When Should the Off-grid Sun Shine at Night? Optimum Renewable Generation and Energy Storage Investments.

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Abstract

Globally, 1.5 billion people live off the grid, their only access to electricity often limited to operationally-expensive fossil fuel generators. Solar power has risen as a sustainable and less costly option, but its generation is variable during the day and non-existent at night. Thanks to recent technological advances, which have made large-scale electricity storage economically viable, a combination of solar generation and storage holds the promise of cheaper, greener, and more reliable off-grid power in the future. Still, it is not yet well-understood how to jointly determine optimal capacity levels for renewable generation and storage. Our work aims to shed light on this question by developing a model of strategic capacity investment in both renewable generation and storage to match demand with supply in off-grid use-cases, while relying on fossil fuel as backup. Despite the complexity of the underlying model, we are able to extract two general results. First, we find that solar capacity and storage capacity are strategic complements, except in cases with very high investment in generation capacity, when they surprisingly turn into strategic substitutes, with implications for long-term investment decisions. Second, we develop a simple heuristic to determine which storage technology, within a given portfolio, can turn a profit in the broadest set of market conditions, and thus is likely to be adopted first. We find that currently, low-efficiency, cheap technologies such as thermal can more easily turn a profit in off-grid applications than high-efficiency, expensive ones such as lithium-ion batteries. We then develop two newsvendor-like approximations of the general model that are analytically tractable, yield precise values for the optimal investment decisions and profit in some cases, and provide bounds to the optimal investment decisions and profits in all other cases. To conclude, we calibrate our models to measure the accuracy of our solutions utilizing real-life data from three geographically-diverse islands, and then use our approximations to provide high-level insights on the role that large-scale storage will play in the years ahead as technology improves, carbon taxes are levied, and solar becomes cheaper.

1 Introduction

About 1.5 billion people worldwide live without connection to modern electricity grids and usually rely on diesel or gasoline generators for their electricity needs, which not only generate dirty energy but are also very expensive to operate (Lam et al. 2019). This problem is quite common in developing countries but is also present in the developed world - whether one looks at islands in Europe or remote villages in the Americas, off-grid power is typically provided through burning fossil fuels, with the same drawbacks of cost and pollution everywhere. While solar has become the cheapest source of electricity in most parts of the world (Lazard 2020), and may seemingly constitute an ideal solution to replace fossil fuel generators in such settings, the sun does not always shine and electricity demand cannot be backlogged. Hence, shifting off-grid energy provision toward renewable generation inevitably means finding ways to match demand with an intermittent supply.

The solution could be the storage of excess generation, to be used at a later time when needed. This is not a new idea: Pumped hydro systems have been utilized in mature energy grids since the late 19^{th} century. However, they are prohibitively expensive for smaller, off-grid applications, they require locations with specific geographic qualities that are rather uncommon, and even then, they only provide a small fraction of the total demand in energy. Thankfully, four concurrent developments in recent years have made multi-hour storage for off-grid applications sought after, technically feasible, and potentially profitable.

The first trend is the ever-decreasing cost of fossil-free technologies, with wind generation costs down 40% and photovoltaic prices down by 70-80%, compared to 2009 (IRENA 2017), rendering renewables increasingly competitive and making the problem of intermittency increasingly pressing.

Second, the cost of non pumped-hydro energy storage has also been decreasing steadily over the past several years. And as technology matures and the cost-benefit ratio improves, more people will take advantage of energy storage solutions. A recent examples of this is an island in American Samoa replacing oil imports with a combination of solar and storage,[1].

The third development is political in nature, with many national and regional governments enacting regulation that requires minimum renewable energy generation ratios in future decades. The island nation of the Maldives aims for 70% renewables by 2030 (World Bank 2020) and the European Union targets 40% by 2030 (EU Commission 2021). Using a different metric, India, home to the world's largest off-grid population, aims for 450GW of renewable generation by 2030 (Frangoul 2021).

The last element, which compounds the previous three, is that carbon emissions are under scrutiny in international treaties, such as the Paris and Katowice climate accords (EU Commission 2015), and further environmental regulations are investigated by economic scholars and (non)governmental institutions (Nordhaus 1994, World Bank 2017).

Taken together, these trends make the provision of renewable-based off-grid energy and storage

[1] https://www.nationalgeographic.com/science/article/tau-american-samoa-solar-power-microgrid-tesla-solarcity

not only politically desirable, but also economically attainable, while potentially offering simultaneously both lower long-term costs and sustainability. Thus, managing the operational aspect of supplying customers with renewable electricity, especially with intermittent generation, is of utmost relevance.

In this paper, we propose a two-stage, stylized model to study the capacity investment decision in storage and renewable generation. In the first stage, a utility provider decides on a combination of renewable generation and storage capacity to serve demand, while we assume that fossil-powered generators already exist as the current generating technology, and can be used as a backup. In the second stage, generation and storage utilization happen over the lifetime of the investment. This model is novel and unique in the literature as it approaches storage differently than traditional time series and computational approaches (Salas and Powell 2018, Cruise et al. 2019). Our analysis offers insights on the strategic relation between generation and storage investment decisions. Specifically, we find that the firm's investment decisions are strategic complements when renewable generation or storage capacities are low, but interestingly, turn into strategic substitutes when generation capacity becomes large. This finding challenges the notion that such investment decisions always support each other.

Some researchers (Diouf and Pode 2015, Kittner et al. 2017) and policy makers (Tsiropoulos I. 2018) suggest that lithium batteries, with their high efficiency and market penetration, may be the future technology of choice. By contrast, we find that technologies such as thermal, that are less efficient but cheaper than lithium batteries, stand to gain the upper hand. Furthermore, we derive a simple heuristic that can be used to determine which storage technology within a given portfolio can turn a profit in the broadest set of market conditions, and thus is likely to be adopted first.

Since in a model that keeps track of the energy stored across all periods —henceforth referred to as "tracking" model —the firm's capacity investments solutions are analytically intractable, we employ two simplifying assumptions and develop two corresponding simplified models for which analytical characterization is possible. In the first of those models, called full-discharge model, we assume that all the energy stored during the day is discharged within the following 24 hours. In the second model, called partial-discharge model, we assume that energy stored in a period is lost if it is not used by the end of the following period. Thanks to either of these assumptions, the T periods of the model under study, which are temporally linked by the stored energy carryover in the tracking model, can be disjoint into T temporally-independent periods (or pairs of periods) with important implications for tractability. In particular, when the cost of fossil fuel backup energy is lower than a given threshold, we are able to derive closed-form solutions for the firm's capacity investment decisions. Beyond this threshold, the solution to our models provide bounds for the optimal storage decision in the "tracking" model. Furthermore, we show via simulation that one of our approximations - the partial-discharge model – constitutes a reasonable proxy for both generation and storage investment decisions across a fairly wide range of realistic problem parameters.

Our model also helps sketch high-level trends regarding the role of storage in the coming years. As storage technologies gradually become cheaper, we find that investment in renewable storage will not happen gradually; rather, there will be a no-investment period, followed by a period of rapid adoption. However, the need for fossil/nuclear energy will likely remain in the medium-to-long term, due to the need of complementing renewables with some amount of non-intermittent generation.

Lastly, we investigate the case in which the back-up generator is downsized following the installation of solar capacity, and therefore cannot fulfill all of demand by itself. In this case, it is optimal for the firm to employ a policy where the generator is run preemptively to ensure that the charge at the end of each period does not fall below a threshold level. Numerical simulation leads to further insights on emissions and renewable investments. For example, reducing backup capacity by as much as 30-40% often leads to no decrease in emissions and may even increase them — the smaller size of the backup generator increases the risk of not meeting all future demand and induces the firm to run it more often (see Section 5.4 for a full discussion).

To summarize, our paper develops a model to jointly determine solar generation and storage for off-grid use cases in the presence of a backup generator, and uses it to (i) solve for the optimal investment decisions and/or derive bounds thereof; (ii) characterize the strategic interaction between generation and storage investments; (iii) derive a simple and effective heuristic to compare different storage technologies; (iv) uncover the consequences of curbing fossil generation on renewable investments, costs, and most importantly emissions; and (v) obtain high-level insights on the role of storage over the coming decades. Overall, our results provide both theoretical and practical insights for policy makers, utilities, and technology startups operating in this space.

2 Literature Review

Given the broad relevance of renewable energy and storage, our paper is at the intersection of multiple research streams. At its core, the investment decision deals with the intricacies of capacity management under uncertainty, an area for which Van Mieghem (2003) provides an excellent review. This stream includes the classic decision of long-term investment, facing market variability (Arrow 2017), but also how decisions change when different options of fulfilling demand are available (Shumsky and Zhang 2009), and how financing impacts such capacity choice (Boyabatlı and Toktay 2011). Wang et al. (2013) point out that such investment decisions are increasingly common as many industries are changing production and distribution practices to become more sustainable.

Thematically, this paper relies on energy research that includes work by Kök et al. (2020) and Wu and Kapuscinski (2013) on the role of renewable intermittency for electricity systems; the impact of emission cost on profitability and technology choice (Drake et al. 2016); the optimal design of feed-in-tariffs (Alizamir et al. 2016); the effect of net-metered energy on a utility's profitability (Sunar and Swaminathan 2018); the capacity effects of different renewable ownership structures

(Agrawal et al. 2019) and energy storage policies (Wu et al. 2012). Additionally, there is a broad field of research on the technical feasibility of renewable grids, from comparing different types of storage (Dunn et al. 2011) over cost-minimal combinations of technologies to achieve high renewable penetration (Budischak et al. 2013), to the long-term impact of large-scale wind energy deployment (Miller and Keith 2018). Beyond this literature, storage investment has also been studied by various papers in economics (Neetzow et al. 2018).

Most papers in the field approach the inherent complexity of storage investment like Jiang et al. (2014), who employ large-scale models and efficient algorithms to optimize over large parameter spaces in order to establish lower bounds on algorithmic solutions. Similarly, Kim and Powell (2011) use parametric models to derive optimal energy commitment conditions in the electricity market. However, it is difficult to extract high-level managerial insights from such computer-guided analyses, given that there are: multiple charge/discharge periods, at least one source of stochasticity, and one must also keep track of the "inventory" of the storage unit, i.e., the charge. Even if solutions are obtainable in closed form in these papers, they typically do not easily lend themselves to interpretations, and make it difficult to develop intuition.

Alternatively, Aflaki and Netessine (2017) employ a higher level of abstraction and aim to derive generalizable, strategic investment insights for renewables using a Newsvendor approach. They conclude that, in the presence of renewable intermittency, an increasing renewable generation share might even increase carbon emissions due to carbon-intensive backup plants. Analogously, Kök et al. (2020) use a Newsvendor-style model to solve a capacity investment problem between conventional and renewable energy sources. They find that flexible, conventional sources and renewables are complements. We use a similarly stylized approach in the context of off-grid energy storage.

To the best of our knowledge, there are currently no papers that consider the strategic role of storage investments. While there are some operational papers on storage in the context of renewable energy, they have a different scope. Qi et al. (2015) look at the combination of grid-interconnection and storage to improve dispatchability of an individual wind farm. They are able to show the existence of lower and upper bounds for storage sizes, but focus more on the grid and deployment aspect than the storage itself, and do not investigate the impact of storage on the overall market. Zhou et al. (2019) study a similar scenario and derive heuristics for storage decisions, obtained from an MDP model. Yang and Nehorai (2014) provide an intricate Lagrangian optimization approach to reduce the complexity of planning generation and storage investments for micro-grids, but only obtain numerical results without generating analytical insights. Luo et al. (2015) calculate the optimal battery capacity in a similar wind-park setting, but the paper is simulation-based and focuses on using storage to bridge the gap between actual and forecasted renewable generation. Schill and Kemfert (2011) focus on the effect of pumped hydro in the German oligopoly market. They find that pumped hydro does not affect a participant's market power and its storage capacity is generally underutilized. Strategic investment analysis was not a part of the paper. Lastly, Song et al. (2012) discuss storage on an individual project level, with emphasis on the state-of-charge of a battery, but they do not consider an entire energy market, backup costs, or the existence of alternative generation technologies. Avci et al. (2014) analyze storage capacity in the context of an electric vehicle charging station focusing not on the combined or total charge but on the optimal number of replacement batteries for a recharge station. The authors employ a repair model to capture the recharging process, as is typical in the spare-parts literature (Muckstadt 2004).

This paper, therefore, expands the existing operations literature on energy storage by presenting a way to jointly model energy storage and intermittent renewable generation capacity investment, while considering backup capacity, charging/discharging efficiency, and emission prices.

3 Model

We aim to capture the strategic trade-off between intermittent renewables combined with storage on one hand, and fossil fuel backup on the other. Operationally, this means making a decision between two technologies: a cheaper and less predictable (renewable) technology and a more expensive, yet always available alternative. Storage can then be thought of as a costly means of reducing the variability of the former option.

3.1 Model Setup

We formulate the problem as a 2-stage, 2-variable newsvendor-like model (but with uncertainty in supply rather than demand). In stage one, the utility makes joint capacity decisions on renewable generation and energy storage. In stage two, demand and generation are realized over T stochastically identical periods: demand is met by employing the capacities from stage one, while supply shortages are met through fossil fuel backup. Figure 1 provides a graphical illustration of the model elements and their relation to each other.

Demand Structure. When modeling storage, the need to consider at least two periods to allow for charging and discharging to occur is inherent. Each of the T periods in the model represents a 24-hour cycle that is further subdivided into two sub-periods, day and night, each lasting 12 hours. This night/day distinction captures the main source of variation in electricity consumption, aligns with the solar generation profile, and simultaneously provides structure to the storage decisions. During the day sub-period, deterministic demand D_H occurs, which is followed by the night, in which deterministic demand D_L occurs. While this is a simplification of real demand patterns, the most important factor governing storage usability is not the absolute level of supply or demand, but the difference between the two, which allows for charging and discharging. We focus on this mismatch by assuming two deterministic demands and variable solar generation, motivated by the fact that in practice variability in supply is much higher than variability in demand. Alternative

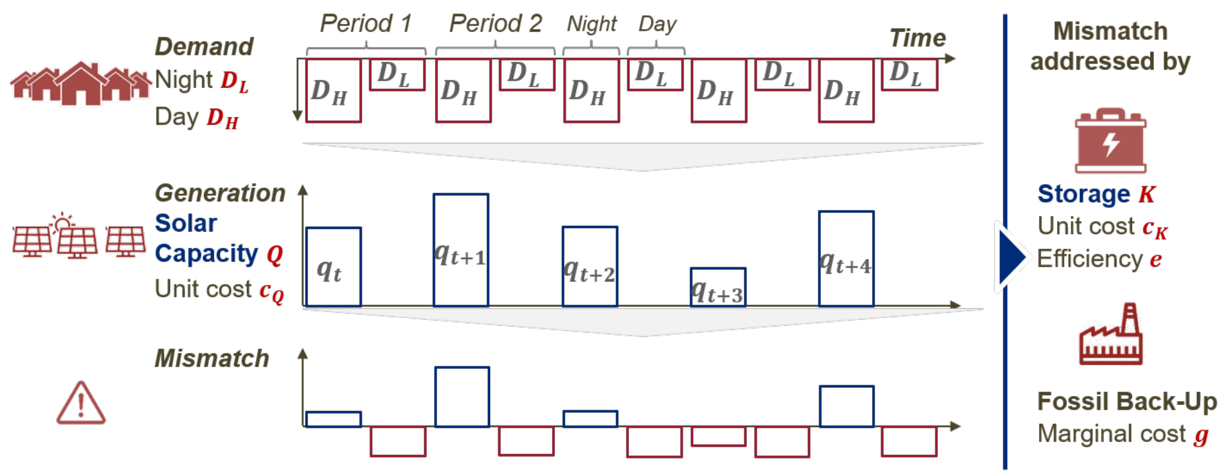


Figure 1: Model Diagram

day/night split lengths can also be accommodated by adding another parameter to the model and adapting the demand accordingly.

Generation Technology. We assume solar generation in this model as it is the cheapest generation source in expectation and in many off-grid use-cases the only feasible renewable solution due to geographical/physical restrictions. In addition, solar panels are more modular than wind turbines, the other frequently-built renewable technology, and can therefore be sized according to the specific needs of the use-case.

Generation during the day is uncertain and is a function of installed generation capacity, Q, while it does not depend on previous period generation or demand. Specifically, we assume for simplicity that daily generation in each period q_t is distributed uniformly $q_t \sim U[0, Q]$. Generation from solar panels at night is naturally 0. This dichotomous nature is the core source of the aforementioned difference between supply and demand, and the reason why we focus on two 12-hour sub-periods for the strategic investment case. Since solar generation will always be lower than energy demand during the night, if any storage charge is to be accumulated for subsequent discharge, the storage unit must be charged by generating more electricity than is demanded, during the day. The unit costs c_Q are linear, average per-period costs of generation capacity. They are obtained by splitting the unit capacity cost across the T days of the assumed investment life-time. We assume that, as is the case in reality, $2c_Q < g$. That is, in expectation (given the uniform distribution of generation) solar is cheaper than the backup technology, as otherwise one would never invest in any solar. Marginal generation costs are 0. Potential renewable subsidies can be priced into the model by calculating the expected subsidies over the lifetime and adjusting the unit costs accordingly.

Storage Technology. Let K be the size of the storage, measured in energy - which is power over time (e.g., MWh) - it can discharge. The storage exhibits cycle efficiency $0 < e \leq 1$, where

$1 - e$ units of energy are lost in each charging/discharging cycle. This efficiency is a core metric for storage technologies, as a perfect system would not lose any energy in the charging/discharging process and return 100% of the originally stored energy. But among other things, secondary reactions in a battery and mechanical losses in thermal systems lead to energy dissipation in real-world installations. Next to unit cost, this factor is of utmost importance when choosing a storage solution, and ranges from 20% to almost 100% in practice, depending on technology and scale (Koohi-Fayegh and Rosen 2020). Unit costs c_K are linear in MWh and are distributed equally across all T periods. Since we measure storage in discharge-able units, we adjust the unit cost as c_K/e to account for the fact that less efficient storage technologies need more capacity to be able to discharge the same amount of energy (to discharge 100 units, a 50% efficient technology needs a capacity of 200).

Brief discussion. Given the fast-paced nature of the energy storage industry, we built the model to capture virtually any type of technology. Advancements in storage technology mostly revolve around three key performance features: unit cost, cycle-efficiency, and number of discharge cycles. Cost and efficiency are directly captured through parameters in the model, while discharge life cycles are indirectly captured by splitting the investment cost over the respective number of days/periods that correspond to the anticipated lifetime.

Backup Technology. The backup capacity is assumed to already exist, typically in the form of a diesel or gasoline generator that traditionally represents the main source of electricity generation in many off-grid scenarios. As this backup burns fossil fuel, we assume a marginal generation cost of g per unit of energy while the generator has the ability to quickly respond to changes in demand. We assume this technology to be able to generate enough electricity to satisfy demand and to be always available (this will be relaxed in Sections 3.3 and 5.4).

Application. This model is applicable to every energy market where solar generation is possible, and generation costs by conventional generators can be estimated[2]. For example, the model can be applied to any off-grid location —islands using diesel-generators to fulfill inhabitants' electricity needs, remote mines burning gas to power operations, villages and small towns in under-developed countries, etc. The reason is that such off-grid locations exhibit known, constant backup costs as they typically have only one type of generator as backup, no merit ordering, and no capacity or energy auctions. As a consequence, the value of solar is easy to compute and equal to the cost of the backup generation it replaces. Lam et al. (2019) estimate that globally 20-30 million of such off-grid sites exist - millions of locations that represent the use-cases we model and that could benefit from the insights we develop.

[2]A benefit of using this model is that all parameters can be easily derived from historic knowledge of demand patterns (average electricity consumption) and publicly available sources (technology and cost parameters)

REIDS, a Singapore-based project, focuses on exactly the energy transition we describe by providing it for islands around Asia and Oceania (Choo 2017). Their business model centers on electrifying or repowering off-grid islands with renewable micro-grids that only rely on diesel-generators as a last resort. In a similar vein, the European Union spearheaded the TILOS project on the eponymous Greek island, where it tests the integration of renewable energy and a natrium-based battery solution (Kaldellis and Zafirakis 2020).

3.2 Objective Functions

The setup we are considering is that of a utility firm simultaneously investing in generation and storage. In Section 3.2.1 we present a model, henceforth referred to as *tracking model*, that keeps track of the energy stored over time as a function of realized generation. This model is useful to tie together the various elements of the model, and derive some structural properties, but is in general too complex to be solved analytically. For this reason, in Sections 3.2.2 and 3.2.3 we introduce two simplified versions of the tracking model that are easier to study and provide useful approximations to the investment decisions from the tracking (time-series) model. The quality of these approximations will be numerically investigated in Section 5.

3.2.1 The Tracking Model.

We begin by describing the charging and discharging process. Let x_t denote the energy stored at the end of period t (and hence the charge at beginning of period $t+1$). We can compute storage at the end of time t using the following expression

$$x_t = \left(\min \left[\left(x_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+ \right)^+ , K \right] - D_L \right)^+ , \quad (1)$$

where $(a)^+ = \max[0, a]$. During the day, there are two possible scenarios: either generation ($q_t \in [0, Q]$) is sufficient to meet daily demand, $q_t \geq D_H$ and unused energy in the amount of $(q_t - D_H)^+$ is charged into storage for later use, allowing $e(q_t - D_H)^+$ of discharge, or generation is insufficient to meet daily demand, $q_t < D_H$, and energy in the amount of $(D_H - q_t)^+$ is discharged to serve unmet demand. During the night, D_L of energy is discharged to serve nightly demand. The formula ensures that the storage charge is never negative or higher than storage capacity K.

The objective function that the firm wants to maximize can be written as the sum of cost-savings from solar and storage across the T periods, minus the capacity cost. Since fossil generation is always available but costly, the economic benefit of each unit of renewable generation, which has zero marginal cost, is equal to the cost g of the fossil backup it replaces. Cost-savings are thus simply equal to the total demand that can be fulfilled, by direct generation or through storage, multiplied by g. To avoid confusion between cost-savings and capacity cost, we will subsequently refer to cost-savings as revenue as they capture the economic benefit that is derived from investing in solar and storage capacity. Equation 2 captures all revenues earned during the day across the T periods.

$$g\ E\Big[\sum_{t=1}^{T}\Big(\min[x_{t-1}+q_t, D_H]\Big)\Big]. \tag{2}$$

The first term in the minimum in Equation 2 is the total renewable energy that is available either through generation q_t in that sub-period or by discharging storage x_{t-1}. The second term D_H is the demand during the day, which is the maximum amount of energy to be fulfilled during the day sub-period.

At night there is no generation, so any replacement of the backup occurs by discharging stored energy, as captured in Equation 3 below.

$$g\ E\Big[\sum_{t=1}^{T}\Big(\min\Big[\big(x_{t-1}+e(q_t-D_H)^+-(D_H-q_t)^+\big)^+, K, D_L\Big]\Big)\Big]. \tag{3}$$

That is, the firm can fulfill demand equal to the minimum of the charge at the end of the day and the nightly demand D_L. Note that, as with the definition for x_t, the charge cannot be negative or exceed storage capacity K.

Lastly, the firm has to pay c_Q for the solar generation capacity Q and c_K/e for the storage unit capacity K each period (total cost divided by all periods), which leaves us with the following objective function for the tracking model, where x_{t-1} is defined as per Equation 1:

$$\begin{aligned}\Pi_{TR}(Q,K) =& g\ E\Big[\sum_{t=1}^{T}\Big(\min[x_{t-1}+q_t, D_H] + \min\Big[\big(x_{t-1}+e(q_t-D_H)^+-(D_H-q_t)^+\big)^+, K, D_L\Big]\Big)\Big]\\ &- T\frac{c_K}{e}K - Tc_QQ,\end{aligned} \tag{4}$$

The objective function of the tracking model is intractable and no closed-form solution for the investment decisions (Q^*_{TR}, K^*_{TR}) can be derived. The main source of complexity comes from adjacent periods being linked to each other through the energy carryover terms x_t. There is, in other words, a positive probability that a unit of charge from period one (or any other period) would get discharged in any subsequent period up to the last one.

Despite not having closed form results for the tracking model, we can still obtain several insights from it by indirectly leveraging some of its properties. We present these insights in the following subsections.

Strategic Complements or Substitutes? In this subsection, we aim to understand if there is a strategy relation between generation and storage capacity. In other words, we study whether investing in either capacity affects the value of investing in the other (see Appendix A.1).

Theorem 1. STRATEGIC INTERACTION BETWEEN INVESTMENT DECISIONS
In the tracking model, renewable generation and energy storage are:

- *Strategic complements at lower levels of capacity investment. Formally* $\partial^2\Pi_{TR}/\partial Q\partial K > 0$ *if* $Q < Q^*_{bor}$ *or* $K < D_L$*, where* $Q^*_{bor} = \sqrt{g(D_L^2/e + 2D_HD_L + D_H^2)/2c_Q}$*;*

- *Strategic substitutes at higher levels of generation capacity, when storage exceeds nightly demand. Formally,* $\exists\ Q'$ *s.t.* $\partial^2\Pi_{TR}/\partial Q \partial K < 0,\ \forall\ K\ > D_L,\ Q > Q'$.

At low levels, capacities in the tracking model are strategic complements because for storage to be profitable, it must occur frequently enough that *generation outstrips demand*, otherwise the storage does not get charged often, and thus cannot justify its cost. An increase in generation therefore leads to an increase in storage because the higher odds of observing excess generation means that a larger battery is needed to store it - we have strategic complementarity.

However, at high-enough levels of generation, we have strategic substitutability. The reason being that for storage to be profitable, it is not sufficient that generation outstrips demand frequently —which ensures that the storage gets charged often —but at the same time it is also important that *demand outstrips generation* frequently, otherwise stored energy is rarely put to use — as it happens when generation is very high. For example, imagine a scenario in which renewable generation capacity is thousands of times higher than demand: there would (nearly) always be more energy generated than demanded, removing any need for storage beyond that of covering nightly demand.

For more details and the strategic investment results for the simplified models, we direct the interested reader to Appendix A.1 (particularly Equations 16 and 18).

Comparing Storage Technologies. Even with the rapid advances in storage technologies over the last years, storage of renewable energy is not yet a profitable investment in all scenarios. However, current trends in many parts of the world (e.g., emission targets issued by governments, increasing calls for a carbon tax) signal that energy storage will likely become a sizeable market in the near future. This means that the storage technologies that exist, or are being developed today, will soon compete for the storage market of tomorrow. Hence, a question of interest is: Which of these technologies is likely to be adopted first? It would thus be useful to establish a criterion that, for any given set of non-Pareto dominated technologies —i.e., a set where no technology is both cheaper and more efficient than another —could determine which technology can turn a profit in the broadest set of market conditions, and is thus more likely to be adopted first. To this end, we first formalize the above discussion and then present our result.

Definition 1. *Storage technology* A (e^A, c_K^A) *is preferred to storage technology* B (e^B, c_K^B) *if and only if* $\Gamma^A \supset \Gamma^B$, *where* $\Gamma^j \triangleq \{(c_Q, D_H, D_L, g) : K^*_{TR}(c_Q, D_H, D_L, g, e^j, c_K^j) > 0\}$ *is the set of non-storage parameters for which the firm finds it optimal to invest in strictly positive storage capacity under storage technology* $j \in \{A, B\}$.

In order to investigate conditions that render one technology preferable over another under Definition 1, we employ a result from the full-discharge model, which we will introduce subsequently and which is equivalent to the tracking model for the parameter space we consider (Theorems 3 and 4). The next lemma (see Appendix A.4 for derivation) will prove useful.

Lemma 1. STORAGE PROFITABLITY

- *The optimal storage investment is positive if and only if the backup cost is higher than the threshold* $g_0 = c_Q + c_K/e + \sqrt{c_Q^2 + 2c_Q c_K/e}$. *Formally* $K^*_{TR} > 0, \iff g > g_0$;
- *The threshold* g_0 *is strictly increasing in the ratio* $\frac{c_K}{e}$, *for any* D_H, D_L, *and* c_Q.

Interestingly, the lemma shows that the backup cost threshold below which storage becomes profitable, g_0, depends on storage technology parameters c_K and e *only through their ratio*, $\frac{c_K}{e}$. We call such ratio the *storage-cost-to-efficiency ratio*. We have the following result:

Theorem 2. COMPARISON OF STORAGE TECHNOLOGIES

a) Storage technology (e^A, c_K^A) *is preferred to* (e^B, c_K^B) *if and only if* $c_K^A/e^A < c_K^B/e^B$;

b) A given storage technology (e^S, c_K^S) *can profitably be invested in iff* $c_K^S/e^S < g - \sqrt{2c_Q g}$.

Theorem 2 provides a necessary and sufficient condition for one storage technology to be preferred over another (point a): a lower cost-to-efficiency ratio $\frac{c_K}{e}$ renders a technology preferred to other technologies with a higher ratio. Moreover, this ratio can also be employed to determine whether a given technology is altogether profitable (point b). Overall, the results in Theorem 2 highlight that it is the storage-cost-to-efficiency ratio that governs the suitability of a given technology as a profitable investment, and that such ratio can be a simple yet quite effective criterion to rank-order technologies from most to least-preferred, in the sense of Definition 1.

We now move from these structural properties to the second objective of our work —the development of simple, tractable solutions for the optimal capacity investments into solar and storage. To this end, we develop two simplified models in the next subsections that approximate the profit and the investment decisions of the tracking model and allow for closed-form investment results. These models will be henceforth referred to as the *full-* and *partial-*discharge models. We examine the quality of these approximations numerically in Section 5.

3.2.2 The Full-Discharge Model.

The full-discharge model rests on Assumption 1.

Assumption 1. *In the full-discharge model, all the energy stored in a period is profitably discharged by the end of the period.*

Under this assumption, each unit of stored energy is discharged and earns revenue equal to g regardless of whether there was enough demand to be served. An important implication of this assumption is that the full-discharge model does not require tracking of stored energy from one period to the next. This approach removes the interdependence between subsequent periods, meaning that we can solve the firm's problem as if the firm had to serve only one period (or, more appropriately, T identical periods). The objective function of the full-discharge model becomes:

$$\Pi_F(Q,K) = T \cdot \left(g\ E\Big[\min[q_t, D_H] + \min[e(q_t - D_H)^+, K] \Big] - \frac{c_K}{e} K - c_Q Q \right). \tag{5}$$

For the above objective function, all partial derivatives and the Hessian can be signed (see Appendix A.3) leading to the following result on the optimal investment decisions.

Theorem 3. Optimal Decisions under the Full-discharge Model

Under the full-discharge model, the objective function is globally concave over its domain. The optimal investment choices are given by:

$$Q_F^* = D_H g \sqrt{\frac{(1-e)e}{2(c_K+c_Q)eg - c_K^2 - e^2 g^2}} \quad K_F^* = \max\left[-D_H e + D_H (ge - c_K) \sqrt{\frac{(1-e)e}{2(c_K+c_Q)eg - c_K^2 - e^2 g^2}}\ , 0 \right].$$

We discuss the results from Theorem 3 after Theorem 4, which relates the full-discharge capacities to that of the tracking model.

Theorem 4. Comparison Between Full-discharge and Tracking Models

- *If $g \le g_0$, the backup cost is too low for the firm to invest in any storage. In that case, the full-discharge model's investment decisions from Theorem 3, and the profit, coincide with the tracking model's. Formally, if $g \le g_0$, $K_F^* = K_{TR}^* = 0$, $Q_F^* = Q_{TR}^*$, and $\Pi_F^* = \Pi_{TR}^*$, where g_0 is given by:*

$$g_0 = c_Q + \frac{c_K}{e} + \sqrt{c_Q^2 + \frac{2c_Q c_K}{e}}.$$

- *If $g_0 < g \le g_F$, the firm's storage investment is positive and the full-discharge model's investment decisions and profit coincide with the tracking model's. Formally, $(Q_F^*, K_F^*) = (Q_{TR}^*, K_{TR}^*)$ and $\Pi_F^* = \Pi_{TR}^*$, if $g_0 < g \le g_F$, where g_F is given by:*

$$g_F = \frac{(em+1)\left(\sqrt{c_Q\left(2c_K(em(m+2)+1) + c_Q(em+1)^2\right)} + c_Q em + c_Q \right) + c_K em(m+2) + c_K}{e(em(m+2)+1)},$$

where $m = D_H/D_L$.

- *If $g > g_F$, storage investment is strictly positive, larger than what is needed to meet nightly demand, and the full-discharge model's storage investment decisions and profit are strictly higher than the tracking model's. Formally, $K_F^* > D_L$, $K_F^* > K_{TR}^*$, and $\Pi_F^* > \Pi_{TR}^*$, if $g > g_F$.*

Theorem 4 contains three elements. First, it establishes the existence of a backup cost threshold, g_0, below which investing in storage is not profitable. Such a threshold increases in storage cost c_K as well as in generation cost c_Q, and decreases in storage efficiency e. In this parameter space, the full-discharge model's investment decisions are exact, that is, they match the tracking model (Appendix A.4). Second, if the backup costs are between g_0 and g_F, storage investment is positive and the full-discharge model's investment decisions coincide with the tracking model. Third, if the backup cost is beyond the threshold g_F, it is optimal for the firm to build at least enough storage capacity to serve all nightly demand; the full-discharge model then is no longer an exact approximation of the tracking model. In this regime, the full-discharge model is still useful, as its investment decisions constitute an *upper bound* to the tracking model's optimal investment decisions.

With this context established, we return to insights previously obtained in Theorem 3 regarding the drivers of generation and storage investments. The optimal generation investment is proportional to daily demand, decreasing in solar cost, and has non-linear relationships with efficiency as well as storage cost. The optimal storage investment also scales with daily demand and is higher when the difference between c_K and g is low enough relative to the storage technology's efficiency e, as this roughly measures the relative cost of serving demand with stored renewables versus fossil backup capacity. If this difference $(ge - c_K)$ is insufficient, optimal storage capacity is zero. Further note that the storage capacity is affected by the same radical expression as generation —this is the indirect impact of solar on storage.

In combination, Theorem 3 shows that a firm may find it optimal to serve demand with renewable generation without investing in storage, but that storage deployment cannot be optimal in the absence of renewable generation.

3.2.3 The Partial-Discharge Model.

When $g > g_F$, the full-discharge model does not provide an exact solution to the tracking model but rather an upper bound to the investment decisions of the firm. Thus, in this section, we develop a second model that can supply additional information regarding optimal investment when the backup cost g is higher than g_F.

The partial-discharge model rests on the following two assumptions.

Assumption 2. *In the partial-discharge model, energy charged in period t expires (i.e., is lost forever) if not used by the end of period $t + 1$.*

Assumption 3. *In the partial-discharge model, demand is met by employing the most recently-generated energy first.*

Note that Assumption 3 entails a LIFO use of energy, meaning that the firm serves demand in a period using energy generated in that period if available, then energy stored in that period if any, and then energy stored in the previous period — in this order of priority. Note also that the need to specify a priority order in the use of energy arises because of Assumption 2 — when energy does not expire, there is no need to treat the energy stored at different times differently.

The use of a LIFO rule for energy consumption, whereby energy that expires at the end of the period is given the lowest priority, may appear sub-optimal, and in fact, it is: Using older energy first would indeed be preferable profit-wise, yet our assumption does *the opposite*. The advantage of such an approach is not to proxy profit as much as possible, but to make the problem more tractable, as we are about to explain.

Under Assumptions 2 and 3, revenues in period t are a function of storage at the end of period $t-1$, x^P_{t-1} (which is important for the accuracy of the partial-discharge model), but are *not* a function of storage at the end of period $t-2$, x^P_{t-2}; nor are they a function of storage in any previous periods. This ensures that revenues in a period are only a function of events occurring in that period and the previous period. This is in stark contrast with the tracking model, where revenues in a period are a function of events occurring in that period *and all previous periods*. Because of this feature, as we are going to show, our assumptions greatly simplify the objective function.

More specifically, Assumption 2 prevents x^P_{t-2} from *directly* affecting revenues in period t, because such energy expires at the end of period $t-1$. However, x^P_{t-2} may still affect revenues in period t *indirectly*, i.e., it may affect x^P_{t-1}, which in turn affects revenues in period t. This can happen in two ways. First, some demand in period $t-1$ may be served using x^P_{t-2} instead of the energy generated in period $t-1$, q_{t-1}, and this alone renders x^P_{t-1} a function of x^P_{t-2}. Alternatively, x^P_{t-2} may occupy storage capacity and prevent energy generated in excess during the day, in period $t-1$, to be stored in the battery, which again affects x^P_{t-1}.

Assumption 3 breaks the indirect dependency between x^P_{t-2} and x^P_{t-1} because the use of x^P_{t-2} has the lowest priority: It is used to meet demand in period $t-1$ only if q_{t-1} is not sufficient, and is kept in storage only if this does not prevent newly-generated energy to also be stored when needed.

Together, Assumptions 2 and 3 imply that the energy available at the beginning of period t, x^P_{t-1}, is given by:

$$x^P_{t-1} = \Big(\min[e(q_{t-1} - D_H)^+, K] - D_L \Big)^+, \tag{6}$$

which is notably *not* a function of x^P_{t-2} or energy stored in any previous period. In particular, this means that revenues in any period t can be computed in expectation by simply knowing the probability distribution of solar generation q_t and q_{t-1}.[3]

To further improve tractability, we modify the revenue terms that capture the energy generated in a period, and discharged in the following period by replacing it with a weakly lower term (see Equation 89 in Appendix B.1). The resulting objective function of the partial-discharge model can thus be written as:

$$\begin{aligned}\Pi_P(Q,K) = T \cdot \Big(g\ E\Big[& \min[q_t, D_H] + \min[e(q_t - D_H)^+, D_L] + \\ & \min[(eq_t - eD_H - D_L)^+, K - D_L, (eD_H + D_L - eq_{t+1})^+]\Big] - \frac{c_K}{e}K - c_Q Q\Big),\end{aligned} \tag{7}$$

[3]We ignore start-up effects (i.e., storage in the very first period would be empty), since they have negligible impact on the overall revenues given that the expected life of a solar panel is 30 years — more than 10,000 days.

where the term multiplied by T is the profit earned in a period, i.e., the cost-saved from solar and storage serving demand.

Theorem 5 characterizes the optimal investment decisions for the firm, under the assumption that storage capacity is not excessively higher than demand, $K < eD_H + 2D_L$. This condition is made for tractability, always holds for $e = 1$, and is confirmed in all of our numerically-simulated scenarios (see Appendix A.5 for more details).

Theorem 5. Optimal Decisions under the Partial-discharge Model

Under the partial-discharge model:

- *if $g_F \leq g \leq g_P$, the border solution is optimal, i.e., $Q_P^* = Q_{bor}^*$ and $K_P^* = K_{bor}^*$,*

$$Q_{bor}^* = \sqrt{\frac{g(D_L^2/e + 2D_H D_L + D_H^2)}{2c_Q}}, \qquad K_{bor}^* = D_L,$$

- *if $g > g_P$, the interior solution is optimal, i.e., $Q_P^* = Q_{int}^*$ and $K_P^* = K_{int}^*$,*

$$Q_{int}^* = \sqrt[3]{-d + \sqrt{d^2 + c^3}} + \sqrt[3]{-d - \sqrt{d^2 + c^3}},$$
$$K_{int}^* = D_L + \frac{1}{2}\left(Qe - \sqrt{4(D_L + D_H e)^2 - 4e(D_L + D_H e)Q + \frac{e(4c_K + eg)Q^2}{g}}\right);$$

where g_P is defined as

$$g_P = \frac{\left(c_K + c_K em(2+m) + 2c_Q(1+em)^2\right)^2}{2c_Q e(1+em)^2(1+em(2+m))},$$

where $m = D_H/D_L$ and c and d are defined in Appendix A.5, Equations 32 and 33.

Theorem 5 characterizes the optimal decisions of the firm under the partial-discharge model. When g is moderately low ($g_F \leq g \leq g_P$), the firm builds only enough storage capacity to fulfill nightly demand. When $g > g_P$, the firm builds more storage capacity than that, potentially allowing excess solar energy during one period to fulfill unmet demand during the next period.

The next theorem compares the investment decisions obtained from the partial-discharge model with those of the tracking model.

Theorem 6. Comparison Between Partial-discharge and Tracking Model

a) The optimal profit under the tracking model, Π_{TR}^*, is always bounded below by its equivalent in the partial-discharge model, $\Pi_P^* \leq \Pi_{TR}^*$;

b) The optimal storage capacity investment under the tracking model for a given value of generation, $K_{TR}(Q)$, is always bounded below by its equivalent in the partial-discharge model, $K_P(Q) \leq K_{TR}(Q)$. When $g_F \leq g \leq g_P$, the optimal storage capacity investment under the tracking model, K_{TR}^*, is always bounded below by its equivalent in the partial-discharge model, $K_P^* \leq K_{TR}^*$.

The partial-discharge model always underestimates the marginal value of storage compared to the tracking model. For this reason, the partial-discharge model's storage investment decision provides a *lower-bound* to the investment decision of the tracking model. This bound holds analytically for $g_F \leq g \leq g_P$ and it holds numerically in all other cases, but is elusive to prove for $g > g_P$ because of the interplay between the two decision variables. The reason for the lower-bounding is that both assumptions underlying the partial-discharge model reduce the value of stored energy, which has a lower duration (Assumption 2) and is employed sub-optimally (Assumption 3), leading the firm to build less storage capacity compared to the tracking model. The decision on generation capacity obtained from the partial-discharge model is, instead, not always lower than the one obtained from the tracking model. We provide a more detailed analysis of this in Section 5.

Remark. Taken together, the full- and partial-discharge models provide important information that can be employed to make investment decisions. At lower backup costs, the full-discharge model is exact, while for higher backup costs, the full-discharge and partial-discharge models respectively provide an upper bound and a lower bound for firm profit and storage investment in the tracking model. We test the accuracy of our two models in Section 5.1.

3.3 Strategic Usage of Capacity-Limited Backup Generator

So far, we always assumed that the backup generator has sufficient capacity to fulfill all demand. However, it is conceivable that there exist off-grid use-cases in which the back-up generation is performed by several combined generators. In those cases, once the solar and storage investment has been made, it may be desirable to retire some of the former back-up generators, and use the remaining capacity strategically.

We study this problem for the tracking model by employing a dynamic programming setup for which we introduce the following notation: Let G denote the backup generator capacity in each sub-period for which we assume $G \leq \min(D_H, D_L)$. Running said generator incurs cost g per unit of energy, but not fulfilling demand incurs cost αg. We assume $\alpha > 1/e$, so that running the generator in combination with storage is at least profitable if meeting demand is guaranteed (i.e. charge storage with the generator during the day to serve demand at night). We thus have to make two decisions each day - how much to run the generator during the day G_{Ht} and how much to run it during the night G_{Lt}. The state of the model consists of the charge at the beginning of the period x_{t-1} and the amount of solar generation in the period q_t. Together we have the two-dimensional state per period $s_t = (x_{t-1}, q_t)$ with time-invariant state space $S = [0, K] \times [0, Q]$. For this setting, we take renewable generation and storage capacity as given.

The objective in the capacity-limited generator scenario is to trade off the cost of running the generator against not meeting demand. Let U_{Ht} and U_{Lt} denote the unmet demand during the day and the night. The per-period cost $c(\cdot)$ and unmet demands are shown in Equation 8.

$$\begin{aligned} c(x_{t-1},q_t,G_{Ht},G_{Lt}) =& g(G_{Ht}+G_{Lt})+\alpha g(U_{Ht}+U_{Lt}). \\ U_{Ht} =& (D_H-G_{Ht}-x_{t-1}-q_t)^+, \\ U_{Lt} =& \Big(D_L-G_{Lt}-\min\Big[\big(x_{t-1}+e(G_{Ht}-D_H+q_t)^+-(D_H-G_{Ht}-q_t)^+\big)^+,K\Big]\Big)^+. \end{aligned} \tag{8}$$

Clearly, the unmet demands are decreasing in storage charge, generation and the back-up decision quantities G_{Ht}, G_{Lt}. Also, given the efficiency loss of the storage solutions, running the generator during the day to fulfill demand at night costs g/e per unit of nightly demand met, making it more expensive than regular back-up operations to meet demand. Crucially, the storage term x_t is what links the state variables and decisions from one period to the next, as we show in Equation 9.

$$x_t = \min\Big[\Big(\min\Big[\big(x_{t-1}+e(q_t+G_{Ht}-D_H)^+-(D_H-q_t-G_{Ht})^+\big)^+,K\Big]+e(G_{Lt}-D_L)^+-(D_L-G_{Lt})^+\Big)^+,K\Big]. \tag{9}$$

The terms in the first minimum account for the storage charge at the end of the day sub-period, while the outer minimum tracks the charge from the end of the day sub-period to the end of the night sub-period. We show in Appendix 7 that because all periods are linked through storage, rather than looking at the two generator decisions G_{Ht}, G_{Lt} separately for every period, it suffices to consider what the charge at the end of the period x_t is supposed to be. We can thus denote the optimal action in each period as the target charge x_t. Conceptually this works, because once the starting charge x_{t-1} and the solar generation realization q_t are known, there is no more uncertainty in period t[4]. Our action space is $X = [0, K]$ and is state- and time-invariant.

Because the periods are linked in this fashion - like an extended version of the tracking model - we can denote the optimal, multi-period cost from t to T with $v_t^*(\cdot)$ using the recursion equation shown in Equation 10.

$$\begin{aligned} v_t(x_{t-1},q_t) =& c(x_{t-1},q_t,x_t(G_{Ht},G_{Lt},\cdot))+E\Big[v_{t+1}(x_t,q_{t+1})\Big], \\ v_t^*(x_{t-1},q_t) =& \min_{x_t\in X}\Big\{c(x_{t-1},q_t,x_t(G_{Ht},G_{Lt},\cdot))+E\Big[v_{t+1}(x_t,q_{t+1})\Big]\Big\}, \\ & \text{s.t. } (9), \end{aligned} \tag{10}$$

where the expectation is taken w.r.t. the random generation realization. The total cost starting from t until the end of the lifetime is equal to the cost in this period $c(\cdot)$ plus the expectation over generation realizations of the next period's cost-function $v_t(x_t, q_{t+1})$.

An important quantity to consider for the optimal generator decision is $\chi \triangleq \min[(x_{t-1}+e(q_t+G-D_H)^+-(D_H-q_t-G)^+)^+,K]-x_t-(D_L-G)$, the maximum amount of energy, beyond a target charge x_t, one can end the night-subperiod with, leading to our next theorem:

[4]We will at times write $x_t(x_{t-1}, q_t, G_{Ht}, G_{Lt})$ to indicate that the action in period t is dependent on the storage charge at the beginning of the period x_{t-1} and the generation realization q_t and the generator decisions.

Theorem 7. Optimal Policy In The Capacitated Generator Setting
In the off-grid setting with a capacitated generator:

- *There exists a unique, optimal policy that is a threshold policy that aims to end a period with an optimal storage charge* x_t^*.
- *For any starting charge* x_{t-1} *and solar realization* q_t*, the optimal generator policy* $\hat{G}_{Ht}, \hat{G}_{Lt}$ *is:*

$$\hat{G}_{Ht} \triangleq \begin{cases} \bar{G}_{Ht} & \text{if } \chi < 0 \\ \bar{G}_{Ht} - \chi/e & \text{if } \chi \in (0, e(\bar{G}_{Ht} - (D_H - q_t)^+)^+] \\ \tilde{G}_{Ht} - \chi & \text{if } \chi \in (e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, \tilde{G}_{Ht}] \\ 0 & \text{if } \chi > \tilde{G}_{Ht} \end{cases}$$

$$\hat{G}_{Lt} \triangleq (G - (\chi - \tilde{G}_{Ht})^+)^+,$$

where $\bar{G}_{Ht}$ *and* $\tilde{G}_{Ht}$ *are defined in Appendix A.7.*

- *The optimal end-of-period-charge* x_t^* *can be lower bounded in closed form (see Equation 75 in Appendix A.7).*

The intuition for the optimal policy is that generating and storing a unit of charge has a constant cost, g/e, and decreasing marginal returns, thus making it optimal to target the end-of-period storage for which the benefit equals the cost.

In Section 5.4 we leverage results from Theorem 7 to provide important insights on the use of a downsized generator and its impact on emissions and renewable investments.

4 Data

After studying the theoretical properties of our model in the previous section, we now use empirical data from differently-sized islands around the world to complement our analytical findings. We chose islands as the studied scenario in this paper because they are off-grid use-cases, are found in most countries, and are clearly geographically isolated from any interconnection or neighbouring generation. We use the real-life data to calibrate our model to i) analyze the quality of the approximation of our full- and partial-discharge models (Section 5.1); ii) derive insights on how changes in technology and policy may impact storage and generation capacity investments over the coming years (Sections 5.2, 5.3, and Appendix B.6); iii) numerically investigate the emission and investment impact of reducing the capacity of the generator (Section 5.4). We begin by describing our data.

Market Data. As islands typically do not have full-fledged utilities, obtaining reliable power data for them is notoriously difficult. We gathered our energy demand and price data from different partners that work with these communities. For La Palma (PAL), we obtained the data from *La Palma Renovable* a EU-backed NGO pursuing the energy transition on the island. For Astypalaia (AST), the time series were kindly shared by Nikos Mamassis who had previously researched the stochasticity of the island's natural resources (Klousakou et al. 2018). For Weno (WEN), the data came from an energy consultancy that was tasked by the state of Micronesia to map a trajectory for future power generation in the country. These islands are characterized by different population sizes and varied ratios of day-to-night demands —see Table 1. The backup cost is rather high in all cases as it is driven by the inefficient generation based on imported oil (but governments typically heavily subsidize electricity prices for consumers). For carbon-emissions, we use data from the Spanish Register of Emissions and Pollutant Sources for La Palma's power plant (for lack of better data, we assume pollution intensity to be the same for all other islands' generators). The data has 10-minute or hourly granularity and consists of time series varying in duration, from several days to a few years.

Table 1: Historic Energy Demand and Price Data from Islands

	Unit	La Palma	Astypalaia	Weno
Demand Day (D_H)	kWh	407,800	7,600	22,900
Demand Night (D_L)	kWh	327,100	9,600	14,400
Demand Ratio ($m = D_H/D_L$)	Numeral	1.25	0.79	1.59
Population	#People	85,000	1,400	14,400
Backup Cost (g)	$/MWh	229	200	205
Backup CO2	tons/MWh	0.72	0.72	0.72

Demands are averages across all observations (day sub-period from 8am-8pm in La Palma, 7am-7pm in Astypalaia and 6am-6pm in Weno) and the backup costs are the average generation costs.

Storage and Renewable Data. For storage technology, we will consider two options. Lithium-ion batteries will be the high cost - high efficiency technology we analyze. One alternative to batteries is thermal energy storage - systems in which energy is stored as heat in various conductive materials ranging from sand over concrete or salt to oils. Typically, these storage solutions have lower levels of efficiency than batteries but they are also less expensive to build. We obtained parameter estimates for the storage technologies from the proprietary research of Kraftblock, a German energy storage start-up. We validated this data against publicly available sources, such as Fu et al. (2018) and reports of contemporary storage installations, as well as Larcher and Tarascon (2015) for storage manufacturing emissions. The solar generation assumptions contain the upfront investment costs and maintenance costs and are in line with the high end of photovoltaic costs in Lazard's Levelized Cost of Energy Analysis as the equipment has to be imported by ship (Lazard

Table 2: Storage Technology and Renewable Generation Data

	Battery	Thermal
\$/MWh	330,000	100,000
Lifetime in days	5,475	10,950
\$/MWh / day	60	9
Efficiency	90%	45%
t CO2/MWh in Production	150	80
t CO2/MWh / day	0.030	0.005

	Solar
\$/MW	1,560,000
Lifetime in days	10,950
\$/MWh/day	11.9
Capacity Factor (r)	25%

2020). To make the two technologies under study comparable, we adjust the cost for the expected lifetime of the technology. Table 2 summarizes storage and renewable generation data.

5 Numerical Analysis

5.1 On the Quality of the Partial- and Full-discharge Approximations

We begin our numerical analysis with an evaluation of the full- and partial-discharge models developed in Section 3.2. We want to understand how good of an approximation each of these models provides, relative to the tracking model, whose solution is obtained through a computer simulation.[5] For all three models, generation patterns are drawn across 10,950 day/night periods (30 years). We benchmark the models across 342 different sets of parameters, for different markets and storage technologies, and varying storage cost, generation cost and backup cost from 50% to 200% of their current values as well as demand ratios from 100% to 200%.

Table 3: Percentage Deviation of Partial- and Full-discharge Model Profit and Capacities Compared to the Tracking Model

		Profit		Generation		Storage	
n = 342	Deviation	Thermal	Battery	Thermal	Battery	Thermal	Battery
Partial-Discharge	average	-4%	0%	1%	2%	-27%	-2%
	median	-4%	0%	1%	2%	-28%	-1%
	largest (magn.)	-6%	-2%	-8%	-1%	-35%	-23%
Full-Discharge	average	39%	47%	76%	42%	18%	196%
	median	39%	51%	79%	43%	10%	209%
	largest (magn.)	49%	70%	95%	102%	130%	281%

Table 3 summarizes the model deviations across all 342 benchmark cases broken down by

[5] We investigate the simulation for storage capacities $K \in [0, 2D_L/e + D_H]$ (upper limit equals max discharge in the partial-discharge model) and generation capacities $Q \in [0, 4D_H + 4D_L/e)]$. For each parameter set, we run the simulation in an evenly-spaced 100x100 capacity grid and select the run with the highest profit.

storage technology, and reports the average, median, and largest deviation (in magnitude) for each. Deviations for both simplified models are calculated relative to the tracking model.

The most important finding is that profit wise, the partial-discharge model is very accurate, and only a few percentage points off relative to the tracking model (worst-case deviations are only -6% and -2% for Thermal and Battery technologies, respectively). The full-discharge model is not nearly as good, with average deviations around 40-50%. This suggests that the full-discharge model, despite being exact for a certain range of game parameters (as per Theorem 3), becomes fairly imprecise outside of that range.

The accuracy of the partial-discharge model carries over from profit to generation, with average and median deviations from the tracking model on the order of 1% to 2%, and worst-case deviation of -8% and -1% for Thermal and Battery technologies, respectively. Gaps increase slightly for storage decisions, with average and median around -30% for Thermal and -1% to -2% for Battery, but even the worst-case deviations are within -35% and -23% respectively.

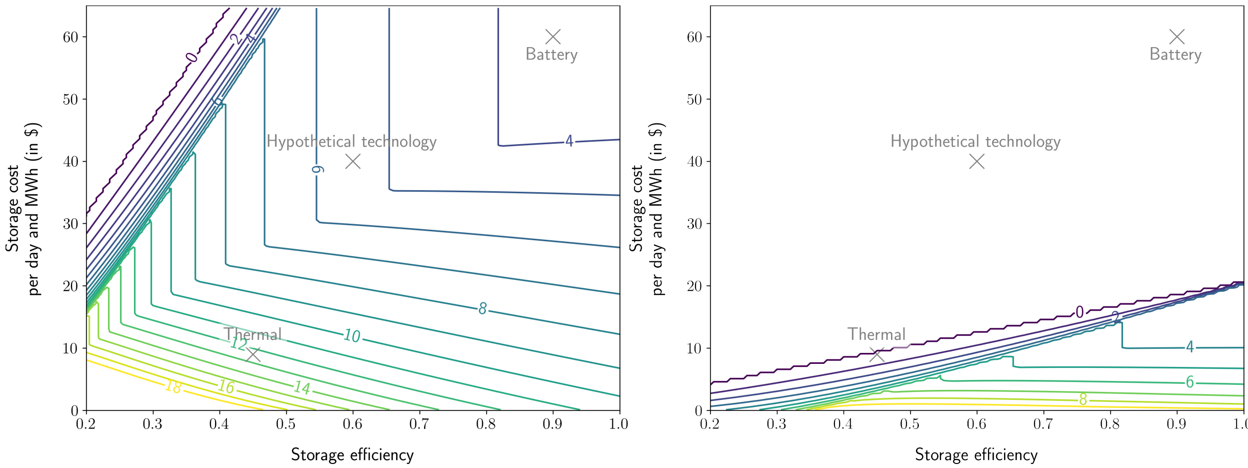
Overall, our numerical analyses confirm how well the the partial-discharge model approximates the tracking model across the wide range of parameters considered. These observations also confirm that the partial- and full-discharge models provide a lower and upper bound, respectively, for the tracking model's storage capacity investment, as discussed in Theorems 4 and 6. For a more detailed discussion of the model's approximations, please see Appendix B.5.

5.2 Improving Storage Technologies

In this section, we employ our partial-discharge model to characterize the optimal storage investment decision under changing technology. One driver of increasing storage penetration is technology improvement —that is, lower capacity cost or higher efficiency. We compared hundreds of hypothetical storage technology combinations of efficiency (30% to 100% in 1% increments) and unit costs ($1 to $65 in $1 increments) for La Palma with and without subsidies (results for AST and WEN are similar to PAL). For each of these hypothetical technologies, we calculated the profit-maximizing capacity investment. Figure 2 plots isocurves for the optimal storage investment as a function of capacity cost and efficiency, while also marking where lithium batteries and thermal technologies are positioned.[6] Importantly, these isocurves plot storage capacity in the amount that can be charged (K/e), rather than the amount that can be discharged, which highlights the investment dynamics as efficiency changes. Three observations are worth noting from these plots.

The first observation pertains to the the complex dynamics of how storage capacity changes as technology improves. Initially, as technology improves (moving from the top-left toward the bottom-right of the plot in Figure 2a) capacity stays equal to zero; the firm does not invest in any storage (the white area above the 0-storage line). As technology further improves and the 0-storage isocurve is crossed, storage becomes profitable. From this point onward, storage grows rapidly in response to small improvements in technology (isocurves are increasingly close to each other) until

[6] As discussed in Section 3.1, these two parameters are enough to capture the key features of any storage technology.



(a) PAL

(b) SUB

Figure 2: Storage Capacity Investment for Hypothetical Technologies in La Palma (with and without subsidies)

it reaches the capacity to fully cover nightly demand. From this level onward, storage capacity dynamics change. In particular, a decrease in cost has no consequences on storage (isocurves are vertical in the plot) until the cost drops below a certain threshold, which makes it profitable to build enough storage to carry energy into the following day. From this point onward, lower costs do increase storage, while higher efficiency has a dual effect: On the one hand, more efficiency makes storage attractive, resulting in more capacity investment, and on the other, more efficiency means that less storage capacity is needed to fulfill the same amount of demand, resulting in less capacity. The net effect of those two drivers can go both ways as evidenced by the different slopes of the isocurves to the right of their vertical parts.

The second observation pertains to the 0-storage isocurve, which identifies all technologies (combinations of capacity cost and efficiency) that would make a firm break-even when investing in a small amount of storage. This isocurve closely matches a line equation of the form $\frac{c_K}{e} = constant$, providing empirical support to the finding in Theorem 2 that comparing technologies based on their storage-cost-to-efficiency ratio constitutes a simple yet powerful way to determine which one can more easily turn a profit (and is thus likely to be adopted first).

The third observation pertains to the usefulness of identifying the 0-storage isocurve. Consider, for example, a storage company developing a new technology aimed at markets like the subsidized island case depicted in Figure 2b. Suppose, for illustrative purposes, that the company had, so far, developed a hypothetical storage technology with a cost of \$40 per MWh per day and 60% efficiency. Based on Figure 2b, the company could easily realize that further efforts to boost efficiency alone would never lead to investment, no matter how large the improvement. The company would then conclude that achieving a lower unit cost should become the priority.

To illustrate the last point in more detail, we plot the 0-storage isocurves for all three islands, and two hypothetical islands in Figure 3. The two hypothetical islands are: (i) an island with the subsidized electricity rate (SUB) that end customers on the islands pay (after accounting for a 75% subsidy), which is closer to backup costs in major grids and therefore allows some high-level insights for (off-grid) scenarios with cheaper, non-renewable options as well; and (ii) a hypothetical island with solar generation cost at 25% of current values. Two dynamics are worth mentioning

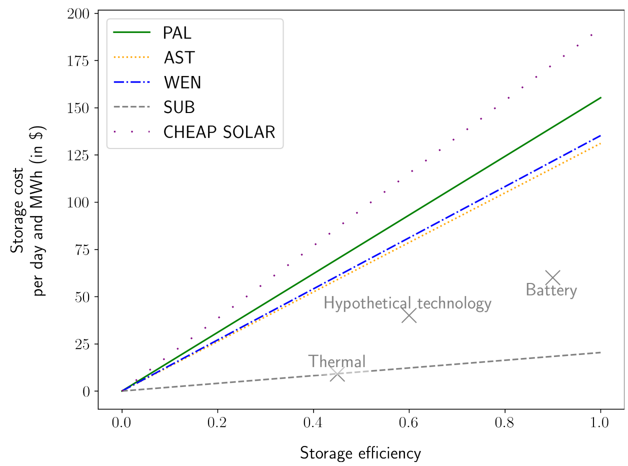


Figure 3: Storage Technology Efficiency Frontier for the Three islands and two Hypothetical Scenarios

in Figure 3. First, one can view these 0-storage isocurves as technology efficiency frontiers for each island. Given a market with its back-up and renewable costs, only storage technologies below the frontier are in the investment consideration set - storage technologies above the frontier are dominated by the choice to not invest in storage. Second, this graph shows how removing the back-up cost subsidies in La Palma would be equal to reducing technology costs by 83% —graphically, the subsidy removal is equivalent to shifting the grey (lowest) line of the subsidized island up to the green (second highest) line of La Palma, thereby increasing the space below the line - i.e. the space of profitable technologies. In comparison, a four-fold reduction in solar costs would only shift the frontier upwards by 24% (equivalent to shifting the green line up to the purple line). This quantifies the magnitude of difference that policy changes can make in investment outcomes, relative to the incremental technological progress.

5.3 Carbon Prices and their Impact on the Adoption of Storage Technologies

In the next two sections, we use the partial-discharge model to derive high-level insights on practically-relevant issues that surround the use of renewables. The following strategic insights are to be taken as characterizations of first-order effects rather than precise estimates of future developments. As evidenced in Theorems 4 and 5, the backup energy cost g has a significant effect on capacity investments. So far, the numerical value for g used in our analysis is based on the average generation cost on the studied islands. However, an increasing number of intergovernmental organizations, federal regulators, and local administrations are vowing to impose or increase some form of carbon taxes, in order to reduce carbon emissions and curb global warming. It is therefore of interest to understand the impact of increased carbon prices (e.g., through a direct tax) on optimal renewable generation and storage capacities since these directly affect emission savings. To this end, we calculate which carbon tax levels would be required to reach enough storage capacity a) for nightly demand (12 hours), b) nightly and daily demand (24 hours), and c) two nights and one daily demand (36 hours). SUB in the table refers to the island with the subsidized backup costs. While intuitively any duration of storage can be achieved through a sufficiently high carbon

Table 4: Carbon Price to Reach Storage Capacity to Cover demand for a Certain Time Period ($/ton of CO2)

Market	**Technology**	**>12h**	**>24h**	**>36h**
PAL	Thermal	$0	$0	>$200
AST	Thermal	$0	$0	>$200
WEN	Thermal	$0	$7	>$200
SUB	Thermal	$34	>$200	>$200
PAL	Battery	$0	>$200	>$200
AST	Battery	$0	>$200	>$200
WEN	Battery	$0	>$200	>$200
SUB	Battery	$97	>$200	>$200

price, Table 4 shows that, depending on the specific market and technology, the tax levels that are required to achieve the same relative capacities differ vastly.

Consider, for example, the level of carbon prices needed for 12 hours of storage to be profitable, for *different markets*. For unsubsidized islands no carbon prices are needed, while for the subsidized island, carbon prices are substantial.

Similarly, consider the level of carbon prices needed for 24 hours of storage to be profitable, for *different technologies*. Thermal storage needs zero or very-low carbon prices in unsubsidized islands, while battery storage is still not profitable even when carbon prices are as high as $200.

These findings point to the fact that it is very important for regulators to consider the implications of carbon pricing or storage subsidies with respect to their idiosyncratic market/technology situation, as these are by no means one-size-fits-all tools.

Another related question is when the technologies will become cheap enough (at current improvement rates) to serve a high share of demand by renewable generation and storage. We investigate this question in Appendix B.6 and find that i) at the subsidized electricity prices of the islands, 70% renewables could be achieved in around five years; ii) without the fossil-fuel subsidies 80% renewable generation would be profitable today; and iii) reaching e.g. 95% renewable share is decades away as replacing back-up capacity becomes increasingly expensive as the renewable share increases. For more details, please see Appendix B.6.

5.4 Strategic Usage of Capacity-Limited Backup Generator - Optimal Policy and Carbon Emissions

We start the numerical analysis of the capacitated generator problem by analyzing the policy threshold x_t^* and the lower bound for which we have an analytical solution. Two parameter choices that have to be made for this scenario are the size of the backup generator capacity G as well as the factor α that captures how costly it is to fail to meet demand, compared with running the generator [7]. In Figure 4 we plot the value for the optimal generator policy threshold, which we obtained numerically, for increasing levels of α, and compare it with our analytical lower-bound from Theorem 7. For this figure, we have assumed $G = 0.5D_H$ (the figure is qualitatively similar for other values of G). Intuitively, the threshold is increasing in α. If α was exactly equal to one,

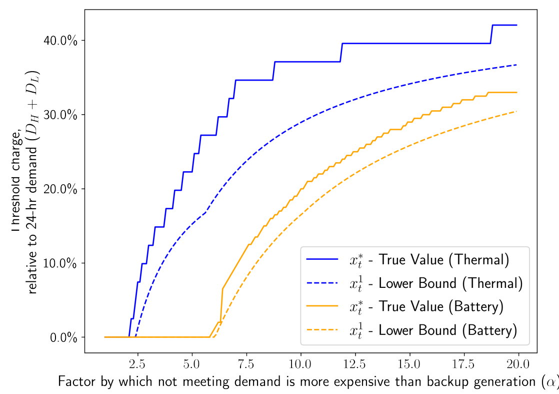


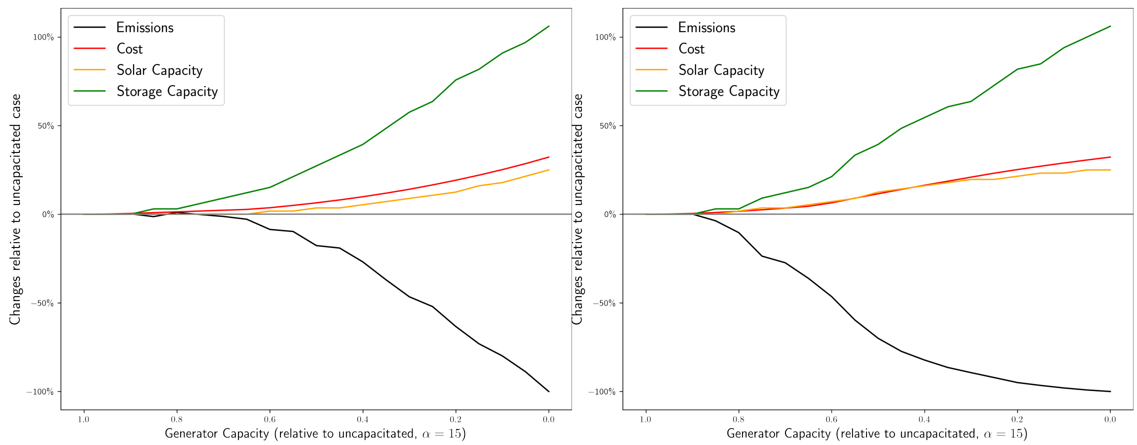
Figure 4: Capacitated Generator Policy Thresholds and Bounds for Battery and Thermal Storage

not meeting demand would incur as much cost as running the generator, so it would be sub-optimal to use the generators to achieve a "buffer" charge in order to increase the odds of meeting future demand. However, as α becomes larger, so does x_t^*, and running the generator to create at least some buffer charge becomes cheaper than potentially not serving demand. As it can be observed in Figure 4, the analytical lower-bound is slightly conservative for thermal when α is in the range 5-8, but it rapidly catches up for higher αs. For batteries, it is remarkably close to the optimal generator threshold regardless of the value of α. Baik et al. (2020) estimate US customers' willingness to pay for electricity during an outage at around \$2 per kWh, which equals an α of 15 when assuming an average electricity price of 13 cents per kWh, suggesting a narrow gap between our lower-bound and the optimal generator threshold for realistic values of α.

Having assessed the quality of the analytical lower-bound for the optimal threshold policy, in the rest of this subsection we aim to understand how the decision to reduce the backup generator

[7] Technically, x_t^* is not stationary, but in practice it is the same for all periods (as the solar randomness is i.i.d.) except the final days before terminal period T. In those final periods, x_t^* is lower as the future value $E[v^*(x_t, q_{t+1})]$ is lower when carrying x_t through to the terminal period becomes more likely. For all calculations in this section, we use and report the average x_t^* over 10,950 periods (i.e. 30 years).

capacity affects emission savings, costs, and capacity investments. Figure 5 shows the impact of curtailing generator capacity on emissions, total costs, solar capacity and generation capacity using Weno's demands and Thermal technology. We compare the case in which the generator is run strategically (in the sense of Theorem 7) to minimize total cost, including cost of unmet demand, (Figure 5, panel a) with the case in which the generator is used myopically, i.e., to simply serve unmet demand without creating a buffer charge (Figure 5, panel b). Note that the two cases are identical when generator capacity is either abundant (there is no point in running the generator preemptively) or zero. Note also that emissions are (trivially) minimized in the latter case (far right in the graphs). We want to highlight six observations from Figure 5 (the effects that we



(a) Strategic Generator Use

(b) Myopic Generator Use

Figure 5: Comparing Outcomes for Different Generator Capacities Under Strategic and Myopic Use

are about to discuss persist across all simulations we have run, unless otherwise specified). First, as one would expect, a comparison of panels (a) and (b) shows that a strategic use of the backup generator leads to lower cost (red line) and higher emissions (black line) compared to a myopic use. This is because the strategic use of the backup generator aims to minimize costs, and its preemptive use (as per Theorem 7) leads to it being used more often compared to a myopic (i.e. passive) use.

Second, even accounting for the above consideration, emission reductions under the strategic use (black line, panel a) are surprisingly low for a wide range of generator capacity. For example, reducing generator capacity by 35% has basically *no impact on emissions*. Moreover, a reduction in generator capacity may actually *increase* emissions. The reason is that a smaller generator is less able to meet unexpected energy shortages, increasing the risk of future unmet demand and thus the need to stock energy preemptively in the sense of Theorem 7.

Third, the way of operating a backup generator affects not just the absolute magnitude of emission levels, but also *when* backup capacity reductions reduce emissions levels (shape of the black line). For example, under the strategic use of generator (panel a), more than half of the reduction in emissions are achieved by reducing backup generator capacity from 30% down to zero (i.e., no backup generator). That is, substantial reductions in emissions are obtained only with very substantial reductions in backup capacity. By contrast, in the myopic case (panel b), most of the emission savings are obtained by cutting backup capacity in half, while reducing it from 30% of demand to zero has nearly no effect on emissions.

Fourth, as the capacity of the back-up generator is further reduced, the optimal storage capacity increases a lot faster, and farther, than solar capacity. For example, getting rid of the generator altogether causes approximately a doubling in storage capacity and only a 20% increase in solar capacity (regardless of how the generator is used). The exact effect sizes differ by market, technology and α, but the general trend persists - additional storage, not additional solar is the capacity investment of choice when facing a limited back-up generator.

Fifth, the cost increase associated with a reduction in backup generator capacity, even a size-able one, is not unreasonably high —e.g., in our simulations the cost increase remains below 25-30% for thermal as shown in Figure 5, but goes up to 45-50% for batteries. Thus, achieving a substantial reduction in emissions at a somewhat reasonable increase in cost is possible, provided that the reduction in backup capacity is replaced by an appropriate increase in storage (and to some extent generation) capacity.

Lastly, if we look at investment decisions as a function of generation capacity, we observe that storage capacity is quite similar in the strategic vs myopic use of the generator. By contrast, solar capacity is higher in the myopic case. This is because a myopic use of a downsized backup generator renders the firm more vulnerable to unexpected energy shortages, and thus prompts the firm to install more solar generation in the first place (yellow line).

Practical Takeaway. In principle, a moderate downsize of backup capacity could seem like a good first step to achieve lower emissions while maintaining sound operations, especially since capacity tends to exhibit decreasing marginal returns. Instead, we find that a moderate downsize of backup capacity (30-40%) has near-zero impact on emissions, and in some cases may even increase emissions —and costs. If a meaningful decrease in emissions is to be achieved, the recommended course of action is a strong reduction in backup capacity, accompanied by a substantial increase in storage capacity and a modest increase in generation capacity, in order to preserve good service levels and keep overall costs in check.

6 Discussion

Our paper provides the first tractable methodological approach in the operations literature to study large-scale storage capacity investment that is used to shift intermittent solar electricity across time, especially between night and day, for off-grid applications. Our results yield several practical take-aways. We show how an investor can use information on demand, cost, and technology to decide on the optimal level of fossil-free generation and storage. We find that these capacity investments are strategic complements at lower capacity levels, but interestingly, they turn into strategic substitutes when renewable generation increases. We then develop two simple models, the full- and partial-discharge, which provide upper and lower bounds for profit and optimal storage investment decisions, with the former yielding exact solutions when the backup cost is low enough, and the latter yielding a pretty good approximation in all other cases. We also establish a simple condition based on the storage-cost-to-efficiency ratio to determine which of the two storage technologies can more easily turn a profit, and is thus likely to be adopted sooner - an approach that can be used by firms to support strategic technology decisions.

Our models also help us derive insights on the role of storage in the coming years. As storage technologies become gradually cheaper, we find that investments in off-grid renewable storage will not happen gradually; rather, there will be a zero-investment period, followed by a period of rapid adoption, followed again by a slower period. Despite the sudden increase in the short-to-medium term, we find that the need for non-intermittent fossil energy (e.g., on islands) will likely remain in the long-term, due to the need of complementing solar power with some amount of flexible, non-intermittent generation. Lastly, our analytical and numerical results show how an off-grid community interested in reducing its emissions can reduce fossil backup capacity and adjust its renewable investment decisions to maintain high service levels and keep costs in check.

However, it must be noted that these findings are based on a stylized model, which tries to identify the over-arching dynamics driving renewable investment choices, but cannot necessarily replicate them in detail. Although designed to provide quick estimates on optimal capacities, the models presented in this paper simplify the demand and generation dynamics observed in practice. Additional layers of complexity could be added by considering stochastic demand and/or costs, higher granularity to compute supply-demand mismatch, consumption changes among electricity customers over time, as well as constraints on location choices or other geographic limitations. Likewise, the engineering and design challenges for storage installations are glanced over as we treat them as a modular investment with known capabilities. These limitations simultaneously present ample opportunities for future research. Understanding how existing fossil generation and social factors impact the adoption of storage capacity, where to locate said investments, and how to size the individual modular components of the combined storage system are all relevant, challenging, and open questions.

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Appendix A Proofs of Theorems

A.1 Proof of Theorem 1

We want to show two properties of the tracking model: First that at low levels of capacity, generation and storage are strategic complements and second that they become strategic substitutes at high levels of generation.

We start by showing strategic complementarity at low levels of capacity. Let storage capacity be smaller than nightly demand ($0 \leq K < D_L$). In this case, one can always discharge any accumulated charge with 100% certainty in the following night. We show in Theorem 4 that for these levels of storage the tracking and the full discharge model are equivalent and we will show subsequently (in Equation 16), that in the full discharge model capacities are always strategic complements. We derive Q^*_{bor} in Appendix A.5 in Equation 41, by showing that for any generation capacity below it $Q < Q^*_{bor}$, we are in the full-discharge model. Notably, this is also true for all values of generation capacity below daily demand $Q \leq D_H + D_L/e$ because these result in optimal storage capacity below nightly demand $K \leq D_L$ and we are again in the full-discharge case.

We now turn to the high-capacity case, which we define as $K \geq D_L$ and for which we introduce the tracking model objective function below. We will show that for sufficiently high generation, strategic substitutability always arises for any given level of storage.

$$\begin{aligned}
&\Pi_{TR}(Q, K) = \\
&g\ E\Big[\sum_{t=1}^{T}\big(\min[q_t, D_H] + \min[x^{TR}_{t-1}, (D_H - q_t)^+] + \min[(x^{TR}_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, D_L]\big)\Big] - \\
&T\frac{c_K}{e}K - Tc_Q Q, \\
&\text{where } x^{TR}_t = (\min[(x^{TR}_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, K] - D_L)^+.
\end{aligned} \tag{11}$$

For notational convenience, we omit the subscript $_{TR}$ in the remainder of the proof, but always work with tracking model.

Because the objective function is intractable, we proceed to prove strategic substitutability without directly inspecting the cross-derivative. Before we begin the proof, we need to introduce one definition:

- Let $LTS \triangleq (K - D_L)^+$ be the long - term - storage capacity, i.e. storage capacity in excess of nightly demand that is used to carry electricity into future periods.

Note that LTS and storage differ by the nightly amount of storage D_L, so LTS can be zero, while storage is still positive.

We now proceed to prove strategic substitutability in five steps:

1. We define the *value of LTS* as the difference in profit between having a given, strictly positive amount of LTS and having no LTS (i.e. $K = D_L$);
2. We upper-bound the value of LTS;
3. We solve the expression for the upper bound;
4. We show that the upper bound on the value of LTS is decreasing in Q and becomes negative when Q is large enough, which implies that for large enough Q the optimal LTS is zero;
5. We connect the optimal LTS capacity of 0 at high capacity levels to strategic substitutability;

Step 1: Defining the value of LTS

We define the value that LTS generates as the difference in profit between having any positive LTS and having no LTS:

$$\begin{aligned}
\Lambda^{LTS}(Q,K) &= \Pi^{LTS}(Q, D_L + LTS) - \Pi^{LTS}(Q, D_L) \\
&= \Big\{ g\ E\Big[\sum_{t=1}^{T} \Big(\min[q_t, D_H] + \min[x_{t-1}, (D_H - q_t)^+] + \\
&\quad \min\left[\left(x_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+ \right)^+, D_L + LTS, D_L \right] \Big) \Big] - T\frac{c_K}{e}(D_L + LTS) - Tc_Q Q \Big\} - \\
&\quad \Big\{ g\ E\Big[\sum_{t=1}^{T} \Big(\min[q_t, D_H] + \min[0, (D_H - q_t)^+] + \\
&\quad \min\left[\left(0 + e(q_t - D_H)^+ - (D_H - q_t)^+ \right)^+, D_L \right] \Big) \Big] - T\frac{c_K}{e} D_L - Tc_Q Q \Big\}, \\
&= \Big\{ g\ E\Big[\sum_{t=1}^{T} \Big(\min[x_{t-1}, (D_H - q_t)^+] + \min\left[\left(x_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+ \right)^+, D_L \right] - \\
&\quad \min\left[\left(e(q_t - D_H)^+ - (D_H - q_t)^+ \right)^+, D_L \right] \Big) \Big] - T\frac{c_K}{e} LTS. \Big\}
\end{aligned} \tag{12}$$

Step 2: Upper bounding the value of LTS

Because this value of LTS is still a multi-period sum of differences and thus difficult to analyze, we will make the following assumption that upper bounds the value of LTS and yields a neat expression for the per-period profit obtained through LTS. We will then show that even this per-period upper bound of LTS value approaches zero for storage capacities over nightly demand.

Assumption: We start every period with the fullest charge possible, i.e. we replace x_{t-1} with $K - D_L = LTS$.

This is an upper bound of the value of LTS, as it replaces the terms in the minimums that are impacted by LTS capacity with the maximum value those terms can attain, thus weakly increasing the value of LTS. Additionally, this re-formulation breaks the inter-temporal connection between periods, allowing us to rephrase the profit as T times the expected profit from the single period. We denote this upper bound by $\bar{\Lambda}^{LTS}(Q,K) > \Lambda^{LTS}(Q,K)$:

$$
\begin{aligned}
&\bar{\Lambda}^{LTS}(Q,K)\\
&=\Big\{g\ E\Big[\sum_{t=1}^{T}\Big(\min[LTS,(D_H-q_t)^+]+\min\Big[\big(LTS+e(q_t-D_H)^+-(D_H-q_t)^+\big)^+,D_L\Big]-\\
&\min\Big[\big(e(q_t-D_H)^+-(D_H-q_t)^+\big)^+,D_L\Big]\Big)\Big]-T\frac{c_K}{e}LTS\Big\},\\
&=T\Big\{g\ E\Big[\Big(\min[LTS,(D_H-q_t)^+]+\min\Big[\big(LTS+e(q_t-D_H)^+-(D_H-q_t)^+\big)^+,D_L\Big]-\\
&\min\Big[\big(e(q_t-D_H)^+-(D_H-q_t)^+\big)^+,D_L\Big]\Big)\Big]-\frac{c_K}{e}LTS\Big\},\\
&=T\Big\{g\Big[\frac{1}{Q}\Big(\frac{D_H^2}{2}-\frac{((D_H-LTS)^+)^2}{2}\Big)\Big]+gE\Big[\min\Big[\big(LTS+e(q_t-D_H)^+-(D_H-q_t)^+\big)^+,D_L\Big]\Big]-\\
&g\Big[\frac{1}{Q}\Big(\frac{D_L^2}{2e}+(Q-D_H-D_L/e)D_L\Big)\Big]-\frac{c_K}{e}LTS\Big\}
\end{aligned}
\tag{13}
$$

Step 3: Deriving the upper-bound expression

The expected value of the second minimum in $\bar{\Lambda}^{LTS}(Q,K)$ is $\min\Big[\Big(LTS+e(q_t-D_H)^+-(D_H-q_t)^+\Big)^+,D_L\Big]$ and takes different values depending on the magnitude of LTS. We further upper-bound the value of LTS, by using the highest possible value this minimum can take, which is D_L.

$$
\begin{aligned}
&\bar{\Lambda}^{LTS}(Q,K)\\
&=T\Big\{g\Big[\frac{1}{Q}\Big(\frac{D_H^2}{2}-\frac{((D_H-LTS)^+)^2}{2}\Big)\Big]+gE\Big[\min\Big[\big(LTS+e(q_t-D_H)^+-(D_H-q_t)^+\big)^+,D_L\Big]\Big]-\\
&g\Big[\frac{1}{Q}\Big(\frac{D_L^2}{2e}+(Q-D_H-D_L/e)D_L\Big)\Big]-\frac{c_K}{e}LTS\Big\},\\
&<T\Big\{g\Big[\frac{1}{Q}\Big(\frac{D_H^2}{2}-\frac{((D_H-LTS)^+)^2}{2}\Big)\Big]+g\Big[D_L\Big]-\\
&g\Big[\frac{1}{Q}\Big(\frac{D_L^2}{2e}+(Q-D_H-D_L/e)D_L\Big)\Big]-\frac{c_K}{e}LTS\Big\},\\
&=T\Big\{g\Big[\frac{1}{Q}\Big(\frac{D_H^2}{2}-\frac{((D_H-LTS)^+)^2}{2}\Big)\Big]+g\Big[D_L\frac{D_H+D_L/e}{Q}\Big]-\frac{c_K}{e}LTS\Big\},
\end{aligned}
\tag{14}
$$

Step 4: Showing that, for any positive LTS, there exists a generation capacity beyond which LTS value becomes negative.

We can now show that for any value of LTS, there exists a generation capacity beyond which this expression is negative - i.e. at sufficiently large generation Q, the value contribution of LTS is negative and the optimal LTS should thus be set to 0.

$$
\begin{aligned}
&T\Big\{g\Big[\frac{1}{Q}\Big(\frac{D_H^2}{2}-\frac{((D_H-LTS)^+)^2}{2}\Big)\Big]+g\Big[D_L\frac{D_H+D_L/e}{Q}\Big]-\frac{c_K}{e}LTS\Big\},\\
&\rightarrow\frac{D_H^2}{2Q}-\frac{((D_H-LTS)^+)^2}{2Q}+D_L\frac{D_H+D_L/e}{Q}=\frac{c_K}{ge}LTS,
\end{aligned}
\tag{15}
$$

The LHS of the second line in Equation 15 is strictly decreasing in Q and goes to zero as $Q \to +\infty$, while the RHS is positive. Therefore, it is always possible to find a capacity Q' high enough so that the LHS is lower than the RHS for any $Q > Q'$. This means that total value of any LTS capacity always becomes negative for sufficiently large generation capacity, and in those cases it would be optimal to set LTS equal to 0.

Step 5: Connecting an optimal LTS capacity of zero to strategic substitutability
As long as it is profitable to install strictly positive LTS for some value of generation capacity (that is, as long as the problem is non-trivial and at least some multi-period storage increases profit) then optimal storage capacity must decrease at some point as we know we have $LTS = 0$ at sufficiently high generation. This implies that the two capacities are strategic substitutes when generation capacity is large enough and concludes the proof.

For completeness, we briefly state strategic complementarity results for both simplified models by studying their respective cross-derivatives:

Full-Discharge Cross-Derivative
We study the cross-derivative of the full discharge model:

$$
\frac{\partial^2\Pi_F(Q,K)}{\partial K\partial Q}=\frac{g}{Q^2}[\frac{K}{e}+D_H]>0. \tag{16}
$$

The cross-derivative in the full-discharge model is clearly always positive: In this model generation and storage capacity are strategic complements.

Partial-Discharge Cross-Derivative
We study the cross-derivative of the partial-discharge discharge model:

$$
\frac{\partial^2\Pi_P(Q,K)}{\partial K\partial Q}=\frac{4D_HD_Lg}{Q^3e}+\frac{2D_H^2g}{Q^3}+\frac{4D_LgK}{Q^3e}-\frac{2gK^2}{Q^3e}-\frac{2D_Lg}{Q^2e}-\frac{D_Hg}{Q^2}+\frac{gK}{Q^2e}. \tag{17}
$$

We show $\frac{\partial^2\Pi_P(Q,K)}{\partial Q\partial K}<0$ if and only if the following hold,

$$
\begin{aligned}
4D_HD_L+2D_H^2e+\frac{4D_LK}{e}-\frac{2K^2}{e}&<Q(2D_L+D_He-K),\\
Q&>2(D_H+\frac{K}{e}),\ \text{or}\\
K&>2D_L+D_He.
\end{aligned}
\tag{18}
$$

The cross-derivative in the partial-discharge model is positive if and only if both capacities are lower than some threshold values identified above. In this model generation and storage capacity are strategic complements unless one of them is high enough, in which case they are strategic substitutes.

A.2 Proof of Theorem 2

We equate the backup cost parameter g to g_0 from Theorem 4 (at g_0, the full-discharge and tracking model are identical). From this equation we can then derive the storage-cost-to-efficiency ratio $\frac{c_K}{e}$:

$$\begin{aligned}
&g_0 = c_Q + \frac{c_K}{e} + \sqrt{c_Q^2 + \frac{2c_Q c_K}{e}}, \\
&(g - c_Q - \frac{c_K}{e})^2 = c_Q^2 + \frac{2c_Q c_K}{e}, \\
&\frac{c_K}{e} = g - \sqrt{2c_Q g}.
\end{aligned} \tag{19}$$

Each storage technology is characterized by the two parameters cost c_K and efficiency e. Together, they affect g_0 only through their ratio c_K/e. Additionally, g_0 is strictly increasing in this ratio c_K/e, so a technology with a higher cost-to-efficiency ratio c_K/e results in a higher threshold back-up g_0 (for a given generation cost c_Q). Conversely, a technology with a lower c_K/e ratio results in a lower g_0. Lastly, note that a lower g_0 means that the range of back-up costs for which the technology is suitable $g \in [g_0, \infty]$ is larger, when g_0 is lower. In other words, a storage technology with lower cost-to-efficiency ratio has more markets/scenarios in which it can be invested in profitably.

A.3 Proof of Theorem 3

The simplified full-discharge, monopoly objective function, per period, is:

$$\Pi_F(Q,K) = \frac{g}{Q}\left[-\frac{D_H^2}{2} - \frac{K^2}{2e} - KD_H\right] + g(D_H + K) - c_K/eK - c_Q Q. \tag{20}$$

From this expression, we derive the first and second partial derivatives as follows:

$$\begin{aligned}
\frac{\partial \Pi_F(Q,K)}{\partial Q} &= \frac{g}{Q^2}\left[\frac{D_H^2}{2} + \frac{K^2}{2e} + KD_H\right] - c_Q, \\
\frac{\partial^2 \Pi_F(Q,K)}{\partial^2 Q} &= -\frac{g}{Q^3}\Big[D_H^2 + \frac{K^2}{e} + 2KD_H\Big], \\
\frac{\partial^2 \Pi_F(Q,K)}{\partial Q \partial K} &= \frac{g}{Q^2}\Big[\frac{K}{e} + D_H\Big], \\
\frac{\partial \Pi_F(Q,K)}{\partial K} &= \frac{g}{Q}\Big[-D_H - \frac{K}{e}\Big] + g - \frac{c_K}{e}, \\
\frac{\partial^2 \Pi_F(Q,K)}{\partial^2 K} &= -\frac{g}{Qe}.
\end{aligned} \tag{21}$$

From here, one can show that the Hessian is negative semi-definite:

$$
\begin{aligned}
H(\Pi_F(Q,K)) &= \begin{bmatrix} \frac{\partial^2 \Pi_F(Q,K)}{\partial Q^2} & \frac{\partial^2 \Pi_F(Q,K)}{\partial Q \partial K} \\ \frac{\partial^2 \Pi_F(Q,K)}{\partial K \partial Q} & \frac{\partial^2 \Pi_F(Q,K)}{\partial K^2} \end{bmatrix}, \\
\frac{\partial^2 \Pi_F(Q,K)}{\partial^2 Q} &< 0, \\
\left| H(\Pi_F(Q,K)) \right| &= \left(-\frac{g}{Q^3}[D_H^2 + \frac{K^2}{e} + 2KD_H] \right) \left(-\frac{g}{Qe} \right) - \left(\frac{g}{Q^2}[\frac{K}{e} + D_H] \right)^2 \geq 0, \\
& \frac{g^2}{eQ^4} D_H^2 (1-e) \geq 0.
\end{aligned} \tag{22}
$$

Hence, the objective function is globally concave over its domain, and we find the unique optimum by setting the first partial derivatives equal to 0.

$$
\begin{aligned}
Q_F^* &= \sqrt{\frac{g}{c_Q} \left[\frac{D_H^2}{2} + \frac{(K_F^*)^2}{2e} + K_F^* D_H \right]}, \\
K_F^* &= \max \left[-D_H e + Q_F^* (e - \frac{c_K}{g}), 0 \right].
\end{aligned} \tag{23}
$$

We can also express these in closed form, which results in slightly less readable expressions that hold as long as $g \leq \sqrt{(2c_K c_Q + c_Q^2)/e^2} + (c_K + c_Q)/e$. For values outside that range, the radicand may become negative:

$$
\begin{aligned}
Q_F^* &= D_H g \sqrt{\frac{(-1+e)e}{c_K^2 - 2(c_K + c_Q)eg + e^2 g^2}}, \\
K_F^* &= \max \left[-D_H e + D_H \left(ge - c_K \right) \sqrt{\frac{(-1+e)e}{c_K^2 - 2(c_K + c_Q)eg + e^2 g^2}} \ , \ 0 \right].
\end{aligned} \tag{24}
$$

A.4 Proof of Theorem 4

A.4.1 Part 1 - g_0

Using the full-discharge monopoly objective function's partial derivative, w.r.t. K and the optimal solution for generation without storage ($Q^*|_{K=0}$), we find the threshold value for the backup cost g, beyond which, the partial derivative becomes positive and the profit-maximizing monopolist would invest in a positive amount of storage. It can be easily verified that $\frac{\partial \Pi_F(Q,K)}{\partial g} > 0$:

$$
\begin{aligned}
\frac{\partial \Pi_F(Q,K)}{\partial K} &= \frac{g}{Q}\Big[-D_H - \frac{K}{e}\Big] + g - \frac{c_K}{e}, \\
Q_F^*(K=0) &= \sqrt{\frac{g}{c_Q}\frac{D_H^2}{2}}, \\
\frac{\partial \Pi_F(Q^*(K=0), K=0)}{\partial K} &= \frac{g}{\sqrt{\frac{g}{c_Q}\frac{D_H^2}{2}}}\Big[-D_H\Big] + g - \frac{c_K}{e} = 0, \\
&- \sqrt{2c_Q g} + g - \frac{c_K}{e} = 0, \\
&g = c_Q + \frac{c_K}{e} + \sqrt{c_Q^2 + \frac{2c_Q c_K}{e}} \triangleq g_0.
\end{aligned} \tag{25}
$$

The profit of the tracking model and the full-discharge for a given Q and K are identical if $g < g_0$ (which implies $K_F^* = 0$). We show this for the more general case of $K \leq D_L$ in Part 3 of this Theorem.

A.4.2 Part 2 - g_F

We know

$$
\begin{aligned}
Q_F^* =& \sqrt{\frac{g}{c_Q}\left[\frac{D_H^2}{2} + \frac{(K_F^*)^2}{2e} + K_F^* D_H\right]}, \\
K_F^* =& \max\left[-D_H e + Q_F^*(e - \frac{c_K}{g}), 0\right].
\end{aligned} \tag{26}
$$

Note that both optimal capacities are weakly increasing in g, we thus aim to find the value for g at which $K_F^* = D_L$. Substituting Q_F^* into K_F^*:

$$
\begin{aligned}
K_F^* =& - D_H e + \sqrt{\frac{g}{c_Q}\left[\frac{D_H^2}{2} + \frac{(K_F^*)^2}{2e} + K_F^* D_H\right]}(e - \frac{c_K}{g}), \\
\frac{(K_F^* + D_H e)^2}{(e - \frac{c_K}{g})^2} =& \frac{g}{c_Q}\left[\frac{D_H^2}{2} + \frac{(K_F^*)^2}{2e} + K_F^* D_H\right], \\
K_F^* =& D_H e \left(-1 + \left(g - \frac{c_K}{e}\right)\sqrt{\frac{(-1+e)e}{c_K^2 - 2(c_K + c_Q)eg + e^2 g^2}}\right).
\end{aligned} \tag{27}
$$

Hence, we can check when $K_F^* = D_L$, which is when:

$$
\begin{aligned}
&D_L = D_H e\left(-1+\left(g-\frac{c_K}{e}\right)\sqrt{\frac{(-1+e)e}{c_K^2-2(c_K+c_Q)eg+e^2g^2}}\right),\\
&g = \frac{(em+1)\left(\sqrt{c_Q\left(2c_K(em(m+2)+1)+c_Q(em+1)^2\right)}+c_Qem+c_Q\right)+c_Kem(m+2)+c_K}{e(em(m+2)+1)} \triangleq g_F,\\
&\text{where } m=\frac{D_H}{D_L}.
\end{aligned}
\tag{28}
$$

The profit of the tracking model and the full-discharge for a given Q and K are identical if $g \leq g_F$ (which implies $K_F^* \leq D_L$). We show this in Part 3 of this Theorem.

A.4.3 Part 3 - Full-Discharge and Tracking Model Equivalence

The tracking model objective function is:

$$
\begin{aligned}
&\Pi_{TR}(Q,K) = \\
&g\sum_{t=1}^{T}\left(\min[q_t,D_H]+\min\left[\, x_{t-1}^{TR},(D_H-q_t)^+\right]+\right.\\
&\left.\min\left[\left(x_{t-1}^{TR}+e(q_t-D_H)^+-(D_H-q_t)^+\right)^+,K,D_L\right]\right)-T\frac{c_K}{e}K-Tc_QQ\\
&\text{where } x_t^{TR}=\left(\min\left[\left(x_{t-1}^{TR}+e(q_t-D_H)^+-(D_H-q_t)^+\right)^+,K\right]-D_L\right)^+.
\end{aligned}
\tag{29}
$$

When $K \leq D_L$, we have that $x_t = 0$ and revenues in any period are stochastically identical and independent from past events by construction, so the problem reduces to study the expected revenue over one period. Since any storage up to D_L will be discharged for sure to meet nightly demand, the tracking model's objective function is the same as the full-discharge model:

$$
\begin{aligned}
&\Pi_{TR}\left(Q,K\leq D_L\right) = g\ E[\min[q_t,D_H]+\min[e(q_t-D_H)^+,K]]-\frac{c_K}{e}K-c_QQ,\\
=&\Pi_F\left(Q,K\leq D_L\right) = \frac{g}{Q}\left[\int_0^{D_H} q\ dq+\int_{D_H}^{Q} D_H\ dq+\int_{D_H}^{D_H+K/e}(q-D_H)\ e\ dq+\int_{D_H+K/e}^{Q} K\ dq\right]-\frac{c_K}{e}K-c_QQ.
\end{aligned}
\tag{30}
$$

When $K > D_L$ (i.e. $g > g_F$), the profit is trivially larger in the full-discharge case, as we assume any excess electricity up to storage capacity K, to be discharged. This can be seen if we consider plugging in the tracking model's optimal solution $(Q_{TR}^*, K_{TR}^* > D_L)$ in the full-discharge model. Because the full-discharge model over-values storage, compared to the tracking model, increasing storage capacity will marginally increase profit $\Pi_F(Q_{TR}^*, K_{TR}^*+\epsilon) > \Pi_F(Q_{TR}^*, K_{TR}^*)$. Furthermore, renewable generation is also marginally over-valued in the full-discharge model, compared to the tracking model $\Pi_F(Q^*_{TR}+\epsilon, K^*_{TR}) > \Pi_F(Q^*_{TR}, K^*_{TR})$. This can be seen by considering the three cases of generation realization (q_t) in a day. Case1: $q_t \leq D_H \rightarrow$ Both models are identical; Case 2: $0 < e(q_t - D_H) \leq K \rightarrow$ In the full-discharge model all generated energy will be sold, in the tracking model there is a chance that not everything will be sold; Case 3: $e(q_t - D_H) > K \rightarrow$ The full-discharge model charges K units of generation and sells all, while the tracking model may start the period with non-empty storage, thereby may not be able to charge as much energy and may not be able to sell the entire charge. In sum, renewable generation is weakly more profitable in the full-discharge model for a given K. Because the objective function is also globally concave and both capacities are strategic complements over the entire parameter space, the full-discharge model's optimal solution will have higher capacities and higher profit than the tracking model's optimal solution.

A.5 Proof of Theorem 5

We start analogously to the full-discharge model by introducing the simplified objective function and its first and second partial derivatives.

$$
\begin{aligned}
\Pi_P(Q,K) =& D_H g + D_L g + \frac{D_H^2 D_L g}{Q^2} + \frac{2D_L^3 g}{3e^2Q^2} + \frac{2D_H D_L^2 g}{eQ^2} - \frac{2D_H D_L g K}{eQ^2} - \frac{D_H^2 g K}{Q^2} - \frac{D_L g K^2}{e^2Q^2} + \\
& \frac{gK^3}{3e^2Q^2} - \frac{D_H^2 g}{2Q} - \frac{2D_H D_L g}{Q} - \frac{2D_L^2 g}{eQ} + \frac{2D_L g K}{Qe} + \frac{D_H g K}{Q} - \frac{gK^2}{2Qe} - c_Q Q - \frac{c_K}{e}K, \\
\frac{\partial \Pi_P(Q,K)}{\partial Q} =& -\frac{2D_H^2 D_L g}{Q^3} - \frac{4D_L^3 g}{3e^2Q^3} - \frac{4D_H D_L^2 g}{eQ^3} + \frac{4D_H D_L g K}{Q^3 e} + \frac{2D_H^2 g K}{Q^3} + \frac{2D_L g K^2}{Q^3 e^2} - \\
& \frac{2gK^3}{3Q^3e^2} + \frac{D_H^2 g}{2Q^2} + \frac{2D_H D_L g}{Q^2} + \frac{2D_L^2 g}{eQ^2} - \frac{2D_L g K}{Q^2 e} - \frac{D_H g K}{Q^2} + \frac{gK^2}{2Q^2 e} - c_Q, \\
\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 Q} =& \frac{6D_H^2 D_L g}{Q^4} + \frac{4D_L^3 g}{Q^4 e^2} + \frac{12 D_H D_L^2 g}{Q^4 e} - \frac{12 D_H D_L g K}{Q^4 e} - \frac{6D_H^2 g K}{Q^4} - \frac{6D_L g K^2}{Q^4 e^2} + \frac{2gK^3}{Q^4 e^2} - \\
& \frac{D_H^2 g}{Q^3} - \frac{4D_H D_L g}{Q^3} - \frac{4D_L^2 g}{eQ^3} + \frac{4D_L g K}{Q^3 e} + \frac{2D_H g K}{Q^3} - \frac{gK^2}{Q^3 e}, \\
\frac{\partial^2 \Pi_P(Q,K)}{\partial Q \partial K} =& \frac{4D_H D_L g}{Q^3 e} + \frac{2D_H^2 g}{Q^3} + \frac{4D_L g K}{Q^3 e^2} - \frac{2gK^2}{Q^3 e^2} - \frac{2D_L g}{Q^2 e} - \frac{D_H g}{Q^2} + \frac{egK}{Q^2}, \\
\frac{\partial \Pi_P(Q,K)}{\partial K} =& -\frac{2D_H D_L g}{Q^2 e} - \frac{D_H^2 g}{Q^2} - \frac{2D_L g K}{Q^2 e^2} + \frac{gK^2}{Q^2 e^2} + \frac{2D_L g}{Qe} + \frac{D_H g}{Q} - \frac{gK}{Qe} - \frac{c_K}{e}, \\
\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 K} =& -\frac{2D_L g}{Q^2 e^2} + \frac{2gK}{Q^2 e^2} - \frac{eg}{Q}.
\end{aligned} \tag{31}
$$

Setting the first partial derivatives equal to 0 gives us the following candidate solution:

$$
\begin{aligned}
Q^*_{int}(K) =& \sqrt[3]{-d+\sqrt{d^2+c^3}} + \sqrt[3]{-d-\sqrt{d^2+c^3}}, \quad \text{where} \\
c =& \frac{g}{3c_Q}\left(-\frac{D_H^2}{2} - 2D_HD_L - \frac{2D_L^2}{e} + 2D_LK + D_HK - \frac{K^2}{2e}\right), \\
d =& \frac{g}{c_Q}\left(D_H^2D_L + \frac{2D_L^3}{3e^2} + 2D_HD_L^2 - \frac{2D_HD_LK}{e} - D_H^2K - \frac{D_LK^2}{e^2} + \frac{K^3}{3e^2}\right),
\end{aligned} \tag{32}
$$

and

$$
K^*_{int}(Q) = D_L + \frac{1}{2}\left(Qe - \sqrt{4(D_L+D_He)^2 - 4e(D_L+D_He)Q + \frac{e(4c_K+eg)Q^2}{g}}\right). \tag{33}
$$

It can be shown that only one interior solution exists through analyzing the Hessian numerically (see Section B.3.3), but here we use geometric properties of the objective function to ensure that we find the global optimum (i.e. profit-maximizing solution). This helps develop the intuition for the problem and lets us highlight several key points around the interplay of storage and solar capacity. Several of the results obtained in this proof will be used in other parts of the paper. We proceed by first determining when the profit-maximizing solution is a border solution and where it is. When the solution cannot be a border solution, we then know that it must be (Q^*_{int}, K^*_{int}), this being the only interior critical point of the objective function.

In order to study the border solutions, it is convenient to show that both second partial derivatives are negative across the relevant parameter space:

We start by showing this for the second partial derivative w.r.t. Q. We know $K \geq D_L$ and $K \leq Q - D_H$ and $Q \geq D_H + \frac{D_L}{e}$.

$$
\begin{aligned}
&\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 Q} < 0, \\
&\frac{6D_H^2D_Lg}{Q^4} + \frac{4D_L^3g}{e^2Q^4} + \frac{12D_HD_L^2g}{eQ^4} - \frac{12D_HD_LgK}{Q^4e} - \frac{6D_H^2gK}{Q^4} - \frac{6D_LgK^2}{Q^4e^2} + \frac{2gK^3}{Q^4e^2} - \\
&\frac{D_H^2g}{Q^3} - \frac{4D_HD_Lg}{Q^3} - \frac{4D_L^2g}{eQ^3} + \frac{4D_LgK}{Q^3e} + \frac{2D_HgK}{Q^3} - \frac{gK^2}{Q^3e} < 0, \\
&6D_H^2D_L + \frac{4D_L^3}{e^2} + \frac{12D_HD_L^2}{e} - \frac{12D_HD_LK}{e} - 6D_H^2K - \frac{6D_LK^2}{e^2} + \frac{2K^3}{e^2} + \\
&Q(-D_H^2 - 4D_HD_L - \frac{4D_L^2}{e} + \frac{4D_LK}{e} + 2D_HK - \frac{K^2}{e}) < 0,
\end{aligned} \tag{34}
$$

We will now show that the expressions in both rows are negative. We start by showing that the expression multiplied by Q in the second row is negative.

$$
\begin{aligned}
& -D_H^2 - 4D_HD_L - \frac{4D_L^2}{e} + \frac{4D_LK}{e} + 2D_HK - \frac{K^2}{e} < 0, \\
& -D_H^2e - 4D_HD_Le - 4D_L^2 + 4D_LK + 2D_HKe - K^2 < 0, \\
& -D_H^2e - 4D_HD_Le - 4D_L^2 + 4D_LK + 2D_HKe - K^2 = 0, \\
& K = 2D_L + D_He \text{ The derivative is positive only outside the parameter space we consider.}
\end{aligned} \tag{35}
$$

This cannot be optimal, as any excess charge in the model is lost after the second period and could never be used.

Having established that this term is negative, we focus on the sum of the terms in the first row, for which we show that there is no value $K \in \left[D_L, 2D_L + eD_H\right]$ (the parameter space we consider for the partial-discharge model) for which this expression is positive, while having $K \leq Q - D_H$, which trivially holds as a higher K would never be fully charged.

$$
\begin{aligned}
& 6D_H^2D_L + \frac{4D_L^3}{e^2} + \frac{12D_HD_L^2}{e} - \frac{12D_HD_LK}{e} - 6D_H^2K - \frac{6D_LK^2}{e^2} + \frac{2K^3}{e^2} = 0, \\
& \text{which has 3 roots:} \\
& K_1^* = D_L - \sqrt{3}\sqrt{D_L^2 + 2D_HD_Le + D_H^2e^2},\ K_2^* = D_L, \\
& K_3^* = D_L + \sqrt{3}\sqrt{D_L^2 + 2D_HD_Le + D_H^2e^2} = D_L + \sqrt{3}(D_L + D_He).
\end{aligned} \tag{36}
$$

From the sign of the K^3 term, we know that the expression is non-negative between the two roots K_2^* and K_3^* and this range includes the entire parameter space we consider.

We now show that the second partial derivative w.r.t. K is also negative.

$$
\begin{aligned}
& \frac{\partial^2\Pi_P(Q,K)}{\partial^2 K} < 0, \\
& -\frac{2D_Lg}{Q^2e^2} + \frac{2gK}{Q^2e^2} - \frac{g}{Qe} < 0, \\
& \frac{\partial^2\Pi_P(Q,K)}{\partial^2 K} < 0 \text{ if } K < \frac{Qe}{2} + D_L \text{ and } K^* < \text{that value.}
\end{aligned} \tag{37}
$$

Having shown that both second partial derivatives are negative across the parameter space, we now turn to find possible candidates for a border solution. In particular, we now argue that the profit-maximizing solution must belong to the set of points $(Q,K) \in \left[\frac{D_L}{e} + D_H, M\right] \times \left[D_L, M\right]$, where M is an arbitrarily large number. Here is why: We only focus on cases with $K \geq D_L$ as this is the parameter space where the partial model is employed (else the full-discharge model yields exact investment decisions). In that space, profit is strictly increasing in Q for $Q < \frac{D_L}{e} + D_H$, hence we can ignore any lower Q. Note also that $lim_{Q\to+\infty}\frac{\partial\Pi_P(Q,K)}{\partial Q} < 0$ and $lim_{K\to+\infty}\frac{\partial\Pi_P(Q,K)}{\partial K} < 0$, hence the profit cannot be maximized when Q or K are too high. Formally:

$$WTS\ \frac{\partial \Pi_P(Q,K)}{\partial K} < 0,\ if K \to \infty:$$
$$\frac{\partial \Pi_P(Q,K\to\infty)}{\partial K} = -\frac{2D_H D_L g}{Q^2 e} - \frac{D_H^2 g}{Q^2} - \frac{2D_L g K}{Q^2 e^2} + \frac{gK^2}{Q^2 e^2} + \frac{2D_L g}{Qe} + \frac{D_H g}{Q} - \frac{gK}{Qe} - \frac{c_K}{e} \tag{38}$$

As $Q > D_H/e + K/e$:

$+ g - g - c_K = -c_K.$

$$\begin{aligned} WTS \quad \frac{\partial \Pi_P(Q,K)}{\partial Q} &< 0, if Q \to \infty: \\ \frac{\partial \Pi_P(Q\to\infty,K)}{\partial Q} &= -c_Q < 0. \end{aligned} \tag{39}$$

It follows that if a border solution for the partial-discharge model exists, it must have either $Q = \frac{D_L}{e} + D_H$ and $K > D_L$, or $K = D_L$ and $Q > \frac{D_L}{e} + D_H$. The former can be ruled out because when $Q = \frac{D_L}{e} + D_H$ there is never excess generation beyond nightly demand D_L to be stored, so profit decreases in K beyond D_L. This means that if a border solution exists, it must be one with $K = D_L$, and such that $\frac{\Pi_P}{\partial Q} = 0$. Let $K = D_L$.

$$\Pi_P(Q,K=D_L) = = D_H g + D_L g - \frac{D_H^2 g}{2Q} - \frac{D_H D_L g}{Q} - \frac{D_L^2 g}{2eQ} - c_Q Q - D_L \frac{c_K}{e} \tag{40}$$

$$\begin{aligned} \frac{\partial \Pi_P(Q,K=D_L)}{\partial Q} &= \frac{D_H^2 g}{2Q^2} + \frac{D_H D_L g}{Q} + \frac{D_L^2 g}{2eQ^2} - c_Q, \\ Q^*_{bor} &= \sqrt{\frac{g(\frac{D_L^2}{e} + 2D_H D_L + D_H^2)}{2c_Q}}. \end{aligned} \tag{41}$$

Hence, in the case we have the border solution $K = D_L$, we know the optimal generation capacity is Q^*_{bor}.

We now show at which value g the interior solution equals the border solution. We then show that the second derivative (differentiating first w.r.t. to K and then w.r.t. g) is strictly increasing in g, which tells us that the parameter g can be used to judge which solution is optimal, the border or the interior.

We now study the first partial derivative w.r.t. K at the border $K = D_L$.

$$
\begin{aligned}
& WTS \frac{\partial \Pi_P(Q, K = D_L)}{\partial K} > 0, \\
& -\frac{2D_H D_L g}{Q^2 e} - \frac{D_H^2 g}{Q^2} - \frac{D_L^2 g}{Q^2 e^2} + \frac{D_L g}{Qe} + \frac{D_H g}{Q} - \frac{c_K}{e} > 0, \\
& Q^2 - Q(\frac{D_L g}{c_K} + \frac{D_H e g}{c_K}) + \frac{2D_H D_L g}{c_K} + \frac{D_H^2 e g}{c_K} + \frac{D_L^2 g}{e c_K} < 0, \\
& \frac{g}{2c_K}(D_L + D_H e) - \sqrt{\frac{(D_L + D_H e)^2 g(-4c_K + ge)}{4c_K^2 e}} < Q < \\
& \frac{g}{2c_K}(D_L + D_H e) + \sqrt{\frac{(D_L + D_H e)^2 g(-4c_K + ge)}{4c_K^2 e}}, \\
& \frac{(D_L + D_H e)}{2c_K}(g - \sqrt{\frac{g(ge - 4c_K)}{e}}) < Q < \frac{(D_L + D_H e)}{2c_K}(g + \sqrt{\frac{g(ge - 4c_K)}{e}}).
\end{aligned}
\tag{42}
$$

If the expression under the radical is negative, $\frac{\partial \Pi_P(Q,K)}{\partial K} < 0$ whenever $K = D_L$ and it follows from concavity w.r.t. K that (Q^*_{bor}, K^*_{bor}) maximizes profit. This happens when $g < \frac{4c_K}{e} \triangleq \underline{g}$.

$$
\begin{aligned}
& ge - 4c_K < 0, \\
& g < 4\frac{c_K}{e} \triangleq \underline{g}.
\end{aligned}
\tag{43}
$$

So for all values of g below the threshold $\underline{g}$, we are certain to have the border solution ($K = D_L$). If instead the expression under the radical is positive, and setting $Q = Q^*_{bor}$, we have:

$$
\begin{aligned}
\frac{(D_L + D_H e)}{2c_K}(g - \sqrt{\frac{g(ge - 4c_K)}{e}}) < \sqrt{\frac{g\left(\frac{D_L^2}{e} + 2D_H D_L + D_H^2\right)}{2c_Q}} &< \frac{(D_L + D_H e)}{2c_K}(g + \sqrt{\frac{g(ge - 4c_K)}{e}}), \\
g &> \frac{\left(2c_Q(D_L + D_H e)^2 + c_K\left(D_L^2 + D_H(D_H + 2D_L)e\right)\right)^2}{2c_Q e(D_L + D_H e)^2\left(D_L^2 + D_H(D_H + 2D_L)e\right)}, \\
&= \frac{\left(c_K + 2c_K em(1+m) + 2c_Q(1+em)^2\right)^2}{2c_Q e(1+em)^2(1+2em(1+m))} \triangleq g_P, \text{where } m = \frac{D_H}{D_L}.
\end{aligned}
\tag{44}
$$

When g is larger than g_P, we have $\frac{\partial \Pi_P(Q^*_{bor}, D_L)}{\partial K} > 0$, hence profit increases at the border solution when storage capacity is increased. Taking the derivative first w.r.t. K and then w.r.t. g confirms that the derivative w.r.t. K increases in g and thus g_P is indeed a threshold.

$$\frac{\partial^2 \Pi_P(Q,K)}{\partial K \partial g} = -\frac{(2D_L + (D_H e - K))(D_H e + K - eQ)}{Q^2 e^2} > 0.$$

The first term is positive and the second negative in the parameter space we consider:

$(2D_L + (eD_H - K)) > 0 \quad \rightarrow \quad K < 2D_L + D_H e.$

Storage capacity larger than this would never be discharged in the partial model

$(D_H + K - Q) < 0 \quad \rightarrow \quad Q > D_H + K/e,$

Any lower generation capacity would never allow to charge the storage capacity fully.

Lastly, it can easily be verified that $\frac{\partial \Pi_P}{\partial K}(K^*_{bor}, Q^*_{bor}, g_P) = 0$. In conjunction with the fact that the derivative w.r.t. K is increasing in g, this completes the proof. We thus have a unique threshold to identify which of the two candidate solutions (border and interior) of the partial-discharge model is profit optimal.

Thus, if $g \geq g_P$, we have the interior solution and if $g \in (g_F, g_P)$, we have the border solution.

A.6 Proof of Theorem 6

We begin the proof by relating the profits of the two models.

$$\begin{aligned}
&\Pi^*_P = \Pi_P(Q^*_P, K^*_P) \leq \Pi_{TR}(Q^*_P, K^*_P),\\
&\text{as the partial-discharge model underestimates profit relative to the tracking model}\\
&\text{(because stored energy expires as per Assumption 2 and is employed sub-optimally as per}\\
&\text{Assumption 3),}\\
&\Pi_{TR}(Q^*_P, K^*_P) \leq \Pi_{TR}(Q^*_{TR}, K^*_{TR}) = \Pi^*_{TR},\\
&\text{as otherwise } (Q^*_{TR}, K^*_{TR}) \text{ would not be optimal. Hence, the partial-discharge model}\\
&\text{underestimates profit relative to the tracking model.}
\end{aligned} \tag{45}$$

We continue the proof by showing that the partial-discharge model always under-predicts optimal storage capacity compared to the tracking model, for a given level of generation capacity. We achieve this by showing that a small increase in storage capacity is marginally less profitable in the partial-discharge model in comparison to the tracking model. Formally, we aim to show that for any given Q, the following holds:

$$\frac{\partial}{\partial K}\Pi_{TR}(Q,K) \geq \frac{\partial}{\partial K}\Pi_P(Q,K) \;\; \forall K. \tag{46}$$

We proceed in 3 steps. In step 1, we re-introduce the models and explain the intuition behind the proof. In step 2, we calculate the value of a marginal unit of storage capacity for both models per period, and in step 3, we compare the marginal value unit of storage capacity of both models and generalize the finding to all periods.

Step 1. We first (re)-introduce the objective functions (for a detailed derivation of getting from the tracking model to the partial-discharge model, please refer to Appendix B.1):

$$\begin{aligned}
&\Pi_{TR}(Q,K) = \\
&g\ E\Big[\sum_{t=1}^{T}\big(\min[q_t, D_H] + \min[\ x_{t-1}^{TR}, (D_H-q_t)^+] + \min[(x_{t-1}^{TR} + e(q_t-D_H)^+ - (D_H-q_t)^+)^+, D_L]\big)\Big] - \\
&T\frac{c_K}{e}K - Tc_QQ, \\
&\text{where } x_t^{TR} = (\min[(x_{t-1}^{TR} + e(q_t-D_H)^+ - (D_H-q_t)^+)^+, K] - D_L)^+. \\
&\Pi_P(Q,K) = g\ E\Big[\sum_{t=1}^{T}\Big(\min[q_t, D_H] + \min[e(q_t-D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, x_{t-1}^P]\Big)\Big] - \\
&T\frac{c_K}{e}K - Tc_QQ, \\
&\text{where } x_t^P = (\min[e(q_t-D_H)^+, K] - D_L)^+.
\end{aligned} \tag{47}$$

Note that the only source of uncertainty in the model comes from generation. Let $\vec{q}$ be a vector of generation realizations over the T periods. We can then rewrite the profit of model $m \in \{P, TR\}$, $\Pi_m(Q,K)$, as

$$\Pi_m(Q,K) = \sum_{t=1}^{T} \mathbb{E}_{\vec{Q}}\left[\pi_t^m(\vec{q},K)\right], \tag{48}$$

where $\pi_t^m(\vec{q},K)$ is the profit earned in period t under model m given a vector of generation realizations $\vec{q}$. Using this notation, note that:

$$\begin{aligned}
\frac{\partial}{\partial K}\Pi_{TR}(Q,K) &\geq \frac{\partial}{\partial K}\Pi_P(Q,K) \quad \text{, iff} \\
\frac{\partial}{\partial K}\mathbb{E}_{\vec{Q}}\left[\pi_t^{TR}(\vec{q},K)\right] &\geq \frac{\partial}{\partial K}\mathbb{E}_{\vec{Q}}\left[\pi_t^{P}(\vec{q},K)\right]\ ,\forall t \quad \text{, iff} \\
\frac{\partial}{\partial K}\int_{\vec{Q}}\pi_t^{TR}(\vec{q},K)f(\vec{q})\ d\vec{q} &\geq \frac{\partial}{\partial K}\int_{\vec{Q}}\pi_t^{P}(\vec{q},K)f(\vec{q})\ d\vec{q}\ ,\forall t,
\end{aligned} \tag{49}$$

where $\vec{Q} = [0,Q]^T$ here denotes the set of all possible vectors of generation realizations ($\vec{q} \in \vec{Q}$) and $f(\cdot)$ is the probability density function of $\vec{q}$ over $\vec{Q}$.

$$
\begin{aligned}
&\frac{\partial}{\partial K}\int_{\vec{Q}} \pi_t^{TR}(\vec{q},K)f(\vec{q})\ d\vec{q}\ \geq\ \frac{\partial}{\partial K}\int_{\vec{Q}} \pi_t^{P}(\vec{q},K)f(\vec{q})\ d\vec{q}\ ,\forall t,\ \text{iff}\\
&\int_{\vec{Q}} \frac{\partial}{\partial K}\pi_t^{TR}(\vec{q},K)f(\vec{q})\ d\vec{q}\ \geq\ \int_{\vec{Q}} \frac{\partial}{\partial K}\pi_t^{P}(\vec{q},K)f(\vec{q})\ d\vec{q}\ ,\forall t.
\end{aligned} \tag{50}
$$

We use the Leibniz integral rule and remark that the limits of integration (a function of solar capacity Q) are invariant to the derivative w.r.t. storage (K). Finally, we have

$$
\begin{aligned}
&\int_{\vec{Q}} \frac{\partial}{\partial K}\pi_t^{TR}(\vec{q},K)f(\vec{q})\ d\vec{q}\ \geq \int_{\vec{Q}} \frac{\partial}{\partial K}\pi_t^{P}(\vec{q},K)f(\vec{q})\ d\vec{q}, \forall t \quad, \text{if}\\
&\frac{\partial}{\partial K}\pi_t^{TR}(\vec{q},K) \geq \frac{\partial}{\partial K}\pi_t^{P}(\vec{q},K)\ \forall t, \forall \vec{q}.
\end{aligned} \tag{51}
$$

Given that the profit in period t, $\pi_t^m(\vec{q},K)$, depends on $\vec{q}$ only through q_t and x_{t-1}^m, we can thus rewrite the above as:

$$
\begin{aligned}
&\frac{\partial}{\partial K}\pi_t^{TR}(\vec{q},K) \geq \frac{\partial}{\partial K}\pi_t^{P}(\vec{q},K)\ \forall t, \forall \vec{q},\ \text{iff}\\
&\frac{\partial}{\partial K}\pi_t^{TR}(q_t,K,x_{t-1}^{TR}) \geq \frac{\partial}{\partial K}\pi_t^{P}(q_t,K,x_{t-1}^{P}),\ \forall t, \forall q_t, \forall x_{t-1}^{TR}, \forall x_{t-1}^{P},\ \text{iff}\\
&\lim_{\epsilon\to 0}\left[\pi_t^{TR}(q_t,K+\epsilon,x_{t-1}^{TR}) - \pi_t^{TR}(q_t,K,x_{t-1}^{TR})\right] \geq\\
&\lim_{\epsilon\to 0}\left[\pi_t^{P}(q_t,K+\epsilon,x_{t-1}^{P}) - \pi_t^{P}(q_t,K,x_{t-1}^{P})\right],\ \forall t, \forall q_t, \forall x_{t-1}^{TR}, \forall x_{t-1}^{P}.
\end{aligned} \tag{52}
$$

Note that it is trivial to show that $x_{t-1}^{TR} \geq x_{t-1}^{P}, \forall \vec{q}$ (see Lemma 2 in Appendix B.1)

Step 2. We thus aim to show that (47) holds. In the interest of readability, we omit the expectation operators in the remaining section. We start by looking at the tracking model.

$$
\begin{aligned}
&\pi_t^{TR}(q_t,K,x_{t-1}^{TR}) =\\
&\min[q_t,D_H] + \min[x_{t-1}^{TR},(D_H-q_t)^+] + \min[(x_{t-1}^{TR} + e(q_t-D_H)^+ - (D_H-q_t)^+)^+, D_L],\\
&= \min[q_t,D_H] + \min[(\min[(x_{t-2}^{TR} - (D_H-q_{t-1})^+)^+ + e(q_{t-1}-D_H)^+, K] - D_L)^+, (D_H-q_t)^+]+\\
&\min[((\min[(x_{t-2}^{TR} - (D_H-q_{t-1})^+)^+ + e(q_{t-1}-D_H)^+, K] - D_L)^+ + e(q_t-D_H)^+ - (D_H-q_t)^+)^+, D_L].\\
&\pi_t^{TR}(q_t,K+\epsilon,x_{t-1}^{TR}) =\\
&\min[q_t,D_H] + \min[(\min[(x_{t-2}^{TR} - (D_H-q_{t-1})^+)^+ + e(q_{t-1}-D_H)^+, K+\epsilon] - D_L)^+, (D_H-q_t)^+]+\\
&\min[((\min[(x_{t-2}^{TR} - (D_H-q_{t-1})^+)^+ + e(q_{t-1}-D_H)^+, K+\epsilon] - D_L)^+ + e(q_t-D_H)^+ - (D_H-q_t)^+)^+, D_L]
\end{aligned} \tag{53}
$$

We will solve for the difference directly, by focusing on cases where the added ϵ storage capacity is used (i.e., part of the smallest element in the minimum). To this end, we analyze the case $(q_t \geq D_H)$ and $(q_t < D_H)$ separately. If $q_t < D_H$:

$$\begin{aligned}
&\pi_t^{TR}(q_t, K+\epsilon, x_{t-1}^{TR}|q_t < D_H) - \pi_t^{TR}(q_t, K, x_{t-1}^{TR}|q_t < D_H) = \\
&\min[(\min[(x_{t-2}^{TR} - (D_H - q_{t-1})^+)^+ + e(q_{t-1} - D_H)^+, K+\epsilon] - D_L)^+, (D_H - q_t)^+] - \\
&\min[(\min[(x_{t-2}^{TR} - (D_H - q_{t-1})^+)^+ + e(q_{t-1} - D_H)^+, K] - D_L)^+, (D_H - q_t)^+] + \\
&\min[((\min[(x_{t-2}^{TR} - (D_H - q_{t-1})^+)^+ + e(q_{t-1} - D_H)^+, K+\epsilon] - D_L)^+ - (D_H - q_t)^+)^+, D_L] - \\
&\min[((\min[(x_{t-2}^{TR} - (D_H - q_{t-1})^+)^+ + e(q_{t-1} - D_H)^+, K] - D_L)^+ - (D_H - q_t)^+)^+, D_L] \\
&= \epsilon\ P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}]P[q_t < D_H + D_L - K|q_t < D_H] + \\
&\epsilon\ P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}]P[q_t < D_H + 2D_L - K|q_t < D_H], \\
&= \epsilon\ P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}](P[q_t < D_H + D_L - K|q_t < D_H] + P[q_t < D_H + 2D_L - K|q_t < D_H]).
\end{aligned} \tag{54}$$

So we are left with the case $q_t \geq D_H$, for which the first two minima drop out when taking the difference, as the first is not dependent on K and the second evaluates to 0.

$$\begin{aligned}
&\pi_t^{TR}(q_t, K+\epsilon, x_{t-1}|q_t \geq D_H) - \pi_t^{TR}(q_t, K, x_{t-1}^{TR}|q_t \geq D_H) = \\
&\min[((\min[(x_{t-2}^{TR} - (D_H - q_{t-1})^+)^+ + e(q_{t-1} - D_H)^+, K+\epsilon] - D_L)^+ + e(q_t - D_H)^+)^+, D_L] - \\
&\min[((\min[(x_{t-2}^{TR} - (D_H - q_{t-1})^+)^+ + e(q_{t-1} - D_H)^+, K] - D_L)^+ + e(q_t - D_H)^+)^+, D_L] \\
&=\ \epsilon\ P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}]P[q_t < D_H + 2D_L - K|q_t \geq D_H].
\end{aligned} \tag{55}$$

We proceed analogously for the partial-discharge model, by first presenting the two functions and then taking the difference.

$$\begin{aligned}
&\pi_t^P(q_t, K, x_{t-1}) = \\
&\min[q_t, D_H] + \min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, (\min[e(q_{t-1} - D_H)^+, K] - D_L)^+]. \\
&\pi_t^P(q_t, K+\epsilon, x_{t-1}^P) = \\
&\min[q_t, D_H] + \min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, (\min[e(q_{t-1} - D_H)^+, K+\epsilon] - D_L)^+].
\end{aligned} \tag{56}$$

When taking the difference, note that the first two minima are not dependent on the storage size K, and thus drop out.

$$\begin{aligned}
&\pi_t^P(q_t, K+\epsilon, x_{t-1}^P) - \pi_t^P(q_t, K, x_{t-1}^P) = \\
&\min[(eD_H + D_L - eq_t)^+, (\min[e(q_{t-1} - D_H)^+, K+\epsilon] - D_L)^+] - \\
&\min[(eD_H + D_L - eq_t)^+, (\min[e(q_{t-1} - D_H)^+, K] - D_L)^+].
\end{aligned} \tag{57}$$

We also split this into two cases, to show that the case $q_t < D_H$ leads to the same result for both models and can be excluded from further analysis.

$$\pi_t^P(q_t, K+\epsilon, x_{t-1}^P | q_t < D_H) - \pi_t^P(q_t, K, x_{t-1}^P | q_t < D_H) = \\ \epsilon\ P[eq_{t-1} > K + eD_H] P[q_t < D_H + 2D_L - K | q_t < D_H] \tag{58}$$

And now the case of $q_t > D_H$:

$$\pi_t^P(q_t, K+\epsilon, x_{t-1}^P | q_t \geq D_H) - \pi_t^P(q_t, K, x_{t-1}^P | q_t \geq D_H) = \\ \epsilon\ P[eq_{t-1} > K + eD_H] P[q_t < D_H + D_L - K | q_t \geq D_H] \tag{59}$$

Step 3. Combining these pieces, we are left with the expression we set out to prove:

$$\begin{aligned}
&\Big(\pi_t^{TR}(q_t, K+\epsilon, x_{t-1}^{TR}) - \pi_t^{TR}(q_t, K, x_{t-1}^{TR})\Big) - \Big(\pi_t^P(q_t, K+\epsilon, x_{t-1}^P) - \pi_t^P(q_t, K, x_{t-1}^P)\Big) = \\
&P[q_t < D_H]\ \epsilon\ \Big[\Big(P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}](P[q_t < D_H + D_L - K | q_t < D_H] + \\
&P[q_t < D_H + 2D_L - K | q_t < D_H])\Big) - \\
&\Big(P[eq_{t-1} > K + D_H e] P[q_t < D_H + 2D_L - K | q_t < D_H]\Big)\Big] + \\
&P[q_t \geq D_H]\ \epsilon\ \Big[\Big(P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}] P[q_t < D_H + 2D_L - K | q_t \geq D_H]\Big) - \\
&\Big(P[eq_{t-1} > K + eD_H] P[q_t < D_H + 2D_L - K | q_t \geq D_H]\Big)\Big] \quad > \\
&P[q_t < D_H]\ \epsilon\ \Big[\Big(P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}] P[q_t < D_H + 2D_L - K | q_t < D_H]\Big) - \\
&\Big(P[eq_{t-1} > K + eD_H] P[q_t < D_H + 2D_L - K | q_t < D_H]\Big)\Big] + \\
&P[q_t \geq D_H]\ \epsilon\ \Big[\Big(P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}] P[q_t < D_H + 2D_L - K | q_t \geq D_H]\Big) - \\
&\Big(P[eq_{t-1} > K + eD_H] P[q_t < D_H + 2D_L - K | q_t \geq D_H]\Big)\Big] > 0
\end{aligned} \tag{60}$$

The first inequality in Equation (60) stems from the fact that we drop the probability term $P[eq_{t-1} > K + eD_H - x_{t-2}^{TR}] P[q_t < D_H + D_L - K | q_t < D_H]$ for the tracking model.

The resulting expression is clearly larger than 0, as $x_{t-2}^{TR} \geq 0$. Thus, the partial-discharge model underestimates the marginal value of storage capacity compared to the tracking model. Note that x_{t-2} does not appear in the partial-discharge model expressions as it is assumed to be lost.

Note that this proof also works for the partial-discharge objective without the simplification we introduce in Equation (89) in Appendix B.1. Without the simplification, all probabilities w.r.t. q_t are the same in the tracking and partial-discharge models, but the probabilities w.r.t. q_{t-1} are still contingent on x_{t-2}^{TR} in the tracking model, and independent of this term in the unsimplified partial-discharge model. As explained above, given that $x_{t-2}^{TR} \geq 0$, this proves the lower marginal benefit of storage in the "unsimplified" partial-discharge model, compared to the tracking model.

Furthermore, note that the optimal solution of the partial-discharge model, K_P^*, is constant ($K_P^* = D_L$) for $g_F \leq g \leq g_P$, while the optimal tracking model storage capacity, K_{TR}^*, is weakly increasing for g in this range and equal to D_L at $g = g_F$ (as shown in Theorem 2). Thus, $K^*_{TR} \geq K^*_P$ for $g_F \leq g \leq g_P$.

A.7 Proof of Theorem 7

We want to derive the optimal policy for the capacitated generator model. In Step 1 of this proof we will thus first show that the value function is convex in the decision we make. We'll show that our decision can be represented by the end-of-storage charge x_t. In Step 2 of this proof we will show that if the value function is convex in this decision, one can find a cost-minimizing storage charge x^*_t with which to end the period, after having started the period with x_{t-1} and observed generation realization q_t. In Step 3 of this proof we will then lower-bound this optimal storage charge x^*_t in closed form. In Step 4 of this proof we will provide comparative statics of this optimal storage charge x^*_t.

A.7.1 Step 1: Proofing Convexity of the Value Function

In any period t, we observe initial charge x_{t-1}, observe generation realization q_t, and then need to decide whether and by how much to run the generator in the day and at night (to meet demand and potentially charge the storage). We now show that, rather than considering daily and nightly backup generation G_{Ht}, G_{Lt} as separate actions every period, it is possible to focus on the period-end-charge x_t as the sole decision to be made, without loss of generality.

We proceed in the following four parts

Part 1: We show that the value function can be written using the end-of-period storage charge x_t as the sole decision, to capture backup generation decisions during day and at night, without loss of generality.

Part 2: We show that the current period's cost function is convex in x_{t-1} and x_t.

Part 3: We show that the value function is convex in the initial storage charge, x_{t-1}.

Part 4: We show that the un-minimized value function is convex in the end-of-period storage charge decision, x_t.

Part 1: Mapping end of period charge x_t into Generator Decisions.
Let $x^+_t \triangleq \min[(x_{t-1} + e(q_t + G - D_H)^+ - (D_H - q_t - G)^+)^+, K]$ be the largest charge one can end the day-sub-period with, for a given x_{t-1} and a given q_t. Let $\bar{G}_{Ht} \triangleq \min((K - x_{t-1} - e(q_t - D_H)^+ + (D_H - q_t)^+)^+/e, G)$ be the minimum between the largest amount that the day-generator can be

used before hitting the storage capacity limit at the end of the day sub-period and the generator capacity. Note that this is the lowest amount of energy the day-generator must produce to reach x_t^+ at the end of the day-subperiod. Let $\tilde{G}_{Ht} \triangleq \min[\bar{G}_{Ht}, (D_H - q_t)^+] + e(\bar{G}_{Ht} - (D_H - q_t)^+)^+$ be demand that the day-generator fulfils when generating $\bar{G}_{Ht}$, plus any storage charge it adds.

We construct a cost-minimizing mapping between the generator decisions and an end-of-period charge x_t, given any combination of starting charge x_{t-1} and solar generation q_t that are feasible[8]. In general, there may be many combinations of generator decisions that reach the same charge x_t with various costs, and even many combinations that reach the same charge with minimal costs. Among this latter group, we choose a mapping that aims to reduce day-time generation before it reduces night-time generation, for expositional and analytical convenience. We first describe the logic of our mapping and then show the resulting mapping.

The Focal Quantity Our mapping focuses on the most amount of energy one can have in storage at the end of the day-subperiod (this is x_t^+, which can be obtained by running the day-time generator at least $\bar{G}_{Ht}$) minus the desired charge at the end the period x_t. We then compare this possible excess charge against the nightly demand that cannot be met by the generator $D_L - G$ (since $G < D_L$). Together, this focal quantity is $\chi \triangleq x_t^+ - x_t - (D_L - G)$. Note that this is equivalent to the definition presented in Section 3.3. In this proof, we will mostly write χ explicitly as the sum of its parts, as we are interested in the dynamics around the individual parts x_t and x_{t-1}(contained in x_t^+).

Priority 1: Avoiding Unmet Demand If $x_t^+ - x_t - (D_L - G) < 0$, we run the generator at night at full capacity (serving one unit of otherwise unmet demand this way costs g and avoids unmet demand costs αg) and the generator during the day as much as needed to avoid unmet demand, without hitting the storage limit (serving one unit of otherwise unmet demand this way avoids costs αg, and generates backup costs g if serving daily demand, which is preferable, and backup costs g/e if serving nightly demand, since energy needs to be charged and discharged). Doing so minimizes the amount of unmet demand, and further minimizes the generation costs we incur to do so. Formally, if $x_t^+ - x_t - (D_L - G) < 0 \implies \hat{G}_{Ht} = \bar{G}_{Ht}, \hat{G}_{Lt} = G$.

Employing Generators in the most cost-efficient way If $x_t^+ - x_t - (D_L - G) \geq 0$, one can meet all demand in a period (with enough backup generation), and it is always optimal to do so. If $x_t^+ - x_t - (D_L - G) > 0$, one no longer needs all generation to fill up storage capacity to achieve the end-of-period charge x_t, and so the focus shifts from minimizing unmet demand (previous case) to reaching x_t at the lowest possible backup generation cost.

To this end, it is important to observe that when one unit of energy from the generator is used to charge storage, a unit of energy yields e units of storage: In other words, increasing storage by one unit costs g/e.

[8] I.e., there exists at least one pair of generator decisions with which one reaches x_t

By contrast, when one unit of energy from the generator is used to meet demand directly (and thus prevent storage discharge), a unit of generator energy effectively yields 1 unit of storage: In other words, increasing storage by one unit costs g. The latter use is more efficient.
Finally, when one unit of energy from the generator is used to charge storage during the day and meet demand at night, a unit of energy achieves the same as e units of storage: here again, increasing storage by one unit costs g/e.
With this in mind, let's see how the optimal usage of the backup generator looks like.

Priority 2: Reducing Charging With Generator If $x_t^+ - x_t - (D_L - G) \in (0, e(\bar{G}_{Ht} - (D_H - q_t)^+)^+]$, we have more starting charge and solar than needed to meet all demand, and if $x_t^+ - x_t - (D_L - G) < e(\bar{G}_{Ht} - (D_H - q_t)^+)^+ > 0$, we also know that the generator is used to create some charge for the storage at the end of the day sub-period. We thus use all available excess energy $x_t^+ - x_t - (D_L - G)$ to reduce the generator charging during the day by as much as possible, while still reaching the x_t target, which is results in $\bar{G} - (x_t^+ - x_t - (D_L - G))/e$. Formally, if $x_t^+ - x_t - (D_L - G) \in (0, e(\bar{G}_{Ht} - (D_H - q_t)^+)^+] \implies \hat{G}_{Ht} = \bar{G} - (x_t^+ - x_t - (D_L - G))/e, \hat{G}_{Lt} = G$.

Priority 3: Reducing Regular Day-time Use of Generator If $x_t^+ - x_t - (D_L - G) \in (e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, \tilde{G}_{Ht}]$, we have so much starting charge and solar $x_t^+ - x_t - (D_L - G) > (e(\bar{G}_{Ht} - (D_H - q_t)^+)^+$ that all generation during day and night is used to directly meet demand in the same sub-period (i.e. the generator is not used to create a charge). In this case, reducing generation during the night and during the day both save g, but our mapping first uses any excess energy to reduce the daily generation. Formally, if $x_t^+ - x_t - (D_L - G) \in (e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, \tilde{G}_{Ht}] \implies \hat{G}_{Ht} = \tilde{G} - (x_t^+ - x_t - (D_L - G)), \hat{G}_{Lt} = G$.

Priority 4: Reducing Regular Night-time Use of Generator Lastly if $x_t^+ - x_t - (D_L - G) > \tilde{G}$, we have so much solar and starting charge that without running the generator during the day we can meet all demand during the day, meet our target x_t and use any spare energy to reduce the generation at night. Formally, if $x_t^+ - x_t - (D_L - G) > \tilde{G} \implies \hat{G}_{Ht} = 0, \hat{G}_{Lt} = (G - (x_t^+ - x_t - (D_L - G) - \tilde{G})^+$.

Given the above steps, the policy achieves end-of-period charge x_t at minimal cost by construction, because it ranks all possible uses of generator's units from the ones with the highest net benefit down to the ones with the worst net benefit, and uses them in this order to achieve the desired end-of-period storage level x_t.

The optimal mapping, analogous to the optimal policy given in Theorem 7, is stated in Equation

62.

$$\mathbb{1}_D \triangleq \begin{cases} 1 \text{ if } x_t^+ - x_t - (D_L - G) < e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, \\ \tilde{G}_{Ht}/\bar{G}_{Ht} \text{ otherwise.} \end{cases}$$
$$\mathbb{1}_X \triangleq \begin{cases} 1 \text{ if } x_t^+ - x_t - (D_L - G) < e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, \\ e \text{ otherwise.} \end{cases} \tag{61}$$

$$\hat{G}_{Ht} \triangleq \left(\bar{G}_{Ht}\mathbb{1}_D - \mathbb{1}_X \frac{(x_t^+ - x_t - (D_L - G))^+}{e}\right)^+,$$
$$\hat{G}_{Lt} \triangleq (G - (x_t^+ - x_t - (D_L - G) - \tilde{G}_{Ht})^+)^+. \tag{62}$$

For the following convexity proof it is useful to write-out the four separate cases that the two indicator functions distinguish. We explicitly state all four cases in Equation 63, which allows the reader to clearly see how the focal quantity $x_t^+ - x_t - (D_L - G)$ impacts the generation decisions.

$$\text{If } x_t^+ - x_t - (D_L - G) \begin{cases} \leq 0 \begin{cases} \hat{G_{Ht}} = \bar{G}_{Ht} \\ \hat{G_{Lt}} = G \end{cases} \\ \in (0, e(\bar{G}_{Ht} - (D_H - q_t)^+)^+] \begin{cases} \hat{G_{Ht}} = \bar{G}_{Ht} - \frac{x_t^+ - x_t - (D_L - G)}{e} \\ \hat{G_{Lt}} = G \end{cases} \\ \in (e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, \tilde{G}_{Ht}] \begin{cases} \hat{G_{Ht}} = \tilde{G}_{Ht} - (x_t^+ - x_t - (D_L - G)) \\ \hat{G_{Lt}} = G \end{cases} \\ > \tilde{G}_{Ht} \begin{cases} \hat{G_{Ht}} = 0 \\ \hat{G_{Lt}} = (G - (x_t^+ - x_t - (D_L - G) - \tilde{G}_{Ht}))^+ \end{cases} \end{cases} \tag{63}$$

Part 2: The Current Period t Cost function is Convex in x_t and x_{t-1}.

With this optimal mapping defined, we want to show the convexity of the cost function in x_t and x_{t-1} and start with the former.

Convexity of $c(x_{t-1}, q_t, x_t)$ in x_{t-1}

Note that the conditions in Equation 63 are ordered based on the magnitude of $x_t^+ - x_t - (D_L - G)$, which is clearly decreasing in x_t.

$$x_t^+ - x_t - (D_L - G) = \min[(x_{t-1} + e(q_t + G - D_H)^+ - (D_H - q_t - G)^+)^+, K] - x_t - (D_L - G),$$
$$\frac{\partial x_t^+ - x_t - (D_L - G)}{\partial x_t} < 0. \tag{64}$$

Increasing x_t will thus decrease $x_t^+ - x_t - (D_L - G)$ and for the different realizations of x_t we will analyze how further increasing x_t impacts cost. Because we are only looking at cost in period

t, aiming to end the period with a higher charge increases cost as we have to run the generator more. Also, to analyze convexity, we assume q_t and x_{t-1} are fixed.

$$x_t \begin{cases} < x_t^+ - D_L - \tilde{G}_{Ht} \implies \hat{G}_{Ht} = \hat{G}_{Lt} = 0 \implies \frac{\partial (G-(x_t^+ - x_t - (D_L - G) - \tilde{G}_{Ht})^+}{\partial x_t} = 0 \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_t} = 0, \\ \in (x_t^+ - D_L - \tilde{G}_{Ht}, x_t^+ - D_L + G - \tilde{G}_{Ht}] \implies \hat{G}_{Ht} = 0, \hat{G}_{Lt} \in (0, G], \\ \qquad \implies \frac{\partial (G-(x_t^+ - x_t - (D_L - G) - \tilde{G}_{Ht}))^+}{\partial x_t} = 1 \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_t} = g, \\ \in (x_t^+ - D_L + G - \tilde{G}_{Ht}, x_t^+ - D_L + G - e(\bar{G}_{Ht} - (D_H - q_t)^+)^+], \\ \qquad \implies \hat{G}_{Ht} \in (0, \bar{G}_{Ht}], \hat{G}_{Lt} = G \implies \frac{\partial (\tilde{G}_{Ht} - (x_t^+ - x_t - (D_L - G))^+}{\partial x_t} = 1 \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_t} = g, \\ \in (x_t^+ - D_L + G - e(\bar{G}_{Ht} - (D_H - q_t)^+)^+], x_t^+ - D_L + G], \\ \qquad \implies \hat{G}_{Ht} \in (0, \bar{G}_{Ht}], \hat{G}_{Lt} = G \implies \frac{\partial (\bar{G}_{Ht} - \frac{x_t^+ - x_t - (D_L - G)}{e})}{\partial x_t} = 1/e \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_t} = g/e, \\ > x_t^+ - D_L + G \implies \hat{G}_{Ht} = \bar{G}, \hat{G}_{Lt} = G \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_t} = \alpha g \text{ or } x_t \text{ is not feasible.} \end{cases} \tag{65}$$

We distinguish 5 ranges for x_t:

i) If $x_t < x_t^+ - D_L - \tilde{G}_{Ht}$, our starting charge and solar generation are such that we need to run neither generator to meet all demands and reach x_t. Marginally raising the end-of-storage goal x_t can still be met without using the generator, thus not impacting cost.

ii) If $x_t \in (x_t^+ - D_L - \tilde{G}_{Ht}, x_t^+ - D_L + G - \tilde{G}_{Ht}]$, raising x_t means that we need to run the generator at night more to meet demand, marginally costing g.

iii) If $x_t \in (x_t^+ - D_L + G - \tilde{G}_{Ht}, x_t^+ - D_L + G - e(\bar{G}_{Ht} - (D_H - q_t)^+)^+]$, there is enough starting charge to cover nightly demand and x_t, but raising x_t means that we use the generator during the day to serve an additional unit of demand that then does not have to be discharged during the day and is available at the end of the night, thus increasing x_t and marginally costing g.

iv) If $x_t \in (x_t^+ - D_L + G - e(\bar{G}_{Ht} - (D_H - q_t)^+)^+], x_t^+ - D_L + G]$, the extra unit of x_t has to be generated by the generator during the day, requiring $1/e$ units of generator energy and marginally costing g/e.

v) If $x_t > x_t^+ - D_L + G$, increasing x_t means that there will more be more unmet demand as the generator is already running at capacity day and night, marginally costing αg.

Since when increasing x_t we either stay within a range or move to another range down the list (e.g., we cannot go from range iii) to range ii) by increasing x_t), then $c(x_{t-1}, q_t, x_t)$ is increasing and convex in x_t, which concludes the proof.

Convexity of $c(x_{t-1}, q_t, x_t)$ in x_{t-1}

We now want to show the convexity of the cost function in x_{t-1}. Note that the conditions in Equation 63 are ordered ,based on the magnitude of $x_t^+ - x_t - (D_L - G)$, which is clearly weakly increasing in x_{t-1}.

$$x_t^+ - x_t - (D_L - G) = \min[(x_{t-1} + e(q_t + G - D_H)^+ - (D_H - q_t - G)^+)^+, K] - x_t - (D_L - G),$$
$$\frac{\partial x_t^+ - x_t - (D_L - G)}{\partial x_{t-1}} \geq 0. \tag{66}$$

Increasing x_{t-1} will thus increase $x_t^+ - x_t - (D_L - G)$ until $x_t^+ = K$. After that, $\partial x_t^+ / \partial x_{t-1} = 0$, so in that case $\partial c(x_{t-1}, q_t, x_t)/\partial x_{t-1} = 0$. Intuitively, if the starting charge in the period becomes so large, that without running the generator during the day the storage is at capacity at the end of the day sub-period, further increasing the starting charge has no value. Because we are looking at cost in period t for a given q_t and x_t, starting the period with a higher storage charge x_{t-1} (weakly) reduces costs as we do not need to run generators as much to end the period with charge x_t.

In the following, we will thus focus on the case where storage capacity is not limiting, without impacting the convexity results (since $c(x_{t-1}, q_t, x_t)$ is increasing in x_{t-1}). For the different realizations of x_{t-1} we will analyze how further increasing x_{t-1} impacts cost. For brevity, we will use $\Delta \triangleq e(q_t + G - D_H)^+ - (D_H - q_t - G)^+$.

$$x_{t-1} \begin{cases} < x_t + D_L - G - \Delta \implies \hat{G}_{Ht} = \bar{G}, \hat{G}_{Lt} = G \implies \frac{\bar{G}_{Ht}}{\partial x_{t-1}} = 0 \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_{t-1}} = -\alpha g, \\ \in (x_t + D_L - G - \Delta, x_t + D_L - G - \Delta + e(\bar{G}_{Ht} - (D_H - q_t)^+)^+] \implies \hat{G}_{Ht} \in (0, \bar{G}_{Ht}], \hat{G}_{Lt} = G, \\ \quad \implies \frac{\partial(\bar{G}_{Ht} - \frac{x_t^+ - x_t - (D_L - G)}{e})}{\partial x_{t-1}} = -1/e \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_{t-1}} = -g/e, \\ \in (x_t + D_L - G - \Delta + e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, x_t + D_L - G - \Delta + \tilde{G}_{Ht}] \implies \hat{G}_{Ht} \in (0, \bar{G}_{Ht}], \\ \quad \hat{G}_{Lt} = G, \implies \frac{\partial(\tilde{G}_{Ht} - \frac{x_t^+ - x_t - (D_L - G)}{e})}{\partial x_{t-1}} = -1 \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_{t-1}} = -g, \\ \in (x_t + D_L - G - \Delta + \tilde{G}_{Ht}, x_t + D_L - \Delta + \tilde{G}_{Ht}] \implies \hat{G}_{Ht} = 0, \hat{G}_{Lt} \in (0, G], \\ \quad \implies \frac{\partial(G - (x_t^+ - x_t - (D_L - G) - \tilde{G}_{Ht}))^+}{\partial x_{t-1}} = -1 \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_{t-1}} = -g, \\ > x_t + D_L - \Delta + \tilde{G}_{Ht} \implies \hat{G}_{Ht} = 0, \hat{G}_{Lt} = 0, \\ \frac{\partial(G - (x_t^+ - x_t - (D_L - G) - \tilde{G}_{Ht}))^+}{\partial x_{t-1}} = 0 \implies \frac{\partial c(x_{t-1}, q_t, x_t)}{\partial x_{t-1}} = 0. \end{cases} \tag{67}$$

We also distinguish 5 ranges for x_{t-1}. Note that the order is reverse compared to x_t as the effect of starting with an extra unit of charge and needing an extra unit of charge are reversed:
i) If $x_{t-1} < x_t + D_L - G - \Delta$, increasing x_{t-1} means that we can meet previously unmet demand, so the generator will run keep running at capacity day and night, but the extra charge marginally saves αg.
ii) If $x_{t-1} \in (x_t + D_L - G - \Delta, x_t + D_L - G - \Delta + e(\bar{G}_{Ht} - (D_H - q_t)^+)^+]$, the extra unit of charge replaces $1/e$ units that the generator hat to produce during the day to create the same charge to serve demand at night, thus the extra charge marginally saves g/e.

iii) If $x_{t-1} \in (x_t + D_L - G - \Delta + e(\bar{G}_{Ht} - (D_H - q_t)^+)^+, x_t + D_L - G - \Delta + \tilde{G}_{Ht}]$, the extra charge allows the generator during the day to run less to meet demand during the day. Thus, the extra charge marginally saves g.
iv) If $x_{t-1} \in (x_t + D_L - G - \Delta + \tilde{G}_{Ht}, x_t + D_L - \Delta + \tilde{G}_{Ht}]$, an extra starting charge means that we need to run the generator at night less to meet demand, thus the extra charge marginally saves g.
v) If $x_{t-1} > x_t + D_L - \Delta + \tilde{G}_{Ht}$, the starting charge and solar generation are such that we need to run neither generator to meet all demands and reach x_t. Further raising the start-of-period charge does not change the generator decision. Thus, the extra charge has no impacting on cost.
Since when increasing x_{t-1} we either stay within a range or move to another range down the list (e.g., we cannot go from range iii) to range ii) by increasing x_{t-1}), then $c(x_{t-1}, q_t, x_t)$ is decreasing and convex in x_{t-1}, which concludes the proof.

We can thus succinctly write the objective function as a cost-to-go function, where the cost from t to T are denoted as v_t and expressed as the current-period cost c_t plus the future cost-to-go-function starting from the next period v_{t+1}. Let $v_t^*(x_{t-1}, q_t)$ denote the optimal/lowest-achievable cost for the current period t and all remaining periods, given a specific charge and generation realization at time t. At the same time, $c(x_{t-1}, q_t, x_t)$ is the cost in period t starting with charge x_{t-1}, observing generation realization q_t, and choosing a desired end-of-period-charge x_t.

$$\begin{aligned} v_t^*(x_{t-1}, q_t) &= \min_{x_t}\{c(x_{t-1}, q_t, x_t) + E[v_{t+1}^*(x_t, q_{t+1})]\} \\ v_t(x_{t-1}, q_t, x_t) &= c(x_{t-1}, q_t, x_t) + E[v_{t+1}^*(x_t, q_{t+1})] \end{aligned} \tag{68}$$

We will analyze this value function for the remainder of the proof.

Part 3: Value Function in period t is convex in x_{t-1}.
In the tracking model, we assume that the storage charge at the end of the terminal period x_T neither has any value nor results in any cost, i.e. $v_T(x_{T-1}, q_T, x_T) = c(x_{T-1}, q_T, x_T)$. As shown in Part 2, $c(x_{T-1}, q_T, x_T)$ is convex in its x_T and x_{T-1}. So the value function in the final period T is convex in its arguments.

$$\begin{aligned} &\text{Induction hypothesis: } v_{t+1}^*(x_t, q_{t+1}) = \min_{x_t}\{c(x_t, q_{t+1}, x_{t+1}) + E[v_{t+2}^*(x_{t+1}, q_{t+2})]\} \text{ is convex in } x_t. \\ &\text{WTS } v_t^*(x_{t-1}, q_t) = \min_{x_t}\{c(x_{t-1}, q_t, x_t) + E[v_{t+1}^*(x_t, q_{t+1})]\} \text{ is convex in } x_{t-1}. \end{aligned} \tag{69}$$

So, we have that

$$
\begin{aligned}
&v_t(x_{t-1}, q_t, x_t) = c(x_{t-1}, q_t, x_t) + E[v^*_{t+1}(x_t, q_{t+1})], \\
&c(x_{t-1}, q_t, x_t) \text{ is convex in } x_t \text{ and } x_{t-1} \text{ as shown above}, \\
&v^*_{t+1}(x_t, q_{t+1}) \text{ is convex in } x_t \text{ by induction assumption}, \\
&\text{Since expectation preserves convexity}, \\
&v_t(x_{t-1}, q_t, x_t) \text{ is jointly convex in } x_{t-1} \text{ and } x_t \text{ being sum of convex functions}, \\
&v^*_t(x_{t-1}, q_t) \text{ is convex in } x_{t-1} \text{ for the Theorem of convexity preservation under minimization.}
\end{aligned}
\tag{70}
$$

Please refer to Hayman and Sobel (1984) for a proof on the Theorem of convexity preservation under minimization invoked above.

Part 4: Un-minimized Value Function in period t is convex in x_t.

As the last step, we want to show that the un-minimized value function is convex in x_t, so that in choosing x_t there exists a unique minimum solution.

$$
\begin{aligned}
&\text{WTS } v_t(x_{t-1}, q_t, x_t) = c(x_{t-1}, q_t, x_t) + E[v^*_{t+1}(x_t, q_{t+1})] \text{ is convex in } x_t. \\
&v_t(x_{t-1}, q_t, x_t) = c(x_{t-1}, q_t, x_t) + E[v^*_{t+1}(x_t, q_{t+1})], \\
&c(x_{t-1}, q_t, x_t) \text{ is convex in } x_t \text{ and } x_{t-1} \text{ as shown above}, \\
&v^*_{t+1}(x_t, q_{t+1}) \text{ is convex as proven in Part 3}, \\
&c(x_{t-1}, q_t, x_t) + E[v^*_{t+1}(x_t, q_{t+1})] \text{ is convex in } x_t \text{ since expectation preserves convexity}, \\
&\rightarrow v_t(x_{t-1}, q_t, x_t) \text{ is convex in } x_t.
\end{aligned}
\tag{71}
$$

A.7.2 Step 2: Characterization of Optimal Policy:

Now that we showed that the cost-function is convex in the arguments, we switch to the optimal policy. Of the two state variables, actions under any policy can only impact the storage charge x_t which will be the focal dimension of our policy - generation is independent of our actions (i.e. sunshine does not depend on our generator operation). We thus choose the generator operation in this period G_{Ht}, G_{Lt} to reach a end-of-period-charge x_t that minimizes the total cost.

Conceptually, the convexity of the cost-function w.r.t. the generator-decision G_{Ht}, G_{Lt} occurs, because the cost for every unit of generation is constant and equal to g. Yet, the benefit (read:

avoided cost) of lowering the future periods' cost $v_{t+1}(x_t(q_t, x_{t-1}, G_{Ht}, G_{Lt}), q_{t+1})$ by running the generator to increase x_t is weakly decreasing in the generation decision. If there is unmet demand in the current period t, running the generator is optimal as $\alpha > 1$. Once all unmet demand of the current period is met, it may or may not be optimal to run the generator to create a buffer charge for the next period, thus reducing the chance of unmet demand in the future.

Hence, there exists a cost-minimizing choice that balances the current-period's generation cost with the future periods' cost of unmet demand, which is the optimal point we are looking for. Because G_{Ht} and G_{Lt} both impact the future period's cost through x_t and cost the same (both incur efficiency loss), we can condense the optimal decision to be the optimal charge to end a period with, henceforth denoted with x_t^*).

$$\text{Set } x_t^*, \text{ so that } -\partial E[v_{t+1}^*(x_t, q_{t+1})]/\partial x_t = g/e. \tag{72}$$

It may be that this optimal charge is 0, if even without any charge the probability weighted cost for not meeting demand is lower than the generator cost (i.e. if $-\partial E[v_{t+1}^*(0, q_{t+1})]/\partial x_t < g/e$).[9]

Because of this characterization, the optimal policy is to run the generator, a) iff in the absence of running the generator, the charge would fall under this threshold x_t^*, and b) importantly also entails to run the generator, such that the final charge equals x_t^*, if possible. Conditional on having established the optimal threshold x_t^*, the optimal generator policy to determine the generator quantities are:

$$\begin{aligned}
G_{H_t}^* &= \min[(D_H - x_{t-1} - q_t)^+ + (D_L + x_t^* - G - \min[x_{t-1} - (D_H - q_t)^+ + e(q_t - D_H)^+, K]^+)^+/e, G]^+, \\
G_{L_t}^* &= \min[D_L + x_t^* - \min[x_{t-1} + e(q_t + G_{Ht} - D_H)^+ - (D_H - q_t - G_{H_t})^+, K]^+, G]^+
\end{aligned} \tag{73}$$

A.7.3 Step 3: Lower Bound the Optimal Threshold

While we are not able to characterize the value of x_t^* in closed form, we now provide a way to lower-bound it.

As mentioned earlier, x_t^* is the amount of charge at which the cost of running the generator to charge the battery g/e is equal to the marginal value of a unit of charge in storage. This value is difficult to calculate as it depends on all future $t+1, ..., T$ periods, which includes $T-t$ random generation realizations. However, we can lower-bound the value of a charge by looking at the value that charge has for a shorter horizon - to start we will investigate the marginal value of storage

[9] Imagine an α close to 1 and a large solar capacity Q, so that not meeting demand incurs a small penalty relative to the generator cost and it is additionally unlikely to not fully meet demand through solar generation.

when only looking one period ahead[10]. Looking one period ahead, the marginal value of storage is equal to the probability of helping meet otherwise unmet demand. Let c^1 denote the marginal value of the x_t unit of storage given a one-period-look-ahead. We explicitly include g from here on again (i.e. stop assuming $g = 1$ as before), to emphasize the back-up cost relation of this decision. However, because the cost of running the generator in c^1 and the expected avoided cost of the future both scale in generator cost, this term cancels out for the optimal policy. Hence, the optimal policy, conditional on a given renewable and storage capacity Q and K is only dependent on the efficiency e and the penalty for unmet demand α.

$$c^1 = -\frac{\partial E_{q_{t+1}}[v_{t+1}(x_t, q_{t+1})]}{\partial x_t} = \alpha g \left(\frac{\frac{(D_L - G - x_t)^+}{e} + (D_H - G - (x_t - D_L + G)^+)^+}{Qe} \right)^+ . \tag{74}$$

The numerator in Equation 74 captures the amount of demand that cannot be met through the generator or storage at night $(D_L - G - x_t)^+/e$ after accounting for the existing storage charge x_t and the efficiency loss if demand at night is met through solar generation. To this quantity, we add the amount of energy not met throughout the day $(D_H - G - (x_t - D_L + G)^+)^+$ and then divide this quantity by solar capacity adjusted by efficiency Qe to obtain the probability that the marginal storage unit x_t was used. Clearly, this probability is decreasing in charge, available solar capacity and backup generator capacity.

Let x_t^1 denote the lower bound of x_t^* that is found by using this one-period lower bound of storage value. Set x_t^1, so that $c_1 = g$, as running the generator directly meets demand in the sub-period and saves one unit of charge in storage. This is a lower bound, as the one-period look ahead is a lower bound of the storage value, so $x_t^1 \leq x_t^*$. This can be done in closed-form, given knowledge of the magnitude of several parameters, in particular the penalty of unmet demand α and the storage efficiency e. We state the optimal solutions stratified by case below and note that x_t^* cannot exceed $K - D_L + G$ as a higher charge cannot be achieved at the end of period.:

$$x_t^1 = \begin{cases} \min\left[\left(D_L + D_H e - (1+e)G - \frac{Qe^2}{a}\right)^+, K - D_L + G\right], \text{if } a \leq \frac{Qe}{D_H - G} \\ \min\left[\left(D_L + D_H - 2G - \frac{Qe}{\alpha}\right)^+, K - D_L + G\right], \text{if } a > \frac{Qe}{D_H - G} \end{cases} \tag{75}$$

A.7.4 Step 4 - Comparative Statics of x_t^*

Even though we cannot characterize the optimal policy x_t^* in closed-form, we can use comparative statics, to further understand the dynamics around this decision. We first re-state and briefly re-organize the condition for x_t^*.

[10]This approach is analogous to how we approximated the storage value in the partial-discharge model by looking at two periods.

$$
\begin{aligned}
x_t^* :=& \frac{\partial - E[v_{t+1}(x_t, q_{t+1})]}{\partial x_t} = g/e \\
& \frac{\partial E\Big[c(x_t, q_{t+1}, x_t) + v_{t+2}(x_{t+1}, q_{t+2})\Big]}{\partial x_t} = -g/e \\
& \frac{\partial E\Big[gG_{Ht+1} + gG_{Lt+1} + \alpha g(D_H - G_{Ht+1} - ex_t - q_{t+1})^+\Big]}{\partial x_t} + \\
& \frac{\partial E\Big[\alpha g\Big(D_L - G_{Lt+1} - \min\Big[\Big(x_t + e(-D_H + G_{Ht+1} + q_{t+1})^+ - (D_H - G_{Ht+1} - q_{t+1})^+\Big)^+, K\Big]\Big)^+ . \Big]}{\partial x_t} + \\
& \frac{\partial E\Big[v_{t+2}(x_{t+1}, q_{t+2})\Big]}{\partial x_t} = -g/e
\end{aligned}
\tag{76}
$$

As next period's charge is weakly increasing in this periods's charge $\partial x_t / \partial x_{t-1} \geq 0$, any impact of x_t^* on next period's cost $c(x_t, q_{t+1}, x_t)$, will directionally be the same as the impact on $v_{t+2}(x_{t+1}, q_{t+2})$.

$$
\text{As } \frac{\partial x_{t+1}}{\partial x_t} \geq 0 \rightarrow \frac{\partial c(x_t, q_{t+1}, x_t)}{\partial x_t} \frac{\partial v_{t+2}(x_{t+1}, q_{t+2})}{\partial x_t} \geq 0 \tag{77}
$$

For comparative statics, it is thus sufficient to verify the impact of any variable change on $\partial c(x_t, q_{t+1}, x_t)/\partial x_t$, which we will subsequently investigate. The optimal x_t^* directionally behaves the same as when checking the condition in Equation 78.

$$
\begin{aligned}
-\frac{g}{e} =& \frac{\partial E\Big[c(x_t, q_{t+1}, x_t)\Big]}{\partial x_t}, \\
\frac{-1}{e} =& \alpha \frac{\partial E\Big[(D_H - G_{Ht+1} - ex_t - q_{t+1})^+\Big]}{\partial x_t} + \\
& \alpha \frac{\partial E\Big[\Big(D_L - G_{Lt+1} - \min\Big[\Big(x_t + e(-D_H + G_{Ht+1} + q_{t+1})^+ - (D_H - G_{Ht+1} - q_{t+1})^+\Big)^+, K\Big]\Big)^+ . \Big]}{\partial x_t}.
\end{aligned}
\tag{78}
$$

Exogenous Choice of Q and K. We begin analyzing the comparative statics for a given choice of capacity, i.e. Q and K are fix. Later, we will lift this assumption and investigate numerically how the comparative statics look if Q and K are endogenous.

Impact of α

$$\frac{\partial -1/e}{\partial \alpha} = 0, \qquad \text{and as } \frac{\partial E\Big[c(x_t, q_{t+1}, x_t)\Big]}{\partial x_t} < 0 \rightarrow \quad \frac{\partial^2 E\Big[c(x_t, q_{t+1}, x_t)\Big]}{\partial x_t \partial \alpha} < 0$$
$$\rightarrow \frac{\partial x_t^*}{\partial \alpha} \geq 0. \tag{79}$$

Clearly, increasing α, the penalty for not meeting demand, increases the amount of cost saved by an additional unit of storage charge. Thus, increasing α increases the optimal buffer charge x_t^*.

Impact of e

$$\begin{aligned}
&-1 = e\,\frac{\partial E\Big[c(x_t, q_{t+1}, x_t)\Big]}{\partial x_t}, \\
&e\,\frac{\partial E\Big[c(x_t, q_{t+1}, x_t)\Big]}{\partial x_t} = -e^2 Pr[D_H - G_{Ht+1} - ex_t - q_{t+1} > 0] - \\
&ePr[(x_t + e(-D_H + G_{Ht+1} + q_{t+1})^+ - (D_H - G_{Ht+1} - q_{t+1})^+)^+ < K \wedge \\
&\Big(D_L - G_{Lt+1} - \min\Big[\Big(x_t + e(-D_H + G_{Ht+1} + q_{t+1})^+ - (D_H - G_{Ht+1} - q_{t+1})^+\Big)^+, K\Big]\Big)^+ > 0].
\end{aligned} \tag{80}$$

While the partial of the LHS does not change as efficiency changes ($\partial - 1/\partial e = 0$), the impact of changing efficiency on the RHS cannot generally be signed, so we need to add the following condition to evaluate the comparative statics.

$$\begin{aligned}
&\text{Let } Pr(A) := Pr\Big[D_H - G_{Ht+1} - ex_t - q_{t+1} > 0\Big], \\
&\text{Let } Pr(B) := Pr\Big[(x_t + e(-D_H + G_{Ht+1} + q_{t+1})^+ - (D_H - G_{Ht+1} - q_{t+1})^+)^+ < K \wedge \\
&\Big(D_L - G_{Lt+1} - \min\Big[\Big(x_t + e(-D_H + G_{Ht+1} + q_{t+1})^+ - (D_H - G_{Ht+1} - q_{t+1})^+\Big)^+, K\Big]\Big)^+ > 0\Big], \\
&\frac{\partial^2 eE\Big[c(x_t, q_{t+1}, x_t)\Big]}{\partial x_t \partial e} = -e^2 \frac{\partial Pr(A)}{\partial e} - 2ePr(A) - e\frac{\partial Pr(B)}{\partial e} - Pr(B), \\
&\text{If } -e^2 \frac{\partial Pr(A)}{\partial e} - 2ePr(A) - e\frac{\partial Pr(B)}{\partial e} - Pr(B) > 0 \quad \rightarrow \quad \frac{\partial x_t^*}{\partial e} \leq 0, \quad \text{else} \quad \frac{\partial x_t^*}{\partial e} < 0.
\end{aligned} \tag{81}$$

Based on the condition established above, x_t^* is either increasing or decreasing in efficiency e. This trade-off occurs as more efficiency reduces the cost of serving energy through storage, but also reduces the demand to be met by storage in the future as the same renewable power covers more demand than before.

Impact of g Lastly, $\partial x_t^*/\partial g = 0$ as mentioned before. With given capacities, the magnitude of the back-up cost is not relevant as one trades off running the generator now versus running it later.

Endogenous Choice of Q and K. This brings us to the second set of comparative statics - how does x_t^* change as one allows renewable and storage capacity Q and K to change as well in response to varying parameters. In the regular capacitated generator model we assume the capacities to be given. An analytical derivation of the comparative statics while endogenizing the storage and solar capacity is intractable. Consider for example, what happens if c_Q is decreased, which would increase Q. First, we would need to know, whether the rest of the parameters were such that storage and solar are strategic complements or strategic substitute. Assuming they are complements, an increase in Q would increase K as well. However, the effect of increasing storage and increasing solar on x_t^* is in different directions: $\partial x_t^*/\partial Q < 0$ and $\partial x_t^*/\partial K > 0$. More generation reduces x_t^* as there is less unmet demand in the future, while an increase in storage increases x_t^*, as more energy can be stored while excess renewables get lost less often.

We thus turn to investigate these numerically. We run each island's data, for generator capacities between 10% and 95% of daily demand, for both storage technologies, and with α values between 1.5 and 15. Based on these specifications, we tested the impact of α, e, c_K, c_Q, and g on x_t^*. We investigate over 1,300 scenarios (combination of parameter values) for each of the following results. We start with the monotone findings:

Impact of α

$$\frac{\partial x_t^*}{\partial \alpha} \geq 0. \tag{82}$$

We were able to confirm the previous analytical insight. Raising α weakly increases investment in generation and storage as well as the optimal buffer charge x_t^* as not meeting demand becomes more expensive.

Impact of c_Q

$$\frac{\partial x_t^*}{\partial c_Q} \geq 0. \tag{83}$$

Increasing solar cost reduces solar capacity, which can increase or decrease storage capacity and thus may have varying direct and indirect effects on x_t^*. Yet, in all runs we performed, the optimal buffer charge x_t^* increased as solar costs increased. The buffer charge is a hedge against the increased likelihood of not having sufficient generation in the coming periods.

Impact of c_K

$$\frac{\partial x_t^*}{\partial c_K} \leq 0. \tag{84}$$

Increasing storage cost reduces storage capacity, and in most cases increases generation capacity. In all runs we performed, the optimal buffer charge x_t^* decreased. Typically, storage capacity is not binding for x_t^*, so a higher c_K lowers K, but rather although every periods now has slightly less storage capacity, every period now also has more generation than before, thus overall reducing the need for buffer capacity.

Impact of e

$$\frac{\partial x_t^*}{\partial e} \geq 0 \text{ for most parameter combinations }, \ \frac{\partial x_t^*}{\partial e} \leq 0 \text{ if } e \lesssim 1 \wedge G \lesssim D_H. \tag{85}$$

We use $\lesssim$ to indicate the LHS being close but not exactly equal to the RHS. Increasing efficiency increases storage capacity and increases or decreases generation capacity. For almost all the runs, increasing efficiency increased the optimal buffer charge x_t^*, but for scenarios with high efficiency values close to 1 and high back-up capacity generators, the optimal buffer charge did decrease. More efficiency means more (effective) generation as less energy is lost when charging, thus making future unmet demand less likely. In most cases this is more than compensated by the fact that an increase in efficiency makes serving demand through storage cheaper.

Impact of g

$$\frac{\partial x_t^*}{\partial g} \geq 0 \text{ if } g \text{ is small and } \frac{\partial x_t^*}{\partial g} \leq 0 \text{if } g \text{ is large.} \tag{86}$$

If g is small, a marginal increase increases capacities (in both generation and storage) and x_t^* increases as not meeting demand is costly. At some point, if g becomes large enough, running the generator becomes so costly that x_t^* starts going down, while especially generation investment goes up to multiples of needed demand.

Appendix B Detailed Derivations and Analysis

B.1 Getting from the Tracking Model to the Partial-Discharge Model

We will proceed in 3 steps. First, we write down the tracking objective function. Second, we introduce the two changes required to transition from the tracking model to the partial model. Third, we prove that after applying the two changes from step 2, the tracking model is equivalent to the partial-discharge model.

Step 1. As introduced in section 3.1, the tracking model's objective function is:

$$
\begin{aligned}
&\Pi_{TR}(Q,K) = \\
&g\ E\Big[\sum_{t=1}^{T}\big(\min[q_t, D_H] + \min[x^{TR}_{t-1}, (D_H - q_t)^+] + \min[(x^{TR}_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, K, D_L]\big)\Big] - \\
&T\frac{c_K}{e}K - Tc_Q Q. \\
&\text{We assume that } K \geq D_L: \\
&\Pi_{TR}(Q,K) = \\
&g\ E\Big[\sum_{t=1}^{T}\big(\min[q_t, D_H] + \min[\ x^{TR}_{t-1}, (D_H - q_t)^+] + \min[(x^{TR}_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, D_L]\big)\Big] - \\
&T\frac{c_K}{e}K - Tc_Q Q, \\
&\text{where } x^{TR}_t = (\min[(x^{TR}_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, K] - D_L)^+.
\end{aligned}
\tag{87}
$$

Clearly, this objective function exhibits the Markov property, as at time t, the future states are only dependent on the state variable/current charge x^{TR}_t and future generation realizations. Furthermore, profit and x^{TR}_{t+1} are both weakly increasing in x^{TR}_t.

Step 2. We now introduce the two changes required to get from the tracking model to the partial-discharge model, in which storage cannot be carried further than 48 hours and most-recently generated energy is used first (Assumptions 2 and 3). One change is adapting the charge terms as follows to reflect these assumptions:

$$
\begin{aligned}
x^{TR}_t &= (\min[(x^{TR}_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, K] - D_L)^+. \\
x^{P}_t &= (\min[e(q_t - D_H)^+, K] - D_L)^+.
\end{aligned}
\tag{88}
$$

Note that in the partial-discharge model, storage at the end of period t, x^P_t, is not a function of storage at the start of period t, x^P_{t-1}, because (i) the energy at the beginning of the period was stored in period $t-1$ during daytime and therefore it expires before the end of period t, and (ii) the energy at the beginning of the period is used to serve nightly demand only when all other energy is depleted. Note also that:

Lemma 2. *For any vector of generation realizations $\vec{q} = [q_1, q_2, ..., q_T]$, the charge in the tracking model is always larger than in the partial-discharge model. Formally, $x^{TR}_t \geq x^P_t\ \forall t$.*

In addition, the second change we introduce is the following simplification to the objective function that, in expectation, weakly decreases the charging terms (hence profit) and improves tractability. We replace:

$$\begin{aligned}
&\min[x^P_{t-1}, (D_H - q_t)^+] + \min[(x^P_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, D_L], \\
\text{with} \quad & \\
&\min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, x^P_{t-1}].
\end{aligned} \tag{89}$$

Specifically, if $q_t \geq D_H$, these two expressions are identical in expectation:

$$\begin{aligned}
&E[\min[x^P_{t-1}, (D_H - q_t)^+] + \min[(x^P_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, D_L]], \\
&= E[\min[(x^P_{t-1} + e(q_t - D_H))^+, D_L]], \\
&= E[\min[x^P_{t-1}, (D_L - e(q_t - D_H))^+] + \min[e(q_t - D_H), D_L]], \\
&= E[\min[e(q_t - D_H), D_L] + \min[(eD_H + D_L - eq_t)^+, x^P_{t-1}]].
\end{aligned} \tag{90}$$

If $q_t < D_H$, the simplification is weakly lower than the original expression, as it reduces one of the values of the minima, but in doing so increases tractability.

$$\begin{aligned}
&E[\min[x^P_{t-1}, (D_H - q_t)^+] + \min[(x^P_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, D_L]], \\
&= E[\min[x^P_{t-1}, (D_H - q_t)^+] + \min[(x^P_{t-1} - (D_H - q_t)^+)^+, D_L]], \\
&= E[\min[D_H + D_L - q_t)^+, x^P_{t-1}]], \\
&> E[\min[(eD_H + D_L - eq_t)^+, x^P_{t-1}]], \\
&= E[\min[0, D_L] + \min[(eD_H + D_L - eq_t)^+, x^P_{t-1}]], \\
&= E[\min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, x^P_{t-1}]].
\end{aligned} \tag{91}$$

In conjunction, making those two changes to the tracking model (different expression for x_t and the objective function simplification) leaves us with the following objective function that we denote C for candidate:

$$\begin{aligned}
&\Pi_C(Q, K) = \\
&g\ E\Big[\sum_{t=1}^{T}\Big(\min[q_t, D_H] + \min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, x^P_{t-1}]\Big)\Big] - T\frac{c_K}{e}K - Tc_Q Q,
\end{aligned}$$

where $x^P_t = (\min[e(q_t - D_H)^+, K] - D_L)^+$. Plugging this into the function results in:

$$\begin{aligned}
&g\ E\Big[\sum_{t=1}^{T}\Big(\min[q_t, D_H] + \min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, (\min[e(q_t - D_H)^+, K] - D_L)^+]\Big)\Big], \\
&- T\frac{c_K}{e}K - Tc_Q Q,
\end{aligned} \tag{92}$$

We will aim to show that this candidate objective function is equivalent to the partial-discharge objective function. For now, notice that the candidate objective function has removed the intertemporal linkages between the periods as profit in period t only depends on the realization of q_t and

q_{t-1}, but no state variable anymore. This sum over T periods is thus, on average, equivalent to multiplying this one period T times.

In the next step, we will introduce a new expression (denoted with $\pi(Q,K)$) that represents the expected quantity of renewable electricity sold from generation and storage during one period t with capacities Q and K, so we can compare the candidate revenue with the partial-discharge revenue. If the revenue is identical for each period, then the sum of revenues is also identical. We abstract from the cost as they are the same in the all considered models. As introduced above, the candidate model is independent of a state variable.

$$\pi_C(Q,K) =$$
$$g\ E\Big[\min[q_t, D_H] + \min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, (\min[e(q_t - D_H)^+, K] - D_L)^+]\Big]. \tag{93}$$

Step 3. We now want to show that the candidate objective function (the tracking model in combination with the adjusted storage term and Assumptions 2 and 3), is equivalent to our partial-discharge model in expectation (see Appendix A.5).

$$\begin{aligned}
&g\ E\Big[\min[q_t, D_H] + \min[e(q_t - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, (\min[e(q_t - D_H)^+, K] - D_L)^+]\Big] \\
=&\frac{g}{Q}\Big(\Big[\int_0^{D_H} q\ dq\Big] + \Big[\int_{D_H}^{\frac{D_L}{e}+D_H} D_H + e(q - D_H)\ dq\Big] + \Big[\int_{\frac{D_L}{e}+D_H}^{Q} D_L + D_H\ dq\Big] + \\
&\frac{1}{Q}\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, K - D_L, (eD_H + D_L - eq_2)^+]\ dq dq_2\Big).
\end{aligned} \tag{94}$$

As explained in Section 3.2.3, the tracking model, and by extension the candidate model, use a supply perspective while the partial-discharge model uses a demand perspective. So in the candidate model, revenue comes from selling today's generation and yesterday's stored excess generation, while in the partial-discharge model it is today's generation and today's excess generation (multiplied by the probability of discharge). To make the models comparable, we change the candidate's generation terms (the first two minima) from q_t to q_{t-1}. As each period has the same generation distribution, this does not change the value of the objective function.

$$\begin{aligned}
&g\ E\Big[\min[q_{t-1}, D_H] + \min[e(q_{t-1} - D_H)^+, D_L] + \min[(eD_H + D_L - eq_t)^+, (\min[e(q_{t-1} - D_H)^+, K] - D_L)^+]\Big] \\
=&\frac{g}{Q}\Big(\Big[\int_0^{D_H} q\ dq\Big] + \Big[\int_{D_H}^{\frac{D_L}{e}+D_H} D_H + e(q - D_H)\ dq\Big] + \Big[\int_{\frac{D_L}{e}+D_H}^{Q} D_L + D_H\ dq\Big] + \\
&\frac{1}{Q}\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, K - D_L, (eD_H + D_L - eq_2)^+]\ dq dq_2\Big).
\end{aligned} \tag{95}$$

Note that $E[\min[q_{t-1}, D_H]] = \frac{1}{Q}\left(\left[\int_0^{D_H} q\ dq\right] + \left[\int_{D_H}^{\frac{D_L}{e}+D_H} D_H\ dq\right] + \left[\int_{\frac{D_L}{e}+D_H}^{Q} D_H\ dq\right]\right)$.
Also note that $E[e\min[e(q_{t-1}-D_H)^+, D_L] = \frac{1}{Q}\left(\left[\int_{D_H}^{\frac{D_L}{e}+D_H} e(q-D_H)\ dq\right] + \left[\int_{\frac{D_L}{e}+D_H}^{Q} D_L\ dq\right]\right)$.

Which leaves us with:

$$\begin{aligned}&E\left[\min[(eD_H + D_L - eq_t)^+, (\min[e(q_{t-1} - D_H)^+, K] - D_L)^+]\right] = \\ &\frac{1}{Q^2}(\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, K - D_L, (eD_H + D_L - eq_2)^+]\ dq dq_2).\end{aligned} \tag{96}$$

Here q_{t-1} equals q and q_t equals q_2 and the expressions are equivalent. We have thus shown how one can get from the tracking model to the partial-discharge model by changing the charging term definition (x_t) as well as making a small simplification for tractability.

For other derivations, it is useful to also define the expected quantity of renewable electricity sold from generation and storage during one period t with capacities Q and K for the tracking model, which additionally requires to specify the starting charge x_{t-1}.

$$\begin{aligned}&\pi_{TR}(Q, K, x_{t-1}^{TR}) = \\ &g\ E\Big[\min[q_t, D_H] + \min[x_{t-1}^{TR}, (D_H - q_t)^+] + \min[(x_{t-1}^{TR} + e(q_t - D_H)^+ - (D_H - q_t)^+)^+, D_L]\Big].\end{aligned} \tag{97}$$

B.2 Concavity of Tracking Model

In this section, we study the tracking model in more detail and proof global concavity of the objective function for a subset of parameters. We also provide results regarding the distribution of the end-of-storage charge x_t. Remember that in the revenue function of the tracking model all periods are linked through x_t, which simultaneously impacts every period's revenue.

$$\begin{aligned}\Pi_{Rev}(Q, K) =& E\Big[\sum_{t=1}^{T}\Big(\min[q_t, D_H] + \min[x_{t-1}, (D_H - q_t)^+] + \\ &\min\left[\left(x_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+\right)^+, K, D_L\right]\Big)\Big], \\ &x_t = \left(\min\left[\left(x_{t-1} + e(q_t - D_H)^+ - (D_H - q_t)^+\right)^+, K\right] - D_L\right)^+.\end{aligned} \tag{98}$$

In order to analyze the concavity of the tracking model (costs are linear in parameters so we only focus on the revenue), we aim to study the Hessian of the objective function, for which we need to know not just the sign, but also the magnitude of all second partial derivatives.

As the storage charge is contained in all derivatives, one cannot solve the derivatives of the tracking model without knowing the distribution of x_t. For that reason, we will use a Markov process approach to capture the probability to end the period with a certain storage charge x_t. Once we have the limiting distribution of that Markov process, we will be able to express the per-period objective function of the tracking model in closed form for the case, when $Q > D_H + K/e$ and $D_L \leq K < 2D_L + D_H$ and $e = 1$. In this parameter space for Q and K, there is positive probability in each period to fully charge or discharge the storage. We will then prove concavity of the tracking model in that case.

B.2.1 Stationary Distribution of x_t - General Case

We use $P(l)$ to indicate the probability of ending the day with charge $l \in [0, K - D_L]$ and put particular emphasis on the special cases of $P(0)$ and $P(K - D_L)$, i.e. ending the day with an empty charge and the highest possible charge. Note that, because nightly demand is always being served, it is not possible to end with any charge higher than $K - D_L$. We begin by writing the transition probabilities between the different states in Table 5.

From (x_{t-1}) \ To (x_t)	0	l	$K - D_L$,
0	$\frac{D_L/e+D_H}{Q}$	$\frac{1}{Qe}$	$1 - \frac{D_H+K/e}{Q}$
l	$\frac{D_L/e+D_H-l}{Q}$	$\begin{cases} \frac{1}{Q} \text{ if } l_{To} \in [0, l] \\ \frac{1}{Qe} \text{ if } l_{To} \in [l, K - D_L] \end{cases}$	$1 - \frac{D_H+(K-L)/e}{Q}$
$K - D_L$	$\frac{D_L(1+1/e)+D_H-K}{Q}$	$\frac{1}{Q}$	$1 - \frac{D_H+D_L/e}{Q}$

Table 5: Transition Probabilities of storage charge x_{t-1} to x_t

We can write them formally into the following system of equations relating the transition probabilities:

$$
\begin{aligned}
P(0) &= \frac{\frac{D_L}{e}+D_H}{Q}P(0)+\frac{D_L(1+\frac{1}{e})+D_H-K}{Q}P(K-D_L)+\int_{\epsilon}^{K-D_L-\epsilon}\frac{D_L/e+D_H-x}{Q}P(x)dx,\\
P(l) &= \frac{1}{Qe}P(0)+\frac{1}{Q}P(K-D_L)+\int_{\epsilon}^{l-\epsilon}\frac{1}{Qe}P(x)dx+\int_{l}^{K-D_L-\epsilon}\frac{1}{Q}P(x)dx,\\
P(K-D_L) &= 1-\frac{D_H+\frac{K}{e}}{Q}P(0)+1-\frac{D_H+\frac{D_L}{e}}{Q}P(K-D_L)+\int_{\epsilon}^{K-D_L-\epsilon}1-\frac{D_H+(K-x)/e}{Q}P(x)dx,
\end{aligned}
$$

Simplifying:

$$
\begin{aligned}
P(0) &= \frac{D_L/e+D_H}{Q}-\frac{1}{Q}\int_0^{K-D_L}xP(x)dx,\\
P(l) &= \frac{1}{Q}+\frac{1-e}{Qe}\int_0^{l-\epsilon}P(x)dx,\\
P(K-D_L) &= 1-\frac{D_H+K/e}{Q}+\frac{1}{Qe}\int_0^{K-D_L}xP(x)dx.
\end{aligned}
\tag{99}
$$

In this general case, it is not possible to derive the limiting distribution of the states in closed form based on these one-step transition equations. In the following, we assume $e = 1$, which results in $P(l) = \frac{1}{Q},\ \forall\, l \in (0, K)$, which allows us to solve for the stationary distribution.

B.2.2 Stationary Distribution with $e = 1$

We write down the same distributions, but for the case of efficiency equal to 1:

$$
\begin{aligned}
P(0) &= \frac{D_L+D_H}{Q}-\frac{1}{Q}\int_0^{K-D_L}xP(x)dx,\\
P(l) &= \frac{1}{Q},\\
P(K-D_L) &= 1-\frac{D_H+K}{Q}+\frac{1}{Q}\int_0^{K-D_L}xP(x)dx,
\end{aligned}
$$

From here we can simplify $P(0)$ and $P(K-D_L)$ to:

$$
\begin{aligned}
P(K-D_L) &= \Big[1-\frac{D_H+K}{Q}+\frac{(K-D_L)^2}{2Q^2}\Big]\Big/\Big[1-\frac{K-D_L}{Q}\Big],\\
P(0) &= \frac{D_L+D_H}{Q}-\frac{(K-D_L)^2}{2Q^2}-\Big[\frac{K-D_L}{Q}\Big]\Big/\Big[1-\frac{K-D_L}{Q}\Big]\Big[1-\frac{D_H+K}{Q}+\frac{(K-D_L)^2}{2Q^2}\Big].
\end{aligned}
\tag{100}
$$

B.2.3 Derivative of x_t w.r.t. storage decision K

From here, we can start tackling the derivative w.r.t. K - i.e., what is the effect that more storage capacity would have on the steady-state distribution of the storage charge?

$$\frac{\partial x_t}{\partial K} = \begin{cases} 1 & \text{if } \max\{f|f \leq t, x_f = K - D_L\} > \max\{h|h \leq t, x_h = 0\}, \\ 0 & \text{otherwise .} \end{cases} \tag{101}$$

Trivially, if $x_t = K - D_L$, then storage capacity at the end of the day-sub-period was binding and more storage would have increased the charge. If $x_t = 0$, more storage capacity would not have increased the charge. The more intricate behavior occurs, if the charge is between the two bounds, i.e. $x_t \in (0, K - D_L)$ at the end of period. In that case, more storage capacity has an effect on the storage charge in period x_t if the last period beforehand during which storage was fully charged has been more recent than the last period during which storage was entirely empty $\max\{f|f \leq t, x_f = K - D_L\} > \max\{h|h \leq t, x_h = 0\}$ (our definition from Equation 101).

We thus look for a general way to express the probability above $\forall x_t \in (0, K - D_L)$. Note that, no matter what the storage charge in the last period was, there is always exactly one generation realization that gets us to a focal charge state $l, l \in (0, K - D_L)$ as can be seen by inspecting Equation 102.

$$\begin{aligned} x_t(e = 1) &= \left(\min\left[\left(x_{t-1} + q_t - D_H\right)^+, K\right] - D_L\right)^+, \\ \rightarrow Pr[x_t = l \in (0, K - D_L)|x_{t-1}]] &= \frac{1}{Q}, \forall x_{t-1}. \end{aligned} \tag{102}$$

Conditional on arriving at particular charge $x_t = l \in (0, K - D_L)$, the probability of having been at a particular charge x_{t-1} in the previous period reduces to the stationary probabilities.

$$Pr[x_{t-1}|x_t = l \in (0, K - D_L)] = \frac{Pr[x_t = l \in (0, K - D_L)|x_{t-1}]Pr[x_{t-1}]}{Pr[x_t = l \in (0, K - D_L)]} = \frac{Pr[x_{t-1}]}{QPr[x_t = l \in (0, K - D_L)]}$$

Leading to the following cases:

$$\begin{aligned} \frac{Pr[x_{t-1} = 0]}{QPr[x_t = l \in (0, K - D_L)]} &= \frac{P(0)]}{QPr[x_t = l \in (0, K - D_L)]} = P(0), \\ \frac{Pr[x_{t-1} \in (0, K - D_L)]}{QPr[x_t = l \in (0, K - D_L)]} &= \frac{Pr(l)}{QPr[x_t = l \in (0, K - D_L)]} = P(l), \\ \frac{Pr[x_{t-1} = K - D_L]}{QPr[x_t = l \in (0, K - D_L)]} &= \frac{P(K - D_L)]}{QPr[x_t = l \in (0, K - D_L)]} = P(K - D_L). \end{aligned} \tag{103}$$

Because the probability of getting to a charge $l \in (0, K - D_L)$ is uniformly $\frac{1}{Q}$ for all prior states (i.e. last period's charges), conditional on being in state l at time t, the probability of having been in a particular state at time $t - 1$ is equal to the long-term transition probabilities. As $\frac{\partial x_t}{\partial K}$ is only non-zero if the storage charge has hit the upper charge limit $K - D_L$ in the past first, we can calculate that probability as the following recursive sum:

$$
\begin{aligned}
\frac{\partial x_t = l \in (0, K-D_L)}{\partial K} &= Pr[\max\{f|f \leq t, x_f = K-D_L\} > \max\{h|h \leq t, x_h = 0\}], \\
&= \frac{Pr[x_{t-1} = K-D_L]}{QPr[x_t = l]} + \frac{Pr[x_{t-1} \in (0, K-D_L)]}{QPr[x_t = l]} \frac{Pr[x_{t-2} = K-D_L]}{QPr[x_{t-1} \in (0, K-D_L)]} + ..., \\
&= P(K-D_L) + \frac{K-D_L}{Q} P(K-D_L) + \frac{(K-D_L)^2}{Q^2} P(K-D_L) + ..., \\
&= P(K-D_L) + P(K-D_L) \sum_{1}^{t-1} \frac{K-D_L}{Q}, \\
&\lim_{t\to\infty} P(K-D_L) + P(K-D_L) \sum_{1}^{t-1} \frac{K-D_L}{Q} = P(K-D_L)\Big[1 + \frac{(K-D_L)/Q}{1-\frac{K-D_L}{Q}}\Big], \\
&= \frac{D_L^2 - 2D_L K + K^2 - 2KQ + 2Q(Q-D_H)}{2(Q+D_L-K)^2}.
\end{aligned}
\tag{104}
$$

Thus, we now have the stationary distribution of charging states and know $\frac{\partial x_t}{\partial K}$ in the limit. Given that the lifetime of our technology is typically on the order of tens of thousands of days, this limit is very accurate.

B.2.4 Revenue Function and Derivatives

The revenue function of the tracking model with $e = 1$ is as follows:

$$
\begin{aligned}
\Pi_{Rev}(Q, K, e=1) =& E\Big[\sum_{t=1}^{T} \Big(\min[q_t, D_H] + \min[x_{t-1}, (D_H - q_t)^+] + \min\left[(x_{t-1} + q_t - D_H)^+, K, D_L\right] \Big)\Big] \\
x_t =& \Big(\min\Big[\Big(x_{t-1} q_t - D_H\Big)^+, K\Big] - D_L\Big)^+,
\end{aligned}
$$

We focus on an individual period t:

$$
\Pi_{Per}(Q, K, e=1) = E\Big[\min[q_t, D_H] + \min[x_{t-1}, (D_H - q_t)^+] + \min\left[(x_{t-1} + q_t - D_H)^+, K, D_L\right]\Big].
\tag{105}
$$

$$
\begin{aligned}
\frac{\partial \Pi_{Per}(Q, K)}{\partial K} =& \frac{1}{Q}\int_0^Q Pr[x_{t-1} < D_H - q_t] E\Big[\frac{\partial x_{t-1} | x_{t-1} < D_H - q_t}{\partial K}\Big] dq_t + \\
& \frac{1}{Q}\int_0^Q Pr[D_H < x_{t-1} \leq D_H + D_L - q_t] E\Big[\frac{\partial x_{t-1} | D_H < x_{t-1} \leq D_H + D_L - q_t}{\partial K}\Big] dq_t, \\
=& \frac{1}{Q}\int_0^Q Pr[x_{t-1} \leq D_H + D_L - q_t] E\Big[\frac{\partial x_{t-1} | x_{t-1} \leq D_H + D_L - q_t}{\partial K}\Big] dq_t, \\
=& \int_{l=0}^{K-D_L} Pr[x_{t-1} = l] Pr[q_t \leq D_H + D_L - l] \frac{\partial x_{t-1} = l}{\partial K} dl.
\end{aligned}
\tag{106}
$$

More storage capacity has a positive impact on revenue in period t, iff the combination of previous period's charge x_{t-1} and current periods generation q_t is insufficient to meet demand and if additionally, more storage capacity would have led to a larger charge in x_{t-1}, i.e., the condition from Equation 101 is met for x_{t-1}.

$$
\begin{aligned}
&\int_{l=0}^{K-D_L} Pr[x_{t-1}=l]Pr[q_t \leq D_H+D_L-l]\frac{\partial x_{t-1}=l}{\partial K}dl,\\
=&P(0)\frac{D_H+D_L}{Q}0+P(K-D_L)\frac{D_H+2D_L-K}{Q}1+\\
&\int_{l=\epsilon}^{K-D_L-\epsilon} Pr[x_{t-1}=l]Pr[q_t \leq D_H+D_L-l]\frac{\partial x_{t-1}=l}{\partial K}dl,\\
=&\Big[1-\frac{D_H+K}{Q}+\frac{(K-D_L)^2}{2Q^2}\Big]/\Big[1-\frac{K-D_L}{Q}\Big]\frac{D_H+2D_L-K}{Q}+\\
&\frac{1}{Q}\frac{D_L^2-2D_LK+K^2-2KQ+2Q(Q-D_H)}{2(Q+D_L-K)^2}\frac{(K-D_L)(D_H+1.5D_L-K/2)}{Q},\\
=&\frac{\left((D_L-K)^2+2(D_H+2D_L-K)Q\right)\left((D_L-K)^2-2(D_H+K)Q+2Q^2\right)}{4Q^2(D_L-K+Q)^2}=\frac{\partial\Pi_{Per}(Q,K)}{\partial K},\\
&\frac{\partial\Pi_{Rev}(Q,K)}{\partial K}=T\frac{\partial\Pi_{Per}(Q,K)}{\partial K}.
\end{aligned}
\tag{107}
$$

From here, we can easily obtain the following two second partial derivatives:

$$
\begin{aligned}
&\frac{\partial^2\Pi_{Rev}(Q,K)}{\partial^2 K}=\\
&T\frac{-(K-D_L)^4+4(K-D_L)^3Q-2\left(2D_H^2+4D_HD_L+5D_L^2-6D_LK+3K^2\right)Q^2+4(D_H+K)Q^3-2Q^4}{2Q^2(D_L-K+Q)^3},\\
&\frac{\partial^2\Pi_{Rev}(Q,K)}{\partial K\partial Q}=T\frac{(K-D_L)^5-4(K-D_L)^4Q+6(K-D_L)^3Q^2}{2Q^3(D_L-K+Q)^3}+\\
&T\frac{2\left(2D_H^2+5D_HD_L+D_L^2-D_HK+3D_LK-2K^2\right)Q^3+2(D_H+2D_L-K)Q^4}{2Q^3(D_L-K+Q)^3}.
\end{aligned}
\tag{108}
$$

B.2.5 Derivative of x_t w.r.t. storage decision Q

We now turn back to the revenue function and investigate the first partial derivative w.r.t. to generation capacity Q. What makes this an intricate derivative to calculate is the charge's x_t dependence on generation capacity Q. An increase in generation capacity increases all past generation realizations and thus weakly increases x_t. We thus start by studying the impact of $\partial x_{t-1}/\partial Q$. One further complexity is that the marginal effect of adding generation capacity Q is different in each period based on the generation realization. $q_t' > q_t$", $\implies$ $\partial q_t'/\partial Q > \partial q_t$"$/\partial Q$. Conceptually, an additional solar panel increases energy generation more on a sunny day than on a cloudy day.

$$
\begin{aligned}
x_t =& \Big(\min \Big[\Big(x_{t-1} + q_t - D_H \Big)^+, K \Big] - D_L \Big)^+, \\
E\Big[\frac{\partial x_t}{\partial Q}\Big] =& E\Big[\sum_{l+1}^{t} \frac{q_t}{Q} \Big] = E\Big[\frac{1}{Q} \sum_{l+1}^{t} q_t, \quad l = \max\{l | l \leq t, x_l \in \{0, K - D_L\}\} \Big], \\
\frac{\partial x_t}{\partial Q} =& \begin{cases} 0 \text{ if } x_t = 0, \\ \frac{1}{Q} \sum_{l+1}^{t} q_t \text{ if } x_t \in (0, K - D_L), \\ 0 \text{ if } x_t = K - D_L, \end{cases} \\
E\Big[\frac{\partial x_t}{\partial Q}\Big] =& \int_{l=0}^{K-D_L} Pr(x_t = l) \frac{\partial x_t = l}{\partial Q} = 0 + 0 + \int_{l=\epsilon}^{K-D_L-\epsilon} Pr(x_t = l) \frac{\partial x_t = l}{\partial Q} = \frac{1}{Q} \int_{l=\epsilon}^{K-D_L-\epsilon} \frac{\partial x_t = l}{\partial Q}.
\end{aligned}
\tag{109}
$$

The impact on the charge in period t of raising generation capacity is that the charge increases proportional to all the generation realizations from the period after the charge last hit the upper capacity limit or was fully discharged. For that, we again need to resort to the conditional transition probabilities (see Equation 103).

$$
\begin{aligned}
&\frac{\partial x_t = l \in (0, K - D_L)}{\partial Q} = E\Big[q_t / Q\Big] + E\Big[\frac{\partial x_{t-1} | q_t}{\partial Q} \Big], \\
=& Pr(0) \frac{l + D_H + D_L}{Q} + 0 + Pr(K - D_L) \frac{2D_L + D_H - K + l}{Q} + 0 + \\
& \int_{j=\epsilon}^{K-D_L-\epsilon} Pr(j) \Big(\frac{D_H + D_L + l - j}{Q} + \frac{\partial x_{t-1} = j}{\partial Q} \Big) dj, \\
=& Pr(0) \frac{l + D_H + D_L}{Q} + Pr(K - D_L) \frac{2D_L + D_H - K + l}{Q} + \\
& \frac{1}{Q} \int_{j=\epsilon}^{K-D_L-\epsilon} \frac{D_H + D_L + l - j}{Q} dj + \frac{1}{Q} \int_{j=\epsilon}^{K-D_L-\epsilon} \frac{\partial x_{t-1} = j}{\partial Q} dj.
\end{aligned}
\tag{110}
$$

We now can go back to Equation 109 and integrate $\partial x_t = l \in (0, K - D_L) \ / \ \partial Q$ over all realizations of $l \in (0, K - D_L)$:

$$
\begin{aligned}
& E\Big[\frac{\partial x_t}{\partial Q}\Big] = \frac{1}{Q} \int_{l=\epsilon}^{K-D_L-\epsilon} \frac{\partial x_t = l}{\partial Q}, \\
& = Pr(0) \frac{1}{Q} \int_{l=\epsilon}^{K-D_L-\epsilon} \frac{l + D_H + D_L}{Q} dl + Pr(K - D_L) \frac{1}{Q} \int_{l=\epsilon}^{K-D_L-\epsilon} \frac{2D_L + D_H - K + l}{Q} dl + \\
& \frac{1}{Q^2} \int_{l=\epsilon}^{K-D_L-\epsilon} \int_{j=\epsilon}^{K-D_L-\epsilon} \frac{D_H + D_L + l - j}{Q} dj dl + \frac{1}{Q} \int_{l=\epsilon}^{K-D_L-\epsilon} \frac{1}{Q} \int_{j=\epsilon}^{K-D_L-\epsilon} \frac{\partial x_{t-1} = j}{\partial Q} dj dl, \\
& = \frac{(K - D_L)(2D_H + 3D_L - K)}{2Q(D_L + Q - K)} + \frac{K - D_L}{Q} E\Big[\frac{\partial x_{t-1}}{\partial Q} \Big], \\
& = \frac{(K - D_L)(2D_H + 3D_L - K)}{2(D_L + Q - K)^2}.
\end{aligned}
\tag{111}
$$

With this derivative calculated, we can now turn to the entire revenue function and take the first and partial derivatives w.r.t. Q:

$$
\begin{aligned}
&\Pi_{Per}(Q,K,e=1) = E\Big[\min[q_t, D_H] + \min[x_{t-1}, (D_H - q_t)^+] + \min\left[(x_{t-1}+q_t-D_H)^+, K, D_L\right]\Big],\\
&= Pr(0)\frac{1}{Q}\int_{q_t=0}^{Q}\min[q_t, D_H+D_L]dq_t + Pr(K-D_L)\frac{1}{Q}\int_{q_t=0}^{Q}\min[q_t+K-D_L, D_H+D_L]dq_t+\\
&\frac{1}{Q^2}\int_{x_{t-1}=\epsilon}^{K-D_L-\epsilon}\int_{q_t=0}^{Q}\min[q_t+x_{t-1}, D_H+D_L]dq_tdx_{t-1},\\
&= \frac{(D_H+D_L)Q}{D_L-K+Q} - \frac{\left(D_H^2+2D_HD_L+2D_L^2-2D_LK+K^2\right)}{2(D_L-K+Q)}+\\
&-\frac{(K-D_L)^4+2(K-D_L)\left(3D_H^2+6D_HD_L+D_L^2+4D_LK-2K^2\right)Q}{12Q^2(D_L-K+Q)}.
\end{aligned}
\tag{112}
$$

$$
\begin{aligned}
\frac{\partial\Pi_{Rev}(Q,K,e=1)}{\partial Q} =&T\Big(-\frac{(K-D_L)^3}{6Q^3} - \frac{\left(2D_H^2+4D_HD_L+D_L^2+2D_LK-K^2\right)}{4Q^2} + \frac{(2D_H+3D_L-K)^2}{4(D_L-K+Q)^2},\\
\frac{\partial^2\Pi_{Rev}(Q,K,e=1)}{\partial^2 Q} =&T\left(\frac{(K-D_L)^3}{2Q^4} + \frac{2D_H^2+4D_HD_L+D_L^2+2D_LK-K^2}{2Q^3} - \frac{(2D_H+3D_L-K)^2}{2(D_L-K+Q)^3}\right).
\end{aligned}
\tag{113}
$$

B.2.6 Hessian and Concavity of Revenue Function

Now that we obtained all the derivatives, we turn to the concavity result. To show concavity of the objective function, the Hessian has to be negative semi-definite. It is easy to show that the second partial derivatives w.r.t. to $\partial^2 K$ and $\partial^2 Q$ are both negative, so we focus on showing that the determinant of the Hessian is always weakly positive.

$$
\begin{aligned}
&WTS\frac{\partial^2\Pi_{Rev}(Q,K,e=1)}{\partial K\partial K}\frac{\partial^2\Pi_{Rev}(Q,K,e=1)}{\partial Q\partial Q} - \frac{\partial^2\Pi_{Rev}(Q,K,e=1)}{\partial Q\partial K}^2 \geq 0,\\
&\frac{(D_H+D_L)^2(-(K-D_L)^4+2(K-D_L)^3Q-4(D_H+D_L)^2Q^2+2(2D_H+3D_L-K)Q^3)}{2Q^5(D_L-K+Q)^3} \geq 0,\\
&-(K-D_L)^4+2(K-D_L)^3Q-4(D_H+D_L)^2Q^2+2(2D_H+3D_L-K)Q^3 \geq 0,\\
&-4(D_H+D_L)^2Q^2+2(2D_H+3D_L-K)Q^3 > 0,\\
&-2(D_H+D_L)^2+(2D_H+3D_L-K)Q > 0,\ \text{As we assume } Q > D_H+K \text{ and the LHS is increasing in } Q:\\
&-2(D_H+D_L)^2+(2D_H+3D_L-K)(D_H+K) > 0,\\
&(K-D_L)(D_H+2D_L-K) \geq 0.
\end{aligned}
\tag{114}
$$

This is a true statement for the parameter space of storage capacity we consider. We were thus able to show that the revenue function of the tracking model for $e = 1$, $D_L < K < 2D_L + D_H$ and $Q > K + D_H$ is concave. As the investment costs are linear and thus concave themselves, the entire tracking objective function is a sum of concave elements and thus concave.

B.3 Partial Discharge Model

As introduced in Section 3.2.3, the objective function of the partial-discharge model is:

$$\begin{aligned}\Pi_P(Q,K) =& \frac{g}{Q}\Big(\left[\int_0^{D_H} q\ dq\right] + \left[\int_{D_H}^{\frac{D_L}{e}+D_H} D_H + e(q-D_H)\ dq\right] + \left[\int_{\frac{D_L}{e}+D_H}^{Q} D_L + D_H\ dq\right] + \\ & \left[\int_{D_H+\frac{D_L}{e}}^{Q} \min(eq - eD_H - D_L, J)dq\right] \\ & \frac{\frac{1}{Q}\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, J, (D_H - q_2)^+ + (D_L - e(q_2 - D_H)^+)^+]\ dqdq_2}{\int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, J]dq}\Big) - \\ & \frac{c_K}{e} D_L - \frac{c_K}{e} J - c_Q Q.\end{aligned} \tag{115}$$

where $J = K - D_L$, $e \in (0, 1]$, and where we restrict our parameter space to $K_F^* \geq D_L$ - since outside of this space the full-discharge model is exact - and to $Q \geq D_H + \frac{D_L}{e}$ - which is equivalent to assuming that generation capacity is not prohibitively expensive.

As previously mentioned, in Appendix B.1, we simplify the objective function by replacing this term

$$\frac{\frac{1}{Q}\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, K - D_L, (D_H - q_2)^+ + D_L - e(q_2 - D_H)^+)^+]\ dqdq_2}{\int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, K - D_L]dq}$$

with this term (116)

$$\frac{\frac{1}{Q}\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, K - D_L, (D_L + eD_H - eq_2)^+]\ dqdq_2}{\int_{D_H+\frac{D_L}{e}}^{Q} \min[eq - eD_H - D_L, K - D_L]dq}.$$

This change further underpredicts storage profitability by reducing one of the values in the minimum operator, and allows improved tractability. With this change, the partial-discharge model we utilize in the paper is:

$$
\begin{aligned}
\Pi_P(Q,K) =& \frac{g}{Q}\Big(\left[\int_0^{D_H} q\ dq\right] + \left[\int_{D_H}^{D_H+\frac{D_L}{e}} D_H + e(q-D_H)\ dq\right] + \left[\int_{D_H+\frac{D_L}{e}}^{Q} D_L + D_H\ dq\right] \\
&+ \left[\int_{D_H+\frac{D_L}{e}}^{Q} \min(eq - eD_H - D_L, J) dq\right] \\
&= \frac{g}{Q}\left(\left[\int_0^{D_H} q\ dq\right] + \left[\int_{D_H}^{\frac{D_L}{e}+D_H} D_H + e(q-D_H)\ dq\right] + \left[\int_{D_H+\frac{D_L}{e}}^{Q} D_L + D_H\ dq\right]\right) + \\
&\frac{ge}{Q^2}\left(\int_0^{Q}\ \int_{D_H+\frac{D_L}{e}}^{Q} \min[q - D_H - \frac{D_L}{e}, \frac{K}{e} - \frac{D_L}{e}, (D_H + \frac{D_L}{e} - q_2)^+] dq dq_2\right) \\
&- \frac{c_K}{e} D_L - \frac{c_K}{e} J - c_Q Q.
\end{aligned} \tag{117}
$$

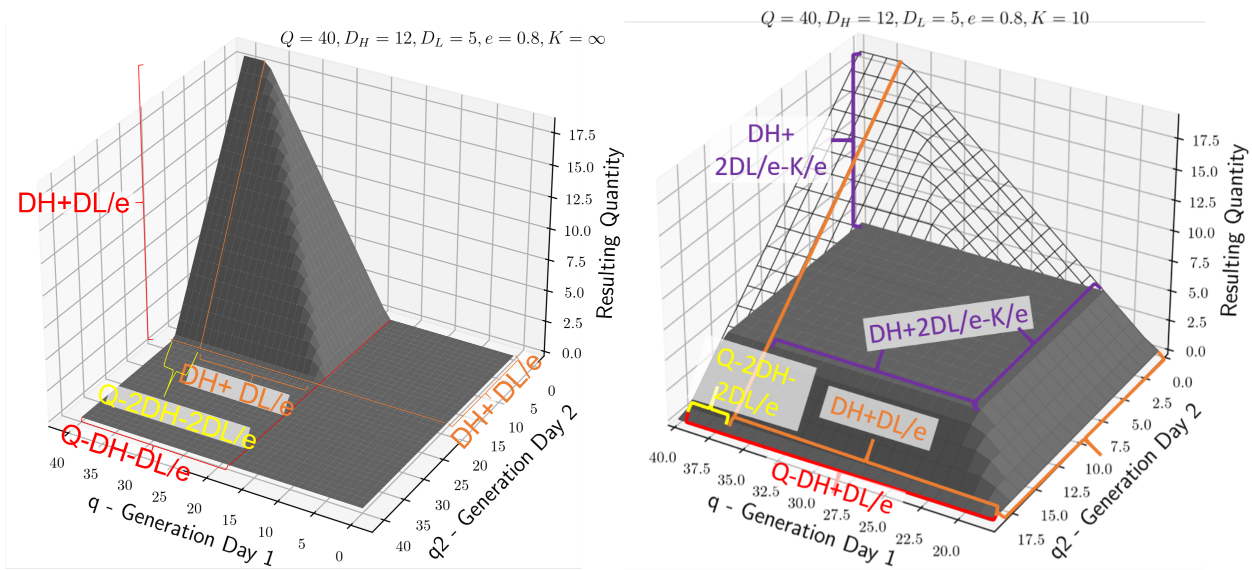
In order to proceed from here, we solve the double integral $\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[q - D_H - \frac{D_L}{e}, \frac{K}{e} - \frac{D_L}{e}, (D_H + \frac{D_L}{e} - q_2)^+] dq dq_2$ and show the geometric intuition below. The expression is solved by accounting for all possible scenarios under which discharge could occur if one was not limited by storage capacity and then deducting all cases that the storage capacity limitation prevents. Note that this term only calculates energy discharged in the next period, while any discharging at night in the same period where the charge occurred is already accounted for in the other terms of the objective function.

For the unlimited storage case, we start by assuming $Q > 2D_H + \frac{2D_L}{e}$, which simplifies the exposition and show in graph 6a) how the discharge quantity (on the z-axis) depends on the generation realizations q (x-Axis) and $q2$ (y-Axis) for some parameters. Two conditions for this quantity to be positive is that 1) one has excess charge on the focal day ($q > D_H + D_L/e$) and un-served demand in the following period ($q2 < D_H + D_L/e$). Lastly, note that the maximum discharge quantity that can be used is $D_H + D_L/e$ as we assume all unused charge to be lost at the end of the second period. Consequently, the discharge quantity without storage limitation is the combination of a pyramid with volume $(D_H + D_L/e)^3/3$ and a prism with volume $(Q - 2D_H - 2D_L/e)(D_H + D_L/e)^2/2$.

After establishing the unlimited discharge quantity, we investigate the effect that limited storage capacity has on said quantity. Note that the integral contains K/e, which is the amount of energy that can be discharged from the storage solution, before the entire expression gets multiplied by the efficiency parameter e outside the parentheses to accounted for efficiency losses. A full discharge of K/e times efficiency e meets K demand. In graph 6 b), we show the cases from graph 6 a) for which the storage quantity was infinite, but now assume a $K \leq eD_H + 2D_L \rightarrow K/e \leq D_H + 2D_L/e$. Again, stored energy beyond this level is always assumed to be lost, thus storage capacity beyond this level is not optimal in this model. The storage addition now limits the realizations on the Z axis - the lower the capacity, the more of the pyramid and prism get "chopped off". We thus subtract a small pyramid with volume $(D_H + 2D_L/e - K/e)^3/3$ and a small prism with volume $(Q - 2D_H - 2D_L/e)(D_H + 2D_L/e - K/e)^2/2$ from the original expression. Hence, the original term can be expressed and then expanded as shown in Equation (118).

$$
\begin{aligned}
&= \frac{ge}{Q^2}\left(\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min[q - D_H - \frac{D_L}{e}, \frac{K}{e} - \frac{D_L}{e}, (D_H + \frac{D_L}{e} - q_2)^+] dq dq_2\right). \\
&= \frac{ge}{Q^2}\Big(\frac{(D_H + D_L/e)^3}{3} - \frac{(D_H + 2D_L/e - K/e)^3}{3} + \frac{(Q - 2D_H - 2D_L/e)(D_H + D_L/e)^2}{2} \\
&\quad - \frac{(Q - 2D_H - 2D_L/e)(D_H + 2D_L/e - K/e)^2}{2}\Big). \qquad (118) \\
&= \frac{ge}{Q^2}\left(\frac{D_H^2 D_L}{e} - \frac{D_H^2 K}{e} + \frac{2D_H D_L^2}{e^2} - \frac{2D_H D_L K}{e^2} - \frac{D_H D_L Q}{e} + \frac{D_H K Q}{e} - \frac{3D_L^2 Q}{2e^2} + \right. \\
&\left. \frac{2D_L^3}{3e^3} - \frac{D_L K^2}{e^3} + \frac{2D_L K Q}{e^2} - \frac{K^2 Q}{2e^2} + \frac{K^3}{3e^3}\right)
\end{aligned}
$$

Even though this exposition assumed $Q > 2D_H + 2D_L/e$, if $Q < 2D_H + 2D_L/e$ the expression remains true as long as $Q \geq D_H + K/e$ (smaller Q would make K sub-optimal as it could never be fully charged) and $K \geq D_L$ (by assumption), but the prism volumes switch their signs. Graphically, the parts of the pyramid (parallel to the orange line in graph 6 a) get removed rather than added. Another way to see this is to contemplate a new pyramid altogether.



(a) (b)

Figure 6: Discharge Quantity with (a) Infinite Storage Capacity and (b) Finite Storage Capacity

For the case, with $Q < 2D_H + 2D_L/e$, lets first consider the case with unlimited storage. We will explain the cases with reference to the volumes of Figure 6 to show how the cases relate. Excess generation on the focal day is always smaller than combined demand on the next day

$q - D_H - D_L/e \leq D_H + D_L/e$, so if $q_2 < 2D_H + 2D_L/e - Q$ all excess generation will be discharged. Graphically speaking this results in a prism with volume $(Q - D_H - D_L/e)^2/2(2D_L/e + 2D_H - Q)$, corresponding to the prism with the yellow edge in Figure 6 a). Note that, relative to that graph, the edge of the prisms turned by 90 degrees counterclockwise in this case, because the limiting factor now is excess electricity from the focal day, not the discharge availability in the next day. All other cases of discharge are contained in a pyramid (corresponding to the orange color pyramid in Figure 6 a) with a volume of $(Q - D_H - D_L/e)^3/3$. If one includes limited storage, the purple pyramid equivalent of Figure 6 b) in this case would have side-length $Q - D_H - K/e$, because $K/e \leq Q - D_H$ are the only cases for which storage can ever be fully charged. This results in a pyramid with volume $(Q - D_H - K/e)^3/3$. Lastly, the prism volume that has to be chopped off is $(Q - D_H - K/e)^2/2(2D_H + 2D_L/e - Q)$, analogous to the previous case.

We now juxtapose the combined expressions of both cases:

$$\begin{aligned}
&\text{Combined expressions for } Q > 2D_H + 2D_L/e:\\
&\Big(\frac{(D_H + D_L/e)^3}{3} - \frac{(D_H + 2D_L/e - K/e)^3}{3} +\\
&\frac{(Q - 2D_H - 2D_L/e)(D_H + D_L/e)^2}{2} - \frac{(Q - 2D_H - 2D_L/e)(D_H + 2D_L/e - K/e)^2}{2}\Big)\\
&\text{Combined expressions for } Q < 2D_H + 2D_L/e:\\
&\Big(\frac{(Q - D_H - D_L/e)^3}{3} - \frac{(Q - D_H - K/e)^3}{3} +\\
&\frac{(2D_H + 2D_L/e - Q)(Q - D_H - D_L/e)^2}{2} - \frac{(2D_H + 2D_L/e - Q)(Q - D_H - K/e)^2}{2}\Big)
\end{aligned} \tag{119}$$

The sums of these expressions are equivalent and we thus continue using the expressions for $Q > 2D_H + 2D_L/e$.

With this derivation completed, we can further collect and simplify terms in the objective function.

$$\begin{aligned}
\Pi_P(Q,K) = \frac{g}{Q}\Bigg(\Bigg[\int_0^{D_H} q\ dq\Bigg] + \Bigg[\int_{D_H}^{\frac{D_L}{e}+D_H} D_H + e(q - D_H)\ dq\Bigg] + \Bigg[\int_{\frac{D_L}{e}+D_H}^{Q} D_L + D_H\ dq\Bigg]\Bigg) +\\
\frac{ge}{Q^2}\Bigg(\int_0^Q \int_{D_H+\frac{D_L}{e}}^{Q} \min(q - D_H - \frac{D_L}{e}, \frac{K}{e} - \frac{D_L}{e}, (D_H + \frac{D_L}{e} - q_2)^+)dqdq_2\Bigg)\\
- c_K D_L - c_K J - c_Q Q,\\
= D_H g + D_L g + \frac{D_H^2 D_L g}{Q^2} + \frac{2D_L^3 g}{3e^2Q^2} + \frac{2D_H D_L^2 g}{eQ^2} - \frac{2D_H D_L g K}{eQ^2} - \frac{D_H^2 g K}{Q^2} - \frac{D_L g K^2}{e^2Q^2} +\\
\frac{gK^3}{3e^2Q^2} - \frac{D_H^2 g}{2Q} - \frac{2D_H D_L g}{Q} - \frac{2D_L^2 g}{eQ} + \frac{2D_L g K}{Qe} + \frac{D_H g K}{Q} - \frac{gK^2}{2Qe} - c_Q Q - \frac{c_K}{e}K.
\end{aligned} \tag{120}$$

$$
\begin{aligned}
\frac{\partial \Pi_P(Q,K)}{\partial Q} =& -\frac{2D_H^2 D_L g}{Q^3} - \frac{4D_L^3 g}{3e^2Q^3} - \frac{4D_H D_L^2 g}{eQ^3} + \frac{4D_H D_L g K}{Q^3 e} + \frac{2D_H^2 g K}{Q^3} + \frac{2D_L g K^2}{Q^3 e^2} - \\
& \frac{2gK^3}{3Q^3e^2} + \frac{D_H^2 g}{2Q^2} + \frac{2D_H D_L g}{Q^2} + \frac{2D_L^2 g}{eQ^2} - \frac{2D_L g K}{Q^2 e} - \frac{D_H g K}{Q^2} + \frac{gK^2}{2Q^2 e} - c_Q, \\
& = Q^3 + Q(-\frac{D_H^2 g}{2c_Q} - \frac{2D_H D_L g}{c_Q} - \frac{2D_L^2 g}{ec_Q} + \frac{2D_L g K}{c_Q e} + \frac{D_H g K}{c_Q} - \frac{gK^2}{2c_Q e}) + \\
& \frac{2D_H^2 D_L g}{c_Q} + \frac{4D_L^3 g}{3c_Q e^2} + \frac{4D_H D_L^2 g}{c_Q} - \frac{4D_H D_L g K}{c_Q e} - \frac{2D_H^2 g K}{c_Q e} - \frac{2D_L g K^2}{c_Q e^2} + \frac{2gK^3}{3c_Q e^2} = 0, \\
& c = \frac{g}{3c_Q}(-\frac{D_H^2}{2} - 2D_H D_L - \frac{2D_L^2}{e} + 2D_L K + D_H K - \frac{K^2}{2e}), \\
& d = \frac{g}{c_Q}(D_H^2 D_L + \frac{2D_L^3}{3e^2} + 2D_H D_L^2 - \frac{2D_H D_L K}{e} - D_H^2 K - \frac{D_L K^2}{e^2} + \frac{K^3}{3e^2}), \\
\rightarrow Q_P^* =& \sqrt[3]{-d + \sqrt{d^2 + c^3}} + \sqrt[3]{-d - \sqrt{d^2 + c^3}}.
\end{aligned}
\tag{121}
$$

$$
\begin{aligned}
\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 Q} =& \frac{6D_H^2 D_L g}{Q^4} + \frac{4D_L^3 g}{Q^4 e^2} + \frac{12 D_H D_L^2 g}{Q^4 e} - \frac{12 D_H D_L g K}{Q^4 e} - \frac{6D_H^2 g K}{Q^4} - \frac{6D_L g K^2}{Q^4 e^2} + \frac{2gK^3}{Q^4 e^2} - \\
& \frac{D_H^2 g}{Q^3} - \frac{4D_H D_L g}{Q^3} - \frac{4D_L^2 g}{eQ^3} + \frac{4D_L g K}{Q^3 e} + \frac{2D_H g K}{Q^3} - \frac{gK^2}{Q^3 e}.
\end{aligned}
\tag{122}
$$

$$
\begin{aligned}
\frac{\partial \Pi_P(Q,K)}{\partial K} =& -\frac{2D_H D_L g}{Q^2 e} - \frac{D_H^2 g}{Q^2} - \frac{2D_L g K}{Q^2 e^2} + \frac{gK^2}{Q^2 e^2} + \frac{2D_L g}{Qe} + \frac{D_H g}{Q} - \frac{gK}{Qe} - \frac{c_K}{e}, \\
\rightarrow K_P^* =& D_L + \frac{1}{2}\left(Qe - \sqrt{4(D_L + D_H e)^2 g - 4e(D_L + D_H e)Q + \frac{e(4c_K + eg)Q^2}{g}}\right).
\end{aligned}
\tag{123}
$$

$$
\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 K} = -\frac{2D_L g}{Q^2 e^2} + \frac{2gK}{Q^2 e^2} - \frac{eg}{Q}. \tag{124}
$$

$$
\frac{\partial^2 \Pi_P(Q,K)}{\partial K \partial Q} = \frac{4D_H D_L g}{Q^3 e} + \frac{2D_H^2 g}{Q^3} + \frac{4D_L g K}{Q^3 e^2} - \frac{2gK^2}{Q^3 e^2} - \frac{2D_L g}{Q^2 e} - \frac{D_H g}{Q^2} + \frac{egK}{Q^2}. \tag{125}
$$

B.3.1 Derivative Proofs for Optimal Solutions

We know $K \geq D_L$ and $K \leq Q - D_H$ and $Q \geq D_H + \frac{D_L}{e}$.

$$
\begin{aligned}
&\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 Q} < 0,\\
&\frac{6D_H^2 D_L g}{Q^4} + \frac{4D_L^3 g}{e^2 Q^4} + \frac{12 D_H D_L^2 g}{eQ^4} - \frac{12 D_H D_L g K}{Q^4 e} - \frac{6D_H^2 g K}{Q^4} - \frac{6 D_L g K^2}{Q^4 e^2} + \frac{2gK^3}{Q^4 e^2} -\\
&\frac{D_H^2 g}{Q^3} - \frac{4D_H D_L g}{Q^3} - \frac{4D_L^2 g}{eQ^3} + \frac{4D_L g K}{Q^3 e} + \frac{2D_H g K}{Q^3} - \frac{gK^2}{Q^3 e} < 0,\\
&6D_H^2 D_L + \frac{4D_L^3}{e^2} + \frac{12 D_H D_L^2}{e} - \frac{12 D_H D_L K}{e} - 6D_H^2 K - \frac{6D_L K^2}{e^2} + \frac{2K^3}{e^2} +\\
&Q(-D_H^2 - 4D_H D_L - \frac{4D_L^2}{e} + \frac{4D_L K}{e} + 2D_H K - \frac{K^2}{e}) < 0,
\end{aligned}
\tag{126}
$$

We first show that the expression multiplied by Q is negative:

$$
\begin{aligned}
&-D_H^2 - 4D_H D_L - \frac{4D_L^2}{e} + \frac{4D_L K}{e} + 2D_H K - \frac{K^2}{e} < 0,\\
&-D_H^2 e - 4D_H D_L e - 4D_L^2 + 4D_L K + 2D_H K e - K^2 < 0,\\
&-D_H^2 e - 4D_H D_L e - 4D_L^2 + 4D_L K + 2D_H K e - K^2 = 0,\\
&K = 2D_L + D_H e \text{ The derivative is positive only outside the parameter space we consider.}
\end{aligned}
\tag{127}
$$

This cannot be optimal, as any excess charge in the model is lost after the second period and could never be used.

Having established that this term is negative, we focus on the sum of the other terms, for which we show that there is no value $K \in [D_L, 2D_L + D_H e]$ for which this expression is positive, while having $Q \geq D_H + K/e$, which we assume.

$$
\begin{aligned}
&6D_H^2 D_L + \frac{4D_L^3}{e^2} + \frac{12 D_H D_L^2}{e} - \frac{12 D_H D_L K}{e} - 6D_H^2 K - \frac{6D_L K^2}{e^2} + \frac{2K^3}{e^2} = 0,\\
&\text{which has 3 roots:}\\
&K_1^* = D_L - \sqrt{3}\sqrt{D_L^2 + 2D_H D_L e + D_H^2 e^2},\ K_2^* = D_L,\\
&K_3^* = D_L + \sqrt{3}\sqrt{D_L^2 + 2D_H D_L e + D_H^2 e^2} = D_L + \sqrt{3}(D_L + D_H e).
\end{aligned}
\tag{128}
$$

$$
\begin{aligned}
&\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 K} < 0,\\
&-\frac{2D_L g}{Q^2 e^2} + \frac{2gK}{Q^2 e^2} - \frac{g}{Qe} < 0,\\
&\frac{\partial^2 \Pi_P(Q,K)}{\partial^2 K} < 0 \text{ if } K < \frac{Qe}{2} + D_L \text{ and } K^* < \text{that value.}
\end{aligned}
\tag{129}
$$

$$
\begin{aligned}
WTS \frac{\partial \Pi_P(Q,K)}{\partial Q} &< 0, if Q \rightarrow \infty, \\
\frac{\partial \Pi_P(Q \rightarrow \infty, K)}{\partial Q} &= -c_Q.
\end{aligned} \tag{130}
$$

$$
\begin{aligned}
&WTS \frac{\partial \Pi_P(Q,K)}{\partial K} < 0, if K \rightarrow \infty, \\
&\frac{\partial \Pi_P(Q, K \rightarrow \infty)}{\partial K} = -\frac{2D_H D_L g}{Q^2 e} - \frac{D_H^2 g}{Q^2} - \frac{2D_L g K}{Q^2 e^2} + \frac{gK^2}{Q^2 e^2} + \frac{2D_L g}{Qe} + \frac{D_H g}{Q} - \frac{gK}{Qe} - \frac{c_K}{e} \\
&\text{As } Q > D_H/e + K/e: \\
&+g - g - c_K = -c_K.
\end{aligned} \tag{131}
$$

B.3.2 Border Solutions

$Q = M$ or $K = M$, where M is a larger number cannot be an optimal border solution as shown by the second derivatives. $Q = D_H + \frac{D_L}{e}$ and $K > D_L$ cannot be optimal as the storage will never be charged. Hence, the only border solution we have to test is $K = D_L$ and $Q > D_H + \frac{D_L}{e}$.

$$
\Pi_P(Q, K = D_L) = = D_H g + D_L g - \frac{D_H^2 g}{2Q} - \frac{D_H D_L g}{Q} - \frac{D_L^2 g}{2eQ} - c_Q Q - D_L \frac{c_K}{e}. \tag{132}
$$

$$
\begin{aligned}
\frac{\partial \Pi_P(Q, K = D_L)}{\partial Q} =& \frac{D_H^2 g}{2Q^2} + \frac{D_H D_L g}{Q} + \frac{D_L^2 g}{2eQ^2} - c_Q, \\
Q_{bor}^* =& \sqrt{\frac{g(\frac{D_L^2}{e} + 2D_H D_L + D_H^2)}{2c_Q}}.
\end{aligned} \tag{133}
$$

Next, we investigate the partial derivative w.r.t. K at the border $K = D_L$

$$\begin{aligned}
&WTS \frac{\partial \Pi_P(Q, K = D_L)}{\partial K} > 0, \\
&-\frac{2D_H D_L g}{Q^2 e} - \frac{D_H^2 g}{Q^2} - \frac{D_L^2 g}{Q^2 e^2} + \frac{D_L g}{Qe} + \frac{D_H g}{Q} - \frac{c_K}{e} > 0, \\
&Q^2 - Q(\frac{D_L g}{c_K} + \frac{D_H e g}{c_K}) + \frac{2D_H D_L g}{c_K} + \frac{D_H^2 e g}{c_K} + \frac{D_L^2 g}{e c_K} < 0, \\
&\frac{g}{2c_K}(D_L + D_H e) - \sqrt{\frac{(D_L + D_H e)^2 g(-4c_K + ge)}{4c_K^2 e}} < Q < \\
&\frac{g}{2c_K}(D_L + D_H e) + \sqrt{\frac{(D_L + D_H e)^2 g(-4c_K + ge)}{4c_K^2 e}}, \\
&\frac{(D_L + D_H e)}{2c_K}(g - \sqrt{\frac{g(ge - 4c_K)}{e}}) < Q < \frac{(D_L + D_H e)}{2c_K}(g + \sqrt{\frac{g(ge - 4c_K)}{e}}).
\end{aligned} \tag{134}$$

If the expression under the radical is negative, $\frac{\partial \Pi_P(Q,K)}{\partial K} < 0$ whenever $K = D_L$, it follows from concavity w.r.t. K that (Q^*_{bor}, K^*_{bor}) maximizes profit. This happens when $g < \frac{4c_K}{e} \triangleq \underline{g_S}$.

Now, setting $Q = Q^*_{bor} = \sqrt{\frac{g(\frac{D_L^2}{e} + 2D_H D_L + 2D_H^2)}{2c_Q}}$:

$$\begin{aligned}
\frac{(D_L + D_H e)}{2c_K}(g - \sqrt{\frac{g(ge - 4c_K)}{e}}) < \sqrt{\frac{g(\frac{D_L^2}{e} + 2D_H D_L + D_H^2)}{2c_Q}} < \frac{(D_L + D_H e)}{2c_K}(g + \sqrt{\frac{g(ge - 4c_K)}{e}}),& \\
g > \frac{\left(2c_Q(D_L + D_H e)^2 + c_K\left(D_L^2 + 2D_H(D_H + D_L)e\right)\right)^2}{2c_Q e(D_L + D_H e)^2\left(D_L^2 + 2D_H(D_H + D_L)e\right)},& \\
= \frac{\left(c_K + 2c_K em(1 + m) + 2c_Q(1 + em)^2\right)^2}{2c_Q e(1 + em)^2(1 + 2em(1 + m))} \triangleq g_P, \text{where } m = \frac{D_H}{D_L}.&
\end{aligned} \tag{135}$$

When g is larger than g_P, we have $\frac{\partial \Pi_P(Q^*_{bor}, D_L)}{\partial K} > 0$, hence the maximum of the function must necessarily be the interior solution (Q^*_{int}, K^*_{int}). Yet, when $g \in (g_F, g_P)$, we have the border solution.

B.3.3 Concavity Proof Using Derivatives Directly

Here, we provide an alternative way of showing concavity of the partial-discharge model by directly analyzing the derivatives, Hessian and their roots. We present this approach second as it does not develop the same intuition about the problem as the geometric approach checking for the various possible (border) solutions, but uses the standard second-partial-derivative test.

$$\Pi_P(Q,K) = D_H g + D_L g + \frac{D_H^2 D_L g}{Q^2} + \frac{2D_L^3 g}{3e^2Q^2} + \frac{2D_H D_L^2 g}{eQ^2} - \frac{2D_H D_L g K}{eQ^2} - \frac{D_H^2 g K}{Q^2} - \frac{D_L g K^2}{e^2Q^2} +$$
$$\frac{gK^3}{3e^2Q^2} - \frac{D_H^2 g}{2Q} - \frac{2D_H D_L g}{Q} - \frac{2D_L^2 g}{eQ} + \frac{2D_L g K}{Qe} + \frac{D_H g K}{Q} - \frac{gK^2}{2Qe} - c_Q Q - \frac{c_K}{e}K. \tag{136}$$

Because $c_Q\ Q$ and $c_K/e\ K$ are concave functions in Q and K, we have to show concavity for the rest of the objective function (see Equation 137). If we can show that, the sum of concave functions results in a concave objective function.

$$\Pi_{P\ excl.\ cost}(Q,K) = D_H g + D_L g + \frac{D_H^2 D_L g}{Q^2} + \frac{2D_L^3 g}{3e^2Q^2} + \frac{2D_H D_L^2 g}{eQ^2} - \frac{2D_H D_L g K}{eQ^2} - \frac{D_H^2 g K}{Q^2} -$$
$$\frac{D_L g K^2}{e^2Q^2} + \frac{gK^3}{3e^2Q^2} - \frac{D_H^2 g}{2Q} - \frac{2D_H D_L g}{Q} - \frac{2D_L^2 g}{eQ} + \frac{2D_L g K}{Qe} + \frac{D_H g K}{Q} - \frac{gK^2}{2Qe},$$

WLOG, we set $g = 1$, $D_L = 1$ and $D_H = mD_L$,

$$\Pi_{Simplified}(Q,K) = m + 1 + \frac{m^2}{Q^2} + \frac{2}{3e^2Q^2} + \frac{2m}{eQ^2} - \frac{2mK}{eQ^2} - \frac{m^2K}{Q^2} - \frac{K^2}{e^2Q^2} +$$
$$\frac{K^3}{3e^2Q^2} - \frac{m^2}{2Q} - \frac{2m}{Q} - \frac{2}{eQ} + \frac{2K}{Qe} + \frac{mK}{Q} - \frac{K^2}{2Qe}. \tag{137}$$

With this formulation, Q and K are expressed as multiples of nightly demand, i.e. $K = 2$ would be storage capacity equal to two nightly demand and $Q = 3 + 2m$ would be generation equal to 3 times nightly demand and two times daily demand. With this re-formulation, we achieve a much more succinct expression of the objective-function, which is helpful in attaining a Hessian that one can analyze/work with. We continue by writing down the first and second partial derivatives for this simplified objective function.

$$\begin{aligned}
\frac{\partial \Pi_{Simplified}(Q,K)}{\partial Q} =& -\frac{2m^2}{Q^3} - \frac{4}{3e^2Q^3} - \frac{4m}{eQ^3} + \frac{4mK}{eQ^3} + \frac{2m^2K}{Q^3} + \frac{2K^2}{e^2Q^3} - \\
& \frac{2K^3}{3e^2Q^3} + \frac{m^2}{2Q^2} + \frac{2m}{Q^2} + \frac{2}{eQ^2} - \frac{2K}{Q^2e} - \frac{mK}{Q^2} + \frac{K^2}{2Q^2e}, \\
\frac{\partial \Pi_{Simplified}(Q,K)}{\partial K} =& -\frac{2m}{eQ^2} - \frac{m^2}{Q^2} - \frac{2K}{e^2Q^2} + \frac{K^2}{e^2Q^2} + \frac{2}{Qe} + \frac{m}{Q} - \frac{K}{Qe}, \\
\frac{\partial^2 \Pi_{Simplified}(Q,K)}{\partial^2 Q} =& \frac{6m^2}{Q^4} + \frac{4}{e^2Q^4} + \frac{12m}{eQ^4} - \frac{12mK}{eQ^4} - \frac{6m^2K}{Q^4} - \frac{6K^2}{e^2Q^4} + \\
& \frac{2K^3}{e^2Q^4} - \frac{m^2}{Q^3} - \frac{4m}{Q^3} - \frac{4}{eQ^3} + \frac{4K}{Q^3e} + \frac{2mK}{Q^3} - \frac{K^2}{Q^3e}, \\
\frac{\partial^2 \Pi_{Simplified}(Q,K)}{\partial^2 K} =& -\frac{2}{e^2Q^2} + \frac{2K}{e^2Q^2} - \frac{1}{Qe}, \\
\frac{\partial^2 \Pi_{Simplified}(Q,K)}{\partial K \partial Q} =& \frac{4m}{eQ^3} + \frac{2m^2}{Q^3} + \frac{4K}{e^2Q^3} - \frac{2K^2}{e^2Q^3} - \frac{2}{Q^2e} - \frac{m}{Q^2} + \frac{K}{Q^2e}.
\end{aligned} \tag{138}$$

We have proven before (see Equations 34 and 37) that the second partial derivatives are concave, so in order to proof concavity of this simplified objective function we now want to show that the Hessian is negative semi-definite over the convex parameter space we consider.[11]

$$
\begin{aligned}
&WTS\ \frac{\partial^2 \Pi_{Simplified}(Q,K)}{\partial^2 Q}\ \frac{\partial^2 \Pi_{Simplified}(Q,K)}{\partial^2 K} - \frac{\partial^2 \Pi_{Simplified}(Q,K)}{\partial Q \partial K}^2 > 0,\\
&-\frac{8}{e^4Q^6} + \frac{8K}{e^4Q^6} - \frac{4K^2}{e^4Q^6} - \frac{24m}{e^3Q^6} + \frac{16Km}{e^3Q^6} - \frac{8K^2m}{e^3Q^6} - \frac{28m^2}{e^2Q^6} + \frac{8Km^2}{e^2Q^6} - \frac{4K^2m^2}{e^2Q^6} - \frac{16m^3}{eQ^6} - \frac{4m^4}{Q^6} +\\
&\frac{4}{e^3Q^5} + \frac{12m}{e^2Q^5} + \frac{2m^2}{e^2Q^5} + \frac{10m^2}{eQ^5} - \frac{2Km^2}{e^2Q^5} + \frac{2Km^2}{eQ^5} + \frac{4m^3}{Q^5} - \frac{m^2}{Q^4} + \frac{m^2}{eQ^4} > 0,\\
&-8 + 8K - 4K^2 - 24em + 16eKm - 8eK^2m - 28e^2m^2 + 8e^2Km^2 - 4e^2K^2m^2 - 16e^3m^3 - 4e^4m^4 + 4eQ +\\
&12e^2mQ + 2e^2m^2Q + 10e^3m^2Q - 2e^2Km^2Q + 2e^3Km^2Q + 4e^4m^3Q + e^3m^2Q^2 - e^4m^2Q^2 > 0,\\
&K^2(-4 - 8em - 4e^2m^2) + K(8 + 16em + 8e^2m^2 - 2e^2m^2Q + 2e^3m^2Q) + 8 - 24em - 28e^2m^2\\
&- 16e^3m^3 - 4e^4m^4 + 4eQ + 12e^2mQ + 2e^2m^2Q + 10e^3m^2Q + 4e^4m^3Q + e^3m^2Q^2 - e^4m^2Q^2 > 0,\\
&K_1 = \tfrac{4-\sqrt{-16(1+em)^6+16e(1+em)^5Q+(1-e)e^3m^2(4+em(8+m+3em))Q^2}+em(8+em(4-(1-e)Q))}{4(1+em)^2},\\
&K_2 = \tfrac{4+\sqrt{-16(1+em)^6+16e(1+em)^5Q+(1-e)e^3m^2(4+em(8+m+3em))Q^2}+em(8+em(4-(1-e)Q))}{4(1+em)^2}.
\end{aligned}
\tag{139}
$$

We know that the Hessian is positive between $K \in (K_1, K_2)$. It can be shown that K_1 and K_2 each have unique, real solutions (i.e. a positive radicand), if $m > 0, 0 < e \leq 1$ and Q is in the aforementioned range. We thus first want to show that $K_1 < 0$, so that the Hessian becomes negative only for storage capacities which we don't consider for the partial-discharge model. Note that we assume for solar generation to cover at least demand in expectation, we have $Q > 2(m+1/e)$.

[11] As a reminder, for the partial-discharge model the parameter space we consider is: Storage capacity is between nightly demand and two days worth of capacity ($D_L \leq K \leq 2D_L + D_H e$) which in our re-formulation is equal to ($1 \leq K \leq 2 + me$). Generation is at least large enough to cover demand in expectation ($Q \geq 2D_L/e + 2D_H$), which in our re-formulation is equal to ($Q \geq 2/e + 2m$)

$K_1 < 0,$

$\frac{4-\sqrt{-16(1+em)^6+16e(1+em)^5Q+(1-e)e^3m^2(4+em(8+m+3em))Q^2}+em(8+em(4-(1-e)Q))}{4(1+em)^2} < 0,$

$4 - \sqrt{-16(1+em)^6 + 16e(1+em)^5Q + (1-e)e^3m^2(4+em(8+m+3em))Q^2}+$

$em(8+em(4-(1-e)Q)) < 0.$

Clearly, the LHS is decreasing in Q, so we substitute the smallest Q possible of $Q = 2m + 2/e$.

$$\begin{aligned}
&= 4 - \sqrt{-16(1+em)^6 + 16e(1+em)^5(2m+2/e) + (1-e)e^3m^2(4+em(8+m+3em))(2m+2/e)^2}+ \\
&em(8+em(4-(1-e)(2m+2/e))) < 0, \\
&= 4 - (4 + 12em + 2em^2 + 10e^2m^2 + 2e^2m^3 + 2e^3m^3) + 8em - 2em^2 + 6e^2m^2 - 2e^2m^3 + 2e^3m^3 < 0, \\
&-4em - 4em^2 - 4e^2m^2 - 4e^2m^3 < 0.
\end{aligned} \tag{140}$$

Thus, K_1 is always negative. We now want to show that K_2 is suitably large, i.e. that $K_2 > 2+me$ so that the Hessian is positive, as long as storage capacity is less than two days. As a reminder, the partial discharge case assumes that all storage is lost after two days, so capacity in excess of that would not be useful in our approximation.

$$\begin{aligned}
&K_2 > 2 + me \\
&\frac{4+\sqrt{-16(1+em)^6+16e(1+em)^5Q+(1-e)e^3m^2(4+em(8+m+3em))Q^2}+em(8+em(4-(1-e)Q))}{4(1+em)^2} \geq 2 + me.
\end{aligned} \tag{141}$$

We show that the LHS in 141 is increasing in Q by showing that some of the positive parts of the derivative are larger in magnitude than the negative parts of the derivative:

$$\begin{aligned}
&\frac{\partial em(8+em(4-(1-e)Q))}{\partial Q} = -(1-e)e^2m^2, \\
&\frac{\partial\sqrt{-16(1+em)^6+16e(1+em)^5Q+(1-e)e^3m^2(4+em(8+m+3em))Q^2}}{\partial Q} \\
&> \frac{\partial\sqrt{(1-e)e^3m^2(4+em(8+m+3em))Q^2}}{\partial Q} = e^{1.5}m\sqrt{(1-e)(4+em(8+m+3em))}. \\
&\text{Comparing the magnitudes of the derivatives:} \\
&|e^{1.5}m\sqrt{(1-e)(4+em(8+m+3em))}| > |-(1-e)e^2m^2|, \\
&e^{1.5}m\sqrt{(1-e)(4+em(8+m+3em))} > (1-e)e^2m^2, \\
&\sqrt{(1-e)(4+em(8+m+3em))} > (1-e)e^{0.5}m, \\
&(1-e)(4+em(8+m+3em)) > (1-e)^2em^2, \\
&4 + 8em + 3e^2m^2 > 0.
\end{aligned} \tag{142}$$

Thus, K_2 is increasing in Q and we again use the lowest possible value of $Q = 2m + 2/e$ to proof $K_2 \geq 2 + me$. Continuing from 141.

$$
\begin{aligned}
&\frac{4+\sqrt{-16(1+em)^6+16e(1+em)^5Q+(1-e)e^3m^2(4+em(8+m+3em))Q^2}+em(8+em(4-(1-e)Q))}{4(1+em)^2} \geq 2+me,\\
&\frac{4+\sqrt{-16(1+em)^6+16e(1+em)^5(2m+2/e)+(1-e)e^3m^2(4+em(8+m+3em))(2m+2/e)^2}+em(8+em(4-(1-e)(2m+2/e)))}{4(1+em)^2} \geq 2+me,\\
&\frac{4+(4+12em+2em^2+10e^2m^2+2e^2m^3+2e^3m^3)+8em-2em^2+6e^2m^2-2e^2m^3+2e^3m^3}{4(1+em)^2} \geq 2+me,\\
&\frac{8+20em+16e^2m^2+4e^3m^3}{4(1+em)^2} \geq 2+me,\\
&2+em \geq 2+me.
\end{aligned}
\tag{143}
$$

Thus, the simplified objective function's is negative semi-definite across the parameter space and thus a concave function. In combination with the concave (linear) cost terms, the objective function of the partial discharge model is thus concave.

B.4 Full Discharge Model

$$
\begin{aligned}
\Pi_F(Q,K) =& \frac{g}{Q}[\int_0^{D_H} q\ dq + \int_{D_H}^{Q} D_H\ dq + \int_{D_H}^{D_H+K/e} (q-D_H)\ e\ dq + \int_{D_H+K/e}^{Q} K\ dq] - \frac{c_K}{e}K - c_Q Q,\\
=& \frac{g}{Q}[\frac{{D_H}^2}{2} + (Q-D_H)D_H + \frac{K^2}{2e} + (Q - D_H - \frac{K}{e})K] - \frac{c_K}{e}K - c_Q Q,\\
=& \frac{g}{Q}[-\frac{D_H^2}{2} - \frac{K^2}{2e} - KD_H] + g(D_H+K) - \frac{c_K}{e}K - c_Q Q.
\end{aligned}
\tag{144}
$$

$$
\begin{aligned}
\frac{\partial \Pi_F(Q,K)}{\partial Q} =& \frac{g}{Q^2}[\frac{D_H^2}{2} + \frac{K^2}{2e} + KD_H] - c_Q,\\
\rightarrow Q_F^* =& \sqrt{\frac{g}{c_Q}[\frac{D_H^2}{2} + \frac{{K_F^*}^2}{2e} + K_F^* D_H]},\\
\rightarrow Q_{F,K=0}^* =& \sqrt{\frac{g}{c_Q}\frac{D_H^2}{2}}.
\end{aligned}
\tag{145}
$$

$$
\frac{\partial^2 \Pi_F(Q,K)}{\partial^2 Q} = -\frac{g}{Q^3}[D_H^2 + \frac{K^2}{e} + 2KD_H]. \tag{146}
$$

$$
\frac{\partial^2 \Pi_F(Q,K)}{\partial Q \partial K} = \frac{g}{Q^2}[\frac{K}{e} + D_H]. \tag{147}
$$

$$
\begin{aligned}
\frac{\partial \Pi_F(Q,K)}{\partial K} =& \frac{g}{Q}[-D_H - \frac{K}{e}] + g - \frac{c_K}{e}, \\
\to K_F^* =& \max[-D_H e + Q_F^*(e - \frac{c_K}{g}), 0].
\end{aligned} \tag{148}
$$

$$
\frac{\partial^2 \Pi_F(Q,K)}{\partial^2 K} = - \frac{g}{Qe}. \tag{149}
$$

B.5 Additional Analysis of the Quality of the Partial- and Full-discharge Approximations

Below, we discuss the results presented in Table 3 from Section 5.1 in more detail and present additional results on how good both models approximate the tracking model as we vary different parameters of the model.

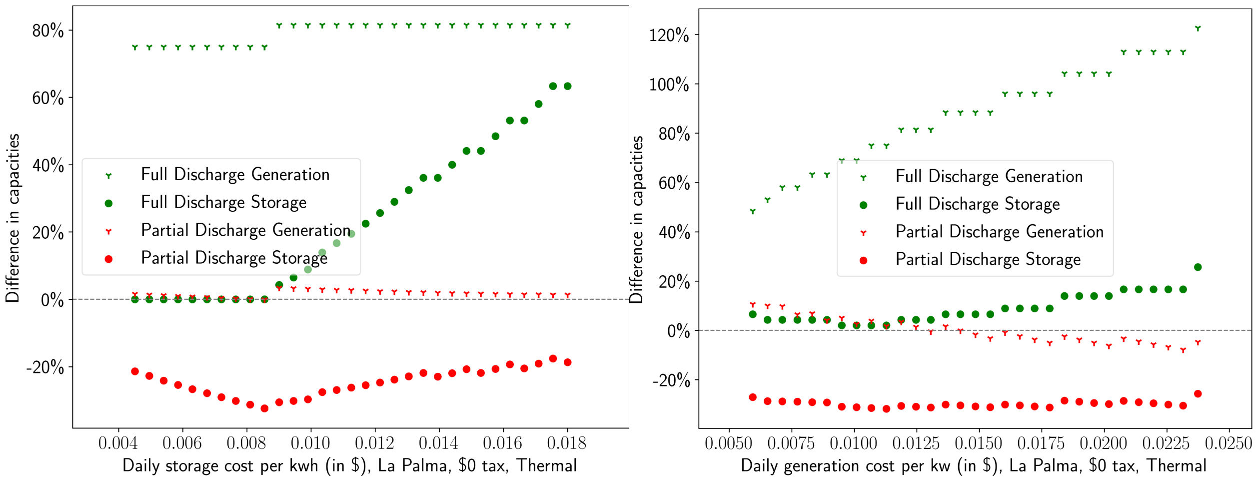
The first and most important observation is that profit-wise, the partial-discharge model is very accurate, and only a few percentage points off relative to the tracking model, with worst-case deviations being only -6% and -2% for Thermal and Battery technologies, respectively. The full-discharge model is not nearly as good, with average deviations around 40-50%. The direction of these deviations is consistent with our analytical findings.

Stepping back, we also observe that overall, the partial-discharge model is much closer to the tracking model than the full-discharge model. This suggests that the full-discharge model, despite being exact for a certain range of game parameters (as per Theorem 3), becomes fairly imprecise outside of that range.

The accuracy of the partial-discharge model carries over from profit to generation, with average and median deviations from the tracking model on the order of 1% to 2%, and worst-case deviation of -8% and -1% for Thermal and Battery technologies, respectively. Gaps increase for storage decisions, with average and median around -30% for Thermal and -2% to -3% for Battery, and worst-case deviations of -35% and -24% respectively. Taken together, these findings point to the partial-discharge model as being more accurate than the full-discharge model.

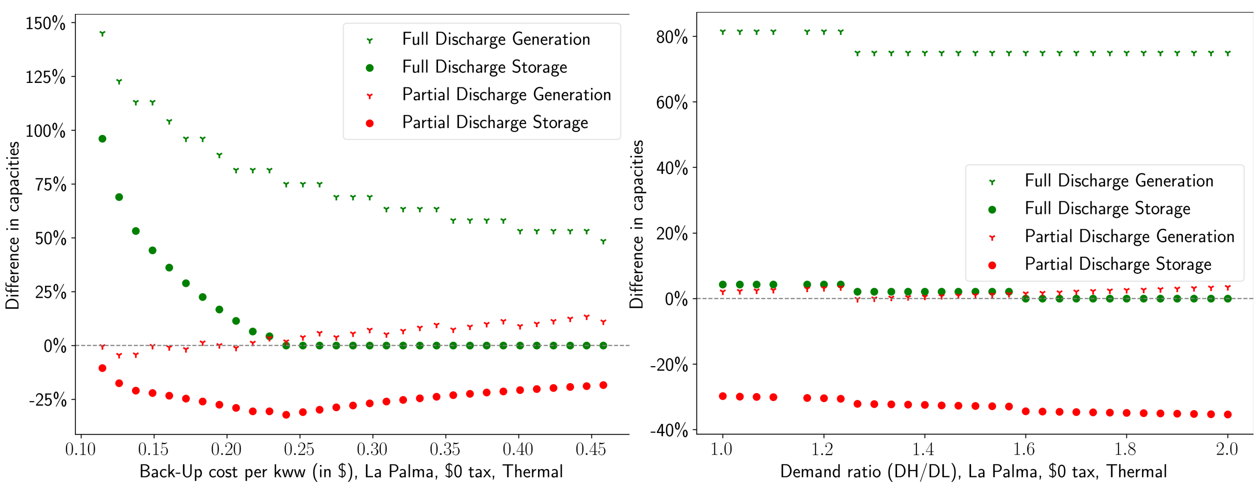
Next, we look at the impact of individual parameters on the approximation quality of our models. Figures 7 and 8 show the model comparison results for La Palma, the largest of the three islands (results for the other markets are similar). Varying the storage and generation costs in Figure 7 confirms that, for a large range of parameters, the partial-discharge model is closer to the simulation results (tracking model) than the full-discharge model.

Figures 8a and 8b demonstrate, once again, the good approximation quality of the partial-discharge model, this time varying the backup cost and demand ratio (D_H/D_L) respectively. All these observations also confirm that the partial- and full-discharge models provide a lower and upper bound, respectively, for the tracking model's storage capacity investment, as discussed in Theorems 4 and 6.



(a) (b)

Figure 7: Storage and Generation Capacity Investment Decisions Under the Partial- and Full-discharge Models, Compared to the Tracking Model, as a Function of Storage Cost (a) and Generation Cost (b)



(a) (b)

Figure 8: Storage and Capacity Investment Decisions Under the Partial- and Full-discharge Models, Compared to the Tracking Model, as a Function of Backup Cost (a) and Demand Ratio (b)

B.6 Decreasing Generation and Storage Cost

Given the consistent decrease in solar generation and storage costs over the past several years, it would be interesting to find out how much cheaper solar generation and storage would need to become in order for a certain fraction of all electricity to be generated from fossil-free sources in a given market.

To this end, we project the current cost-reduction rates of 7% per year for solar and 8% for storage going forward,[12] and analyze how long it would take to profitably reach 70%, 80%, and 90% of renewable generation (i.e., only using the backup for 30%, 20%, or 10% of demand, respectively). Our investigation should therefore yield reasonably good predictions on the evolution of renewable generation rates on islands, in the absence of governmental intervention.

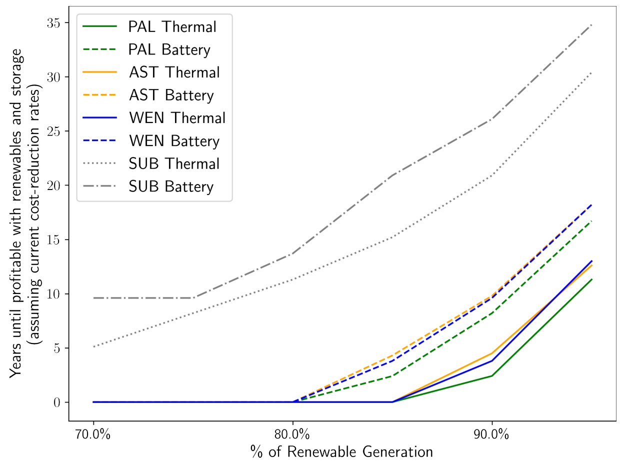


Figure 9: Time Until a Percentage of Renewable Generation Becomes Profitable

Figure 9 shows three core findings under a conservative assumption of zero carbon tax. First, at the comparatively high, unsubsidized electricity prices of the islands, 80% of renewable penetration would already be profitable today. Second, with subsidies, we are five years away from seeing 70% of generation being met by solar and thermal storage (10+ years with batteries). The third, and potentially most important long-term insight, is that while 70-80% penetration may be right around the corner, moving renewable penetration closer to 100% will instead take a considerable amount of time, with or without subsidies. Achieving 95% of renewable generation is multiple decades away in said markets and even at the unsubsidized prices will take more than a decade, depending on technology. This pattern showcases the increasing difficulty of fully replacing the flexible, fossil backup even at very high levels of renewables and storage.

[12]Using historical cost reductions is the best proxy for future cost reductions (while we acknowledge that these reduction rates might not persist exactly at these levels, it should be noted that the overall downward trend for costs has been stable and consistent for more than a decade).