The Timing and Location of Entry in Growing Markets: Subgame Perfection at Work –Online Appendix–

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A Price Competition

Monopoly. Consider a monopolist located at $x \in [0, \frac{1}{2}]$. Taking its location as given, the monopolist sets its price p to maximize profits. Define $\underline{z}$ and $\overline{z}$, where $\underline{z} \leq \overline{z}$, to be the solutions to $a - b(z-x)^2 - p = 0$. That is,

$$\underline{z} = x - \sqrt{\frac{a-p}{b}}, \quad \overline{z} = x + \sqrt{\frac{a-p}{b}}. \tag{1}$$

Note that $\underline{z}$ and $\overline{z}$ are well-defined for all $p \leq a$. Moreover, we have

$$\underline{z} \geq 0 \Leftrightarrow p \geq a - bx^2, \quad \overline{z} \leq 1 \Leftrightarrow p \geq a - b(1-x)^2 \tag{2}$$

and

$$a - bx^2 \geq a - b(1-x)^2 \Leftrightarrow x \in [0, \frac{1}{2}]. \tag{3}$$

Consumers in the set $[\underline{z}, \overline{z}] \cap [0,1]$ prefer good x over the outside good. Demand is thus given by

$$D(p) = \begin{cases} 0 & \text{if} \quad a \leq p, \\ 2\sqrt{\frac{a-p}{b}} & \text{if} \quad a - bx^2 \leq p < a, \\ x + \sqrt{\frac{a-p}{b}} & \text{if} \quad a - b(1-x)^2 \leq p < a - bx^2, \\ 1 & \text{if} \quad p < a - b(1-x)^2. \end{cases} \tag{4}$$

The first case is obvious, the second case corresponds to $0 \leq \underline{z} \leq \overline{z} \leq 1$, the third to $\underline{z} < 0 \leq \overline{z} \leq 1$, and the fourth to $\underline{z} < 0 < 1 < \overline{z}$. We proceed to calculate profits on a case-by-case basis.

Case 1. Clearly, $\pi^* = 0$.

Case 2. The monopolist solves

$$\max_{p \geq 0} 2\sqrt{\frac{a-p}{b}}(p-c). \tag{5}$$

Solving the FOC yields

$$p^* = c + \frac{2}{3}(a-c) \tag{6}$$

and thus

$$\pi^* = \frac{4}{3}\sqrt{\frac{a-c}{3b}}(a-c). \tag{7}$$

We clearly have $p^* < a$. Moreover,

$$a - bx^2 \leq p^* \Leftrightarrow x \geq \sqrt{\frac{a-c}{3b}}. \tag{8}$$

Case 3. The monopolist solves

$$\max_{p \geq 0} \left(x + \sqrt{\frac{a-p}{b}} \right) (p-c). \tag{9}$$

Solving the FOC yields

$$p^* = c + \frac{2}{3}(a-c) + \frac{2bx}{9} \left(\sqrt{x^2 + \frac{3(a-c)}{b}} - x \right) \tag{10}$$

and thus

$$\pi^* = \frac{2}{3}x(a-c) + \frac{2b}{27} \left(\left(x^2 + \frac{3(a-c)}{b} \right)^{\frac{3}{2}} - x^3 \right). \tag{11}$$

We have

$$\begin{aligned} & a - b(1-x)^2 \leq p^* < a - bx^2 \\ \Leftrightarrow \quad & 0 < -7x^2 - 2x\sqrt{x^2 + \frac{3(a-c)}{b}} + \frac{3(a-c)}{b} \leq 9 - 18x. \end{aligned} \tag{12}$$

Case 4. Clearly, $\pi^* = a - c - b(1-x)^2$.

While it is in general not possible to determine which of the four cases is associated with a profit maximum, our intuition suggests that the monopolist chooses to fully cover the market whenever gross surplus is sufficiently high. This is confirmed by the following proposition.

Proposition 1. *Let $\frac{a-c}{b} > 3$. Then the monopolist sets price $p^* = a - b(1-x)^2$ and makes profits $\pi^* = a - c - b(1-x)^2$.*

Proof. Case 2 is clearly ruled out. Recall that case 3 requires $-7x^2 - 2x\sqrt{x^2 + 3\xi} + 3\xi - 9 + 18x \leq 0$, where $\xi = \frac{a-c}{b}$. Differentiating the LHS of this inequality with respect to ξ yields

$$3 \left(1 - \frac{x}{\sqrt{x^2 + 3\xi}} \right) > 0. \tag{13}$$

Moreover, the LHS is zero at $\xi = x^2 - 4x + 3 \leq 3$. This rules out case 3. Recall that case 1 yields profits of zero. Profits in case 4 attain their minimum of $a - c - b$ at $x = 0$. Hence, profits in case 4 exceed those in case 1 whenever $\frac{a-c}{b} > 1$. □

Consider a consumer located at z. His utility is $a - b(z-x)^2 - p^*$. Integrating over all consumers yields instantaneous consumer surplus

$$\sigma^M(x) = \int_0^1 a - b(z-x)^2 - p^* dz = b\left(\frac{2}{3} - x \right). \tag{14}$$

Consider an omnipotent social planner that controls price and thus sets $p = c$ (or any other price for that matter because the price is a transfer from the consumers to the monopolist). Instantaneous social surplus, defined as gains from trade net of transportation costs, is

$$\omega^M(x) = \int_0^1 a - c - b(z-x)^2 dx = a - c - b\left(x^2 - x + \frac{1}{3}\right). \tag{15}$$

Duopoly. There are two firms $i \in \{1,2\}$. Firm i is located at $x_i \in [0,1]$. We assume $x_1 \leq x_2$. Taking both locations as given, firm i sets its price p_i to maximize profits. Define $\tilde{z}$ to be the solution to $a - b(z-x_1)^2 - p_1 = a - b(z-x_2)^2 - p_2$. That is,

$$\tilde{z} = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2b(x_2 - x_1)}. \tag{16}$$

For now we assume that $\tilde{z} \in [0,1]$ and that the market is fully covered. Later on we provide conditions such that this is the case in equilibrium.

Consumers in the set $[0,\tilde{z}]$ prefer firm 1 over firm 2, consumers in the set $[\tilde{z},1]$ prefer firm 2 over firm 1. Demand is thus given by $D_1(p_1,p_2) = \tilde{z}$ and $D_2(p_1,p_2) = 1 - \tilde{z}$. The profit maximization problems of firms 1 and 2 are $\max_{p_1 \geq 0} D_1(p_1,p_2)(p_1 - c)$ and $\max_{p_2 \geq 0} D_2(p_1,p_2)(p_2 - c)$, respectively. Solving the system of FOCs yields

$$p_1^* = c + \frac{b(x_2 - x_1)}{3}(2 + x_1 + x_2), \tag{17}$$

$$p_2^* = c + \frac{b(x_2 - x_1)}{3}(4 - x_1 - x_2) \tag{18}$$

and thus

$$\pi_1^* = \frac{b(x_2 - x_1)}{18}(2 + x_1 + x_2)^2, \tag{19}$$

$$\pi_2^* = \frac{b(x_2 - x_1)}{18}(4 - x_1 - x_2)^2. \tag{20}$$

It remains to ensure that the prices in equations (17) and (18) constitute an equilibrium.

Proposition 2. *Let $\frac{a-c}{b} > \frac{5}{4}$. Then the duopolists set prices $p_1^* = c + \frac{b(x_2-x_1)}{3}(2+x_1+x_2)$ and $p_2^* = c + \frac{b(x_2-x_1)}{3}(4-x_1-x_2)$ and make profits $\pi_1^* = \frac{b(x_2-x_1)}{18}(2+x_1+x_2)^2$ and $\pi_2^* = \frac{b(x_2-x_1)}{18}(4-x_1-x_2)^2$.*

Proof. Note that at p_1^* and p_2^* we have $\tilde{z} = \frac{1}{3} + \frac{1}{6}(x_1 + x_2) \in [0,1]$. It remains to check that all consumers strictly prefer one of the inside goods over the outside good.[1] It suffices to consider the consumers located at $z \in \{0, \tilde{z}, 1\}$. At $z = 0$ we have

$$a - bx_1^2 - p_1^* > 0 \Leftrightarrow \frac{a-c}{b} > \frac{2}{3}x_1^2 - \frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_2^2, \tag{21}$$

[1] We use strict inequalities here to rule out so-called kinked equilibria (?).

where the RHS of the second inequality has a maximum of 1. At $z = 1$ we have

$$a - b(1 - x_2)^2 - p_2^* > 0 \Leftrightarrow \frac{a - c}{b} > \frac{1}{3}x_1^2 - \frac{4}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_2^2 + 1, \tag{22}$$

where the RHS again has a maximum of 1. Finally, at $z = \tilde{z}$ we have

$$\begin{aligned} & a - b(\tilde{z} - x_1)^2 - p_1^* = a - b(\tilde{z} - x_2)^2 - p_2^* > 0 \\ \Leftrightarrow \quad & \frac{a - c}{b} > \frac{13}{16}x_1^2 - \frac{11}{9}x_1 - \frac{5}{18}x_1x_2 + \frac{7}{9}x_2 + \frac{13}{16}x_2^2 + \frac{1}{9}, \end{aligned} \tag{23}$$

where the RHS has a maximum of $\frac{5}{4}$. □

Consider first a consumer located at $z \leq \tilde{z}$. Her utility is $a - b(z - x_1)^2 - p_1^*$. Next consider a consumer located at $z \geq \tilde{z}$. Her utility is $a - b(z - x_2)^2 - p_2^*$. Integrating over all consumers yields instantaneous consumer surplus

$$\begin{aligned} \sigma^D(x_1, x_2) &= \int_0^{\tilde{z}} a - b(z - x_1)^2 - p_1^* dz + \int_{\tilde{z}}^1 a - b(z - x_2)^2 - p_2^* dz \\ &= a - c - \frac{b}{3} + \frac{b(x_2 - x_1)}{36}(x_1 + x_2)^2 - \frac{b}{9}\left(4x_1^2 - 11x_1 + 2x_2 + 5x_2^2\right). \end{aligned} \tag{24}$$

Consider an omnipotent social planner that controls prices and thus sets $p_1 = p_2 = c$. Consumers to the left of $\frac{x_1+x_2}{2}$ are served by firm 1, consumers to the right by firm 2. Instantaneous social surplus therefore is

$$\begin{aligned} \omega^D(x_1, x_2) &= \int_0^{\frac{x_1+x_2}{2}} a - c - b(z - x_1)^2 dz + \int_{\frac{x_1+x_2}{2}}^1 a - c - b(z - x_2)^2 dz \\ &= a - c - \frac{b}{3} + \frac{b(x_2 - x_1)}{4}(x_1 + x_2)^2 + bx_2(1 - x_2). \end{aligned} \tag{25}$$

B The Leader's Problem

To characterize the two subsets $\underline{X}_1(t_1)$ and $\overline{X}_1(t_1)$, consider the equation $t_1 = m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)$. Recall that $m' > 0$ and $\frac{\partial \pi_2^D}{\partial x_1} < 0$ if $x_1 \leq x_2$. Hence, provided that it exists, the solution $\tilde{x}_1(t_1)$ to this equation is unique. Moreover, we have $\tilde{x}_1' > 0$ and, by construction, $\tilde{x}_1(t_1) = 0$ at $t_1 =$

$m^{-1}\left(\frac{rF}{\pi_2^D(0,1)}\right)$ and $\tilde{x}_1(t_1) = \frac{1}{2}$ at $t_1 = m^{-1}\left(\frac{rF}{\pi_2^D(\frac{1}{2},1)}\right)$. $\underline{X}_1(t_1)$ and $\overline{X}_1(t_1)$ are therefore given by

$$\underline{X}_1(t_1) = \begin{cases} \varnothing & \text{if} \quad t_1 \in \left[0, m^{-1}\left(\frac{rF}{\pi_2^D(0,1)}\right)\right), \\ [0, \tilde{x}_1(t_1)] & \text{if} \quad t_1 \in \left[m^{-1}\left(\frac{rF}{\pi_2^D(0,1)}\right), m^{-1}\left(\frac{rF}{\pi_2^D(\frac{1}{2},1)}\right)\right], \\ [0, \frac{1}{2}] & \text{if} \quad t_1 \in \left(m^{-1}\left(\frac{rF}{\pi_2^D(\frac{1}{2},1)}\right), \infty\right), \end{cases} \tag{26}$$

$$\overline{X}_1(t_1) = \begin{cases} [0, \frac{1}{2}] & \text{if} \quad t_1 \in \left[0, m^{-1}\left(\frac{rF}{\pi_2^D(0,1)}\right)\right), \\ [\tilde{x}_1(t_1), \frac{1}{2}] & \text{if} \quad t_1 \in \left[m^{-1}\left(\frac{rF}{\pi_2^D(0,1)}\right), m^{-1}\left(\frac{rF}{\pi_2^D(\frac{1}{2},1)}\right)\right], \\ \varnothing & \text{if} \quad t_1 \in \left(m^{-1}\left(\frac{rF}{\pi_2^D(\frac{1}{2},1)}\right), \infty\right). \end{cases} \tag{27}$$

We consider the two subsets $\underline{X}_1(t_1)$ and $\overline{X}_1(t_1)$ in turn.

Case 1: Immediate Entry by the Follower. On the set $\underline{X}_1(t_1)$, we have $t_2^*(t_1, x_1) = t_1$, which is independent of x_1, and the leader's problem simplifies to

$$\max_{x_1 \in \underline{X}_1(t_1)} \int_{t_1}^{\infty} e^{-rt}\pi_1^D(x_1, 1)m(t)dt - e^{-rt_1}F. \tag{28}$$

As long as $t_1 < \infty$, this is equivalent to maximizing instantaneous profits. The solution to the leader's problem on the set $\underline{X}_1(t_1)$ is thus $x_1^\circ(1) = 0$.

Case 2: Deferred Entry by the Follower. On the set $\overline{X}_1(t_1)$, we have $t_2^*(t_1, x_1) = m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)$, which is independent of t_1, and the leader's problem simplifies to

$$\begin{aligned} \max_{x_1 \in \overline{X}_1(t_1)} \quad & \int_{t_1}^{m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)} e^{-rt}\pi^M(x_1)m(t)dt \\ & + \int_{m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)}^{\infty} e^{-rt}\pi_1^D(x_1, 1)m(t)dt - e^{-rt_1}F. \end{aligned} \tag{29}$$

The derivative with respect to x_1 is

$$\begin{aligned} \frac{\partial(.)}{\partial x_1} = \; & \int_{t_1}^{m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)} e^{-rt}\pi^{M\prime}(x_1)m(t)dt \\ & - e^{-rm^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)}\pi^M(x_1)\frac{(rF)^2}{\pi_2^D(x_1,1)^3}\frac{1}{m'\left(m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)\right)}\frac{\partial\pi_2^D(x_1,1)}{\partial x_1} \\ & + \int_{m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)}^{\infty} e^{-rt}\frac{\partial\pi_1^D(x_1,1)}{\partial x_1}m(t)dt \\ & + e^{-rm^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)}\pi_1^D(x_1,1)\frac{(rF)^2}{\pi_2^D(x_1,1)^3}\frac{1}{m'\left(m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right)\right)}\frac{\partial\pi_2^D(x_1,1)}{\partial x_1}. \end{aligned} \tag{30}$$

The leader's location decision is governed by three considerations. First, by moving towards the center of the market, the leader increases its profits from the monopoly phase since $\pi^{M\prime} > 0$ (first term). Second, the leader decreases its profits from the duopoly phase since $\frac{\partial \pi_1^D}{\partial x_1} < 0$ (third term). Third, by moving towards the center of the market, the leader deters entry by the follower. This increases the duration of the monopoly phase (second term) and decreases the duration of the duopoly phase (fourth term) since $\frac{\partial \pi_2^D}{\partial x_1} < 0$. The net effect is positive since $\pi^M(x_1) > \pi_1^D(x_1, 1)$.

Let $\overline{X}_1^*(t_1)$ denote the set of solutions to the leader's problem on the set $\overline{X}_1(t_1)$. While the solution to the leader's problem may not be unique, we can determine how $\overline{X}_1^*(t_1)$ shifts with t_1. The following lemma provides conditions under which $\overline{X}_1^*(t_1)$ shifts to the left.

Lemma 1. *Suppose $t_1 < t_1'$, $x_1 \in \overline{X}_1^*(t_1)$, and $x_1 < x_1'$. If $x_1 \in \overline{X}_1(t_1')$, then $x_1' \notin \overline{X}_1^*(t_1')$. Moreover, if $x_1 \in \text{int}\overline{X}_1(t_1')$, then $x_1 \notin \overline{X}_1^*(t_1')$.*

Proof. The cross-partial derivative of V_1 with respect to x_1 and t_1 is

$$\frac{\partial^2 V_1}{\partial x_1 \partial t_1} = -e^{-rt_1}\pi^{M\prime}(x_1)m(t_1) \leq 0 \tag{31}$$

with strict inequality whenever $t_1 > 0$.

To see that $x_1' \notin \overline{X}_1^*(t_1')$ note that

$$\begin{aligned}
& V_1\left(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1',1)}\right), x_1', 1\right) - V_1\left(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right), x_1, 1\right) \\
= & \int_{x_1}^{x_1'} \frac{\partial V_1\left(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(s,1)}\right), s, 1\right)}{\partial x_1} ds \\
< & \int_{x_1}^{x_1'} \frac{\partial V_1\left(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(s,1)}\right), s, 1\right)}{\partial x_1} ds \\
= & V_1\left(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1',1)}\right), x_1', 1\right) - V_1\left(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right), x_1, 1\right) \\
\leq & \ 0,
\end{aligned} \tag{32}$$

where the first inequality follows since the cross-partial derivative of V_1 with respect to x_1 and t_1 is negative whenever $t_1 > 0$ and the second inequality follows because $x_1 \in \overline{X}_1^*(t_1)$ (i.e., x_1 is optimal at t_1) and $x_1 < x_1'$ implies $x_1' \in \overline{X}_1(t_1)$ (i.e., x_1' is feasible at t_1). Since $x_1 \in \overline{X}_1(t_1')$ (i.e., x_1 is feasible at t_1') and $V_1\left(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1',1)}\right), x_1', 1\right) < V_1\left(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right), x_1, 1\right)$, x_1' cannot be a solution to the leader's problem at t_1'. This proves the first part of the claim.

To prove the second part of the claim, note that if $x_1 \in \text{int}\overline{X}_1(t_1') \subseteq \text{int}\overline{X}_1(t_1)$, then we have

$$0 = \frac{\partial V_1\left(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right), x_1, 1\right)}{\partial x_1} > \frac{\partial V_1\left(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right), x_1, 1\right)}{\partial x_1}. \tag{33}$$

Hence, x_1 cannot be a solution to the leader's problem at t_1'. □

C SPE

Region 5. Consider the subgame starting at time $t_1 \in \left(m^{-1}\left(\frac{144rF}{25b}\right), \infty\right)$. We have $L(t_1) = V_1(t_1, t_1, 0, 1) = V_2(t_1, 0, 1) = F(t_1)$. Moreover, we have $L' = F' < 0$ and, in fact, $\lim_{t_1 \to \infty} L(t_1) = \lim_{t_1 \to \infty} F(t_1) = 0$ because of discounting for all $t_1' > t_1$. This implies

$$L(t_1) > L(t_1') = F(t_1'). \tag{34}$$

Hence, it is optimal for a firm to enter at time t_1 and location $x_1 \in X_1^*(t_1) = \{0\}$. Moreover, it is optimal to enter irrespective of the opponent's strategy.

Region 4. Consider the subgame starting at time $t_1 \in \left[m^{-1}\left(\frac{2rF}{b}\right), m^{-1}\left(\frac{144rF}{25b}\right)\right]$. Note that

$$L(t_1) \geq F(t_1) \tag{35}$$

for all t_1. The reason is that the leader is always free to locate at the extreme of the market, which then causes the follower to enter immediately and leads to equal payoffs for both firms. More formally, we have $L(t_1; x_1) \geq L(t_1; 0) = F(t_1; 0) \geq F(t_1; x_1)$ for all $x_1 \in X_1^*(t_1)$ and all t_1. Moreover, as Proposition 3 below shows, we have

$$L(t_1) > L(t_1') \tag{36}$$

for all $t_1' > t_1$. Hence, we have

$$L(t_1) > \max\left\{L(t_1'), F(t_1')\right\} \tag{37}$$

for all $t_1' > t_1$, and it is optimal for a firm to enter at time t_1 and location $x_1 \in X_1^*(t_1)$ irrespective of the opponent's strategy.

Note that, to the extent that $X_1^*(t_1)$ is not a singleton, there may be multiplicity, but this multiplicity has no impact on the outcome of the SPE because, as we will show, the time of first entry is prior to t_1.

Proposition 3. *Suppose $t_1, t_1' \in \left[m^{-1}\left(\frac{2rF}{b}\right), m^{-1}\left(\frac{144rF}{25b}\right)\right]$ and $t_1 < t_1'$. Then $L(t_1) > L(t_1')$.*

Proof. Note that

$$L(t_1) = \max\left\{V_1(t_1, t_1, 0, 1), \max_{x_1 \in \overline{X}_1(t_1)} V_1\left(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1, 1)}\right), x_1, 1\right)\right\}. \tag{38}$$

We show that both arguments of the maximum operator are decreasing.

Consider the first argument. Differentiating V_1 with respect to t_1 yields

$$-e^{-rt_1}\pi_1^D(0,1)m(t_1) + re^{-rt_1}F = -e^{-rt_1}\left(\pi_1^D(0,1)m(t_1) - rF\right) \leq 0 \tag{39}$$

whenever $t_1 \geq m^{-1}\left(\frac{rF}{\pi_1^D(0,1)}\right) = m^{-1}\left(\frac{2rF}{b}\right)$ with strict inequality whenever $t_1 > m^{-1}\left(\frac{rF}{\pi_1^D(0,1)}\right) = m^{-1}\left(\frac{2rF}{b}\right)$.

Consider the second argument. Suppose to the contrary that the second argument is nondecreasing. Then there exists $x_1 \in \overline{X}_1^*(t_1)$ and $x_1' \in \overline{X}_1^*(t_1')$ such that

$$V_1(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1,1)}\right), x_1, 1) - V_1(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1',1)}\right), x_1', 1) \leq 0. \tag{40}$$

Since $\overline{X}_1(t_1') \subseteq \overline{X}_1(t_1)$, this implies

$$V_1(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1',1)}\right), x_1', 1) - V_1(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1',1)}\right), x_1', 1) \leq 0. \tag{41}$$

Using A to denote the LHS of the above inequality, we have

$$A \equiv \int_{t_1}^{t_1'} e^{-rt}\pi^M(x_1')m(t)dt - e^{-rt_1}F + e^{-rt_1'}F. \tag{42}$$

To establish a contradiction, it remains to show that $A > 0$. Clearly, $A = 0$ for $t_1' = t_1$. Differentiating A with respect to t_1' yields

$$e^{-rt_1'}\pi^M(x_1')m(t_1') - re^{-rt_1'}F = e^{-rt_1'}\left(\pi^M(x_1')m(t_1') - rF\right) > 0 \tag{43}$$

whenever $t_1' > m^{-1}\left(\frac{rF}{\pi^M(x_1')}\right) \in \left[m^{-1}\left(\frac{rF}{a-c-\frac{b}{4}}\right), m^{-1}\left(\frac{rF}{a-c-b}\right)\right]$. Our assumption that $t_1' \in \left[m^{-1}\left(\frac{2rF}{b}\right), m^{-1}\left(\frac{144rF}{25b}\right.\right.$ ensures this. Thus $A > 0$ whenever $t_1' > t_1$. □

Region 3. Consider the subgame starting at time $t_1 \in \left(m^{-1}\left(\frac{rF}{a-c-b}\right), m^{-1}\left(\frac{2rF}{b}\right)\right)$. Corollary 1 says that

$$F(t_1) \leq F(t_1') \tag{44}$$

for all $t_1' > t_1$. Moreover, Proposition 4 below shows that

$$L(t_1) > L(t_1') \tag{45}$$

for all $t_1' > t_1$. From equation (35) we also know that

$$L\left(m^{-1}\left(\frac{2rF}{b}\right)\right) \geq F\left(m^{-1}\left(\frac{2rF}{b}\right)\right). \tag{46}$$

It follows that

$$L(t_1) \geq F(t_1) \tag{47}$$

for all t_1. (In fact, the inequality is strict.) Hence, we again have

$$L(t_1) > \max\left\{L(t_1'), F(t_1')\right\} \tag{48}$$

for all $t_1' > t_1$, and it is optimal for a firm to enter at time t_1 and location $x_1 \in X_1^*(t_1)$ irrespective of the opponent's strategy. The comment on multiplicity made at the end of our discussion of region 4 again applies.

Proposition 4. *Suppose* $t_1, t_1' \in \left(m^{-1}\left(\frac{rF}{a-c-b}\right), m^{-1}\left(\frac{2rF}{b}\right)\right)$ *and* $t_1 < t_1'$. *Then* $L(t_1) > L(t_1')$.

Proof. Similar to the proof of Proposition 3. □

Region 2b. Consider the subgame starting at time $t_1 \in \left[t_1^*, m^{-1}\left(\frac{rF}{a-c-b}\right)\right]$. We have $L(t_1) > L(t_1')$ by Assumption 3 and, by Corollary 1, $F(t_1) \leq F(t_1')$ for all $t_1' > t_1$. Consequently, we can repeat the argument for region 3 given above to establish that it is optimal for a firm to enter at time t_1 and location $x_1 \in X_1^*(t_1)$ irrespective of the opponent's strategy. The comment on multiplicity made at the end of our discussion of region 4 again applies.

Region 1. While $L(.)$ is decreasing in regions 3, 4, and 5, $L(.)$ is increasing in region 1.

Proposition 5. *Suppose* $t_1, t_1' \in \left[0, m^{-1}\left(\frac{rF}{a-c-\frac{b}{4}}\right)\right)$ *and* $t_1 < t_1'$. *Then* $L(t_1) < L(t_1')$.

Proof. Note that

$$L(t_1) = \max_{x_1 \in \overline{X}_1(t_1)} V_1(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1, 1)}\right), x_1, 1). \tag{49}$$

Suppose to the contrary that this is nonincreasing. Then there exists $x_1 \in \overline{X}_1^*(t_1)$ and $x_1' \in \overline{X}_1^*(t_1')$ such that

$$V_1(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1, 1)}\right), x_1, 1) - V_1(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1', 1)}\right), x_1', 1) \geq 0. \tag{50}$$

Since $\overline{X}_1(t_1) = \overline{X}_1(t_1') = \left[0, \frac{1}{2}\right]$, this implies

$$V_1(t_1, m^{-1}\left(\frac{rF}{\pi_2^D(x_1, 1)}\right), x_1, 1) - V_1(t_1', m^{-1}\left(\frac{rF}{\pi_2^D(x_1, 1)}\right), x_1, 1) \geq 0. \tag{51}$$

Using A to denote the LHS of the above inequality, we have

$$A \equiv \int_{t_1}^{t_1'} e^{-rt}\pi^M(x_1)m(t)dt - e^{-rt_1}F + e^{-rt_1'}F. \tag{52}$$

To establish a contradiction, it remains to show that $A < 0$. Clearly, $A = 0$ for $t_1' = t_1$. Differentiating A with respect to t_1' yields

$$e^{-rt_1'}\pi^M(x_1)m(t_1') - re^{-rt_1'}F = e^{-rt_1'}\left(\pi^M(x_1)m(t_1') - rF\right) < 0 \tag{53}$$

whenever $t_1' < m^{-1}\left(\frac{rF}{\pi^M(x_1)}\right) \in \left[m^{-1}\left(\frac{rF}{a-c-\frac{b}{4}}\right), m^{-1}\left(\frac{rF}{a-c-b}\right)\right]$. Our assumption that $t_1' \in \left[0, m^{-1}\left(\frac{rF}{a-c-\frac{b}{4}}\right)\right)$ ensures this. Thus $A < 0$ whenever $t_1' > t_1$. □

D Additional Proofs

Proof of Proposition 1. Fix x_1. We start by showing that $t_2^*(x_1) < \infty$. Note that $\lim_{t_2\to\infty} V_2(t_2, x_1, x_2) = 0$. Suppose now that the follower enters at time T in location $x_2^\circ(x_1)$. This guarantees it instantaneous profits of at least $\pi_2^D(x_1, x_2^\circ(x_1))$ from time T on. Note that $\pi_2^D(x_1, x_2^\circ(x_1)) \geq \pi_2^D(\frac{1}{2}, 1)$. The NPV of the follower's payoffs is thus at least

$$\int_T^\infty e^{-rt}\pi_2^D(\frac{1}{2}, 1)m(t)dt - e^{-rT}F \geq e^{-rT}\left(\frac{\pi_2^D(\frac{1}{2}, 1)m(T)}{r} - F\right) > 0, \tag{54}$$

where the last inequality is due to Assumption 2. Hence, the follower enters in finite time. This in turn implies that any solution to its problem entails maximizing its instantaneous profits. Thus $x_2^*(x_1) \in x_2^\circ(x_1)$.

The second derivative of V_2 with respect to t_2 is

$$\frac{\partial^2 V_2}{\partial t_2^2} = re^{-rt_2}\pi_2^D(x_1, x_2)m(t_2) - e^{-rt_2}\pi_2^D(x_1, x_2)m'(t_2) - r^2e^{-rt_2}F. \tag{55}$$

Note that

$$\frac{\partial V_2}{\partial t_2} = 0 \Rightarrow \frac{\partial^2 V_2}{\partial t_2^2} = -e^{-rt_2}\pi_2^D(x_1, x_2)m'(t_2) < 0 \tag{56}$$

for all $x_2 \in x_2^\circ(x_1)$. Hence, V_2 is strictly quasiconcave in t_2 for all $x_2 \in x_2^\circ(x_1)$, which ensures that $t_2^*(x_1)$ is uniquely pinned down. Moreover, at $t_2 = 0$, equation (10) reduces to

$$rF > 0 \tag{57}$$

because of Assumption 1. Evaluating equation (10) at $t_2 = T$ yields

$$-e^{-rT}\pi_2^D(x_1, x_2)m(T) + re^{-rT}F \leq -e^{-rT}\left(\pi_2^D(\frac{1}{2}, 1)m(T) - rF\right) < 0 \tag{58}$$

for all $x_2 \in x_2^\circ(x_1)$, where the first inequality follows because $\pi_2^D(\frac{1}{2}, 1)$ is the minmax instantaneous profit of firm 2 and the second inequality follows because of Assumption 2. It follows that equation (10) has a zero in the interval $(0, T)$ for all $x_2 \in x_2^\circ(x_1)$. This zero in turn determines $t_2^*(x_1)$. □

Proof of Proposition 2. The second derivative of V_1 with respect to t_1 is

$$\frac{\partial^2 V_1}{\partial t_1^2} = re^{-rt_1}\pi^M(x_1)m(t_1) - e^{-rt_1}\pi^M(x_1)m'(t_1) - r^2 e^{-rt_1}F. \tag{59}$$

Note that

$$\frac{\partial V_1}{\partial t_1} = 0 \Rightarrow \frac{\partial^2 V_1}{\partial t_1^2} = -e^{-rt_1}\pi^M(x_1)m'(t_1) < 0 \tag{60}$$

for all x_1. Hence, V_1 is strictly quasiconcave in t_1 for all x_1, which ensures that t_1^* is uniquely pinned down. Moreover, at $t_1 = 0$, equation (13) reduces to

$$rF > 0 \tag{61}$$

because of Assumption 1. Evaluating equation (13) at $t_1 = T$ yields

$$-e^{-rT}\pi^M(x_1)m(T) + re^{-rT}F < -e^{-rT}\left(\pi_2^D(\frac{1}{2},1)m(T) - rF\right) < 0, \tag{62}$$

where the first inequality follows because $\pi^M(x) > \pi_i^D(x_1, x_2)$ and the second inequality follows because of Assumption 2. It follows that equation (13) has a zero in the interval $(0, T)$ for all x_1. This zero in turn determines t_1^*. □

Proof of Proposition 3. The leader is always free to choose time $t_2^*(0) - \epsilon$ and location 0, where ϵ is small but positive. Hence,

$$V_1(t_1^*, t_2^*(x_1^*), x_1^*, 1) \geq A + B, \tag{63}$$

where

$$A = \int_{t_2^*(0)-\epsilon}^{t_2^*(0)} e^{-rt}\pi^M(0)m(t)dt - e^{-r(t_2^*(0)-\epsilon)}F + e^{-rt_2^*(0)}F, \tag{64}$$

$$B = \int_{t_2^*(0)}^{\infty} e^{-rt}\pi_1^D(0,1)m(t)dt - e^{-rt_2^*(0)}F. \tag{65}$$

Since $\pi_1^D(0,1) = \pi_2^D(0,1)$, the leader makes the same profits as the follower during the duopoly phase. Hence, $B = V_2(t_2^*(0), 0, 1) \geq V_2(t_2^*(x_1^*), x_1^*, 1)$, where the inequality follows from applying the envelope theorem to the follower's problem in equation (9) and the fact that $\frac{\partial \pi_2^D}{\partial x_1} < 0$ if $x_1 \leq x_2$. It remains to show that $A > 0$ for some $\epsilon > 0$. Clearly, $A = 0$ if $\epsilon = 0$. Differentiating A with respect to ϵ and evaluating the result at $\epsilon = 0$ yields

$$e^{-rt_2^*(0)}\pi^M(0)m(t_2^*(0)) - re^{-rt_2^*(0)}F > e^{-rt_2^*(0)}\left(\pi_2^D(0,1)m(t_2^*(0)) - rF\right) = 0, \tag{66}$$

where the inequality follows because $\pi^M(x) > \pi_i^D(x_1, x_2)$ and the equality follows because of

Proposition 1. The claim follows. □

Proof of Proposition 4. Similar to the proof of Proposition 1. □

Proof of Proposition 6. Similar to the proof of Proposition 3. □

Proof of Proposition 7. (i/ii) Note that $\max_{x_1 \in X_1^*(\hat{t}_1)} F(\hat{t}_1; x_1) \geq \min_{x_1 \in X_1^*(\hat{t}_1)} F(\hat{t}_1; x_1) \geq L(\hat{t}_1)$. If $X_1^*(\hat{t}_1)$ is not a singleton, then the first inequality is strict; if $F(\hat{t}_1) > L(\hat{t}_1)$, then the last inequality is strict. Consider $t_1 = \hat{t}_1 + \epsilon$, where ϵ is small but positive. Since $F(.)$ is nondecreasing, we have $F(t_1) > L(\hat{t}_1)$. Using the fact that $L(.)$ is continuous, this implies $F(t_1) \geq L(t_1)$ provided that ϵ is sufficiently small. Hence, $\hat{t}_1$ cannot be the sup of the set in question.

(iii) Suppose that $X_1^*(\tilde{t}_1)$ is a singleton. Then we have $\min_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1) = \max_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1) \geq L(\tilde{t}_1)$. It follows that $\hat{t}_1 \geq \tilde{t}_1$, a contradiction.

(iv) Suppose that $\max_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1) > L(\tilde{t}_1)$ and consider $t_1 = \tilde{t}_1 + \epsilon$, where ϵ is small but positive. Since $F(.)$ is nondecreasing, we have $F(t_1) > L(\tilde{t}_1)$. Using the fact that $L(.)$ is continuous, this implies $F(t_1) \geq L(t_1)$ provided that ϵ is sufficiently small. Hence, $\hat{t}_1 > \tilde{t}_1$, a contradiction.

(v) Suppose not. Then there exists $\bar{\epsilon} > 0$ such that $\tilde{t}_1 + \epsilon \in \tilde{T}_1$ for all $0 \leq \epsilon < \bar{\epsilon}$. Using properties (iii) and (iv), we have

$$\max_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1) = L(\tilde{t}_1) > \min_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1), \tag{67}$$

$$\max_{x_1 \in X_1^*(\tilde{t}_1+\epsilon)} F(\tilde{t}_1 + \epsilon; x_1) = L(\tilde{t}_1 + \epsilon) > \min_{x_1 \in X_1^*(\tilde{t}_1+\epsilon)} F(\tilde{t}_1 + \epsilon; x_1). \tag{68}$$

Since $F(.)$ is nondecreasing, this implies

$$L(\tilde{t}_1) = \max_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1) \leq \min_{x_1 \in X_1^*(\tilde{t}_1+\epsilon)} F(\tilde{t}_1 + \epsilon; x_1) < L(\tilde{t}_1 + \epsilon) \tag{69}$$

for all $0 < \epsilon < \bar{\epsilon}$. Taking the limit as $\epsilon \to 0$ and using the fact that $L(.)$ is continuous yields

$$\max_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1) = \min_{x_1 \in X_1^*(\tilde{t}_1)} F(\tilde{t}_1; x_1), \tag{70}$$

a contradiction, since $X_1^*(\tilde{t}_1)$ is not a singleton. □

Proof of part (ii) of Theorem 1. We again focus on regions 1 and 2a in what follows. Working backwards through time, consider the subgame starting at time $t_1 \in \left[\tilde{t}_1, t_1^*\right]$. If a firm enters first at time t_1 and location $\min X_1^*(t_1)$ according to the prescribed strategy, then it gets $L(t_1)$. If the firm deviates from the prescribed strategy, then its rival enters first and the firm gets $F(t_1; \min X_1^*(t_1))$. We claim that $L(t_1) \geq F(t_1; \min X_1^*(t_1))$, so that the firm has no incentive to deviate from the prescribed strategy.

To see this, recall that $\max_{x_1 \in X_1^*(t_1)} F(t_1; x_1) = F(t_1; \min X_1^*(t_1))$. Suppose first that $t_1 \in \tilde{T}_1$. Then $\max_{x_1 \in X_1^*(t_1)} F(t_1; x_1) = L(t_1)$ by part (iv) of Proposition 7 and the claim follows. Next suppose that $t_1 \notin \tilde{T}_1$. Then $\max_{x_1 \in X_1^*(t_1)} F(t_1; x_1) < L(t_1)$ by construction (see equation (20)) and the claim follows.

Continuing to work backwards through time, consider the subgame starting at time $t_1 \in [0, \tilde{t}_1)$. If a firm does not enter according to the prescribed strategy, then it gets $L(\tilde{t}_1) = F(\tilde{t}_1; \min X_1^*(\tilde{t}_1))$, where the equality follows from part (iv) of Proposition 7 and $\max_{x_1 \in X_1^*(t_1)} F(t_1; x_1) = F(t_1; \min X_1^*(t_1))$. If the firm deviates from the prescribed strategy and enters first at time t_1 at location $x_1 \in [0, \frac{1}{2}]$, then it gets at most $L(t_1)$ (and $L(t_1)$ if $x_1 \in X_1^*(t_1)$). We have $L(t_1) < L(\tilde{t}_1)$ for all $t_1 < \tilde{t}_1$ because $L(.)$ is increasing in region 1 and in region 2a by Assumption 3. Taken together, we have

$$L(t_1) < L(\tilde{t}_1) = F(\tilde{t}_1; \min X_1^*(\tilde{t}_1)). \tag{71}$$

Hence, the firm has no incentive to deviate from the prescribed strategy. □

Proof of Proposition 8. The "cutting" equilibria in part (i) of Theorem 1 entail rent equalization by part (ii) of Proposition 7. In line with our focus on the threshold for market size at the time of first entry, we change variables from entry times $\left(t_1^{SPE}(\gamma), t_2^{SPE}(\gamma)\right)$ to market sizes $\left(m_1^{SPE}(\gamma) = m(t_1^{SPE}(\gamma)), m_2^{SPE}(\gamma) = m(t_2^{SPE}(\gamma))\right)$. We establish below that $\frac{\partial m_1^{SPE}(\gamma)}{\partial \gamma} < 0$, conditional on entry locations $\left(x_1^{SPE}(\gamma), x_2^{SPE}(\gamma) = 1\right)$. To simplify the notation in what follows we omit superscripts and indices.

Because there is deferred entry by the follower in regions 1, 2, and 3, we have $m_1 < m_2$ and

$$H_2(m_1, m_2, \gamma) = m_2 - \frac{rF}{\pi_2^D(x_1, 1)} = 0. \tag{72}$$

Using equations (6), (7), and (8), we write the condition for rent equalization as

$$\begin{aligned} H_1(m_1, m_2, \gamma) = &\left(\frac{\pi^M(x_1)}{r-\gamma} m_1 - F\right) e^{-rm^{-1}(m_1)} \\ &- \left(\frac{\pi^M(x_1) - \pi_1^D(x_1, 1) + \pi_2^D(x_1, 1)}{r-\gamma} m_2 - F\right) e^{-rm^{-1}(m_2)} = 0, \end{aligned} \tag{73}$$

where $t = m^{-1}(m) = \frac{1}{\gamma} \ln m$ is the inverse of $m = m(t) = e^{\gamma t}$. Plugging in yields

$$H(m_1, \gamma) = \left(\frac{\pi^M}{r-\gamma} m_1 - F\right) m_1^{-\frac{r}{\gamma}} - \left(\frac{\pi^M - \pi_1^D + \pi_2^D}{r-\gamma} \frac{rF}{\pi_2^D} - F\right) \left(\frac{rF}{\pi_2^D}\right)^{-\frac{r}{\gamma}} = 0, \tag{74}$$

where we have suppressed the dependency of instantaneous profits on entry locations to simplify the notation. Equation (74) determines how m_1 changes with γ.

We begin by establishing that, given γ, equation (74) has a unique solution in m_1 that is attained in the interval $\left[\frac{(r-\gamma)F}{\pi^M}, \frac{rF}{\pi^M}\right]$. Because $\pi^M > \pi_1^D$, $\pi_2^D > 0$, and $0 < \gamma < r$, we have

$$\frac{\pi^M - \pi_1^D + \pi_2^D}{r-\gamma}\frac{rF}{\pi_2^D} - F = \left(\left(\frac{\pi^M - \pi_1^D}{\pi_2^D} + 1\right)\frac{r}{r-\gamma} - 1\right)F > 0. \tag{75}$$

It follows that at a solution to equation (74), it must be that

$$\frac{\pi^M}{r-\gamma}m_1 - F > 0 \Longleftrightarrow m_1 > \frac{(r-\gamma)F}{\pi^M}. \tag{76}$$

Turning from the lower bound to the upper bound, we have

$$\frac{\partial H}{\partial m_1} = -\pi^M \frac{1}{\gamma} m_1^{-\frac{r}{\gamma}} + F\frac{r}{\gamma}m_1^{-\frac{r}{\gamma}-1} = m_1^{-\frac{r}{\gamma}-1}\frac{1}{\gamma}\left(-\pi^M m_1 + rF\right). \tag{77}$$

Hence, $H(m_1, \gamma)$ attains its maximum at $\widehat{m}_1 = \frac{rF}{\pi^M}$ and is increasing (decreasing) to the left (right) of $\widehat{m}_1$. Moreover, $\widehat{m}_1 < \frac{rF}{\pi_2^D}$ because $\pi^M > \pi_2^D$.

We have $\lim_{m_1 \to 0+} H(m_1, \gamma) = -\infty$. We also have $H\left(\frac{rF}{\pi_2^D}, \gamma\right) \geq 0$, with strict inequality as long as $\pi_1^D > \pi_2^D$, and therefore $H\left(\widehat{m}_1, \gamma\right) > 0$. It follows that equation (74) has a unique solution that is attained in the interval $\left[\frac{(r-\gamma)F}{\pi^M}, \frac{rF}{\pi^M}\right]$.

Next we characterize how the solution in m_1 to equation (74) changes with γ. We obtain $\frac{\partial m_1}{\partial \gamma}$ from the implicity function theorem as

$$\frac{\partial m_1}{\partial \gamma} = -\frac{\partial H/\partial \gamma}{\partial H/\partial m_1}. \tag{78}$$

Hence, showing that $\frac{\partial m_1}{\partial \gamma} < 0$ amounts to showing that $\frac{\partial H}{\partial m_1}$ and $\frac{\partial H}{\partial \gamma}$ have the same sign.

We have $\frac{\partial H}{\partial m_1}$ in equation (77). Because the solution to equation (74) is attained in the interval $\left[\frac{(r-\gamma)F}{\pi^M}, \frac{rF}{\pi^M}\right]$, it follows that $\frac{\partial H}{\partial m_1} \geq 0$, with strict inequality as long as $m_1 < \frac{rF}{\pi^M}$.

Turning from $\frac{\partial H}{\partial m_1}$ to $\frac{\partial H}{\partial \gamma}$, we have

$$\begin{aligned}\frac{\partial H}{\partial \gamma} &= \frac{\pi^M}{(r-\gamma)^2}m_1^{-\frac{r}{\gamma}+1} + \left(\frac{\pi^M}{r-\gamma}m_1 - F\right)\frac{r}{\gamma^2}m_1^{-\frac{r}{\gamma}}\ln m_1 \\ &-\frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2}\frac{rF}{\pi_2^D}\left(\frac{rF}{\pi_2^D}\right)^{-\frac{r}{\gamma}} - \left(\frac{\pi^M - \pi_1^D + \pi_2^D}{r-\gamma}\frac{rF}{\pi_2^D} - F\right)\frac{r}{\gamma^2}\left(\frac{rF}{\pi_2^D}\right)^{-\frac{r}{\gamma}}\ln\left(\frac{rF}{\pi_2^D}\right).\end{aligned} \tag{79}$$

Using equation (74) to rewrite equation (79) yields

$$\begin{aligned}\frac{\partial H}{\partial \gamma} &= \frac{\pi^M}{(r-\gamma)^2}m_1^{-\frac{r}{\gamma}+1} - \frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2}m_2^{-\frac{r}{\gamma}+1} \\ &+ \left(\frac{\pi^M - \pi_1^D + \pi_2^D}{r-\gamma}m_2 - F\right)\frac{r}{\gamma^2}m_2^{-\frac{r}{\gamma}}\ln\left(\frac{m_1}{m_2}\right),\end{aligned} \tag{80}$$

where $m_2 = \frac{rF}{\pi_2^D}$. Our goal is to provide a sufficient condition for $\frac{\partial H}{\partial \gamma} > 0$. To this end, we bound the above expression for $\frac{\partial H}{\partial \gamma}$ from below. We have

$$\frac{\partial H}{\partial \gamma} = m_2^{-\frac{r}{\gamma}+1}\left(\frac{\pi^M}{(r-\gamma)^2}\left(\frac{m_1}{m_2}\right)^{-\frac{r}{\gamma}+1} - \frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2} + \left(\frac{\pi^M - \pi_1^D + \pi_2^D}{r-\gamma} - \frac{F}{m_2}\right)\frac{r}{\gamma^2}\ln\left(\frac{m_1}{m_2}\right)\right) \tag{81}$$

$$\geq m_2^{-\frac{r}{\gamma}+1}\left(\frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2}\left(\frac{m_1}{m_2}\right)^{-\frac{r}{\gamma}+1} - \frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2} + \left(\frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2} - \frac{F}{(r-\gamma)m_2}\right)\frac{r(r-\gamma)}{\gamma^2}\ln\left(\frac{m_1}{m_2}\right)\right) \tag{82}$$

$$= m_2^{-\frac{r}{\gamma}+1}\frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2}\left(\left(\frac{m_1}{m_2}\right)^{-\frac{r}{\gamma}+1} - 1 + \left(1 - \frac{F(r-\gamma)}{(\pi^M - \pi_1^D + \pi_2^D)m_2}\right)\frac{r(r-\gamma)}{\gamma^2}\ln\left(\frac{m_1}{m_2}\right)\right), \tag{83}$$

where the inequality follows from $\pi_1^D \geq \pi_2^D$. Recall that $\ln\left(\frac{m_1}{m_2}\right) < 0$. Hence, to further bound $\frac{\partial H}{\partial \gamma}$ from below, we have to replace $1 - \frac{F(r-\gamma)}{(\pi^M - \pi_1^D + \pi_2^D)m_2}$ by something larger or, equivalently, $\frac{F(r-\gamma)}{(\pi^M - \pi_1^D + \pi_2^D)m_2}$ by something smaller. We know that $m_1 \leq \frac{rF}{\pi^M}$ or, equivalently, that $F(r-\gamma) \geq \frac{\pi^M m_1 (r-\gamma)}{r}$. Hence,

$$\frac{\partial H}{\partial \gamma} \geq m_2^{-\frac{r}{\gamma}+1}\frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2}\left(\left(\frac{m_1}{m_2}\right)^{-\frac{r}{\gamma}+1} - 1 + \left(1 - \frac{\pi^M}{\pi^M - \pi_1^D + \pi_2^D}\frac{m_1}{m_2}\frac{r-\gamma}{r}\right)\frac{r(r-\gamma)}{\gamma^2}\ln\left(\frac{m_1}{m_2}\right)\right). \tag{84}$$

To further bound $\frac{\partial H}{\partial \gamma}$ from below, we again have to replace $1 - \frac{\pi^M}{\pi^M - \pi_1^D + \pi_2^D}\frac{m_1}{m_2}\frac{r-\gamma}{r}$ by something larger or, equivalently, $\frac{\pi^M}{\pi^M - \pi_1^D + \pi_2^D}$ by something smaller. Because $\pi_1^D \geq \pi_2^D$, we have that $\frac{\pi^M}{\pi^M - \pi_1^D + \pi_2^D} \geq 1$. Hence,

$$\frac{\partial H}{\partial \gamma} \geq m_2^{-\frac{r}{\gamma}+1}\frac{\pi^M - \pi_1^D + \pi_2^D}{(r-\gamma)^2}\left(\left(\frac{m_1}{m_2}\right)^{-\frac{r}{\gamma}+1} - 1 + \left(1 - \frac{m_1}{m_2}\frac{r-\gamma}{r}\right)\frac{r(r-\gamma)}{\gamma^2}\ln\left(\frac{m_1}{m_2}\right)\right). \tag{85}$$

The term in brackets is a function of $0 < \frac{\gamma}{r} < 1$ and $0 < \frac{m_1}{m_2} < 1$. It is easily checked numerically that the term in brackets is positive if $\frac{m_1}{m_2} < \frac{1}{5}$ irrespective of the value of $\frac{\gamma}{r}$ or if $\frac{m_1}{m_2} < \frac{1}{4}$ and $\frac{\gamma}{r} < \frac{17}{20}$. Hence, $\frac{\partial H}{\partial \gamma} > 0$ if one of these two conditions holds.

To complete the proof, we provide a sufficient condition for $\frac{m_1}{m_2} < \frac{1}{5}$. Because $m_1 \leq \frac{rF}{\pi^M(x_1)}$ from our earlier analysis of equation (74) and $m_2 = \frac{rF}{\pi_2^D(x_1,1)}$, where we make explicit the dependency of instantaneous profits on entry locations, we have

$$\begin{aligned}\frac{m_1}{m_2} \leq \frac{\pi_2^D(x_1,1)}{\pi^M(x_1)} = \frac{\frac{b(1-x_1)}{18}(4-x_1-1)^2}{a-c-b(1-x_1)^2} &= \frac{(1-x_1)(3-x_1)^2}{18\left(\frac{a-c}{b}-(1-x_1)^2\right)} \\ &\leq \frac{1}{2\left(\frac{a-c}{b}-1\right)},\end{aligned} \tag{86}$$

where the last inequality uses $x_1 \in [0, \frac{1}{2}]$. Hence, a sufficient condition for $\frac{m_1}{m_2} < \frac{1}{5}$ is that $\frac{a-c}{b} > \frac{7}{2}$. □

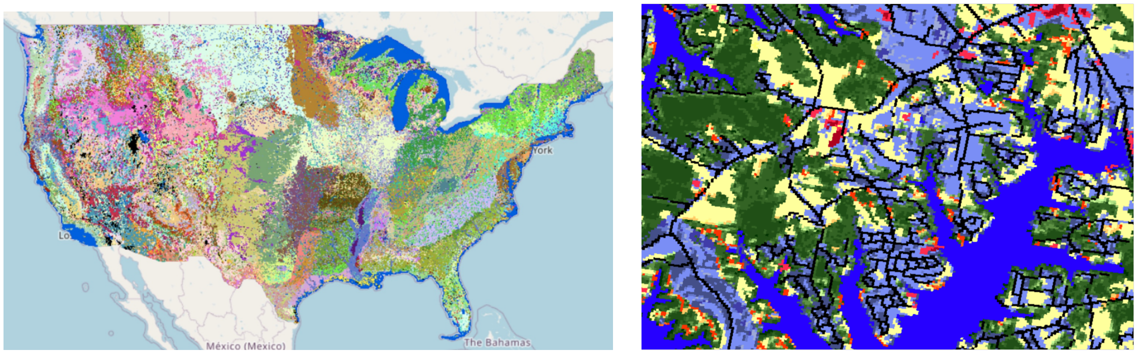
E Data Details

Annual average daily traffic (AADT) is the number of vehicles that pass through a given highway segment during a day, averaged over a calendar year. The segments vary in length and are not aligned with exits and intersections. To allocate AADT to markets, we consider all segments that are within 1000m of any road crossing in the market. Along a highway, we first average the traffic volumes of multiple segments, with each segment's length as weight. This gives the average traffic volume for that highway in the market. We then sum the average traffic volumes of all highways in the market.

For land prices, we use the landpriceindex field from the 2018 LANDDATA.MSA file available from the American Enterprise Institute at `https://www.aei.org/housing/land-price-indicators/`. The land price index normalizes land prices in different MSAs relative to an average value of one. We use the land price for the MSA that is closest to a market in terms of Euclidean distance.

Vegetation data are from the United States Geological Survey LANDFIRE Data Distribution Site at `https://landfire.gov/viewer/viewer.html`. We use vegetation data for 2001, the middle of our sample period. Figure 1 shows these data for the U.S. and for a close-up. The map visualization tool has both a detailed legend and provides the ability to click anywhere on the map to get detailed vegetation data. Measurement tools for distance are also included. Two undergraduate research assistants visually assessed the proportion of each vegetation type within 500m and 1000m of the market center. The correlations in the vegetation measures are between 0.56 and 0.80 and we use the average for each market. We use these proportions to calculate the area (in square kilometers) occupied by each vegetation type. Table 1 provides summary statistics.

Figure 1: 2001 vegetation data. Blue = water, purple/red/black = developed, yellow = field and grass, green = forest, brown=shrub and herb



(a) U.S. (b) Close-up

	Mean	Std. dev.
	500m	
water (sq. km)	0.019	0.037
developed (sq. km)	0.398	0.158
field and grass (sq. km)	0.17	0.139
forest (sq. km)	0.143	0.122
shrub and herb (sq. km)	0.052	0.091
	1000m	
water (sq. km)	0.125	0.166
developed (sq. km)	1.245	0.605
field (sq. km)	0.811	0.592
forest (sq. km)	0.714	0.535
shrub and herb (sq. km)	0.229	0.372
N	1632	

Table 1: 2001 vegetation data summary statistics.