Simple Policies for Managing Flexible Capacity

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October 2014

Abstract

In many production scenarios, a fixed capacity is shared flexibly between multiple products. To manage such multi-product systems, firms need to make two sets of decisions. The first one requires setting an inventory target for each product and the second decision requires dynamically allocating the scarce capacity among the products. It is not known how to make these decisions optimally. In this paper, we propose easily implementable policies that have both theoretical and practical appeal. We first suggest simple and intuitive allocation rules that determine how such scarce capacity is shared. Given such a rule, we calculate the optimal inventory target for each product. We demonstrate analytically that our policies are optimal under two asymptotic regimes represented by high service levels (i.e. high shortage costs) and heavy traffic (i.e. tight capacity). We also demonstrate that our policies outperform current known policies over a wide range of problem parameters. In particular, the cost savings from our policies become more significant as the capacity gets more restrictive.

Keywords: Flexible Capacity, Multiple Products, Allocation Rules, Asymptotic Optimality.

1 Introduction

In many industries such as auto-manufacturing, semiconductors, consumer electronics and pharmaceuticals, a firm's ability to carefully manage its flexible capacity is often a significant factor for its long-term success. Firms that are able to manage flexible capacity efficiently can operate with smaller capacities to satisfy varying market demand across several products. The focus of this

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paper is to provide simple decision rules for managing flexibility efficiently - more specifically, rules for determining how limited capacity can be dynamically allocated across several products.

To achieve our goal, we study a firm that produces multiple products in every period, using a shared resource with limited capacity. We represent the firm's decisions using a periodically reviewed stochastic inventory model. Production occurs at the beginning of each period. A random demand (for each product) occurs during the period. For all products, the unsatisfied demand at the end of any period is backordered. Linear holding and shortage costs are assessed for all products at the end of every period.

We explore the objective of minimizing the long-run average cost per period as the performance measure. This optimization problem comprises of two sets of related decisions. The first one involves setting the target level for each product, and the second requires an allocation rule that determines how the *scarce* capacity is shared among the products. It is well known that performing these two tasks optimally is difficult (more details on this difficulty in the next section). Therefore, in this work, we propose implementable policies that have both theoretical and practical appeal.

Given the mathematical difficulty of analyzing our problem, we take an approach similar in spirit to papers that study limiting regimes of such stochastic control problems. Limiting regimes yield insights on the structure of optimal policies. Policies constructed using this structural insight are then empirically shown to perform well in non-limiting regimes. In the same spirit, we first suggest an intuitive *class* of allocation rules called *weighted balancing rules*. These rules are parametrized by a *weight* for each product, and they determine how the scarce capacity in any period is shared amongst multiple products. For every rule in this class, the optimal target level for each product is obtained directly from an application of the newsvendor formula – we refer to the combination of weighted balancing rules with these target levels as *weighted balancing policies*.

To provide theoretical validity to this class of policies, we study two different asymptotic regimes represented by (*i.*) service levels approaching one (i.e., when shortages are prohibitively expensive), and (*ii.*) utilization approaching one (i.e., when there is little slack between capacity and expected aggregate demand). For each of these two regimes, we identify an allocation rule (i.e. a vector of weights) within our class which is asymptotically optimal under certain assumptions. To investigate how our class of policies performs, in general, we study a set of problems that span a wide range of costs, demand variabilities and capacity utilizations, which are not in the asympototic regime.

1.1 Our Approach

We focus on a class of policies called *stationary base-stock policies*, which we define below. There is a target or base-stock level corresponding to each product. This target is constant, i.e., stationary across periods. At the beginning of a period, let us assume that the inventory level of each product will be at most equal to that product's base-stock level. (It will be easy to see that our definition of a stationary base-stock policy is such that this assumption is satisfied in every period if it is satisfied in the first period). The difference between the inventory level and the base-stock level is called the "opening shortfall". If the aggregate shortfall of all products is smaller than the capacity limit, we produce enough of each product to raise its inventory to its base-stock level. The resulting shortfall ("ending shortfall") is thus zero for every product. If the aggregate shortfall exceeds the capacity, then the entire production capacity is used in such a way that the inventory level of each product does not exceed its base-stock level. This concludes our definition of a stationary base-stock policy which is known to be optimal for the two-product case, albeit, under the finite horizon (Evans, 1967).

We note that, in our discussion of stationary base-stock policies, we have not described *how* the capacity is allocated to the different products in any period in which capacity is insufficient (scarce) for all products to reach their base-stock levels. We refer to such a description as an *allocation rule*. Clearly, even if we restrict our attention to the class of stationary base-stock policies, calculating the exact base-stock levels in an optimal policy within this class entails an understanding of the optimal allocation rule in periods when capacity is scarce. Thus, *even* within this class, the optimal policy involves two interdependent sets of decisions, namely, base-stock levels (production policy) and an allocation rule. The lack of knowledge of the structure of the optimal allocation rule is thus the main stumbling block. To resolve this difficulty, we suggest and work with a set of policies that *decouple* the base-stock and allocation decisions.

The class of policies we advocate is the following. For any given vector of base-stock levels, we raise all inventory levels to the base-stock levels whenever it is feasible to do so. This is possible in periods in which the aggregate shortfall does not exceed the capacity. In periods in which the capacity is insufficient, all the capacity is used. Only in such periods, the allocation rule becomes relevant. An important aspect of the class of allocation rules proposed by us is that, in any period, the *only* state information these decisions require is the opening shortfall of each product (equivalently, the allocation decisions depend on the inventory levels only through the shortfalls). In other words, for a given vector of opening shortfalls, the allocation decisions remain the same for any choice of base-stock levels. For any such allocation rule, the stationary distribution of the vector of ending shortfalls in a period is independent of the base stock levels. This finding has an important implication – the optimal base-stock level for each product can be computed using the newsvendor formula applied to the convolution of that product's demand and its ending shortfall.

Our approach is the following: We restrict attention to simple choices of allocation rules within the aforementioned class, and we choose the base-stock vector corresponding to any particular allocation rule optimally. In the following paragraph, we describe the allocation rules which we propose in detail. We subsequently explain the benefits of our approach.

We use a family of allocation rules which we refer to as *weighted balancing rules*. These rules work as follows. Each product is assigned a strictly positive weight which is constant through time. Next, at the beginning of each period, we rank order the products based on their weighted shortfalls (i.e. the shortfall divided by the weight). We then take the highest ranked product (i.e., the one with the largest weighted shortfall), and use the capacity to bring its weighted shortfall to be equal to the weighted shortfall of the second highest product. Next, we allocate capacity to both these products simultaneously until their weighted shortfalls coincide with the third highest product. We continue this procedure with subsequent products until the entire capacity is exhausted. As mentioned earlier, for any vector of weights, the base-stock level for each product is chosen optimally. This completes the description of a *weighted balancing policy*, given the vector of weights.

We now discuss the issue of choosing the weight vector. One special choice is that all weights are equal to 1 - we call the resulting allocation rule as the *symmetric rule*. At the other extreme are choices of the following type: There is some permutation $\{(1), (2), \ldots, (N)\}$ of the N products such that the weight for (1) $<<$ the weight for (2) $<< \ldots <<$ the weight for (N) (here, we use $<<$ to mean "much smaller than"). Intuitively, such a choice mimics the *priority rule*, i.e. the rule which devotes all the available capacity to (1) until its shortfall is zero and then devotes all the remaining capacity to (2) until its shortfall is zero and so on. Later, we will prove that this priority rule, can be approximated by a suitable weighted balancing policy, for every beginning shortfall. We will show under certain assumptions that the symmetric rule is asymptotically optimal in high service level regimes while the priority rule is asymptotically optimal in heavy traffic. But, in general, the heuristic we propose searches over the space (more precisely, a grid) of weight vectors and picks the best vector – we will refer to the policy of using this weight vector along with the corresponding optimal base-stock levels as the *search policy.* Thus, for every problem instance, the search policy is at least as good as the two asymptotically optimal rules mentioned above; therefore, this policy also has the desired optimality property in both the asymptotic regimes. We conclude this section by summarizing the benefits of the class of weighted balancing policies.

1. In the single product case, our policy (when there is only one product, there is only one policy in this class) is optimal.

2. When all products are symmetric (i.e. they have identical costs and demand distributions), we show (in §4) formally that our policy with symmetric weights is optimal.

3. In high service level regimes, our policy with symmetric weights is asymptotically optimal.

4. In heavy traffic (i.e. when utilization approaches one), the policy with weights chosen to mimic a priority policy is asymptotically optimal.

2 Related Literature

We note that the single product capacitated inventory problem is a special case of our problem. It is well known that a modified base stock policy is optimal for the single product problem as noted in Federgruen and Zipkin (1986).

Not much is known about the the problem with multiple products and limited capacity due to two sets of difficulties – computational and theoretical. From a computational perspective, a dynamic programming approach to solve this problem becomes intractable due to the curse of dimensionality. Providing simple and cost-effective heuristics which scale well to problems with many products is valuable - we will see that the policies we propose have these desirable attributes.

The theoretical difficulty is as follows. In the finite horizon dynamic program for the single product problem, the cost-to-go function is convex, and that guarantees the optimality of base-stock policies. The cost-to-go function can be shown to be convex even for the multi-product problem; however, this only guarantees the existence of a minimizer (interpreted as the vector of optimal *after-order* inventory levels) but it does not guarantee the optimality of *base-stock policies*. (We will show in Theorem 5 that base-stock policies are optimal for the special case in which all products have identical demand distributions and costs).

Moreover, even temptingly simple and intuitive statements do not follow from convexity. For instance, one would imagine that the optimal policy possesses the following property which base-stock policies satisfy: If the inventory levels of all products at the beginning of a period are smaller than their optimal after-order inventory levels, then the inventory level of every product after ordering will be no larger than the optimal after-order inventory level. The careful reader will note that this property does not follow from convexity. In fact, a description of the optimal policy has so far been provided only for the two product case (that too, only for the finite or infinite horizon discounted cost problems, not the average cost problem) by Shaoxiang (2004) who expands on the early work by Evans (1967). For this case, Shaoxiang shows that the optimal policy is a base-stock policy. For the two product case, our weighted balancing rules can be viewed as linear approximations of the monotone switching curve in Shaoxiang (2004).

DeCroix and Arreola-Risa (1998) study the periodic review multi-product problem under both the finite horizon and the infinite horizon discounted cost criteria. They prove the optimality (for the finite horizon) of base-stock policies for the special case where all products are identical both in costs and demand distributions, and when the inventory level of each product in the first period is below its target level. For the general case, they provide a heuristic, but there are no results on the asymptotic performance of those policies.

Aviv and Federgruen (2001) study a multi-product inventory system in which production occurs in two stages in every period. In the first stage, a common product ("blank") is produced, and, in the second stage, blanks are converted into finished products. They present a heuristic and a lower bound for allocating blanks to develop insights for *delayed differentiation*. When the lead times at both stages are zero, their problem is the same as ours. While their heuristic policy is optimal for the single product problem, it can be verified that it is not optimal even for the multi-product problem with symmetric products – this is because the base-stock levels used are obtained by solving a relaxed problem. As with DeCroix and Arreola-Risa's heuristic, there are no results on the asymptotic performance of this policy.

While our focus is on periodically reviewed systems (i.e. discrete time), there are counterparts in make-to-stock queues - we briefly review this literature. Ha (1997) studies a special case with two products and shows several structural properties of the optimal policy – these results are similar to Shaoxiang's results in the discrete time case. The other papers in the area (Zipkin, 1995; Veatch and Wein, 1996; Rubio and Wein, 1996; Pena-Perez and Zipkin, 1997) study multi-product systems in the framework of multi-class make-to-stock queues – that is, the entire attention is on the class of base-stock policies and on finding good policies within this class.

This body of work uses a combination of heavy traffic analysis and computational tests to motivate and evaluate various choices of base-stock levels and allocation rules. Most works in this literature stream assume *Poisson* demand processes. Among these papers, Pena-Perez and Zipkin (1997) and Veatch and Wein (1996) are closely related to our paper, as explained below.

Pena-Perez and Zipkin (1997) argue that a specific *priority rule* is asymptotically optimal under certain assumptions for systems in "heavy traffic", i.e. systems where the aggregate demand rate is close to the production capacity rate. Their asymptotic analysis is based on the results of Wein (1992) and uses diffusion approximations. In this paper, we show a parallel result in periodic inventory models with two main strengths: (a) our notion of asymptotic optimality is *strong* (i.e. the difference between the cost of the priority policy and the optimal cost is bounded, while the optimal cost itself approaches infinity in heavy traffic) whereas their notion is *weak* (i.e. the ratio between the cost of the priority policy and the optimal cost approaches one) and (b) our proof is from first principles and does not rely on diffusion approximations.

Veatch and Wein (1996) propose and evaluate *index rules* - these rules suggest that when production occurs, it should be devoted to the product with the lowest index at that time. Our weighted balancing rules are analogous to "linear" index rules.

We conclude this section with the following summary on how our work contributes to the literature on multi-product inventory systems. Our paper first proves our weighted balancing policies are optimal under high service levels. We also prove the asymptotic optimality of our policies in heavy traffic - an approach typically employed in queueing models. Both these theoretical results are new to the literature with very sparse theoretical findings. Particulary, our weighted balancing policies *reduce* to the optimal policy for the two special cases for which the optimal policy is known, namely, the single product case and the symmetric, multi-product case. Finally, we will demonstrate in Section 7 that our policies consistently perform better than the existing approaches in this literature.

3 Model Description

We index the products by n, $1 \le n \le N$. The holding and backorder cost associated with product n in \$/unit/period are h^n and b^n, respectively. Periods are indexed by $t \ge 1$. In period t, the net-inventory, x_t^n (inventory on hand minus backorders) for each product n is observed and the production quantity, q_t^n, for each product is decided. The total production quantity $q_t = \sum_{n=1}^N q_t^n$ is constrained from above by a capacity limit κ. Next, the demand, D_t^n for each product n is observed. Finally, the cost C_t incurred for this period is computed based on the inventory levels and backorder levels at the end of the period as follows:

$$C_t \;=\; \sum_{n=1}^{N} \left(h^n \cdot (x_t^n + q_t^n - D_t^n)^+ \;+\; b^n \cdot (D_t^n - x_t^n - q_t^n)^+\right) \;.$$

The optimization problem that we are interested in solving is that of minimizing the long run average cost when the set of admissible (or feasible) policies is the set of all non-anticipatory policies. A formal definition of this problem follows. A non-anticipatory policy π is described by a set of vector-valued functions $\{\pi_t : t = 1, 2, \ldots,\}$ where $q_t^n = \pi_t^n(\mathbf{x}_t)$; here, $\mathbf{x}_t$ is the state vector $(x_t^1, \ldots, x_t^N)$ in period t and π_t is a function from $\Re^n \to \Re^{n,+}$. Let Π denote the set of all non-anticipatory policies π such that the capacity constraint

$$\sum_{n=1}^{N} \pi_t^n(\mathbf{x}) \le \kappa \text{ for all } \mathbf{x} \in \Re^n \text{ and for all } t \in \{1, 2, \ldots\}$$

is satisfied. If C_t^π denotes the cost incurred by the system in period t when the system follows the policy π, our performance measure is

$$C^\pi = \lim_{T\to\infty} \sup E[\sum_{t=1}^{T} C_t^\pi]/T \;.$$

The optimal long run average cost is defined as $C^* = \inf_{\pi\in\Pi} C^\pi$.

Throughout the paper, we assume that the sequence of random vectors $\{\mathbf{D}_t\}$ is independent and identically distributed across time periods, where $\mathbf{D}_t = (D_t^1, \ldots, D_t^N)$. Note that we allow for the demands of the products to be correlated. We use D^n to denote a random variable with the same distribution as the single period demand for product n and D to denote a random variable with the same distribution as the aggregate single period demand. Let $\mu^n = E[D^n]$. We also assume that capacity exceeds aggregate expected demand, i.e., $\mu := \sum_{n=1}^N \mu^n < \kappa$, which is a necessary condition for the existence of a policy with a finite long-run average cost. Finally, the aggregate demand in a period can exceed capacity with positive probability, i.e., $P\left(\sum_{n=1}^N D^n > \kappa\right) > 0$. When the above condition does not hold, we can decompose our problem into a set of N newsvendor problems.

Let Π_{BS} denote the subset of stationary base-stock policies described at the beginning of Section 1.1. We now introduce some notation specific to Π_{BS}.

Let S^n denote the target or base-stock level for product n, and $\mathbf{S}$ denote the vector of base-stock levels. In our analysis of stationary base-stock policies, we assume that $x_1^n \leq S^n$ for all n. Let $W_t^n = S^n - x_t^n$; we refer to W_t^n as the opening shortfall for n in period t. Let V_t^n denote the ending shortfall, i.e. shortfall after ordering. So, $V_t^n = W_t^n - q_t^n$. By definition of a base-stock policy, the following condition holds:

$$\text{if } \sum_{n=1}^N W_t^n \leq \kappa \ , \quad \text{then } q_t^n \ = \ W_t^n \ \text{ for all } n \ .$$

That is, all inventory levels are raised to their respective targets, if that is feasible. Otherwise, the entire capacity is used for production without the inventory level of any product exceeding its target, i.e.,

$$\text{if } \sum_{n=1}^N W_t^n > \kappa \ , \quad \text{then } \sum_{n=1}^N q_t^n \ = \ \kappa \ \text{ and } q_t^n \leq W_t^n \text{ for all } n \ .$$

Notice that the exact manner in which the capacity is allocated among products in such periods has not been completely specified yet. We will specify these allocation rules shortly.

Let Π_{BS-B} denote the set of stationary base-stock policies in which the weighted balancing allocation rule is followed. We will refer to these as weighted balancing policies. A verbal description of these allocation rules was given in Section 1.1. Clearly, Π_{BS-B} is a subset of Π_{BS}. A mathematical description of a policy in this class follows.

Weighted Balancing Allocation: Rank the products according to the 'weighted' shortfalls $\{W_t^j/\alpha^j\}$, where α^j is the weight corresponding to product j. Let $\boldsymbol{\alpha} = (\alpha^1, \ldots, \alpha^N)$. The symmetric rule chooses $\boldsymbol{\alpha} = \mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)$. Let $\tilde{n}$ denote the product with the n^{th} largest value of the weighted shortfall, W_t^j/α^j, breaking ties arbitrarily. Allocate production to product $\tilde{n} = 1$ until its weighted shortfall equals that of $\tilde{n} = 2$, or until the capacity is exhausted. Using the remaining capacity, allocate production to products $\tilde{1}$ and $\tilde{2}$ (proportionally based on their weights so that their weighted shortfalls are always equal) until their weighted shortfalls equal that of $\tilde{3}$, or until the capacity is exhausted. This process is continued until the entire capacity available in the period is exhausted. While the description above applies when inventory and production quantities are real-valued, a simple uniform randomization scheme can be used to define the policy when these quantities are integer-valued. Note that any policy $\pi \in \Pi_{BS-B}$ is completely specified by a pair $(\mathbf{S}, \boldsymbol{\alpha})$ where $\mathbf{S}$ is a vector of base-stock levels $\mathbf{S}$ and $\boldsymbol{\alpha}$ is a vector of weights.

Priority Allocation: Let $\{(1), (2), \ldots, (N)\}$ denote any permutation of $\{1, 2, \ldots, N\}$. Then, a priority rule defined by this permutation works as follows: In every period, allocate production to (1) until its shortfall is zero or the entire capacity is consumed; then, proceed to (2) and do the same until all product shortfalls are zero or the capacity is consumed. Intuitively, this rule can be imitated by a weighted balancing rule which assigns the weights α^js in an extremely disparate fashion. We show this formally in Section 4.

4 Weighted Balancing Policies: Preliminaries

We start by explaining the connection between weighted balancing policies and the known structural properties of the optimal policy for the special case of two products. Shaoxiang (2004) shows for the infinite horizon, discounted cost version of this problem that an optimal policy satisfies the following: There exists a base-stock vector (S^1, S^2) such that once the inventory vector reaches a point which is componentwise smaller than (S^1, S^2), then the inventory vector in every subsequent period is also smaller than (S^1, S^2). Thus, the effective state space (i.e. possible inventory vectors) is $(-\infty, S^1] \times (-\infty, S^2]$; so, it is sufficient (for our purposes since we consider the average cost version of the problem) to study the optimal policy within this "rectangle". Within this region, the optimal policy is completely described by a monotone switching curve or function $\overline{x}^2(x^1)$.

Our weighted balancing policies work exactly like the optimal policy except that they replace $\overline{x}^2(x^1)$ with the function $(\frac{\alpha_2}{\alpha_1}) \cdot x^1$. In other words, computing the optimal policy involves finding or searching for the function $\overline{x}^2(x^1)$, i.e. the optimal "switching curve" within the space of all increasing functions, whereas, the best weighted balancing policy is found by searching for the best ratio $\frac{\alpha_2}{\alpha_1}$.

Consider a stationary base-stock policy with base-stock levels $S^1, \ldots, S^N$ for the N products. Let $V_t = \sum_{n=1}^{N}(S^n - x_t^n - q_t^n)^+$ denote the aggregate ending shortfall in period t. Let $D_t = \sum_{n=1}^{N} D_t^n$ similarly denote the aggregate demand the system faces in period t. We begin by making a simple observation about the aggregate shortfall process. All proofs are relegated to the appendix.

Lemma 1. *Consider any policy in Π_{BS} with some base-stock vector $\mathbf{S}$. Assume $x_1^n = S^n$ for all n. The evolution of the aggregate shortfall process $\{V_t\}$, is described by the recursive equation $V_{t+1} = (V_t + D_t - \kappa)^+$. Moreover, (i) the distribution of V_t is independent of $\mathbf{S}$ for all t, (ii) V_t converges in distribution to a limiting random variable V_∞ as $t \to \infty$ and (iii) the distribution of V_∞ is also independent of $\mathbf{S}$.*

Proof of Lemma 1 is straightforward and is omitted for the sake of brevity. We now make a few observations about the vector of individual shortfalls of the products.

Lemma 2. *Consider any policy in Π_{BS-B} defined by a base-stock vector $\mathbf{S}$ and a weight vector $\boldsymbol{\alpha}$. Assume $x_1^n = S^n$ for all n. Then, (i) the distribution of the vector of ending shortfalls is independent of $\mathbf{S}$ for all t, (ii) the sequence of distributions of this vector converges to a limiting distribution as $t \to \infty$ and (iii) this limiting distribution is also independent of $\mathbf{S}$.*

Proof of Lemma 2 is straightforward and is omitted for the sake of brevity. For any policy $\pi \in \Pi_{BS-B}$ defined by the pair $(\mathbf{S}, \boldsymbol{\alpha})$, we use $\mathbf{V}_t^{\boldsymbol{\alpha}}$ to denote the vector of shortfalls in period t. (Note that it is not necessary to include the argument $\mathbf{S}$ in the notation for the shortfall vector since its distribution does not depend on $\mathbf{S}$.) Let $\mathbf{V}^{\boldsymbol{\alpha}}$ denote the limiting distribution of $\mathbf{V}_t^{\boldsymbol{\alpha}}$; thus, $V^{\boldsymbol{\alpha},n}$ is the *steady-state* shortfall of product n. Let $\Phi^{\boldsymbol{\alpha},n}$ denote the distribution of the convolution of $V^{\boldsymbol{\alpha},n}$ and D^n. Let

$$\mathbf{S}^{\boldsymbol{\alpha}*} = (S^{\boldsymbol{\alpha}*,1}, \dots, S^{\boldsymbol{\alpha}*,N}), \text{ where } S^{\boldsymbol{\alpha}*,n} = (\Phi^{\boldsymbol{\alpha},n})^{-1}\left(\frac{b^n}{b^n + h^n}\right).$$

We will now show that the base-stock vector $\mathbf{S}^{\boldsymbol{\alpha}*}$ is the optimal choice of $\mathbf{S}$ within the subset of those policies in Π_{BS-B} that use the weight vector $\boldsymbol{\alpha}$.

Lemma 3. *Consider the class of weighted balancing policies, Π_{BS-B}. Within the subclass of policies which use the weight vector $\boldsymbol{\alpha}$, the policy with the base-stock vector $\mathbf{S}^{\boldsymbol{\alpha}*}$ is optimal.*

Next, we discuss the special case in which all products are "symmetric", i.e. identical in terms of cost parameters and demand distributions. We are able to make stronger statements about the optimal policy for this special case. We first formally state our assumption.

Assumption 1. *The following conditions hold. (a) $h^n = h$ and $b^n = b$ for all n. (b) $(D^1, \dots, D^N)$ has a symmetric distribution, that is, the joint distribution of $(D^1, \dots, D^N)$ is identical to the joint distribution of $(D^{\theta(1)}, \dots, D^{\theta(N)})$ for any permutation $(\theta(1), \dots, \theta(N))$ of $(1, \dots, N)$.*

Theorem 4. *Consider the policy in Π_{BS-B} defined by a base-stock vector $\mathbf{S}$ and the weight vector $\mathbf{1}$. Assume that $x_1^n = S^n$. Under Assumption 1 (b), the following statements hold.*
(i) The distribution of $\mathbf{V}_t^{\mathbf{1}}$ is symmetric for all t.
(ii) The distribution of $\mathbf{V}_\infty^{\mathbf{1}}$, the limiting random vector mentioned in Lemma 2, is symmetric.

Next, we show that the policy in Π_{BS-B} that uses the symmetric allocation rule and the corresponding optimal base-stock vector (as defined in Lemma 3) is optimal over all policies, not just base-stock policies, when all products are identical. This result is the average cost version of Theorem 3 of DeCroix and Arreola-Risa (1998)[1], which pertains to the finite horizon and infinite horizon discounted cost problems.

[1]We note that the conclusion of Theorem 3 of DeCroix and Arreola-Risa (1998) is not completely correct. For example, they claim the following: if, in some period, some products have inventory levels which exceed their optimal base-stock levels and if it is feasible to raise the inventory levels of the other products to their optimal base-stock levels, then the optimal policy is to not produce any of the products in the former category while bringing the other products' inventories to their optimal base-stock levels. This claim is incorrect because the optimal inventory level for a product after ordering depends non-trivially on the inventory levels of the other products since the cost-to-go function is *not* separable even though the single period cost function is separable with respect to the inventory levels of the products. However, we note that their claims are correct for every inventory vector in which every component is below its corresponding optimal base-stock level. The above comments also apply to Theorem 1 of their paper.

Theorem 5. *Consider the policy in* Π_{BS-B} *with the weight vector* $\mathbf{1}$ *and the base-stock vector* $\mathbf{S}^{\mathbf{1}*}$. *Under Assumption 1, this policy minimizes the long run average cost per period* $\lim_{T\to\infty}\sup E[\sum_{t=1}^{T} C_t^{\pi}]/T$ *over* Π*, the class of all non-anticipatory policies.*

In the next result, we show that shortfalls under the priority policy can be approximated by policies in Π_{BS-B}.

Lemma 6. *Let* $((1),(2),\ldots,(N))$ *denote any permutation of* $\{1,2,\ldots,N\}$. *Consider any given shortfall vector* $\mathbf{W}$ *(before ordering) in any period. Let* $\boldsymbol{\alpha}_m$ *be defined by* $\alpha_m^{(1)} = 1$ *and* $\alpha_m^{(j)} = m \cdot \alpha_m^{(j-1)}$ *for* $j \in \{2,\ldots,N\}$. *Let* $\mathbf{V}^P$ *and* $\mathbf{V}^{\boldsymbol{\alpha}_m}$ *denote the shortfall vectors after ordering under the priority rule (with priorities* $(1) > (2) > \ldots > (N)$*) and the weighted balancing rule (with weight vector* $\boldsymbol{\alpha}_m$*), respectively. Then, for every* $\epsilon > 0$*, there exists a sufficiently large* M *such that* $|\mathbf{V}^{\boldsymbol{\alpha}_m} - \mathbf{V}^P| < \epsilon$ *for all* $m > M$*, where* $|(u_1,u_2,\ldots,u_n)| = max\{u_1,u_2,\ldots,u_n\}$.

5 High Service Level Asymptotics

We show that if the joint distribution of demands for all the products is symmetric and the holding costs for all products are identical, then the best base-stock policy under the symmetric allocation rule is asymptotically optimal along a sequence of problems in which the backorder costs are scaled by a factor β that approaches ∞. In more practical terms, when the cost parameters are such that service levels for all products are high (in any reasonable policy), the best base-stock policy under the symmetric allocation rule is close to being optimal. We note that we do *not* restrict the backorder cost parameters for the products to be identical in this analysis. We proceed to state our assumptions formally, and then present our analysis.

Assumption 2. *The following conditions hold.*
(a) All products have identical holding costs, that is, $h^n = h$ *for all* $n \in \{1,\ldots,N\}$.
(b) $(D^1,\ldots,D^N)$ *has a symmetric distribution.*

When the demand vector has a symmetric distribution, let us employ $C^*(h,b)$ to denote the optimal long run average cost of a system in which *all* the products have the same holding cost parameter h and the same backorder cost parameter b. When the backorder costs are not identical (which might generally be the case), we use $C^*(h,\mathbf{b})$ to denote the same except that $\mathbf{b}$ represents the vector of backorder costs over all the products. We denote the long-run average cost of the policy in Π_{BS-B} with parameters $(\mathbf{S}^{\mathbf{1}*}, \mathbf{1})$ as $C^{\mathbf{1}*}(h, \mathbf{b})$. Finally, we denote the lowest backorder cost parameter in $\mathbf{b}$ by $min(\mathbf{b})$ and the average of all the individual itemwise backorder costs by $avg(\mathbf{b})$.

In what follows, we note that $C^*(h, b)$ can be evaluated using Theorem 4. In our analysis, we use the cost $C^*(h, b)$ as a basis for cost comparisons across various policies because we know how it can be computed. In fact, we know that

$$C^*(h, b) \;=\; N \cdot L(h, b, V_\infty^{\mathbf{1},1} + D^1) \;, \tag{1}$$

where (i) $V_\infty^{\mathbf{1},1}$ is the marginal distribution of any component of the vector $\mathbf{V}_\infty^{\mathbf{1}}$ (recall that the distribution of $\mathbf{V}_\infty^{\mathbf{1}}$ is symmetric when the distribution of $(D^1, \ldots, D^N)$ is symmetric) and (ii) $L(h, b, X)$ is the optimal cost of a single product newsvendor problem with holding and penalty cost parameters h and b respectively, and facing a demand distribution of X, i.e.,

$$L(h, b, X) \;=\; \min_y \; h \cdot E[(y - X)^+] \;+\; b \cdot E[(X - y)^+] \;.$$

Before proceeding to the details of the analysis leading to the asymptotic optimality result of Theorem 8, we outline the main steps. In Lemma 7, we show that $C^*(h, avg(\mathbf{b}))$ and $C^*(h \min(\mathbf{b}))$ are upper and lower bounds, respectively, on $C^{\mathbf{1}*}(h, \mathbf{b})$ - the long run average cost of the optimal symmetric policy. Notice that both the bounds are optimal costs of systems in which the products are symmetric in costs. (Recall that throughout this section we assume that the product demands are symmetric.) Thus, we can express these bounds as the optimal costs of certain newsvendor problems involving the convolution of demands and shortfalls as explained in the previous paragraph. Our goal is to show that the ratio $\frac{C^{\mathbf{1}*}(h, \beta \cdot \mathbf{b})}{C^*(h, \beta \cdot \mathbf{b})}$ approaches 1 as β approaches ∞. Thus, it is sufficient to show that the ratio of the optimal costs of the two newsvendor problems alluded to above converges to 1 since one of these optimal costs is an upper bound on the numerator of the ratio of interest and the other is a lower bound on its denominator. We establish this convergence in the proof of Theorem 8 by making use of a result in Huh et al. (2009) (presented in our appendix as Lemma 12) for the standard newsvendor problem under a mild distributional assumption on demand. Since the newsvendor problems we are interested in involve the convolution of a product's demand and its shortfall, we have to demonstrate that this convolution also satisfies their assumption - this step is done in Lemma 13 of our appendix.

Lemma 7. *Under Assumption 2, the following inequalities hold:*

$$C^*(h, min(\mathbf{b})) \;\leq\; C^*(h, \mathbf{b}) \;\leq\; C^{\mathbf{1}*}(h, \mathbf{b}) \;\leq\; C^*(h, avg(\mathbf{b}))\ . \tag{2}$$

We are now ready to derive an upper bound on the ratio $\frac{C^{\mathbf{1}*}(h,\mathbf{b})}{C^*(h,\mathbf{b})}$ and show that this ratio approaches 1 as $\mathbf{b}$ is scaled by a factor β which approaches ∞. This is the asymptotic optimality result that we have been referring to all along - for this result, we assume that the demand distribution for every product is IFR, i.e. has an increasing failure rate which is a condition satisfied by several common distributions.

Theorem 8. *Under Assumption 2, the increase in cost due to using the symmetric allocation rule and its corresponding optimal base-stock vector relative to the optimal cost can be bounded as follows:*

$$\left(\frac{C^{\mathbf{1}*}(h, \mathbf{b})}{C^*(h, \mathbf{b})}\right) \;\leq\; \left(\frac{C^*(h, avg(\mathbf{b}))}{C^*(h, min(\mathbf{b}))}\right)\ .$$

Moreover, if the common marginal distribution of the random variables $\{D^j\}$ is an IFR distribution, this ratio converges to 1 *as the backorder cost parameters grow, in the following sense:*

$$\lim_{\beta\to\infty}\left(\frac{C^{\mathbf{1}*}(h, \beta\mathbf{b})}{C^*(h, \beta\mathbf{b})}\right) \;=\; 1\ .$$

6 Heavy Traffic Asymptotics

In this section, we assume without loss of generality that the products are numbered in such a way that $h^1 \geq h^2 \geq \ldots \geq h^N$. We show that when $b^N = \min\{b^j\}$, the priority rule which assigns priorities based on the order $(1, 2, \ldots, N)$ is asymptotically optimal in heavy traffic, i.e. as the capacity κ approaches the expected aggregate demand $E[\sum_1^N D^j]$. As mentioned earlier, Pena-Perez and Zipkin (1997) argued that such a result holds in continuous time systems by appealing to diffusion approximations based on Wein (1992). Our proof, as we will see, is from first principles and does not use such approximations. Moreover, our result is that the asymptotic optimality discussed above holds in the *strong* sense whereas Pena-Perez and Zipkin use it in the *weak* sense.

We say that a policy π is asymptotically optimal in the weak sense along a sequence of systems indexed by n if the optimal cost approaches ∞ as n approaches ∞ and the ratio between the cost of π and the optimal cost approaches one. Furthermore, if the absolute difference between the cost of π and the optimal cost is bounded, we say π is strongly asymptotically optimal.

To proceed with our asymptotic analysis, we first introduce some notation. Let $C^*(\mathbf{h}, \mathbf{b}, \kappa)$ be the optimal long run average cost of our inventory system when the holding cost vector is $\mathbf{h}$, the backorder cost vector is $\mathbf{b}$ and the capacity is $\kappa \in (\mu, \infty)$. Let $C^*(h, b, \kappa)$ be the same as $C^*(\mathbf{h}, \mathbf{b}, \kappa)$ when $\mathbf{h} = (h, h, \ldots, h)$ and $\mathbf{b} = (b, b, \ldots, b)$. Let $C^P(\mathbf{h}, \mathbf{b}, \kappa)$ denote the long run average cost of the priority policy, P, which assigns priority based on the order $(1, 2, \ldots, N)$ and uses the corresponding optimal base-stock levels according to Lemma 3.

We present a preliminary lemma on the asymptotic behavior of the optimal cost $C^*(\mathbf{h}, \mathbf{b}, \kappa)$ using a well known result due to Kingman (1962) that a suitably scaled distribution of the waiting time in a single server queue converges to an exponential distribution in heavy traffic.

Lemma 9. *As the capacity κ approaches the expected aggregate demand μ, the optimal cost approaches ∞, i.e.*

$$\lim_{\kappa \downarrow \mu} C^*(\mathbf{h}, \mathbf{b}, \kappa) = \infty \ .$$

Next, we present our assumption on the cost parameters formally before stating and proving our asymptotic result in Theorem 10.

Assumption 3. *The cost parameters satisfy the following conditions: $h^1 \geq h^2 \geq \ldots \geq h^N$ and $b^N = \min\{b^j : 1 \leq j \leq N\}$.*

Theorem 10. *Under Assumption 3, the following statement holds: There exists a finite constant $\overline{M} < \infty$ such that*

$$C^P(\mathbf{h}, \mathbf{b}, \kappa) - C^*(\mathbf{h}, \mathbf{b}, \kappa) \leq \overline{M} \text{ for all } \kappa > \mu \ .$$

Therefore, $\lim_{\kappa \downarrow \mu} \frac{C^P(\mathbf{h}, \mathbf{b}, \kappa)}{C^*(\mathbf{h}, \mathbf{b}, \kappa)} = 1$.

7 Policy Performance and Results

Theorem 8 establishes that, as the backorder costs grow (or required service levels increase), the optimal cost under the symmetric allocation rule asymptotically approaches the optimal cost when the holding costs and demand distributions of all products are identical. While this result is of theoretical interest, it is also important to benchmark our policy.

Lower Bound for Benchmarking: Since the optimal cost is virtually impossible to calculate for a large set of problem instances due to the curse of dimensionality associated with dynamic programming, we require an easily computed lower bound on the optimal cost. Although we already have such a lower bound in Lemma 7 for the case of symmetric demands, we require a more generally applicable lower bound because the case of asymmetric demands is also included in our numerical investigation. We state such a lower bound in Lemma 11.

Let $G^n(x) = h^n \cdot E[(x-D^n)^+] + b^n \cdot E[(D^n-x)^+]$ be the expected single period newsvendor cost function for product n. We now develop a lower bound on the optimal long run average cost by using the free balancing relaxation (see, for example, Eppen and Schrage (1981) or Aviv and Federgruen (2001)). Let $F_1(y)$ be defined as follows: $F_1(y) = \min_y \sum_{n=1}^{N} G^n(y^n)$ s.t. $\sum_{n=1}^{N} y^n = y$. Note that the computation of $F_1(y)$ can be done quite efficiently using a greedy algorithm to solve the optimization problem above. We can now construct a lower bound on the optimal long run average cost using the function $F_1(\cdot)$. Recall that V_∞ is the limiting random variable of the stochastic process representing aggregate ending shortfalls, i.e. $\{V_t\}$. We employ this limiting distribution to derive a lower bound on the optimal cost.

Lemma 11. *Let $LB_1 = \min_S E[F_1(S - V_\infty)]$. Then, LB_1 is a lower bound on the optimal long run average cost over Π, the class of all non-anticipatory policies.*

7.1 Existing Heuristics

We now describe the heuristics of DeCroix and Arreola-Risa (1998) and Aviv and Federgruen (2001), and compare their heuristics with our weighted balancing approach.

The heuristic of DeCroix and Arreola-Risa (1998) is a stationary base-stock policy. We now explain how their base-stock levels are chosen and what their allocation rule is. Let $(S^1, \ldots, S^N)$ denote the vector of base-stock levels for the N products. For any such vector, the allocation rule they use in every period in which capacity is insufficient to raise the inventory levels of all products to their base-stock levels is that of "balancing" the ratios $\{x^n/S^n\}$, where x^n is the net-inventory of product n at the beginning of the period. That is, allocate capacity to the product with the lowest value of this ratio until this ratio equals the next highest ratio; from then, allocate capacity to these two products until their ratios equal the next highest ratio and so on, until the capacity is exhausted. It still remains to specify how the base-stock vector is chosen. This is done as follows. For every $n \in \{1, \ldots, N\}$, let z^n denote the newsvendor level for product n. That is, $z^n = \max\{\arg\min_y G^n(y^n)\}$. For products $n \in \{2, \ldots, N\}$, let $\gamma^n = z^n/z^1$. Let $f(S^1)$ denote the long run average cost of using the policy with the base-stock vector $(S^1, S^1 \cdot \gamma^2, S^1 \cdot \gamma^3, \ldots, S^1 \cdot \gamma^N)$ and the allocation rule described above. The prescribed value of S^1 is that which minimizes $f(\cdot)$ and the prescribed value of S^n for any $n \neq 1$ is $S^1 \cdot \gamma^n$. Note that the evaluation of $f(S^1)$ for a given value of S^1 requires the computation of the steady state distribution of the shortfall vector. The computational effort for our weighted balancing approach is just the effort required to obtain this distribution. However, the heuristic above requires evaluating $f(S^1)$ over an entire search set for S^1, whereas we compute the steady state distribution of the shortfall vector *only once*. Finally, as evidenced by our numerical experiments, our policy significantly outperforms the policy described above.

The heuristic of Aviv and Federgruen (2001) is also a stationary base-stock policy. Let $(S^1, \ldots, S^N)$ denote the vector of base-stock levels for the N products, the computation of which we discuss after discussing their allocation rule, which is a myopic allocation rule. That is, in every period in which there is not enough capacity for the inventory levels of all the products to attain their base-stock levels, the vector of inventory levels after ordering, say $(y^1, \ldots, y^N)$, is chosen to be a solution to the following optimization problem: $\min_{\mathbf{y}} \sum_{n=1}^N G^n(y^n)$ s.t $y^n \geq x^n \ \forall \ n$ and $\sum_{n=1}^N (y^n - x^n) = \kappa$, where x^n is the net inventory of product n at the beginning of the period. The base-stock vector $(S^1, \ldots, S^N)$ is chosen as the solution to the optimization problem $\min \sum_{n=1}^N G^n(S^n)$ s.t. $\sum_{n=1}^N S^n = s$, where $s = \arg\min_S E[F_1(S - V_\infty)]$. Recall that $F_1(y) = \min_y \sum_{n=1}^N G^n(y^n)$ s.t. $\sum_{n=1}^N y^n = y$. In terms of computational effort, this heuristic also requires the computation of the steady state distribution of the aggregate shortfall, in order to obtain the function F_1; thus, the AF method is comparable to our weighted balancing policies in terms of computational effort. However, the AF heuristic is *not* guaranteed to be optimal even in the symmetric case because the base-stock levels are not chosen optimally. Our policy provides optimal policies in the symmetric case.

7.2 Policy under Weighted Balancing

Recall that we have established the optimality of the symmetric policy (a weighted balancing policy with weights of 1) and of the priority policy (a weighted balancing policy with extremely different weights) in the asymptotic regimes of high service levels and heavy traffic, respectively. Motivated by the fact that these two policies are very different in terms of their weight vectors, we propose searching over the space of weight vectors. While an exhaustive search for the weights would involve searching over the $N-1$ dimensional space of positive reals, we design a one dimensional search using a weight vector which is prescribed by m similar to Lemma 6, to find the best weighted balancing policy (i.e. the policy with the lowest cost). In our tables, we will refer to this policy as the "Search" policy or simply as *our* policy.

We conducted several computational experiments and compared the performance of our policy, with those of the heuristics of DeCroix and Arreola-Risa (1998) (which we refer to as the "DA heuristic" in the tables) and Aviv and Federgruen (2001) (which we refer to as the "AF heuristic" in the tables). We also compared our heuristic policy against the priority policy (represented as "Pri" in the tables).

7.3 Computational Design

In the computational study, we report problems with three products, i.e. $N = 3$. Nevertheless, there are several attributes and features of importance in the multi-product problem, viz. the capacity available, the mean and the standard deviation of the demands of individual products, the holding and penalty costs of each product. To carefully calibrate the performance of our policy, we have to demonstrate the performance with respect to each parameter in the model. To achieve this end, we build our computational experiments by adding more complexity to the problem in each subsection, and demonstrate the improved performance of our policy.

We report a summary of our computational results run for a large range of Erlang (k, λ) demands, since by using Erlang demand with appropriate k and λ, we can arbitrarily approximate any continuous demand distribution up to the first two moments (mean and variance).

$\diamond$ In Section 7.4, we explore our policy under asymmetric demands (but symmetric costs), and show that our policy performs better as the demands across different products get more asymmetric.

$\diamond$ In Section 7.5, we explore the impact of asymmetric penalty costs but with symmetric demands. To make costs asymmetric, we hold holding costs identical, and vary backorder cost $\{b^n\}$ to construct problems with different service levels, while also depicting cost asymmetry among the products. We notice the performance of our policy improves (i) as the capacity gets scarcer, and (ii) as the penalty costs become more asymmetric.

$\diamond$ In Section 7.6, we combine the effect of asymmetric demands and asymmetric penalty costs. Moreover, products with high penalty costs may have high (or low) demand variability.

$\diamond$ In Section 7.7, we allow for all the parameters, i.e., the holding costs, the penalty costs and the demand distributions to be asymmetric, and find that our policy outperforms the existing policies consistently.

$\diamond$ In Section 7.8, we show that for the cases with high asymmetry (both in demand and costs), our policy performs well when capacity is ample, and then demonstrate that as the total capacity becomes scarce (i.e, κ decreases), our policy does increasingly better.

To summarize, we establish that our policy is asymptotically optimal, while showing that it consistently outperforms other approaches in the literature.

7.4 Effect of Asymmetric Demands

We begin with a base case (the first instance in Table 1), where the capacity is held at 48, and perturb only the demand distributions. (The products continue to be symmetric, except for their demand distributions).

To create systematic demand asymmetry effects, we hold the k-parameter of the Erlang distribution identical across all products and vary λ. Since the mean demand of the Erlang(k, λ) distribution is $\frac{k}{\lambda}$ and the variance is $\frac{k}{\lambda^2}$, by increasing the λ parameter, we decrease the mean demand and the demand variance of a product. For the experiments shown in Table 1, in each successive line, we increase λ for product 1 and decrease λ for product 3, while holding all other parameters constant. Therefore, as we progress down Table 1, the variance to mean ratio for product 1's demand decreases while the ratio increases for product 3. Whenever we vary the demand, we will repeat this scheme for all computational experiments that follow in the paper.

$\mathbf{b} = (3,3,3)$			Costs					% gap of heuristic			
λ^1	λ^2	λ^3	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
1	1	1	49.5	56.9	53.1	53.8	53.1	7.2%	7.3%	0.0%	1.4%
1.1	1	0.9	49.7	57.3	53.5	54.3	53.5	7.6%	7.2%	0.0%	1.6%
1.2	1	0.8	50.5	58.7	54.7	55.8	54.7	8.3%	7.3%	0.1%	2.0%
1.3	1	0.7	52.0	61.3	56.9	58.5	56.9	9.4%	7.7%	0.0%	2.8%
1.4	1	0.6	54.2	65.5	60.5	62.9	60.5	11.6%	8.4%	0.0%	4.0%
1.5	1	0.5	57.9	72.9	66.7	70.5	66.6	15.0%	9.4%	0.1%	5.8%
$\mathbf{b} = (10,10,10)$											
1	1	1	84.2	97.2	88.5	89.8	88.5	5.1%	9.8%	0.0%	1.5%
1.1	1	0.9	84.5	97.3	89.0	90.5	89.0	5.3%	9.4%	0.0%	1.8%
1.2	1	0.8	85.4	98.8	90.3	92.2	90.3	5.7%	9.3%	0.0%	2.1%
1.3	1	0.7	86.7	101.5	92.6	95.1	92.6	6.8%	9.6%	0.0%	2.7%
1.4	1	0.6	88.1	108.2	95.8	99.3	95.8	8.7%	13.0%	0.0%	3.8%
1.5	1	0.5	89.5	123.0	105.0	107.5	104.0	16.2%	18.2%	0.9%	3.3%
$\mathbf{b} = (15,15,15)$											
1	1	1	96.5	111.4	100.9	102.5	100.9	4.6%	10.4%	0.0%	1.5%
1.1	1	0.9	96.7	111.3	101.3	102.8	101.3	4.8%	9.8%	0.0%	1.5%
1.2	1	0.8	97.3	112.4	102.5	104.1	102.5	5.3%	9.7%	0.1%	1.6%
1.3	1	0.7	98.0	116.0	104.2	106.6	104.2	6.3%	11.3%	0.1%	2.3%
1.4	1	0.6	98.0	124.4	107.3	110.0	106.7	8.9%	16.7%	0.6%	3.1%
1.5	1	0.5	98.3	141.5	118.2	123.0	117.8	19.8%	20.1%	0.3%	4.3%

Table 1: Cost behavior as the demands become asymmetric. All products have the same holding cost $h = 1$ and backorder cost (as shown). We have $k^1 = k^2 = k^3 = 11$, λs as indicated. Thus, Product 1 (Product 3) has a lower (higher) mean demand and lower (higher) variance, when compared to product 2. The total capacity is 48.

We note three successive sub-tables in Table 1. Within each sub-table, the demand is made more asymmetric for a given penalty cost. Across sub-tables, we progressively increase the penalty costs.

Our policy performs significantly better than the priority policy in all the cases. In fact, as the penalty costs increase, the relative superior performance of our policy is more pronounced. Also observe that our policy performs better than the AF heuristic. Note that the performance improves as the demands become more asymmetric. We also note that our cost performance is comparable to the DA heuristic under asymmetric demands and symmetric costs. (In most cases, the difference is negligible; our costs are always better than DA but within 1.0% difference).

7.5 Effect of Asymmetric Penalty Costs

To explore the effect of asymmetric backorder costs, we begin with our base case and examine the performance of our policy when the demand distributions for all products are identical, but the backorder costs are asymmetric. The mean demand and variance are fixed as in the first line of Table 1.

In Tables 2 and 3, we gradually increase the backorder cost of product 1 successively, while decreasing the backorder cost of product 3, thus making the newsvendor fractiles of the products more asymmetric. Again, throughout the computations in Table 2, the demand distributions of the products are kept symmetric, and the capacity is kept fixed at the same level. In Table 3, we follow the scheme as in Table 2, except that the available overall capacity is lower.

$k = 12, \lambda = 1$			Costs of Heuristics					% gap of our policy			
b^1	b^2	b^3	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
4.5	3	2	50.4	59.9	56.7	57.0	56.2	11.5%	6.5%	0.9%	1.4%
6	3	1.5	50.9	59.0	58.2	57.0	56.3	10.6%	4.8%	3.4%	1.2%
8	3	1	51.0	57.4	59.9	55.8	55.7	9.2%	3.0%	7.6%	0.3%
9	3	1	52.1	58.5	61.4	57.1	56.9	9.2%	2.9%	8.0%	0.4%
12	3	0.75	53.3	58.8	64.2	57.7	57.5	7.9%	2.2%	11.7%	0.3%
15	3	0.6	54.4	59.2	66.7	58.4	58.2	7.0%	1.8%	14.6%	0.4%

Table 2: We consider three products with identical Erland(k, λ) demands. While the holding costs are identical, the backorder cost of product 1, b^1 is progressively increased down the table, and the backorder cost of product 3, b^3 is decreased. The total capacity is $K = 48$.

From Table 2, our performance is significantly better than the priority policy. As one would expect, as the costs become asymmetric, the priority policy improves, but our heuristic continues to outperform the priority policy. For asymmetric costs, we observe that our policy performs significantly better than the DA heuristic. As the cost difference between the highest and the lowest backorder costs of all products $\{\max\{b^j\} - \min\{b^j\}\}$ increases, we note that our heuristic performs much superior to the DA heuristic, providing as much as a 14.6% cost difference. When examined against the AF heuristic, our performance is consistently slightly better.

Suppose the aggregate capacity becomes tighter under asymmetric costs. How does our policy perform? To address this question, we re-run the tests in Table 3 by decreasing the total capacity from $K = 48$ (Table 2) to $K = 44$. As the capacity becomes tighter (as the capacity utilization increases from 75% to 82%), our policy performance improves significantly on *every* instance. For

$K = 44$			Costs of Heuristics					% gap of our policy			
b^1	b^2	b^3	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
4.5	3	2	53.1	68.3	63.9	64.4	62.5	17.7%	9.3%	2.2%	3.1%
6	3	1.5	53.2	66.0	66.1	63.2	62.0	16.5%	6.5%	6.7%	2.0%
8	3	1	52.7	62.8	69.3	61.1	60.4	14.6%	4.1%	14.8%	1.3%
9	3	1	53.9	64.0	71.4	62.1	61.5	14.1%	4.0%	16.0%	0.9%
12	3	0.75	54.8	63.3	75.9	61.9	61.5	12.2%	2.9%	23.4%	0.7%
15	3	0.6	55.6	63.1	79.8	62.2	61.6	10.8%	2.4%	29.5%	0.9%

Table 3: Cost behavior as the backorder costs become asymmetric. Same computational design as in Table 2, except that the total capacity is reduced to 44.

instance, comparing the first rows of Table 2 and 3, as the capacity got tighter, our policy improves from over the priority heuristic. As before, the performance of our policy is better than the AF heuristic in all instances, with the performance gap nearly *doubling* from Table 2. Our policy also significantly outperforms the DA heuristic. In fact, in the last case reported in Table 3, which is the most asymmetric case, we outperform the DA heuristic by nearly 30% (improving from 16.6%, when capacity was ample in Table 2).

Finally, the improved performance of our policy relative to the lower bound as asymmetry increases is significant, given that the optimal allocation rule structure remains unknown for the multi-product capacitated problem.

7.6 Effect of Asymmetric Demand and Penalty Costs

While the previous two subsections focus on problems with 1) asymmetric demands but symmetric costs, and 2) asymmetric costs but symmetric demands, we now study problems where *both* demands and costs are asymmetric. We report a set of experiments in Table 4. In Table 4, we construct a set of experiments in which both the demand distributions and penalty costs are different for the three products. Within each sub-table in the Table 4, we sequentially increase the demand asymmetry. Across sub-tables, we steadily increase penalty costs, repeating tests.

The performance of our policy in relation to the lower bound, is steady at about 13-19% across all sub-tables. This gap increases slightly as the demand becomes asymmetric, and the penalty costs increase.

Our policy performs significantly better than the priority policy in all cases (on average about 40% or better). Compared to the DA heuristic, our policy performs better (i) as the demand

Backorder Costs (1, 5, 10)			Costs					% gap of our policy			
λ_1	λ_2	λ_3	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
1	1	1	59.7	98.2	78.3	68.3	67.6	13.2%	44.7%	15.6%	0.9%
1.1	1	0.9	58.5	99.9	77.9	67.3	66.7	14.0%	49.7%	16.7%	0.9%
1.2	1	0.8	58.5	107.5	81.1	68.0	67.3	15.0%	59.6%	20.5%	1.1%
1.3	1	0.7	60.4	125.8	90.0	71.3	70.6	16.9%	78.2%	27.5%	1.1%
1.4	1	0.6	67.9	170.2	114.7	81.4	80.1	18.0%	112.3%	43.1%	1.6%
(2, 5, 12)											
1	1	1	67.6	108.0	84.1	79.4	77.6	14.8%	37.9%	8.3%	2.4%
1.1	1	0.9	66.9	110.2	83.9	79.1	77.8	16.3%	41.7%	7.9%	1.7%
1.2	1	0.8	67.8	118.8	87.5	81.3	79.6	17.4%	49.3%	9.9%	2.1%
1.3	1	0.7	71.7	138.9	96.6	87.3	85.3	19.0%	62.8%	13.2%	2.3%
1.4	1	0.6	84.4	186.9	121.6	103.2	100.7	19.3%	85.5%	20.7%	2.4%
(3, 6, 15)											
1	1	1	75.8	119.6	92.1	89.6	87.0	14.8%	35.9%	4.6%	2.1%
1.1	1	0.9	75.4	122.3	92.3	89.7	87.8	16.4%	39.4%	5.1%	2.2%
1.2	1	0.8	76.9	131.9	95.9	92.7	90.3	17.4%	46.1%	6.2%	2.6%
1.3	1	0.7	82.1	154.2	105.9	100.7	97.5	18.8%	58.1%	8.6%	3.3%
1.4	1	0.6	98.3	206.5	132.9	120.3	117.1	19.1%	76.3%	13.5%	2.7%

Table 4: Cost behavior as both costs and demands become asymmetric. The computational experiments are structured similar to those in Table 1 except that the product backorder costs are also asymmetric and $k^1 = k^2 = k^3 = 12$, $\lambda^j, j = 1, 2, 3$ are indicated as above. Available aggregate capacity is 44.

becomes more asymmetric for given penalty costs, and (ii) as the costs decreases given the same demand characteristics. Finally, our policy performance is better than the AF heuristic. Furthermore, as the penalty costs increase, the relative performance of our policy improves.

In Table 4, the items that had higher backorder costs had lower demand variability. In Table 5, the backorder costs are reversed for the three products such that the product with higher backorder cost also faces demand with higher variability. In general, our policy performs strongly compared to the existing heuristics in these asymmetric cases. In fact, when compared to the symmetric demand and symmetric cost cases discussed before, the performance of our heuristic is now consistently better than the DA and the AF heuristic.

Table 5 indicates that as the backorder costs diverge further, the performance of our policy improves relative to the Priority, the DA and the AF heuristics, when compared with the same asymmetric demand case (see corresponding rows in Table 4). As products' demand variances diverge, our heuristic provides a cost performance of about, 4% or better than the Priority heuristic, 9% or better than the DA heuristic, and roughly 1% or better than the AF heuristic.

Backorder Costs (10, 5, 1)			Costs					% gap of our policy			
λ_1	λ_2	λ_3	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
1	1	1	59.7	70.4	78.2	68.26	67.6	13.2%	4.2%	15.6%	0.9%
1.1	1	0.9	61.7	73.5	80.8	70.98	70.3	13.9%	4.7%	15.0%	1.0%
1.2	1	0.8	64.7	78.8	87.3	75.80	74.6	15.3%	5.5%	16.9%	1.6%
1.3	1	0.7	69.2	87.3	101.4	83.53	81.7	18.1%	6.9%	24.1%	2.2%
1.4	1	0.6	77.8	103.4	134.9	98.14	95.0	22.1%	8.8%	42.0%	3.3%
(12, 5, 2)											
1	1	1	67.5	83.4	84.1	79.39	77.6	15.0%	7.4%	8.3%	2.3%
1.1	1	0.9	69.4	86.4	86.8	81.80	80.1	15.4%	7.9%	8.3%	2.1%
1.2	1	0.8	72.6	92.5	93.0	86.83	85.0	17.1%	8.8%	9.4%	2.2%
1.3	1	0.7	78.1	103.3	107.6	95.69	93.6	19.8%	10.4%	15.0%	2.3%
1.4	1	0.6	90.6	125.4	143.0	115.59	112.4	24.1%	11.6%	27.2%	2.9%
(15, 6, 3)											
1	1	1	75.7	95.6	91.9	89.50	87.0	14.9%	14.8%	5.6%	2.9%
1.1	1	0.9	77.5	98.8	94.7	92.33	89.4	15.4%	10.6%	6.0%	3.3%
1.2	1	0.8	80.7	105.6	101.4	97.35	94.4	17.0%	11.9%	7.4%	3.2%
1.3	1	0.7	86.8	118.2	116.8	107.19	103.9	19.7%	13.8%	12.5%	3.2%
1.4	1	0.6	102.6	144.8	154.9	132.67	127.7	24.5%	13.5%	21.4%	3.9%

Table 5: Cost behavior of our policy as both costs and demands become asymmetric. Available aggregate capacity is 44.

7.7 Asymmetric Demands, Holding, and Penalty costs

In this section, we examine general asymmetric problems with varying asymmetric holding costs, penalty costs and demand distributions.

Table 6 shows how our policy compares to the Priority, the DA and the AF heuristics. Our policies generally continue to perform better than the extant heuristics. Compared to the Priority policy, our heuristic provides about 9% or more cost savings. In fact, as the costs increase (in the lower sub-table of Table 6), we have the cost savings of our heuristic increase to 12% relative to the priority heuristic. The performance increases as the product demands become more asymmetric. Comparing our policy to the DA heuristic, we find that our policy improves as the demand becomes more asymmetric and for lower penalty costs. Nevertheless, our policy significantly outperforms the DA policy with a cost benefit ranging from from 3.5% to 23.5%.

In Table 7, we increase the holding costs to $(1.2, 1.0, 0.8)$, thus the problems become more asymmetric, in both penalty and holding costs. Our policy continues to outperform other heuristics. This observation persists at higher penalty costs.

$\mathbf{b} = (15, 6, 3)$ $\mathbf{h} = (1.1, 1, 0.9)$			Costs					% gap of our policy			
λ_1	λ_2	λ_3	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
1	1	1	77.1	96.0	93.3	90.8	88.0	14.1%	13.9%	6.0%	3.1%
1.1	1	0.9	79.1	99.6	96.6	93.9	90.8	14.8%	9.7%	6.4%	3.4%
1.2	1	0.8	82.8	106.7	103.2	99.8	96.2	16.2%	10.9%	7.3%	3.8%
1.3	1	0.7	89.4	119.4	119.5	110.2	106.2	18.8%	12.4%	12.5%	3.7%
1.4	1	0.6	105.3	145.8	159.7	134.6	129.3	22.8%	12.8%	23.5%	4.1%
$\mathbf{b} = (20, 10, 5)$ $\mathbf{h} = (1.1, 1, 0.9)$			Costs					% gap of our policy			
1	1	1	90.7	115.3	106.9	107.0	103.3	13.9%	11.6%	3.4%	3.5%
1.1	1	0.9	92.6	119.3	109.8	110.1	106.0	14.5%	12.6%	3.7%	3.9%
1.2	1	0.8	96.3	127.7	117.1	116.3	111.6	15.9%	14.4%	5.0%	4.3%
1.3	1	0.7	103.6	143.1	135.0	128.4	123.0	18.7%	16.4%	9.8%	4.4%
1.4	1	0.6	125.1	175.9	178.1	161.9	152.9	22.2%	15.0%	16.5%	5.8%

Table 6: Cost behavior of our policy all parameters become asymmetric. i.e, backorder costs, holding costs and the demand distributions are all asymmetric. Available aggregate capacity is 44.

$\mathbf{b} = (15, 6, 3)$ $\mathbf{h} = (1.2, 1, 0.8)$			Costs					% gap of our policy			
λ_1	λ_2	λ_3	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
1	1	1	78.2	96.2	94.5	91.9	88.8	13.6%	8.3%	6.3%	3.5%
1.1	1	0.9	80.6	100.1	98.2	95.3	92.0	14.1%	8.9%	6.8%	3.7%
1.2	1	0.8	84.6	107.4	105.4	101.6	97.8	15.6%	9.9%	7.8%	3.9%
1.3	1	0.7	91.7	120.3	123.1	112.9	108.0	17.8%	11.3%	13.9%	4.5%
1.4	1	0.6	107.9	146.2	164.0	136.3	130.4	20.8%	12.1%	25.7%	4.5%
$\mathbf{b} = (20, 10, 5)$ $\mathbf{h} = (1.2, 1, 0.8)$			Costs					% gap of our policy			
1	1	1	91.7	114.9	107.8	108.0	103.8	13.3%	10.7%	3.8%	4.1%
1.1	1	0.9	94.1	119.3	111.2	111.5	106.9	13.6%	11.6%	4.0%	4.3%
1.2	1	0.8	98.2	127.8	118.8	118.8	112.9	15.0%	13.2%	5.2%	5.2%
1.3	1	0.7	106.0	143.2	137.0	130.6	124.6	17.5%	15.0%	10.0%	4.9%
1.4	1	0.6	127.2	175.0	182.5	162.4	153.5	20.7%	14.0%	18.9%	5.8%

Table 7: Cost behavior of our policy when parameters, i.e, backorder costs, holding costs and the demand distributions are all asymmetric. The parameters are identical to the Table 6 except for holding cost parameters that are more asymmetric.

7.8 The Effect of Capacity

In this section, we explore the effect of capacity κ on asymmetric problem scenarios with different backorder costs, holding costs, and demands. In Tables 8 and 9, we sequentially decrease the capacity such that the utilization increases from 73.3% ($K = 60$) to 97.78% (for $K = 45$) and demonstrate that our policy is very efficient in allocating scarce capacity resource amongst the products.

In Table 8, product 1 has the lowest variability and product 3 has the highest variability. As the capacity falls from 60 to 45, the relative performance of our policy improves consistently. Against the priority heuristic, the relative cost advantage of our heuristic improves from 6% to 9%. Against the DA heuristic, as the capacity gets tighter, the relative advantage increases from 6% to as high as 46%. Similarly, against the AF heuristic, the performance improves from 0.3% to about $3-5\%$.

$k=12, \lambda=(1.5,1,0.5)$ $\mathbf{b}=(15,6,3)$ $\mathbf{h}=(1.1,1,0.9)$	Costs					% gap			
Capacity	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
60	78.7	88.7	86.3	83.8	83.6	6.2%	6.1%	3.3%	0.3%
58	78.9	91.0	89.5	85.8	85.2	8.0%	6.8%	5.0%	0.6%
56	79.4	94.1	94.4	88.6	87.7	10.5%	7.4%	7.7%	1.1%
54	80.3	98.5	99.9	92.9	91.2	13.6%	8.0%	9.6%	1.9%
52	82.1	104.7	108.4	98.8	96.4	17.4%	8.6%	12.4%	2.4%
50	86.1	114.3	121.5	109.0	104.8	21.7%	9.1%	15.9%	3.9%
48	95.2	130.1	144.9	125.4	118.9	24.9%	9.4%	21.9%	5.5%
46	120.2	161.6	197.3	156.1	147.9	23.0%	9.3%	33.4%	5.5%
45	150.0	194.0	262.5	185.7	179.3	19.5%	8.2%	46.4%	3.5%

Table 8: Cost behavior of our policy as the total capacity becomes tighter for asymmetric demand and costs: Erlang(k, λ) with $k = 12$ and $\lambda_1 = 1.5, \lambda_2 = 1.0, \lambda_3 = 0.5$.

We also note that as the capacity gets tighter, the relative difference between our policy and the lower bound increases. This is due to the weakened nature of the lower bound under high utilization. When capacity is unlimited the multi-product problem decomposes into N individual newsvendor problems. In this case, the balancing heuristic is optimal and the lower bound coincides with the optimal cost. As the capacity gets tighter, the issue of allocating capacity becomes important and the lower bound benefits from the fact that it allows for costless redistribution of inventories in each period. In any case, the relative performance of our heuristic continues to improve as the capacity becomes tighter.

In Table 9, we follow the same schematic as in Table 8, except that the pattern of asymmetric demands are now reversed. Product 1 has the highest and Product 3 has the lowest variance to mean ratio. As the capacity falls from 60 to 45, again the performance of our policy improves consistently. Against the priority heuristic, the relative cost advantage of our heuristic improves

$k = 12, \lambda = (0.5, 1, 1.5)$ $\mathbf{b} = (20, 10, 5)$ $\mathbf{h} = (1.2, 1, 0.8)$	Costs					% gap			
Capacity	LB	Pri	DA	AF	Search	LB	Pri	DA	AF
60	86.6	99.3	96.3	93.9	93.3	7.7%	6.4%	3.2%	0.7%
58	86.9	102.4	100.1	96.7	95.6	10.0%	7.1%	4.8%	1.2%
56	87.5	106.6	104.3	100.3	98.9	13.0%	7.8%	5.5%	1.4%
54	88.8	112.4	110.4	105.9	103.5	16.5%	8.6%	6.7%	2.2%
52	91.5	120.7	119.7	114.2	110.3	20.6%	9.4%	8.5%	3.6%
50	97.2	133.4	133.5	127.3	120.8	24.3%	10.4%	10.5%	5.4%
48	110.1	154.0	159.0	149.5	138.3	25.6%	11.4%	14.9%	8.1%
46	144.5	195.5	216.7	190.6	175.5	21.5%	11.4%	23.5%	8.6%
45	184.3	238.6	290.1	234.9	216.8	17.6%	10.1%	33.8%	8.8%

Table 9: Cost behavior of our policy as the total capacity becomes tighter. The asymmetric demands are *reversed* from the previous table

Capacity	Priority			DA			AF			Search		
60	60	35	24	60	29	16	60	29	16	60	32	20
58	60	36	26	60	29	16	60	29	16	60	33	21
56	60	37	28	64	31	17	60	30	16	60	34	22
54	60	39	32	68	33	18	60	30	17	60	35	26
52	60	41	38	72	35	19	60	31	18	63	37	29
50	60	43	47	82	40	22	60	33	20	66	40	34
48	60	47	64	93	45	25	60	37	24	70	44	47
46	60	52	101	122	59	33	60	46	44	75	49	77
45	60	56	139	150	73	40	60	48	79	77	52	113

Table 10: Base-stock levels under different policies for instances in Table 9.

from 6.4% to 10.1%. Against the DA heuristic, as the capacity gets tighter, the advantage increases from 3.2% to as high as 34%. Similarly, against the AF heuristic, the performance of our heuristic improves from 0.7% to about 8.8%, as the capacity becomes tighter. To summarize, under scarce capacity our approach does 9% or better cost-wise against *every* extant heuristic.

In Table 10, we show the base-stock levels for the scenarios reported in Table 9. In general, it appears that the priority policy assigns a significantly higher base-stock for product 3 (which is cheapest to hold). On the other hand, the DA heuristic chooses inventories such that a significantly higher base stock is assigned to Product 1. Our Search policy and the AF heuristic both choose base-stock levels that are in between those chosen under the Priority and the DA heuristics. It appears that the AF heuristic chooses weakly lower base-stocks for the products, compared to our policy. These differences are more pronounced as the capacity becomes tighter. It also appears that our policy outperforms significantly better than other policies when the capacity is scarce, by setting up the base-stock parameters appropriately. The difference in the base-stock levels in our policy and those in the other heuristics may be possibly due to the better allocation approach used in our policy.

Although it is hard to characterize the structure of the optimal policy decisions, we consider a simple scenario which illustrates the different decisions made under the policies which we study in this paper, using the last line $K = 45$ in Table 9) as an instance. Let the beginning inventory levels in some period be $(70, 70, 80)$ for the three products. Under the priority policy, we have to produce 59 units for product 3 and none for products 1 and 2. Due to limited capacity, shortfalls continue to exist (for product 3). Under the DA heuristic, we have to produce 80 units of product 1, 3 units for product 2 and none for product 3. Even in this case, shortfalls continue to exist, and surely for product 1, since capacity available is 45 but 80 units have to be produced. Under the AF heuristic, all items are above their base stock level under the heuristic, so the entire capacity goes unused. However, under our policy, 33 units of product 3 are produced. There is no shortfall. In this scenario, the DA and priority heuristics allow for too much shortfall for different products, and under the AF heuristic, the capacity may go unused (compared to the Search policy). Our policy tries to find a balance between excessive shortfalls (due to high base-stock levels) and low utilization (due to low base-stock levels).

7.9 Inference from the Computational Study

- When all product attributes (i.e., demand and costs) are symmetric, our policy and the DA heuristic are optimal. The AF heuristic is sub-optimal.

- In virtually all of the problem instances we computed, our policy *significantly outperforms* all the extant heuristics. The only instances in which the approaches have comparable performances are those instances with low utilization (i.e., amply capacity).

- As the capacity gets tighter, our heuristic consistently outperforms other heuristics. It performs significantly better than the priority heuristic in all cases. It consistently outperforms the AF heuristic, and by as much as 13% when the products are asymmetric and capacities are tight. Our performance is also significantly better than the DA heuristic, except for a few symmetric cases with low utilization when the two heuristics are comparable. When the products and demands are asymmetric, it is possible that our policy saves more than 40% in costs.

8 Concluding Remarks

We have developed an intuitive, theoretically appealing and implementable policy for managing finite flexible capacity shared by multiple products. To implement our allocation policies, one just needs to examine their current shortfalls to determine the allocation of capacity amongst different products. In addition to being simple and intuitive to implement, our policies (a) have the theoretical appeal of being asymptotically optimal at high service levels and at high utilization levels, and (b) perform well when flexible capacity is most valuable (i.e., scarce capacity, varying demands). Nevertheless, there are several challenging questions that are left unanswered. Very little is known about the structure of the optimal policy when products are asymmetric. Perhaps benchmarking our policies against the optimal policy is a calibration step in that direction. Another important factor is the possibility of correlated demand structure among the products. We leave such directions to future research.

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Appendix

Proof of Lemma 3

Consider the subclass of policies mentioned in the statement of the lemma. Once the base-stock vector $\mathbf{S}$ is chosen for a policy within this class, the policy is entirely specified. The long run average cost of this policy is $\sum_{n=1}^{N} E[h^n \cdot (S^n - V^{\boldsymbol{\alpha},n} - D^n)^+ \; + \; b^n \cdot (V^{\boldsymbol{\alpha},n} + D^n - S^n)^+]$. Since the distribution of $\mathbf{V}^{\boldsymbol{\alpha}}$ does not depend on the base-stock vector, the expression above is separable in $(S^1, \ldots, S^N)$; thus, the optimal value of S^n is simply the minimizer of the "newsvendor-type" expression within the summation above. The desired result is immediate.

Proof of Theorem 4

By assumption $V_1^{\mathbf{1},n} = 0$ for all n. This establishes statement (i) for $t = 1$. Under the symmetric allocation rule, if the distribution of $\mathbf{V}_t^{\mathbf{1}}$ is symmetric across n for some t and the distribution of D_t^n is also symmetric across n, then the distribution of $\mathbf{V}_{t+1}^{\mathbf{1}}$ will also be symmetric. Statement (i) follows for all t by induction. Statement (ii) is a direct consequence of statement (i).

Proof of Lemma 6

Without loss of generality, we assume that the priority order $(1), (2), \ldots, (N)$ is $1, 2, \ldots, N$. When the capacity is not binding (i.e. $\sum_j^N (W^j) \leq \kappa$), the shortfalls after ordering are zero under both rules (for any m). Thus, the statement holds for any m. Similarly, if the shortfall before ordering, W^j, is zero for any j, then the shortfalls after ordering $V^{P,j}$ and $V^{\boldsymbol{\alpha}_m,j}$ are both zero. Thus, it is sufficient to consider the case where $\mathbf{W}$ is strictly positive in every component.

When capacity is binding, there exists some $k, 1 \leq k < N$ such that $\sum_1^k W^j \leq \kappa$ and $\sum_1^{k+1}(W^j) > \kappa$. Then, under the priority policy $V^{P,j} = 0 \; \forall \; j = 0, \ldots, k$, $V^{P,k+1} = W^{k+1} + \kappa - \sum_1^k W^j$, and $V^{P,j} = W^{P,j} \; \forall \; j = k+2, \ldots, N$. Let us define $\beta = W^{k+1} + \kappa - \sum_1^k W^j$, i.e. $\beta = V^{P,k+1}$.

Let M be large enough that $W^{k+2}/M^{k+1} \geq W^{k+3}/M^{k+2} \geq \ldots \geq W^N/M^{N-1}$. Let $\tilde{\epsilon} \in (0, \epsilon/k)$ and let $\tilde{\epsilon} \leq \min\{W^1, \ldots, W^k, \beta/k\}$. Moreover, let M be large enough that $\tilde{\epsilon} \geq W^{k+1}/M^k$, $\tilde{\epsilon}/M \geq W^{k+1}/M^k$, …, $\tilde{\epsilon}/M^{k-1} \geq W^{k+1}/M^k$ and $(\beta - k \cdot \tilde{\epsilon})/M^k \geq W^{k+1}/M^k$. (All the inequalities above except the first and the last are redundant - but we present them here for ease of verification of our next claim). These inequalities ensure that even if the first $k+1$ components of the shortfall vector before ordering were reduced to $(\tilde{\epsilon}, \ldots, \tilde{\epsilon}, \beta - k \cdot \tilde{\epsilon})$, the weighted balancing rule defined by the vector $\boldsymbol{\alpha}_m$ prefers to allocate the next incremental amount of capacity to the first $k+1$ products and not the products in $\{k+2, \ldots, N\}$.

It is now easy to verify that $\mathbf{V}^{\boldsymbol{\alpha}_m}$ satisfies the following inequalities for all $m \geq M$: $V^{\boldsymbol{\alpha}_m,j} = W^j = V^{P,j}$ for all $j \in \{k+2, k+3, \ldots, N\}$, $V^{\boldsymbol{\alpha}_m,j} \in [0, \tilde{\epsilon}] = [V^{P,j}, V^{P,j} + \tilde{\epsilon}]$ for all $j \in \{1, 2, \ldots, k\}$, and $V^{\boldsymbol{\alpha}_m,k+1} \in [\beta - k \cdot \tilde{\epsilon}, \beta] = [V^{P,k+1} - k \cdot \tilde{\epsilon}, V^{P,k+1}]$. The proof of the lemma is complete from the fact that $\tilde{\epsilon} \leq \epsilon/k$.

Online Technical Appendix

Proof of Theorem 5

Lemma 3 establishes the optimality of the base-stock vector $\mathbf{S}^{\mathbf{1}*}$ for policies in Π_{BS-B} that use the weight vector $\mathbf{1}$. It remains to show that the policy in Π_{BS-B} defined by the base-stock vector $\mathbf{S}^{\mathbf{1}*}$ and the weight vector $\mathbf{1}$ is an optimal policy when all policies in Π are considered.

Let us first consider the finite horizon discounted cost problem with a discount factor $\gamma \in (0, 1]$ and a planning horizon of T periods, that is, the problem of minimizing $E[\sum_{t=1}^{T} \gamma^t \cdot C_t]$ over Π. This finite horizon dynamic program can be represented through the cost-to-go functions $\{f^{\gamma}_{t,T} : t = 1, \ldots, T\}$ as follows:

$$
\begin{aligned}
f^{\gamma}_{t,T}(\mathbf{x}) \;\; &= \;\; \min_{\mathbf{y}} \; \sum_{n=1}^{N} \left(h^n \cdot E[(y^n - D^n)^+] \; + \; b^n \cdot E[(D^n - y^n)^+] \right) + \gamma \cdot E[f^{\gamma}_{t+1,T}(\mathbf{y} - \mathbf{D})] \\
& \qquad \text{s.t.} \quad \mathbf{y} \; \geq \; \mathbf{x} \quad \text{and} \quad \sum_{n=1}^{N} y^n \leq \sum_{n=1}^{N} x^n + \kappa \; ,
\end{aligned}
$$

where $f^{\gamma}_{T+1,T}(\mathbf{x}) := 0$ for all $\mathbf{x}$.

It is fairly easy to show using induction that under Assumption 1, the function $f^{\gamma}_{t,T}$ is convex and symmetric. Using standard dynamic programming arguments, we can establish the pointwise convergence of the finite horizon cost-to-go functions $\{f^{\gamma}_{1,T}(\mathbf{x})\}$ to $\{f^{\gamma}(\mathbf{x})\}$ the cost-to-go function of the infinite horizon, discounted cost dynamic program (defined for $\gamma \in (0, 1)$) represented below:

$$
f^{\gamma}(\mathbf{x}) \;\; = \;\; \min_{\mathbf{y}} \; g^{\gamma}(\mathbf{y}) \tag{3}
$$

$$
\text{s.t.} \quad \mathbf{y} \; \geq \; \mathbf{x} \quad \text{and} \quad \sum_{n=1}^{N} y^n \leq \sum_{n=1}^{N} x^n + \kappa \; , \tag{4}
$$

where $g^{\gamma}(\mathbf{y}) = \sum_{n=1}^{N} (h^n \cdot E[(y^n - D^n)^+] \; + \; b^n \cdot E[(D^n - y^n)^+]) + \gamma \cdot E[f^{\gamma}(\mathbf{y} - \mathbf{D})]$. The infinite horizon discounted cost optimal policy is defined by a selector $\mathbf{y}^{\gamma*}(\mathbf{x})$ such that for every $\mathbf{x}$, the vector $\mathbf{y}^{\gamma*}(\mathbf{x})$ is a solution to the above minimization problem. The convergence of $\{f^{\gamma}_{1,T}(\mathbf{x})\}$ to $\{f^{\gamma}(\mathbf{x})\}$ ensures that g^{γ} is also convex and symmetric. The convexity and symmetry of g^{γ} implies the existence of a vector $\mathbf{S}^{\gamma*}$ such that (a) it minimizes $g^{\gamma}(\mathbf{y})$ and (b) all its components are identical; let us denote this identical base-stock value for all components as $S^{\gamma*}$.

Next, we claim that the symmetric allocation rule applied in combination with the base-stock vector $\mathbf{S}^{\gamma*}$ is an optimal policy for the infinite horizon, discounted cost problem defined in (3)-(4) when $\mathbf{x} \leq \mathbf{S}^{\gamma*}$. There are two cases to study. The first case is the following: $\mathbf{x}$ is such that $\mathbf{x} \leq \mathbf{S}^{\gamma*}$

and $\sum_{n=1}^{N} x^n + \kappa \geq \sum_{n=1}^{N} S^{\gamma *}$. From such an inventory state $\mathbf{x}$, before ordering, we know that it is feasible to reach $\mathbf{S}^{\gamma *}$ after ordering. Moreover, this action is optimal since $\mathbf{S}^{\gamma *}$ minimizes $g^{\gamma}(\mathbf{y})$. The second case is the following: $\mathbf{x}$ is such that $\mathbf{x} \leq \mathbf{S}^{\gamma *}$ and $\sum_{n=1}^{N} x^n + \kappa < \sum_{n=1}^{N} S^{\gamma *}$. Convexity of the function $g^{\gamma}(\mathbf{y})$ ensures that there is an optimal solution such that $\sum_{n=1}^{N} y^n = \sum_{n=1}^{N} x^n + \kappa$. Moreover, it is easy to show the following inequality by using the convexity and symmetry of $g^{\gamma}(\cdot)$: Let $\mathbf{y}$ and $\tilde{\mathbf{y}}$ be two vectors such that $\sum_{n=1}^{N} y^n = \sum_{n=1}^{N} \tilde{y}^n = N \cdot \overline{y}$ for some constant $\overline{y}$, and such that, for every n, $y^n \in [\tilde{y}^n, \overline{y}]$ if $\tilde{y}^n \leq \overline{y}$ and $y^n \in [\overline{y}, \tilde{y}^n]$ if $\tilde{y}^n \geq \overline{y}$. Then, $g^{\gamma}(\tilde{\mathbf{y}}) \geq g^{\gamma}(\mathbf{y})$. In words, this inequality asserts that, given two inventory vectors $\tilde{\mathbf{y}}$ and $\mathbf{y}$ which are equal in terms of their aggregate inventories, the vector $\mathbf{y}$ which is more symmetric (i.e. whose components are closer to the average $\overline{y}$) is preferable with respect to the cost function $g^{\gamma}(\cdot)$. Since all products have the same mean demands per period, notice that the policy of using the symmetric allocation rule and the base-stock vector $\mathbf{S}^{\gamma *}$ leads to the "most symmetric" vector $\mathbf{y}$ among all vectors such that (4) holds. Thus, the inventory vector $\mathbf{y}$ chosen by this policy is optimal over all $\mathbf{y}$ satisfying the constraints above. This completes the proof of the claim.

Now we return to the infinite horizon average cost problem. Schäl (1993) shows that, under certain conditions, the sequence of infinite horizon discounted cost optimal policies converges to an infinite horizon average cost optimal policy as the discount factor γ approaches 1. A straightforward extension of Huh et al. (2011) leads to Schäl's conditions for our multi-product problem, and details are available on request.

Thus the above mentioned convergence of discounted cost optimal policies to an average cost optimal policy holds in our case. This implies that there exists a vector $\mathbf{S}^*$, in which all components are identical, such that the symmetric allocation rule applied in combination with the base-stock vector $\mathbf{S}^*$ is an average cost optimal policy. Finally, we know from Lemma 3 that, within Π_{BS-B}, the optimal base-stock vector corresponding to the weight vector $\mathbf{1}$ is $\mathbf{S}^{\mathbf{1}*}$. Thus, $\mathbf{S}^{\mathbf{1}*}$ is a valid choice for $\mathbf{S}^*$; this completes the proof.

Proof of Lemma 7

The first inequality is trivial to establish because the cost incurred by any policy in any period when the backorder costs are given by $\mathbf{b}$ exceed the corresponding quantity when all backorder costs are $min(\mathbf{b})$. The second inequality follows from the definition of $C^*(h, \mathbf{b})$ and $C^{\mathbf{1}*}(h, \mathbf{b})$ as the optimal cost over all policies and the cost of the optimal weighted balancing policy, respectively. We now show the third inequality. From Theorem 5, we know that

$$C^{\mathbf{1}*}(h, avg(\mathbf{b})) = C^*(h, avg(\mathbf{b})) \ .$$

Observe that $C^{\mathbf{1}*}(h, \mathbf{b})$ is a constant with respect to permutations to $\mathbf{b}$ due to the symmetric demand assumption and the symmetric nature of the symmetric allocation rule. The average of all possible permutations of $\mathbf{b}$ is

$$avg(\mathbf{b}) \cdot (1, 1, \ldots, 1) \ .$$

Since the single period function is linear with respect to $\mathbf{b}$ for any given state and action, it is easy to show that, for any policy π, $C^{\pi}(h, \mathbf{b})$ is concave with respect to $\mathbf{b}$. This implies that

$$C^{\mathbf{1}*}(h, \mathbf{b}) \ \leq \ C^{\mathbf{1}*}(h, avg(\mathbf{b})) \ .$$

Recalling that $C^{\mathbf{1}*}(h, avg(\mathbf{b})) = C^{*}(h, avg(\mathbf{b}))$, we have

$$C^{\mathbf{1}*}(h, \mathbf{b}) \ \leq \ C^{*}(h, avg(\mathbf{b})) \ .$$

Proof of Theorem 8

The first statement follows directly from Lemma 7. We now prove the asymptotic limit result by invoking and proving two related results.

Lemma 12 (Huh et al. (2009))**.** *Let X be a random variable such that $\overline{M} = \sup\{x : P(X \leq x) < 1\}$ and $\lim_{x \uparrow \overline{M}} \ \frac{E[X - x \ | \ X > x]}{x} \ = \ 0$, where $\overline{M} \in \Re^{+} \cup \{\infty\}$. Then,*

$$\lim_{\beta \to \infty} \left(\frac{L(h, \beta \cdot b', X)}{L(h, \beta \cdot b, X)} \right) \ = \ 1 \ \ \textit{for all} \ \ (h, b', b) \ .$$

Note that any probability distribution with an increasing failure rate (IFR) satisfies the condition in Lemma 12. Next, we show that, for every product, the convolution of the steady state shortfall and demand distributions satisfies the condition in Lemma 12.

Lemma 13. *Under Assumption 2, the following statement holds for all $j \ \in \ \{1, \ldots, N\}$: The convolution of $V_{\infty}^{\mathbf{1},j}$ and D^{j} is unbounded, i.e. $P(V_{\infty}^{\mathbf{1},j} + D^{j} < x) < 1$ for all x. Moreover, if the common marginal distribution of the random variables $\{D^{j}\}$ is an IFR distribution, then,*

$$\lim_{x \to \infty} \ \frac{E[(V_{\infty}^{\mathbf{1},j} + D^{j}) - x \ | \ (V_{\infty}^{\mathbf{1},j} + D^{j}) > x]}{x} \ = \ 0 \ \ \textit{for all} \ j \ .$$

Proof: Recall that $P(D > \kappa) > 0$. Since the steady state distribution of the aggregate shortfall V_{∞} is the same as that of the waiting time in a G/D/1 queue, it is easy to verify that the random variable V_{∞} is unbounded. Under symmetric demands, we know from Theorem 4 that the distributions of all the random variables $V_{\infty}^{\mathbf{1},j}$ are identical. By definition, V_{∞} has the same distribution as the convolution $\sum_{j=1}^{N} V_\infty^{\mathbf{1},j}$. Since V_∞ is unbounded, it follows that the random variables $V_\infty^{\mathbf{1},j}$ are also unbounded; this implies that the random variables $V_\infty^{\mathbf{1},j} + D^j$ are also unbounded which proves the first part of the lemma. We now proceed to the second part.

It is well known (see Bryson and Siddiqui (1969)) that if a non-negative random variable X has a finite mean and an IFR distribution, then its mean residual life $E[X-x|x>x]$ is decreasing and therefore bounded by the mean, $E[X]$. Thus, for every j, we know that the mean residual life of D^j is bounded. Next, we use conditional expectations and observe that

$$E[(V_\infty^{\mathbf{1},j} + D^j) - x \mid (V_\infty^{\mathbf{1},j} + D^j) > x] = E_{V_\infty^{\mathbf{1},j}}\left[E_{D^j}[v + D^j - x \mid D^j > x - v] | V_\infty^{\mathbf{1},j} = v\right] .$$

Dividing both sides by x and taking the limit as $x \to \infty$, we obtain

$$\begin{aligned}
&\lim_{x\to\infty} \frac{E[(V_\infty^{\mathbf{1},j} + D^j) - x \mid (V_\infty^{\mathbf{1},j} + D^j) > x]}{x} \\
&= \lim_{x\to\infty} \frac{E_{V_\infty^{\mathbf{1},j}}\left[E_{D^j}[v + D^j - x \mid D^j > x - v] | V_\infty^{\mathbf{1},j} = v\right]}{x} \\
&\le \lim_{x\to\infty} \frac{E_{V_\infty^{\mathbf{1},j}}\left[\left(1(x \ge v)\cdot E[D^j] + 1(x < v)\cdot(E[D^j] + v - x)\right) | V_\infty^{\mathbf{1},j} = v\right]}{x} ,
\end{aligned}$$

where $1(\cdot)$ is the indicator operator; the inequality follows from the fact that the mean residual life of D^j is bounded by its unconditional mean. The expression on the right side of the inequality can be bounded above by

$$\lim_{x\to\infty} \frac{E[D^j + V_\infty^{\mathbf{1},j}]}{x} = 0 \text{ because } E[D^j] < \infty \text{ and } E[V_\infty^{\mathbf{1},j}] < \infty \text{ (since } E[D] < \kappa \text{ by assumption).}$$

This proves the desired result. $\square$

We know from (1) that

$$\begin{aligned}
C^*(h, avg(\mathbf{b})) &= N \cdot L(h, avg(\mathbf{b}), V_\infty^{\mathbf{1},1} + D^1) \quad \text{and} \\
C^*(h, min(\mathbf{b})) &= N \cdot L(h, min(\mathbf{b}), V_\infty^{\mathbf{1},1} + D^1) \, .
\end{aligned}$$

Therefore,

$$\left(\frac{C^{\mathbf{1}*}(h,\mathbf{b})}{C^*(h,\mathbf{b})}\right) \le \left(\frac{L(h, avg(\mathbf{b}), V_\infty^{\mathbf{1},1} + D^1)}{L(h, min(\mathbf{b}), V_\infty^{\mathbf{1},1} + D^1)}\right) .$$

The desired asymptotic result now follows directly from Lemma 12 and Lemma 13. This completes the proof of Theorem 8.

Proof of Lemma 9

Proof. First, we observe that

$$C^*(\mathbf{h},\mathbf{b},\kappa) \geq C^*(h,b,\kappa)\ ,\ \text{if}\ \ 0\ <\ h\ \leq\ h^j\ \forall\, j\ \ \text{and}\ \ 0\ <\ b\ <\ b^j\ \forall\, j\ , \tag{5}$$

where $C^*(h,b,\kappa)$ is the optimal cost of a system in which all products have the same holding cost h and the same backorder cost b. Thus, it suffices to show that, for any $h > 0$ and $b > 0$,

$$\lim_{\kappa \downarrow \mu} C^*(h,b,\kappa) = \infty\ .$$

Next, let us define $V_\infty(\kappa)$ as the steady state version of the aggregate shortfall process $\{V_t(\kappa)\}$ defined by the recursion $V_{t+1}(\kappa) = (V_t(\kappa) + D - \kappa)^+$ (recall that $D = \sum_{j=1}^N D^j$). We claim that

$$C^*(h,b,\kappa) \geq \min_S h \cdot E[(S - V_\infty(\kappa) - D)^+] + b \cdot E[(D + V_\infty(\kappa) - S)^+]\ . \tag{6}$$

The proof of the claim is the following: Consider any feasible policy in the multi-product system. We can use this policy to construct a feasible policy in the "aggregate system" whose optimal long run average cost is represented on the right side of (6) such that the cost in the latter system (and therefore, the long run average cost) is smaller than that in the former system every period. This is done by ordering, in the latter system, the sum of the quantities ordered for all the products in the former system – the fact that the cost in the latter system is smaller in every period follows from the inequalities

$$\sum_{j=1}^N (x^j - d^j)^+ \geq \left(\sum_{j=1}^N (x^j - d^j)\right)^+ \ \text{and}\ \sum_{j=1}^N (d^j - x^j)^+ \geq \left(\sum_{j=1}^N (d^j - x^j)\right)^+ .$$

This proves the claim. Thus, it only remains to show that

$$\lim_{\kappa \downarrow \mu} \min_S h \cdot E[(S - V_\infty(\kappa) - D)^+] + b \cdot E[(D + V_\infty(\kappa) - S)^+] = \infty.$$

To show this, we first note that replacing D by its expectation, μ, in the expression within the limit above we obtain a lower bound on that expression (this is a consequence of Jensen's inequality and the convexity of the function $(x)^+$). Letting $\tilde{S} = S - \mu$, it is sufficient to show that

$$\lim_{\kappa \downarrow \mu} \min_{\tilde{S}} h \cdot E[(\tilde{S} - V_\infty(\kappa))^+] + b \cdot E[(V_\infty(\kappa) - \tilde{S})^+] = \infty. \tag{7}$$

Next, observe that the recursion for $\{V_t(\kappa)\}$ is the same as that for the waiting time process for a $G/G/1$ queue in which the inter-arrival times are deterministic and equal to κ and the service time for the t^{th} customer is D_t. We know from Kingman (1962) that the distribution of the random variable $[\frac{(\kappa-\mu)}{\sigma^2}] \cdot V_\infty(\kappa)$ converges to an exponential distribution with mean 1/2, i.e.,

$$\lim_{\kappa \downarrow \mu} P\left(\frac{(\kappa-\mu)}{\sigma^2} \cdot V_\infty(\kappa) \geq z\right) = e^{-2z} \ , \quad \text{for all} \ \ z \ \geq \ 0 \ ,$$

where σ^2 is the variance of the aggregate demand D. We can verify using straight forward calculus that this implies that

$$\begin{aligned} & \lim_{\kappa \downarrow \mu} \min_{S'} h \cdot E\left[(S' - \frac{(\kappa-\mu)}{\sigma^2} \cdot V_\infty(\kappa))^+\right] + b \cdot E\left[(\frac{(\kappa-\mu)}{\sigma^2} \cdot V_\infty(\kappa) - S')^+\right] \\ = \ & (h/2) \cdot \ln\left((b+h)/h\right). \end{aligned} \tag{8}$$

It is easy to verify that the desired equality in (7) follows directly from (8). □

Proof of Theorem 10

Proof. The second statement follows directly from the first statement and Lemma 9. We proceed to show the first statement. Our plan is to find an upper bound on $C^P(\mathbf{h}, \mathbf{b}, \kappa)$ and a lower bound on $C^*(\mathbf{h}, \mathbf{b}, \kappa)$ and show that the difference between these bounds is finite for all κ.

Let $S(\kappa)$ be defined as $\arg\min_S h^N \cdot E[(S - D - V_\infty(\kappa))^+] + b^N \cdot E[(D + V_\infty(\kappa) - S)^+]$. Now, consider a policy π which uses the same priority rule as P but uses the following non-optimal base-stock levels:

$$S^j = 0 \text{ for all } j < N \text{ and } S^N = S(\kappa) \ .$$

Let $C^\pi(\mathbf{h}, \mathbf{b}, \kappa)$ ($C^{\pi,N}(h^N, b^N, \kappa)$) denote the long run average cost for the system (product N) under π given the respective parameters. Since P uses the optimal base-stock levels under the given priority allocation rule and π does not, we obtain the following relations:

$$\begin{aligned} C^P(\mathbf{h}, \mathbf{b}, \kappa) \ & \leq \ C^\pi(\mathbf{h}, \mathbf{b}, \kappa) \\ & = \ \sum_{j=1}^{N-1} b^j \cdot E[D^j + V_\infty^{P,j}] + C^{\pi,N}(h^N, b^N, \kappa) \ . \end{aligned} \tag{9}$$

The equality above follows from the fact that under π, there is never any inventory of products 1 through $N-1$ on hand and from the fact that the shortfall process under π is the same as that under P. From (9) and Assumption 3, it follows that

$$\begin{aligned} C^P(\mathbf{h},\mathbf{b},\kappa) &\le b^1 \cdot \sum_{j=1}^{N-1} \left(E[D^j + V_\infty^{P,j}]\right) + C^{\pi,N}(h^N, b^N, \kappa) \\ &= b^1 \cdot \sum_{j=1}^{N-1} \left(E[D^j + V_\infty^{P,j}]\right) \\ &\quad + h^N \cdot E[(S(\kappa) - D^N - V_\infty^{P,N}(\kappa))^+] + b^N \cdot E[(D^N + V_\infty^{P,N}(\kappa) - S(\kappa))^+] \; . \end{aligned} \tag{10}$$

The inequality above provides an upper bound on $C^P(\mathbf{h},\mathbf{b},\kappa)$.

Next, we proceed to identify a lower bound on $C^*(\mathbf{h},\mathbf{b},\kappa)$. By Assumption 3, we have

$$C^*(\mathbf{h},\mathbf{b},\kappa) \ge C^*(h^N, b^N, \kappa) \; . \tag{11}$$

Now, observe that $C^*(h^N, b^N, \kappa)$ is the optimal cost of a multi-product inventory system in which all products have identical holding and shortage costs. We have shown in the proof of Lemma 9 that this quantity exceeds the optimal cost of a single product inventory system with a holding cost h^N, backorder cost b^N, capacity κ and demand distribution D. That is,

$$\begin{aligned} C^*(h^N, b^N, \kappa) &\ge \min_S h^N \cdot E[(S - D - V_\infty(\kappa))^+] + b^N \cdot E[(D + V_\infty(\kappa) - S)^+] \; , \\ &= h^N \cdot E[(S(\kappa) - D - V_\infty(\kappa))^+] + b^N \cdot E[(D + V_\infty(\kappa) - S(\kappa))^+] \; . \end{aligned} \tag{12}$$

Let us define $V_\infty^{P,[1,N-1]}$ as $\sum_{j=1}^{N-1} V_\infty^{P,j}$ and $D^{[1,N-1]}$ as $\sum_{j=1}^{N-1} D^j$. Now, comparing (10) and (12) and using (11), we can write

$$\begin{aligned} & C^P(\mathbf{h},\mathbf{b},\kappa) - C^*(\mathbf{h},\mathbf{b},\kappa) \\ \le\; & b^1 \cdot \sum_{j=1}^{N-1} \left(E[D^j + V_\infty^{P,j}]\right) + h^N \cdot E[D^{[1,N-1]} + V_\infty^{P,[1,N-1]}(\kappa)] \\ =\; & (b^1 + h^N) \cdot E[D^{[1,N-1]} + V_\infty^{P,[1,N-1]}(\kappa)] \; . \end{aligned} \tag{13}$$

Notice that $V_\infty^{P,[1,N-1]}(\kappa)$ is the steady state distribution of the stochastic process $\{V_t^{P,[1,N-1]}(\kappa)\}$ which evolves according to the recursion

$$V_{t+1}^{P,[1,N-1]}(\kappa) = \left(V_t^{P,[1,N-1]}(\kappa) + D_t^{[1,N-1]} - \kappa\right)^+ \; .$$

Since $\mu > E[D^{[1,N-1]}]$, it is easy to see that $\overline{V} := \lim_{\kappa \downarrow \mu} E[V_\infty^{P,[1,N-1]}(\kappa)]$ exists and is finite. Thus, we obtain $C^P(\mathbf{h}, \mathbf{b}, \kappa) - C^*(\mathbf{h}, \mathbf{b}, \kappa) \le \overline{M} := (b^1 + h^N) \cdot E[D^{[1,N-1]} + \overline{V}] < \infty$ for all $\kappa > \mu$. This completes the proof of the theorem. $\square$

Proof of Lemma 11

Consider any non-anticipatory policy π. Let y_t^π denote the aggregate inventory level after ordering in period t, when this policy is followed. Similarly let $\mathbf{y}_t^\pi$ ($\mathbf{x}_t^\pi$) denote the vector of inventory levels after (before) ordering in period t and let C_t^π be the cost incurred in that period. Thus, $E[C_t^\pi] = \sum_{n=1}^N G^n(y_t^{\pi,n})$. Therefore, we know from the definition of F_1 that

$$E[C_t^\pi] \ge F_1(y_t^\pi) \ .$$

$$\Rightarrow \inf_{\pi \in \Pi} \lim_{T \to \infty} \sup \frac{E\left[\sum_{t=1}^T C_t^\pi\right]}{T} \ge \inf_{\pi \in \Pi} \lim_{T \to \infty} \sup \frac{E\left[\sum_{t=1}^T F_1(y_t^\pi)\right]}{T} \ .$$

Note that Π is the class of non-anticipatory policies satisfying the constraints $\mathbf{y_t^\pi} \ge \mathbf{x_t^\pi}$ and $\sum_{n=1}^N y_t^{\pi,n} \le \sum_{n=1}^N x_t^{\pi,n} + \kappa$, in every period. Let Π' denote the larger class of policies which are non-anticipatory and require that only the second constraint, i.e. the capacity constraint, is satisfied in every period. This implies that

$$\inf_{\pi \in \Pi} \lim_{T \to \infty} \sup \frac{E\left[\sum_{t=1}^T F_1(y_t^\pi)\right]}{T} \ge \inf_{\pi \in \Pi'} \lim_{T \to \infty} \sup \frac{E\left[\sum_{t=1}^T F_1(y_t^\pi)\right]}{T} \ .$$

The quantity on the right side of the above inequality is nothing but the long run average optimal cost for a single product inventory problem with a capacity limit of κ and an expected single period cost $F_1(\cdot)$, which is a convex function. We know from Federgruen and Zipkin (1986) and Huh et al. (2011) that a base-stock policy is optimal for this problem. Thus, we obtain

$$\inf_{\pi \in \Pi'} \lim_{T \to \infty} \sup \frac{E\left[\sum_{t=1}^T F_1(y_t^\pi)\right]}{T} = \min_S E[F_1(S - V_\infty)]$$

using the strong law of large numbers for Markov Chains (see Resnick 1992 for details). This leads to the desired result that

$$\inf_{\pi \in \Pi} \lim_{T \to \infty} \sup \frac{E\left[\sum_{t=1}^T C_t^\pi\right]}{T} \ge \min_S E[F_1(S - V_\infty)] \ .$$