ricardian feature of tax changes is that tax changes are generally not lump-sum and thus have incentive effects that would interfere with Ricardian Equivalence. Moreover, Ricardian Equivalence does not apply to changes in government debt that arise from changes in government purchases. A challenge for future empirical work would be to find a way to focus on changes in government debt that come closer to reflecting Ricardian tax changes rather than changes in government purchases or distortionary taxes.

7. Conclusion

Two assumptions provide a simple framework for analyzing the crowding-out coefficient in Ricardian economies: (1) Ricardian equivalence holds so that all households increase their positions in riskless assets by the amount of the current lump-sum tax cut they receive; and (2) no household will simultaneously borrow and hold positive amounts of any riskless assets. Under these two assumptions, the crowding-out coefficient for small tax cuts equals the fraction of the aggregate current tax cut that accrues to households that borrow. That fraction can be any rational number in the unit interval. Existing analyses of Ricardian equivalence typically add a third assumption: (3) all households receive identical tax cuts in the current period. Imposing this third assumption, in addition to the two assumptions listed above, implies that the crowding-out coefficient equals the fraction of households that borrow. Using data from the Survey of Consumer Finances, I have shown that this fraction is about 0.75, so the three assumptions together imply that the crowding-out coefficient should be 0.75. However, empirical estimates of the crowding-out coefficient are around 0.5.

One can view the disparity between the value of 0.75 implied by the three assumptions and the value of 0.5 that is econometrically estimated to be an informal rejection of the joint hypothesis that all three assumptions hold. This disparity invites investigation of how
departures from these assumptions would change the value of the crowding-out coefficient in the model. In this paper, I have explored departures from the first and third assumptions. Specifically, if one replaces the first assumption by an assumption that all households increase their positions in riskless assets by two-thirds—rather than the full amount—of the current lump-sum tax cut they receive, the model predicts a crowding-out coefficient of 0.5, which matches the econometrically-estimated value.

I also explored the effect of departing from assumption (3), while maintaining assumptions (1) and (2) intact. Specifically, I have replaced the assumption that all households receive identical lump-sum tax cuts with the assumption that all households receive a tax cut in proportion to the share of aggregate taxes paid by their own income quintile. Because households in higher income quintiles pay higher taxes per household and have a higher incidence of borrowing, this modification increases the crowding-out coefficient in the model. That is, it exacerbates the discrepancy between the model and the econometrically-estimated crowding-out coefficient. The crowding-out coefficient in the model could be reduced by assuming that within each income quintile, non-borrowing households pay higher taxes than borrowing households. In fact, if each non-borrowing household faces a tax change six times as large as the tax change faced by each borrowing household in its own income quintile, then the model predicts a crowding-out coefficient of 0.5. However, such a large disparity of tax cuts within income quintiles seems counter to the spirit of the tax changes that are the focus of Ricardian Equivalence.

Future research might usefully be directed at relaxing the second of three assumptions above. That is, it may prove fruitful to allow households in the model to borrow and hold positive gross positions in riskless assets at the same time. The challenge is to understand and tractably model the determinants of a household’s simultaneous riskless borrowing and holding of positive gross positions in riskless assets. In addition, further extensions of the model could go beyond household balance sheets to examine the assets and liabilities held
simultaneously by financial intermediaries.

References


<table>
<thead>
<tr>
<th>Population Shares of Borrower Categories After Tax Change</th>
<th>Tax Increase ($\tau &gt; 0$)</th>
<th>Tax Decrease ($\tau &lt; 0$)</th>
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<td>Incumbent borrowers</td>
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<td>$G(\tau; 1)$</td>
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<td>Former borrowers</td>
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<td>$G(0; 1) - G(\tau; 1)$</td>
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<tr>
<td>Non-borrowers</td>
<td>$1 - G(\tau; 1)$</td>
<td>$1 - G(0; 1)$</td>
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<tr>
<td>Total</td>
<td>1</td>
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Table 1: Population Shares of Borrower Categories
| Year | Lowest Income Quintile | Second Income Quintile | Middle Income Quintile | Fourth Income Quintile | Highest Income Quintile | All: $G(0; 1)$  
$= (1/5) \sum \beta_j$  
($\times 100$) |
<table>
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<td>47.1</td>
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<td>78.1</td>
<td>86.2</td>
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Table 2: Percentage of Households That Borrow. The first five columns are the percentages of households in each income quintile that borrow. The final column is the percentage of borrowers in the overall population, as reported in the 2013 Chartbook of the Federal Reserve’s Survey of Consumer Finances. It may differ from the average of the reported values for the five income quintiles due to rounding.
<table>
<thead>
<tr>
<th>Year</th>
<th>Lowest Income Quintile</th>
<th>Second Income Quintile</th>
<th>Middle Income Quintile</th>
<th>Fourth Income Quintile</th>
<th>Highest Income Quintile</th>
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<td>2010</td>
<td>0.4</td>
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<td>9.1</td>
<td>17.6</td>
<td>68.8</td>
<td>85.1</td>
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Table 3: Percentage of Federal Tax Liabilities. The first five columns are the percentages of overall Federal tax liabilities paid by households in each income quintile. The final column is the percentage of overall Federal tax liabilities paid by households that borrow, assuming that within each income quintile, individual household tax liability is uncorrelated with borrowing status.
\[
\eta T_{j,N} = \eta T_{j,B}
\]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>( G(0; Z) )</td>
<td>0.918</td>
<td>0.851</td>
<td>0.745</td>
<td>0.663</td>
<td>0.598</td>
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<td>0.500</td>
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Table 4: Within-Quintile Heterogenous Taxes, 2010.

\( G(0; Z) \) is the percentage of an aggregate tax cut received by households that borrow, assuming that within the \( j - th \) income quintile, the average tax cut received by non-borrowers, \( T_{j,N} \), is \( \eta > 0 \) times as large as the average tax cut received by borrowers, \( T_{j,B} \).
Appendix

Proof of Lemma 1. Proof of statement 1: If $b_P(i) < 0$, then equations (2), (3), (4), and (5) imply $a^H(i) = a^F(i) = b^G(i) = 0$, so equation (1) implies $A(i) = b_P(i) < 0$. Proof of statement 2: If $A(i) < 0$, then $b_P(i) < 0$, which implies $a^H(i) = a^F(i) = b^G(i) = 0$, so $b_P(i) = A(i) < 0$. If $A(i) = 0$, then $b_P(i)$ cannot be negative because if $b_P(i)$ were negative, then statement 1 would imply $A(i) = b_P(i) < 0$, which would contradict $A(i) = 0$. If $A(i) = 0$, then $b_P(i)$ cannot be positive because equations (1) and (2) would imply $A(i) \geq b_P(i) > 0$, which contradicts $A(i) = 0$. Therefore, if $A(i) = 0$, then $b_P(i) = 0 = A(i)$.

Proof of Proposition 6. First consider the case in which $\tau > 0$. Substitute equation (11) into the definition of the crowding-out coefficient and use the fact that $\int_{0 \leq A_0(i) < \tau z(i)} A_0(i) dF(i)$

$$= \int_{0 < A_0(i) < \tau z(i)} A_0(i) dF(i)$$

to obtain

$$\theta(\tau) = -\frac{1}{\tau} \int_{0 < A_0(i) < \tau z(i)} A_0(i) dF(i) + \int_{A_0(i) < \tau z(i)} z(i) dF(i) \leq \int_{A_0(i) < \tau z(i)} z(i) dF(i) = G(\tau; Z),$$

where the inequality follows from $\tau > 0$, so that $\frac{1}{\tau} \int_{0 \leq A_0(i) < \tau z(i)} A_0(i) dF(i) \geq 0$. This first equality in this equation can be written as

$$\theta(\tau) = \int_{A_0(i) \leq 0} z(i) dF(i) + \int_{0 < A_0(i) < \tau z(i)} \left( z(i) - \frac{A_0(i)}{\tau} \right) dF(i).$$

The first integral on the right hand side is $G(0^+; Z)$ and the second integral is non-negative so $\theta(\tau) \geq G(0^+; Z)$. Therefore, $G(0; Z) \leq \theta(\tau) \leq G(\tau; Z)$, if $\tau > 0$.

Now consider the case in which $\tau < 0$. Substitute equation (10) into the definition of the crowding-out coefficient to obtain

$$\theta(\tau) = G(\tau; Z) + \int_{\tau z(i) \leq A_0(i) < 0} \frac{A_0(i)}{\tau z(i)} z(i) dF(i) \geq G(\tau; Z),$$
where the inequality follows from the fact that $\frac{A_0(i)}{\tau z(i)} > 0$ for $\tau z (i) \leq A_0 (i) < 0$. Use

$$\int_{\tau z(i) \leq A_0(i) < 0} \frac{A_0(i)}{\tau z(i)} z(i) dF(i)$$

$$= \int_{\tau z(i) \leq A_0(i) < 0} \left( \frac{A_0(i)}{\tau z(i)} - 1 + 1 \right) z(i) dF(i)$$

$$= - \int_{\tau z(i) \leq A_0(i) < 0} \left( 1 - \frac{A_0(i)}{\tau z(i)} \right) z(i) dF(i) + (G(0; Z) - G(\tau; Z))$$

to rewrite

$$\theta(\tau) = G(\tau; Z) + \int_{\tau z(i) \leq A_0(i) < 0} \frac{A_0(i)}{\tau z(i)} z(i) dF(i)$$

as

$$\theta(\tau) = G(0; Z) - \int_{\tau z(i) \leq A_0(i) < 0} \left( 1 - \frac{A_0(i)}{\tau z(i)} \right) z(i) dF(i) \leq G(0; Z),$$

where the inequality follows from the fact $1 - \frac{A_0(i)}{\tau z(i)} \geq 0$ for $\tau z (i) \leq A_0 (i) < 0$. Therefore,

$G(\tau; Z) \leq \theta(\tau) \leq G(0; Z)$, if $\tau < 0$. ■

**Proof of Corollary 7.** This corollary follows directly from Proposition 1 and the facts that $\lim_{\tau \to 0} G(\tau; Z) = G(0; Z)$ and $\lim_{\tau \to 0} G(\tau; Z) \equiv G(0^+; Z)$. ■

**Proof of Proposition 8.** Consider nonzero values of $\tau_1$ and $\tau_2$ with $\tau_2 > \tau_1$. There are three cases, depending on whether both $\tau_1$ and $\tau_2$ are positive, both are negative, or $\tau_2 > 0$ and $\tau_1 < 0$.

Case I: $\tau_2 > \tau_1 > 0$. Use the expression for $\theta(\tau)$ in Lemma 5 to obtain

$$\theta(\tau) = - \frac{1}{\tau} \int_{0 \leq A_0(i) < \tau z(i)} A_0(i) dF(i) + \int_{A_0(i) < \tau z(i)} z(i) dF(i), \quad \text{for } \tau > 0.$$

Evaluate $\theta(\tau)$ at $\tau_1$ and $\tau_2$ and rearrange to obtain

$$\theta(\tau_1) = - \frac{1}{\tau_1} \int_{0 \leq A_0(i) < \tau_1 z(i)} A_0(i) dF(i) + \int_{A_0(i) < \tau_1 z(i)} z(i) dF(i)$$

Evaluate $\theta(\tau)$ at $\tau_1$ and $\tau_2$ and rearrange to obtain

$$\theta(\tau_1) = - \frac{1}{\tau_1} \int_{0 \leq A_0(i) < \tau_1 z(i)} A_0(i) dF(i) + \int_{A_0(i) < \tau_1 z(i)} z(i) dF(i)$$

36
and

\[
    \theta(\tau_2) = \frac{1}{\tau_2} \int_{0 \leq A_0(i) < \tau_2 z(i)} A_0(i) dF(i) + \int_{A_0(i) < \tau_2 z(i)} z(i) dF(i)
\]

\[
    = \frac{1}{\tau_2} \left( \int_{0 \leq A_0(i) < \tau_1 z(i)} A_0(i) dF(i) + \int_{\tau_1 z(i) \leq A_0(i) < \tau_2 z(i)} A_0(i) dF(i) \right)
\]

\[
    + \int_{A_0(i) < \tau_2 z(i)} z(i) dF(i).
\]

7 Subtract \( \theta(\tau_1) \) from \( \theta(\tau_2) \) and rearrange to obtain

\[
    \theta(\tau_2) - \theta(\tau_1) = \left( \frac{1}{\tau_1} - \frac{1}{\tau_2} \right) \int_{0 \leq A_0(i) < \tau_1 z(i)} A_0(i) dF(i)
\]

\[
    - \frac{1}{\tau_2} \int_{\tau_1 z(i) \leq A_0(i) < \tau_2 z(i)} A_0(i) dF(i)
\]

\[
    + \int_{\tau_1 z(i) \leq A_0(i) < \tau_2 z(i)} z(i) dF(i).
\]

8 To show that \( \theta(\tau_2) - \theta(\tau_1) \geq 0 \), it suffices to show that \( \frac{1}{\tau_1} - \frac{1}{\tau_2} > 0 \), which follows from \( \tau_2 > \tau_1 > 0 \). Therefore, \( \theta(\tau_2) \geq \theta(\tau_1) \) if \( 0 < \tau_1 < \tau_2 \).

Case II: \( \tau_1 < \tau_2 < 0 \). Use the expression for \( \theta(\tau) \) in Lemma 5 to obtain

\[
    \theta(\tau) = \frac{1}{\tau} \int_{\tau z(i) \leq A_0(i) < 0} A_0(i) dF(i) + \int_{A_0(i) < \tau z(i)} z(i) dF(i), \quad \text{for } \tau < 0.
\]

2 Evaluate \( \theta(\tau) \) at \( \tau_2 \) and \( \tau_1 \) and rearrange to obtain

\[
    \theta(\tau_1) = \int_{A_0(i) < \tau_1 z(i)} z(i) dF(i)
\]

\[
    + \frac{1}{\tau_1} \int_{\tau_1 z(i) \leq A_0(i) < \tau_2 z(i)} A_0(i) dF(i)
\]

\[
    + \frac{1}{\tau_1} \int_{\tau_2 z(i) \leq A_0(i) < 0} A_0(i) dF(i).
\]
Therefore, for \( \tau_2 \) and rearrange to obtain

\[
\theta (\tau_2) - \theta (\tau_1) = \int_{\tau_1 z(i) \leq A_0(i) < \tau_2 z(i)} z(i) dF(i) + \frac{A_0(i)}{\tau_1} \int_{\tau_2 z(i) \leq A_0(i) < 0} \frac{A_0(i)}{\tau_2} z(i) dF(i) \geq 0,
\]

because \( 1 - \frac{A_0(i)}{\tau_1 z(i)} \geq 0 \) for \( \tau_1 z(i) \leq A_0(i) < 0 \) so the first integral is non-negative, (2) \( \frac{A_0(i)}{\tau_2 z(i)} > 0 \) for \( \tau_2 z(i) \leq A_0(i) < 0 \) so the second integral is non-negative, and (3) \( \frac{\tau_1 - \tau_2}{\tau_1} > 0 \).

Therefore, \( \theta (\tau_2) \geq \theta (\tau_1) \) if \( \tau_1 < \tau_2 < 0 \).

Case III: \( \tau_1 < 0 < \tau_2 \). Use the expression for \( \theta (\tau) \) in Lemma 5 to obtain

\[
\theta (\tau_1) = \frac{1}{\tau_1} \int_{0 \leq A_0(i) < \tau_1 z(i)} A_0(i) dF(i) + \int_{A_0(i) < \tau_1 z(i)} z(i) dF(i), \quad \text{for } \tau_1 < 0.
\]

and

\[
\theta (\tau_2) = -\frac{1}{\tau_2} \int_{0 \leq A_0(i) < \tau_2 z(i)} A_0(i) dF(i) + \int_{A_0(i) < \tau_2 z(i)} z(i) dF(i), \quad \text{for } \tau_2 > 0.
\]

Therefore, for \( \tau_1 < 0 < \tau_2 \),

\[
\theta (\tau_2) - \theta (\tau_1) = -\frac{1}{\tau_2} \int_{0 \leq A_0(i) < \tau_2 z(i)} A_0(i) dF(i) + \int_{\tau_1 z(i) \leq A_0(i) < \tau_2 z(i)} z(i) dF(i) \\
- \frac{1}{\tau_1} \int_{\tau_1 z(i) \leq A_0(i) < 0} A_0(i) dF(i)
\]

which can be rearranged to obtain

\[
\theta (\tau_2) - \theta (\tau_1) = -\frac{1}{\tau_2} \int_{0 \leq A_0(i) < \tau_2 z(i)} A_0(i) dF(i) + \int_{0 \leq A_0(i) < \tau_2 z(i)} z(i) dF(i) \\
+ \int_{\tau_1 z(i) \leq A_0(i) < 0} z(i) dF(i) - \frac{1}{\tau_1} \int_{\tau_1 z(i) \leq A_0(i) < 0} A_0(i) dF(i).
\]
Therefore,

\[ \theta (\tau_2) - \theta (\tau_1) = \left[ \frac{1}{\tau_2} \int_{0 \leq \tau_0(i) < \tau_2 z(i)} (\tau_2 z(i) - \tau_0(i)) \, dF(i) \right. \\
\left. - \frac{1}{\tau_1} \int_{\tau_1 z(i) \leq \tau_0(i) < 0} (\tau_0(i) - \tau_1 z(i)) \, dF(i) \right] \geq 0 \]

since \( \frac{1}{\tau_2} > 0, \frac{1}{\tau_1} < 0, \int_{0 \leq \tau_0(i) < \tau_2 z(i)} (\tau_2 z(i) - \tau_0(i)) \, dF(i) \geq 0, \) and \( \int_{\tau_1 z(i) \leq \tau_0(i) < 0} (\tau_0(i) - \tau_1 z(i)) \, dF(i) \geq 0. \) Therefore, \( \theta (\tau_2) \geq \theta (\tau_1) \) if \( \tau_1 < 0 < \tau_2. \)

Combining the results of Cases I, II, and III proves that \( \theta (\tau_2) \geq \theta (\tau_1) \) for nonzero \( \tau_2 > \tau_1. \]

**Proof. of Proposition 10.** Divide both sides of equation (15) by \( b^T \) and use the definition of \( \lambda \) to obtain \( G (0; 1) \lambda = 1 - G (0^+; 1). \) Next use the definition of \( \omega_0 \) to obtain \( G (0; 1) \lambda = 1 - \omega_0 - G (0; 1), \) which implies \((1 + \lambda) G (0; 1) = 1 - \omega_0, \) and hence \( G (0; 1) = \frac{1 - \omega_0}{1 + \lambda}. \)

Corollary 7 implies that when \( Z = 1, \theta (0^-) = G (0; 1), \) which implies \( \theta (0^-) = \frac{1 - \omega_0}{1 + \lambda}. \) To calculate \( \theta (0^+), \) once again use the definition of \( \omega_0 \) to rewrite \( G (0; 1) \lambda = 1 - G (0^+; 1), \) but eliminate \( G (0; 1) \) rather than \( G (0^+; 1), \) to obtain \( (G (0^+; 1) - \omega_0) \lambda = 1 - G (0^+; 1), \) which implies \( G (0^+; 1) = \frac{1 + \omega_0 \lambda}{1 + \lambda}. \) Corollary 7 implies that when \( Z = 1, \theta (0^+) = G (0^+; 1), \) which implies \( \theta (0^+) = \frac{1 + \omega_0 \lambda}{1 + \lambda}. \]

**Proof. of Proposition 12.** If \( \tau > 0, \) divide both sides of equation (23) by \( \tau \) and use Definition 11 to obtain

\[ \Theta_N (\tau, \gamma) = (1 - \gamma) \left( \begin{array}{c} \int_{A_0(i) < (1 - \gamma) \tau z(i)} z(i) \, dF(i) \\ - \int_{0 \leq A_0(i) < (1 - \gamma) \tau z(i)} A_0(i) \frac{z(i)}{(1 - \gamma) \tau z(i)} \, dF(i) \end{array} \right), \] 

if \( \tau > 0. \)

Lemma 5 implies that

\[ \theta ((1 - \gamma) \tau) = \left( \begin{array}{c} \int_{A_0(i) < (1 - \gamma) \tau z(i)} z(i) \, dF(i) \\ - \int_{0 \leq A_0(i) < (1 - \gamma) \tau z(i)} A_0(i) \frac{z(i)}{(1 - \gamma) \tau z(i)} \, dF(i) \end{array} \right), \] 

if \( \tau > 0, \)

so \( \Theta_N (\tau, \gamma) = (1 - \gamma) \theta ((1 - \gamma) \tau), \) for \( \tau > 0. \)
If \( \tau < 0 \), divide both sides of equation (24) by \( \tau \) and use Definition 11 to obtain

\[
\Theta_N(\tau, \gamma) = (1 - \gamma) \left( \int_{A_0(i) < (1-\gamma)\tau z(i)} z(i) \, dF(i) + \int_{(1-\gamma)\tau z(i) \leq A_0(i) < 0} \frac{A_0(i)}{(1-\gamma)\tau z(i)} z(i) \, dF(i) \right), \text{ if } \tau < 0.
\]

Lemma 5 implies that for \( \tau < 0 \),

\[
\theta((1-\gamma)\tau) = \int_{A_0(i) < (1-\gamma)\tau z(i)} z(i) \, dF(i) + \int_{(1-\gamma)\tau z(i) \leq A_0(i) < 0} \frac{A_0(i)}{(1-\gamma)\tau z(i)} z(i) \, dF(i), \text{ if } \tau < 0,
\]

so \( \Theta_N(\tau, \gamma) = (1 - \gamma) \theta((1 - \gamma) \tau) \), if \( \tau < 0 \). Therefore, \( \Theta_N(\tau, \gamma) = (1 - \gamma) \theta((1 - \gamma) \tau) \), both for \( \tau > 0 \) and for \( \tau < 0 \).

Finally, \( \Theta_N(0^+, \gamma) \equiv \lim_{\tau \searrow 0} \Theta_N(\tau, \gamma) = \lim_{\tau \searrow 0} (1 - \gamma) \theta((1 - \gamma) \tau) = (1 - \gamma) \theta(0^+) \) and \( \Theta_N(0^-, \gamma) \equiv \lim_{\tau \nearrow 0} \Theta_N(\tau, \gamma) = \lim_{\tau \nearrow 0} (1 - \gamma) \theta((1 - \gamma) \tau) = (1 - \gamma) \theta(0^-). \)