A model of hysteresis arising from social interaction within a firm

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Abstract. This paper subscribes to the view that a key distinguishing feature of a firm is its social nature. We present a model in which hysteresis arises from the social interactions between employees. Employees have a simple response to incentives in the form of the pay available outside the firm relative to that available within the firm. Allowing for social interaction, whereby employees are influenced by the effort levels of fellow employees, leads to the distinctive effects, such as lazy relay responses to incentives, associated with hysteresis.

1. Introduction

There is a huge literature on why some economic activities are conducted by way of "markets", and some by "firms". In this paper we follow the account of "the" firm that stresses the social nature of the interactions between the employees in that mode of organizing economic activity. Social relationships obviously pertain to markets as well. Think of the social interactions between buyers and sellers in an auction market, for example, as presided over by the entity conducting the auction, which is itself often a firm. There is a spectrum here, with the firm involving closer, more continuous, social contacts between employees than is the case with individuals interacting during market transactions: the "no person is an island" precept applies more strongly to firms than to markets.

The strongly social nature of the interactions between employees has been taken to be the distinctive feature of firms in a variety of accounts. Simon has stressed the willingness to accept authority by a firms employees [1]; Bowles the ability of a firm to shape the private preferences of its employees into social ones [2]; Casadesus-Masanell the ability of a firm to imbue a work ethic in employees [3]; Holmstrom and Milgrom the incentives to multitasking by employees within a firm [4]; and finally, but by no means exhaustively, Akerlof and Kranton have stressed the sense of identity, or belonging, which a firm can engender in its employees [5].

In the present paper we take as given that such social forces are central to the fabric of a firm [6] and ask what happens as a result of such social interactions within a firm. Our conjecture is that social interactions within a firm give rise to hysteresis in the way the employees within the firm respond to the incentives offered within the firm relative to those available outside the firm. A manifestation of this would be that there is a range of relative pay incentives within which employees will not change their behavioral responses, in the form of quitting the firm, for example.

The term hysteresis comes from the Greek to be late or come behind, and was coined by Ewing to describe the behavior of electromagnetic fields in ferric metals [7]. If a magnetizing force is applied to such metals, then reversed, the metals do not revert to their previous characteristics: they remain changed. As a general systems property [8], hysteresis implies that the outputs of a system do not just depend on the contemporaneous values of the inputs applied, but also retain a memory, such as of the non-dominated extremum values of the inputs applied in the past. Thus there are effects that remain after the immediate causes are removed: history is not bunk. If our conjecture that the social interactions within a firm lead to hysteresis is correct, the firm will display a form of temporary inertia, in that employees participation decisions will not respond to all changes to relative pay incentives; and the pool of employees who opt to remain working within the firm will be shaped by a memory of the relative pay incentives experienced in the past.

The rest of this paper is structured as follows. In the next section we outline a model of hysteresis, which arises when employees, who have the same responses to relative pay rate incentives, can observe the effort levels of fellow employees. We then construct probabilistic versions of this model and discuss their dynamic properties.

2. A model of hysteresis within a firm

Initially, the term hysteresis was coined to describe results obtained in experiments on electromagnetic fields in ferric metals. The need for this new term was felt by Ewing because he thought that hysteresis effects would apply to a wider range of phenomena [7]. This prophecy has been borne out in the wide range of areas of inquiry in which hysteresis has been discovered, ranging from physics to biology, materials science to mechanics and from electronics to economics (see the entries in [9]). In the literature on firms Ford speculated that organizations might display hysteresis in managerial intensity when they move from expansionary to contractionary modes [10], but there appears to have been little by way of follow up to this conjecture in the literature on organizations or management. Ford suggested that participation decisions were one possible reason why organizational hysteresis might arise, but did not offer a formal model of such decisions.

The present paper aims to fill this gap by providing a formal model of how hysteresis can arise in a firm as a result of the social interactions involved when employees can observe the effort levels of their fellow employees.

The formalization of hysteresis as a general systems property was made by Krasnosel'skii and Pokrovskii [8]. The most widely used model of hysteresis is that derived from the work of Preisach [11], in which hysteresis at the aggregate or systems level arises from elements at the micro level that are heterogeneous and respond non-linearly to a common input shock. In such a world the aggregate output of the system displays remanence, in that the application and removal of an input shock will not see the systems output return to the status quo ante; and the output of the system displays an erasable, selective memory of the non-dominated extremum values of the shocks experienced.

A widely used specification of non-linear responses at the micro level is the non-ideal or lazy relay. In such a relay the individual elements do not respond to an input shock until some value, α, is reached, in which case the behavior, changes from some state to another, say low to high; and does not switch from the high to the low state until some value of the input shock, β, is reached, $\beta < \alpha$. Thus there is a range of values for the input i, $\beta < i < \alpha$, inside which behavior will not change in the face of variation in i. Hence in order to know whether the individual element is in a high or low state it is necessary to know the history of i, which explains whether the range $\beta < i < \alpha$ has been approached from above or below. Heterogeneity can be introduced by allowing the α and β switching points to be different between the elements. The systems output y would then depend not just on the contemporaneous value of i, but also on the non-dominated extremum values of i experienced in the past [12].

The model of hysteresis proposed in the present paper has a more parsimonious specification of the elements, in this context employees, in the system. We assume that employees are homogeneous in the sense that they respond in the same way to an incentive in the form of the pay rate offered by the firm relative to what they could earn by not being employed by the firm. The rate of pay that pertains outside the firm is held constant, the relative pay incentive being varied by changing the rate of pay offered to employees within the firm. The other parsimonious feature is to employ a Heaviside step function, instead of a non-ideal or lazy relay, to describe how the employees initially respond to relative pay incentives. Thus there is a single trigger α, for relative pay, with values of relative pay greater than α, leading the individuals to choose to be employed within the firm; and values of relative pay less than or equal to α, leading the individuals to choose not to be employed by the firm. The intriguing result is that if employees within the firm can observe the effort levels of other employees, and react by raising/lowering their effort levels in sympathy with the effort levels of fellow employees, hysteresis arises. This hysteresis takes the form of a lazy relay between incentives and the number of individuals who elect to remain employed within the firm. If this lazy relay applies, the way would then be open to use Preisach-type models to analyze conglomerate firms as composed of operational units characterized by such processes.

3. Hysteresis emerging from interaction

The following construction can be seen as a prototype model for emerging hysteretic response that results from interaction of non-hysteretic responses [13]. More complex systems of interacting agents will produce more complex hysteric response to exogenous stimuli in a similar fashion [14].

Consider two identical agents which respond to incentive i according to the simple threshold rule. Each agent, at any given time, adopts one of the two states: either the high performance state $s = 1$ or the low performance state $s = 0$. When $i \leq \alpha$, the agents adopt the state $s = 0$; when $i > \alpha$, they adopt the state $s = 1$. In other words, if we introduce the Heaviside step function

$$H_\alpha(x) = \begin{cases} 0, & x \leq \alpha \\ 1, & x > \alpha \end{cases},$$

we simply have

$$s = H_\alpha(i) \tag{1}$$

at all times, where s is the state of either agent. Here α is a threshold, which presents a sufficient level of incentive for an agent to adopt the high performance state. As i (the input) is varied, both agents switch from one state to the other each time the input crosses the threshold value α, see Fig. 1(b). There is no hysteresis or memory in this response. A counterpart of such response, which allows the performance to increase continuously with incentive, can be modeled by the piecewise linear function

$$s = \begin{cases} 0, & i \leq \alpha \\ k^{-1}(i - \alpha), & \alpha < i < \alpha + k \\ 1, & i \geq \alpha + k \end{cases}, \tag{2}$$

see Fig. 1(a).

The above agents act independently. However, if agents affect each other, hysteresis may emerge. Suppose that the "percepted incentive" (or, the total incentive) for the first agent is

$$\tilde{i}_1 = i + ks_2,$$

where $k > 0$ is some coefficient, which is smaller than α. Similarly, assume that the total incentive for the second agent is

$$\tilde{i}_2 = i + ks_1.$$

Now, the switching rules become

$$s_1 = H_\alpha(\tilde{i}_1), \qquad s_2 = H_\alpha(\tilde{i}_2).$$

In other words, the response of the agents to variations of the applied incentive i is determined by the implicit coupled equations

$$s_1 = H_\alpha(i + ks_2), \qquad s_2 = H_\alpha(i + ks_1). \tag{3}$$

We need to find a solution s_1, s_2 of this system as a "function" of the given input i (although, as a matter of fact, this is not a function, but an operator with memory).

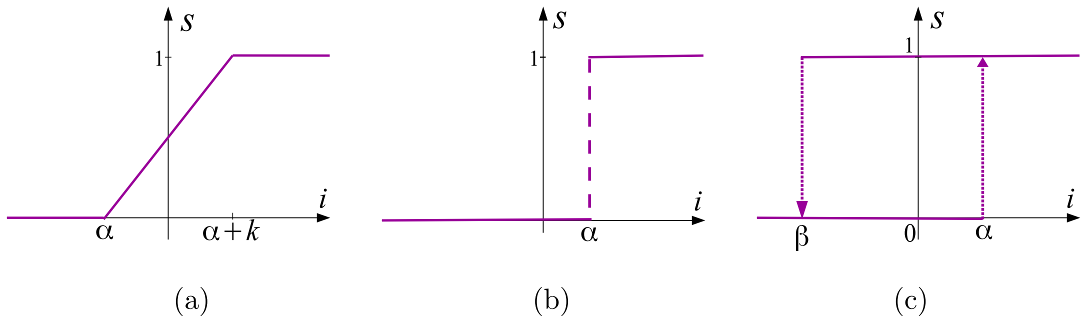


Figure 1. (a) Piecewise linear response. (b) Heaviside step function. (c) Non-ideal relay response (rectangular hysteresis loop).

The first observation is that if $s_1 = s_2$ at some moment in time, then the states of the two agent will remain equal at all later times. Therefore, we can consider the common state $s = s_1 = s_2$ of the two agents, and the system reduces to one implicit equation

$$s = H_\alpha(i + ks). \tag{4}$$

It turns out that this equation defines the non-ideal relay with two thresholds, also known as an elementary rectangular hysteresis loop [13, 15]. That is, i is the input and s is the output of the non-ideal relay, with s switching from 0 to 1 and backwards at two different threshold values of i, see Fig. 1(c).

Indeed, assume that initially $i = 0$ and $s = 0$, and let i increase. As i exceeds the value α, the right hand side becomes equal to 1, hence the state s becomes $s = 1$. That is, one switching threshold remains equal to α. From this moment (and until the state switches to zero again), the input $\tilde{i} = i + ks$ of H_α becomes equal to $\tilde{i} = i + ks = i + k$, hence $s = H_\alpha(i + k)$. The right hand side will become zero when the sum $i + k$ reaches the value α, i.e. when i reaches the value $\alpha - k$. Hence, the second switching threshold is $\beta = \alpha - k$. There is a difference between the switching thresholds for the transitions from 0 to 1 and backwards, and this difference equals to k. As k measures the strength of the affect of one agent on the other, we can say that a stronger interaction results in a larger separation of thresholds in the response—the more interaction, the more hysteresis.

4. A stochastic switching model

It seems realistic to assume that thresholds in individuals are not deterministic, but rather transitions of an individual between the states are a stochastic process. That is, given a certain input (e.g. pay rate), switching the state is a random event characterized by a certain transition probability rate. Here we develop the model described in the previous subsection to include uncertainty in switching events. A simple Markov chain model with transition probability rates depending on the pay rate i is suitable for our purposes.

Assume that the probability rate for the transition from state $s = 0$ to state $s = 1$ for an individual is $p_{0,1}(i)$. This probability rate should naturally be an increasing function of i. Similarly, let us assume that the probability rate $p_{1,0}(i)$ for the transition from state $s = 1$ to $s = 0$ decreases with i. We treat i as a control variable (input) that can change with time, i.e. $i = i_t$, hence the transition probability rates become functions of time. This defines a continuous time Markov chain with two states and time-dependent transition probability rates. Denoting by π_t^s the probability to find the agent in state s at time t, we can write the master equation for the evolution of the probability distribution vector $\pi_t = (\pi_t^0, \pi_t^1)$:

$$\frac{d\pi_t}{dt} = \pi_t P(i_t), \tag{5}$$

where

$$P(i) = \begin{pmatrix} -p_{0,1}(i) & p_{0,1}(i) \\ p_{1,0}(i) & -p_{1,0}(i) \end{pmatrix} \tag{6}$$

and $\pi_t^0 + \pi_t^1 = 1$ at all times.

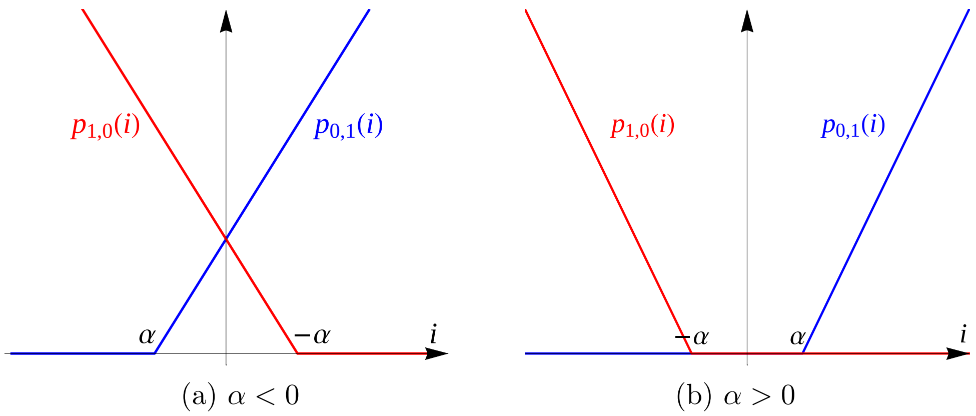


Figure 2. Probability of transition from state $s = 0$ to state $s = 1$ (blue) and vice versa (red) as a function of the stimulus i for one individual.

Next, we need to define the functions $p_{0,1}(i)$ and $p_{1,0}(i)$. As before, we assume that there are no transitions from state $s = 0$ to state $s = 1$ if the pay rate stays below a threshold value α. For the values of i exceeding α, we simply assume that the transition probability increases linearly with i. This leads to the law

$$p_{0,1}(i) = \gamma\,(i - \alpha)H_\alpha(i) \tag{7}$$

with the steepness coefficient (slope) $\gamma > 0$. Again, for simplicity, we assume symmetry in the law of transitions between the states by setting

$$p_{1,0}(i) = p_{0,1}(-i) = \gamma\,(-i - \alpha)H_\alpha(-i), \tag{8}$$

see Fig. 2. The stochastic model (5)–(8) with $\alpha \le 0$ replaces and extends the simple deterministic switching rule (1). The non-positive α corresponds to the absence of hysteresis in individuals. The same model with $\alpha > 0$ extends the deterministic switching rule based on the non-ideal relay [8]. That is, a positive α corresponds to the hypothesis that hysteresis is present in the response of individuals to the incentive, for example, due to interaction of individuals within a firm.

The model (3) of two interacting agents can be extended to include uncertainty in decision making along the same lines. Here, we have a Markov chain with four states, $(0,0),(0,1),(1,0),(1,1)$, with the first digit denoting the state of the first agent and the second digit standing for the state of the second agent. Assuming that decision making of an agent is affected by the other agent, we define the transition probability rates from state $(0,s)$ to state $(1,s)$ and from state $(s,0)$ to state $(s,1)$ by

$$p_{(0,s),(1,s)}(i) = p_{(s,0),(s,1)}(i) = \gamma\,(i+ks-\alpha)H_\alpha(i+ks), \tag{9}$$

where s is either 0 or 1 (cf. (3)). Similarly, the probability rates for transitions from state $(1,s)$ to $(0,s)$ and from state $(s,1)$ to $(s,0)$ are defined by

$$p_{(1,s),(0,s)}(i) = p_{(s,1),(s,0)}(i) = \gamma\,(-i-ks-\alpha)H_\alpha(-i-ks). \tag{10}$$

Finally, we set

$$p_{(0,0),(1,1)}(i) = p_{(0,1),(1,0)}(i) = p_{(1,0),(0,1)}(i) = p_{(1,1),(0,0)}(i) = 0. \tag{11}$$

Then, the evolution of the four-dimensional probability distribution vector $\pi_t = (\pi_t^{(0,0)}, \pi_t^{(0,1)}, \pi_t^{(1,0)}, \pi_t^{(1,1)})$ is determined by the master equation

$$\frac{d\pi_t}{dt} = \pi_t \hat{P}(i_t), \tag{12}$$

where the non-diagonal entries $p_{(s_1,s_2),(\sigma_1,\sigma_2)}$ of the 4×4 transition rate matrix $\hat{P}(i)$ are defined by equations (9)–(11) and the diagonal entries are determined by the condition that the row sum for each row of $\hat{P}$ is zero. System (12) serves as a probabilistic analog of the deterministic model (3).

5. Dynamics of the stochastic model

The following results illustrate the response of the stochastic model presented in the previous section to a periodically varying incentive and highlight the dependence of this response on the model and input parameters.

Fig. 3 presents time varying probabilities π_t^0 and π_t^1 to find an agent in state $s=0$ (black) and state $s=1$ (red), respectively, generated by master equation (5) with the periodic input $i=\sin t$ for a positive threshold value α and a steep slope γ.

Fig. 4 shows the response of model (5) to a slowly varying periodic input depending on the threshold parameter α (see Fig. 2). An agent spends part of the period in the "pure" state $s=0$ ($\pi^0=1$), part of the period in the "pure" state $s=1$ ($\pi^0=0$), and part in a "mixed" state. Each loop with $\alpha>0$ departs from the line $\pi^0=0$ at the point $i=\alpha$ and lands at the line $\pi^0=1$ for a slightly larger value of i; similarly, it departs from the line $\pi^0=1$ when $i=-\alpha$ and lands at the line $\pi^0=0$ for a slightly lower value of α. Hence, for $\alpha>0$, we observe an almost rectangular hysteresis loop (brown, red),

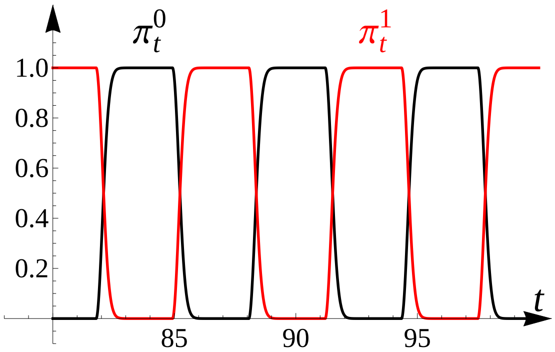


Figure 3. Periodic solution (π_t^0, π_t^1) of master equation (5) with the periodic input $i = \sin t$. Parameters are $\alpha = 0.1$, $\gamma = 15$.

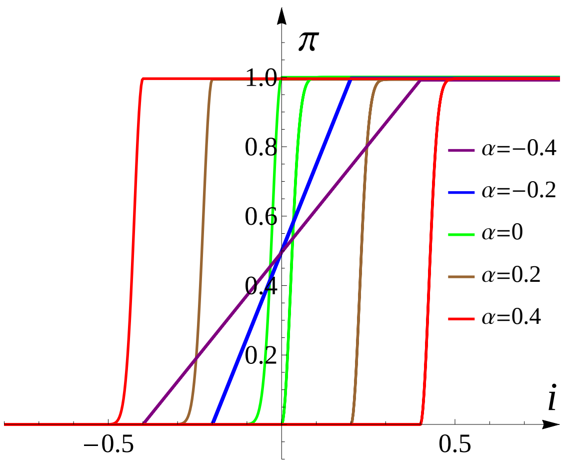
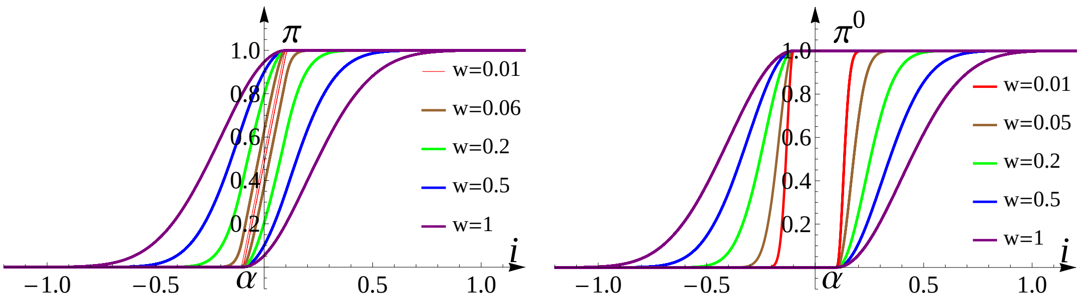


Figure 4. Input-output loops of model (5) (with $\gamma = 15$) generated by the slowly varying periodic input $i = \sin(0.01t)$ for different values of the threshold α.

which widens with α. For $\alpha < 0$, the loop degenerates to a piecewise linear curve (blue, purple) shown in Fig. 7(a), and no hysteresis is observed.

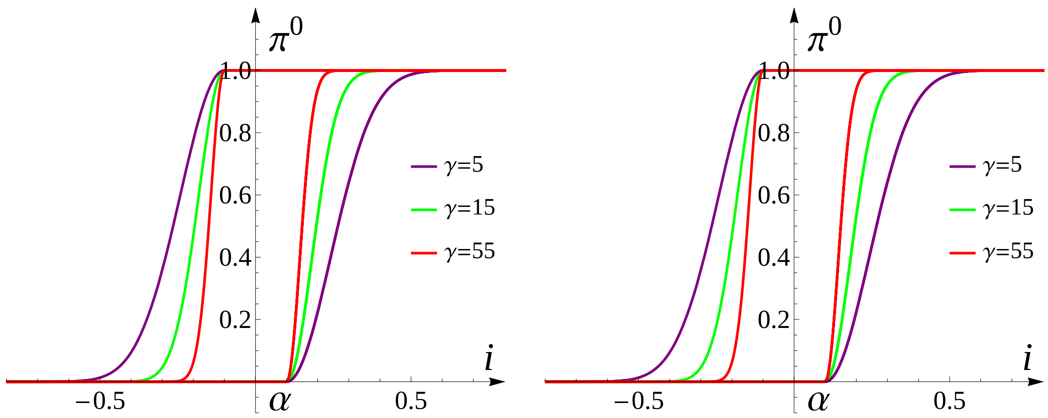
The shape of the input-output loop depends on the rate of change of the input, see Fig. 5 where the input is $i = 1.3\sin(wt)$. The left and right panels correspond to $\alpha < 0$ and $\alpha > 0$, respectively. One can see that the slower the input (smaller frequency w), the narrower the loop is. In particular, one can notice the "fake" hysteresis for higher frequencies on the left panel, which disappears for lower frequencies when the loop shrinks to almost a line (red). The "true" hysteresis is manifested by the low frequency loop (red) on the right panel. In the limit of low frequencies, the loop approaches the shape shown in Fig. 7—the piecewise linear (non-hysteretic) saturation function for $\alpha < 0$, and the rectangular hysteresis loop for $\alpha > 0$.

The slope γ of the transition probability rates (7) and (8) controls the shape of the



(a) $\alpha = -0.1$ (b) $\alpha = 0.1$

Figure 5. Input-output loops of model (5) (with $\gamma = 15$) generated by the periodic input $i = 1.3\sin(wt)$ for different frequencies w.



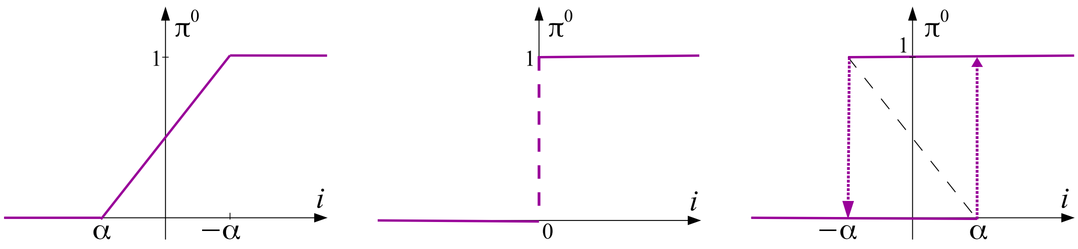
(a) $\alpha = -0.2$ (b) $\alpha = 0.2$

Figure 6. Input-output loops of model (5) generated by a periodic input $i(t) = \sin(0.1t)$ for different slopes γ of the functions $p_{01}(i), p_{10}(i)$ shown in Fig. 2.

loop in a similar fashion. As Fig. 6 shows, the steeper the slope γ, the closer the loop to the limit case shown in Fig. 7 is. That is, increasing γ results in the same effect as decreasing the input frequency w.

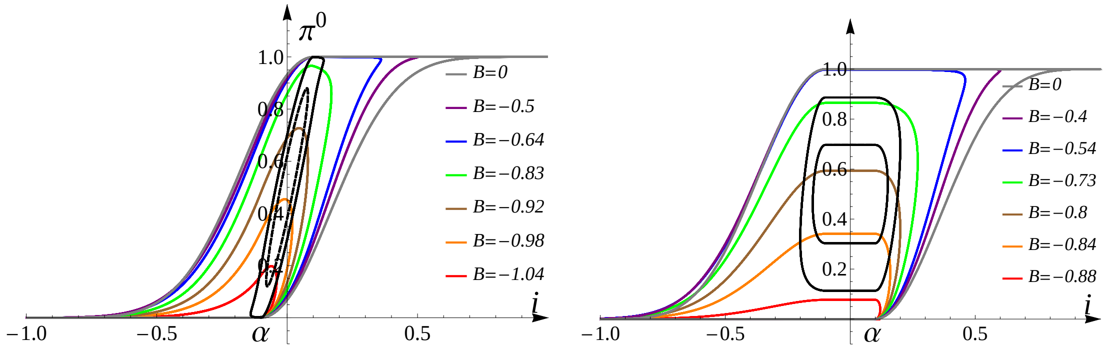
The limit shapes of the plots in Figs. 5 and 6 as $w \to 0$ and $\gamma \to 0$ are shown in Fig. 7 (cf. Fig. 1). For $\alpha < 0$, the probability π^0 to find an agent in state $s = 0$ is a piecewise linear function $\pi^0(i)$ shown on panel (a). There is no hysteresis. For $\alpha > 0$, the dependence of π^0 on i is described by the non-ideal relay operator (an elementary rectangular hysteresis loop) with two different thresholds $\pm\alpha$ for switching from state $s = 0$ ($\pi^0 = 1$) to $s = 1$ ($\pi^0 = 0$) and vice versa.

Fig. 8 presents the periodic solution of model (5) with the periodic input $i(t) =$



(a) $\alpha < 0$ (b) $\alpha = 0$ (c) $\alpha > 0$

Figure 7. Response of model (5) to a slowly varying input $i(t)$ in the limit when $\gamma \to \infty$ and $i'(t) \to 0$.

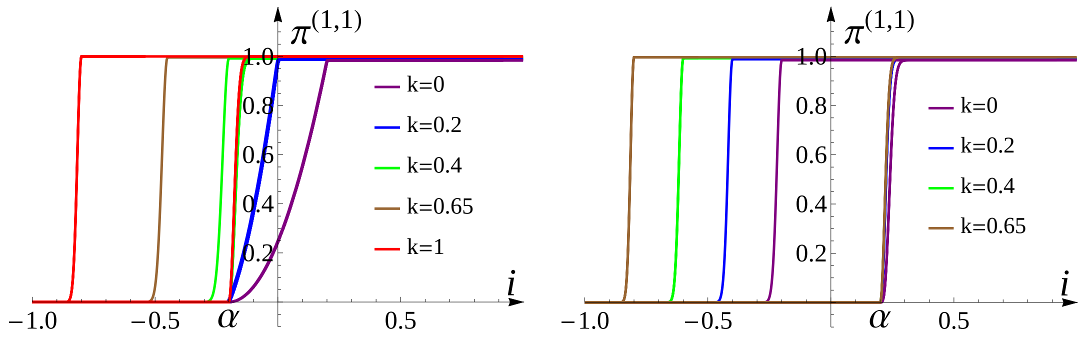


(a) $\alpha = -0.1$ (b) $\alpha = 0.1$

Figure 8. Input-output loops of model (5) (with $\gamma = 15$) generated by the periodic input $i(t) = B + R\sin t$ for different values of B.

$B + R\sin t$ for different values of B. The maximum and the minimum of the input are equal to $x_M = B + R$ and $x_m = B - R$, respectively. When $x_M < \alpha$, the agent resides at the pure state $s = 1$ ($\pi^0 = 0$) at all times. For $x_M > \alpha$, a loop appears and grows with x_M (which grows with B), see red, orange, brown, green curves on both panels. For larger values of B, the loop resides in each of the pure states $s = 0, 1$ for part of the period (blue, purple, and gray curves). Black loops that remain in the mixed state for all times, that is they do not touch the lines $s = 0, 1$, correspond to inputs with $B = 0$ and a sufficiently small amplitude R.

Finally, in Fig. 9, we consider the behavior of model (12) with two interacting agents for different values of the interaction strength k. The figure shows the dependence of the probability $\pi_t^{(1,1)}$ on the slowly varying input $i(t) = \sin(wt)$. The probabilities $\pi_t^{(0,1)}$ and $\pi_t^{(1,0)}$ are close to zero at all times, hence $\pi_t^{(0,0)} + \pi_t^{(1,1)} \approx 1$ for all t. In the case $\alpha < 0$



(a) $\alpha = -0.2$

(b) $\alpha = 0.2$

Figure 9. Input-output loops of model (12) of two interacting agents generated for different values of the interaction strength k. The figure shows the dependence of the probability $\pi_t^{(1,1)}$ on the slowly varying input $i(t) = \sin(wt)$ with $w = 0.01$.

(left panel), for smaller values of k, the dependence is described by a sigmoid saturated function $\pi^{(1,1)}(i)$, no hysteresis is observed (blue and purple lines). For larger k, the hysteresis loop appears due to interaction of the agents. The right (ascending) segments of all the loops almost coincide, while the width of the loop grows with k (green, brown, and red lines). In the case $\alpha > 0$ (right panel), the hysteresis loop is present for all $k \geq 0$. Again the right segment of the loop is almost the same for all k, and the loop grows wider with increasing interaction strength k.

6. Conclusions

We have proposed a probabilistic switching system, which can potentially be used for modeling the behavioral response of employees to the incentives offered within a firm. Variants of the model include or ignore social interactions between the employees as a factor that can affect their behavior. In the limit of slow variations of the input, the model either reduces to a memoryless instantaneous functional relationship between the input and the output or gives rise to hysteresis in the form of a non-ideal relay operator (rectangular hysteresis loop), depending on the parameters used. In particular, a sufficiently strong interaction between the agents can induce hysteresis, which is not observed in non-interacting individuals. On the other hand, if hysteresis is present in individuals, the interaction can enhance it. Faster variations of the input produce wider input-output loops, which look like a hysteretic response but may shrink to a memoryless functional response curve as the rate of input variations slows down.

Acknowledgments

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