Online Appendix

Simple Policies for Managing Flexible Capacity

Ganesh Janakiraman, Mahesh Nagarajan, Senthil Veeraraghavan

ganesh@utdallas.edu, mahesh.nagarajan@sauder.ubc.ca, senthilv@wharton.upenn.edu

Proof of Lemma 1

Note that $V_1^n = 0$ for all n by assumption. This establishes statement (i) for $t = 1$. Notice that in Π_{BS-B}, for any $\boldsymbol{\alpha}$, the allocation rule makes production decisions which can be computed by knowing the opening shortfalls in a period. So, the ending shortfalls in period t, for any $t > 1$, depend exclusively on the ending shortfalls in period $t-1$, the demands in period $t-1$ and κ. This proves statement (i) for any t. Note that any weighted balancing allocation rule, i.e. any allocation rule in Π_{BS-B}, is monotone with respect to the shortfalls in the sense that if the shortfall of a product is perturbed non-negatively in period t, the shortfall of that product is perturbed non-negatively in period $t+1$. Then, (ii) follows from Lemma 1 of Loynes (1962) and the remark following that lemma. Statement (iii) follows directly from (i) and (ii). To show (iv), observe that $V_1^n = 0$ for all n by assumption. Therefore, $V_1^{agg} = 0$. Now, consider period $t+1$ for any $t \geq 0$. The aggregate opening shortfall in period $t+1$ is $V_t^{agg} + D_t^{agg}$. If this is smaller than κ, all inventory levels are raised to the respective base-stock levels and thus V_{t+1}^n is zero for each n. Thus, $V_{t+1}^{agg} = 0$ in this case. If $V_t^{agg} + D_t^{agg} > \kappa$, the entire production capacity is used, i.e. $\sum_{n=1}^N q_{t+1}^n = \kappa$ and none of the inventory levels exceeds the corresponding base-stock level. So, the new aggregate shortfall, V_{t+1}^{agg} equals $\sum_{n=1}^N (V_t^n + D_t^n - q_{t+1}^n)$ which can be written as $V_t^{agg} + D_t^{agg} - \kappa$. Combining the two cases, we get $V_{t+1}^{agg} = (V_t^{agg} + D_t^{agg} - \kappa)^+$.

Proof of Lemma 2

Consider the subclass of policies mentioned in the statement of the lemma. Once the base-stock vector $\mathbf{S}$ is chosen for a policy within this class, the policy is entirely specified. The long run average cost of this policy is

$$\sum_{n=1}^{N} E[h^n \cdot (S^n - V_\infty^{\boldsymbol{\alpha},n} - D^n)^+ \; + \; b^n \cdot (V_\infty^{\boldsymbol{\alpha},n} + D^n - S^n)^+] \; .$$

Since the distribution of $\mathbf{V}_\infty^{\boldsymbol{\alpha}}$ does not depend on the base-stock vector, the expression above is separable in $(S^1, \ldots, S^N)$; thus, the optimal value of S^n is simply the minimizer of the "newsvendor-type" expression within the summation above. The desired result is immediate.

Proof of Lemma 3

By assumption $V_1^{\mathbf{1},n} = 0$ for all n. This establishes statement (i) for $t = 1$. Under the symmetric allocation rule, if the distribution of the vector $\mathbf{V}_t^{\mathbf{1}}$ is exchangeable for some t and the distribution of the demand vector in period t is also exchangeable across n, then the distribution of $\mathbf{V}^{\mathbf{1}}_{t+1}$ will also be exchangeable. Statement (i) follows for all t by induction. Statement (ii) is a direct consequence of statement (i).

Proof of Theorem 4

Lemma 2 establishes the optimality of the base-stock vector $\mathbf{S}^{\mathbf{1}*}$ for policies in Π_{BS-B} that use the weight vector $\mathbf{1}$. It remains to show that the policy in Π_{BS-B} defined by the base-stock vector $\mathbf{S}^{\mathbf{1}*}$ and the weight vector $\mathbf{1}$ is an optimal policy when all policies in Π are considered.

Let us first consider the finite horizon discounted cost problem with a discount factor $\gamma \in (0, 1]$ and a planning horizon of T periods, that is, the problem of minimizing $E[\sum_{t=1}^{T} \gamma^t \cdot C_t]$ over Π. This finite horizon dynamic program can be represented through the cost-to-go functions $\{f^{\gamma}_{t,T} : t = 1, \dots, T\}$ as follows:

$$\begin{aligned} f^{\gamma}_{t,T}(\mathbf{x}) \quad &= \quad \min_{\mathbf{y}} \sum_{n=1}^{N} \left(h^n \cdot E[(y^n - D^n)^+] \; + \; b^n \cdot E[(D^n - y^n)^+] \right) + \gamma \cdot E[f^{\gamma}_{t+1,T}(\mathbf{y} - \mathbf{D})] \\ & \qquad \text{s.t.} \quad \mathbf{y} \; \geq \; \mathbf{x} \quad \text{and} \quad \sum_{n=1}^{N} y^n \leq \sum_{n=1}^{N} x^n + \kappa \; , \end{aligned}$$

where $f^{\gamma}_{T+1,T}(\mathbf{x}) := 0$ for all $\mathbf{x}$.

It is fairly easy to show using induction that under Assumption 1, the function $f^{\gamma}_{t,T}$ is convex and symmetric. Using standard dynamic programming arguments, we can establish the pointwise convergence of the finite horizon cost-to-go functions $\{f^{\gamma}_{1,T}(\mathbf{x})\}$ to $\{f^{\gamma}(\mathbf{x})\}$ the cost-to-go function of the infinite horizon, discounted cost dynamic program (defined for $\gamma \in (0, 1)$) represented below:

$$f^{\gamma}(\mathbf{x}) \quad = \quad \min_{\mathbf{y}} \; g^{\gamma}(\mathbf{y}) \tag{3}$$

$$\text{s.t.} \quad \mathbf{y} \; \geq \; \mathbf{x} \quad \text{and} \quad \sum_{n=1}^{N} y^n \leq \sum_{n=1}^{N} x^n + \kappa \; , \tag{4}$$

where $g^{\gamma}(\mathbf{y}) = \sum_{n=1}^{N} (h^n \cdot E[(y^n - D^n)^+] \; + \; b^n \cdot E[(D^n - y^n)^+]) + \gamma \cdot E[f^{\gamma}(\mathbf{y} - \mathbf{D})]$. The infinite horizon discounted cost optimal policy is defined by a selector $\mathbf{y}^{\gamma*}(\mathbf{x})$ such that for every $\mathbf{x}$, the vector $\mathbf{y}^{\gamma*}(\mathbf{x})$ is a solution to the above minimization problem. The convergence of $\{f^{\gamma}_{1,T}(\mathbf{x})\}$ to $\{f^{\gamma}(\mathbf{x})\}$ ensures that g^{γ} is also convex and symmetric. The convexity and symmetry of g^{γ} implies the existence of a vector $\mathbf{S}^{\gamma*}$ such that (a) it minimizes $g^{\gamma}(\mathbf{y})$ and (b) all its components are identical; let us denote this identical base-stock value for all components as $S^{\gamma*}$. Using the convexity and symmetry of g^{γ}, it is also straightforward to show through the K.K.T. conditions that the symmetric allocation rule applied in combination with the base-stock vector $\mathbf{S}^{\gamma*}$ is an optimal policy for the infinite horizon, discounted cost problem defined in (3)-(4) when $\mathbf{x} \leq \mathbf{S}^{\gamma*}$.

Let us now return to the infinite horizon average cost problem. Schäl (1993) shows that, under certain conditions, the sequence of infinite horizon discounted cost optimal policies converges to an infinite horizon average cost optimal policy as the discount factor γ approaches 1. Huh et al. (2011) refer to this convergence as the preservation property and verify Schäl's conditions for the single product capacitated problem. A straightforward extension of their analysis can be used to verify Schäl's conditions for our multi-product problem, and is available on request.

Thus the above mentioned convergence of discounted cost optimal policies to an average cost optimal policy holds in our case. This implies that there exists a vector $\mathbf{S}^*$, in which all components are identical, such that the symmetric allocation rule applied in combination with the base-stock vector $\mathbf{S}^*$ is an average cost optimal policy. Finally, we know from Lemma 2 that, within Π_{BS-B}, the optimal base-stock vector corresponding to the weight vector $\mathbf{1}$ is $\mathbf{S}^{\mathbf{1}*}$. Thus, $\mathbf{S}^{\mathbf{1}*}$ is a valid choice for $\mathbf{S}^*$; this completes the proof.

Proof of Lemma 5

Without loss of generality, we assume that the priority order $(1), (2), \ldots, (N)$ is $1, 2, \ldots, N$. When the capacity is not binding (i.e. $\sum_j^N (W^j) \leq \kappa$), the shortfalls after ordering are zero under both rules (for any m). Thus, the statement holds for any m.

Similarly, if the shortfall before ordering, W^j, is zero for any j, then the shortfalls after ordering $V^{P,j}$ and $V^{\boldsymbol{\alpha}_m,j}$ are both zero. Thus, it is sufficient to consider the case where $\mathbf{W}$ is strictly positive in every component.

When capacity is binding, there exists some $k, 1 \leq k < N$ such that $\sum_1^k W^j \leq \kappa$ and $\sum_1^{k+1}(W^j) > \kappa$. Then, under the priority policy $V^{P,j} = 0 \ \forall \ j = 0, \ldots, k$, $V^{P,k+1} = W^{k+1} + \kappa - \sum_1^k W^j$, and $V^{P,j} = W^{P,j} \ \forall \ j = k+2, \ldots, N$. Let us define $\beta = W^{k+1} + \kappa - \sum_1^k W^j$, i.e. $\beta = V^{P,k+1}$.

Let M be large enough that $W^{k+2}/M^{k+1} \geq W^{k+3}/M^{k+2} \geq \ldots \geq W^N/M^{N-1}$. That is, $k+2$ is the product with the largest weighted shortfall before ordering among products $\{k+2, \ldots, N\}$.

Let $\tilde{\epsilon} \in (0, \epsilon/k)$ and let $\tilde{\epsilon} \leq \min\{W^1, \ldots, W^k, \beta/k\}$. Moreover, let M be large enough that $\tilde{\epsilon} \geq W^{k+1}/M^k$, $\tilde{\epsilon}/M \geq W^{k+1}/M^k$, $\ldots$, $\tilde{\epsilon}/M^{k-1} \geq W^{k+1}/M^k$ and $(\beta - k \cdot \tilde{\epsilon})/M^k \geq W^{k+1}/M^k$. (All the inequalities above except the first and the last are redundant - but we present them here for ease of verification of our next claim). These inequalities ensure that even if the first $k+1$ components of the shortfall vector before ordering were reduced to $(\tilde{\epsilon}, \ldots, \tilde{\epsilon}, \beta - k \cdot \tilde{\epsilon})$, the weighted balancing rule defined by the vector $\boldsymbol{\alpha}_m$ prefers to allocate the next incremental amount of capacity to the first $k+1$ products and not the products in $\{k+2, \ldots, N\}$.

It is now easy to verify that $\mathbf{V}^{\boldsymbol{\alpha}_m}$ satisfies the following inequalities for all $m \geq M$:
$V^{\boldsymbol{\alpha}_m,j} = W^j = V^{P,j}$ for all $j \in \{k+2, k+3, \ldots, N\}$,
$V^{\boldsymbol{\alpha}_m,j} \in [0, \tilde{\epsilon}] = [V^{P,j}, V^{P,j} + \tilde{\epsilon}]$ for all $j \in \{1, 2, \ldots, k\}$, and
$V^{\boldsymbol{\alpha}_m,k+1} \in [\beta - k \cdot \tilde{\epsilon}, \beta] = [V^{P,k+1} - k \cdot \tilde{\epsilon}, V^{P,k+1}]$. The proof of the lemma is complete from the fact that $\tilde{\epsilon} \leq \epsilon/k$.

Proof of Lemma 6

The first inequality is trivial to establish because the cost incurred by any policy in any period when the backorder costs are given by $\mathbf{b}$ exceed the corresponding quantity when all backorder costs are $\min(\mathbf{b})$. The second inequality follows from the definition of $C^*(h, \mathbf{b})$ and $C^{\mathbf{1}*}(h, \mathbf{b})$ as the optimal cost over all policies and the cost of the optimal weighted balancing policy, respectively. We now show the third inequality. From Theorem 4, we know that

$$C^{\mathbf{1}*}(h, \operatorname{avg}(\mathbf{b})) = C^*(h, \operatorname{avg}(\mathbf{b})) \ .$$

Observe that $C^{\mathbf{1}*}(h, \mathbf{b})$ is a constant with respect to permutations to $\mathbf{b}$ due to the assumption of an exchangeable demand distribution and the symmetric nature of the symmetric allocation rule. The average of all possible permutations of $\mathbf{b}$ is

$$\operatorname{avg}(\mathbf{b}) \cdot (1, 1, \ldots, 1) \ .$$

Since the single period function is linear with respect to $\mathbf{b}$ for any given state and action, it is easy to show that, for any policy π, $C^{\pi}(h, \mathbf{b})$ is concave with respect to $\mathbf{b}$. This implies that

$$C^{\mathbf{1}*}(h, \mathbf{b}) \ \leq \ C^{\mathbf{1}*}(h, \operatorname{avg}(\mathbf{b})) \ .$$

Recalling that $C^{\mathbf{1}*}(h, \operatorname{avg}(\mathbf{b})) = C^*(h, \operatorname{avg}(\mathbf{b}))$, we have

$$C^{\mathbf{1}*}(h, \mathbf{b}) \ \leq \ C^*(h, \operatorname{avg}(\mathbf{b})) \ .$$

Proof of Theorem 7

The first statement follows directly from Lemma 6. We now prove the asymptotic limit result by invoking a known result.

Lemma 13 (Huh et al. (2009))**.** *Let X be a random variable such that $\overline{M} = \sup\{x : P(X \leq x) < 1\}$ and $\lim_{x \uparrow \overline{M}} \ \frac{E[X - x \ | \ X > x]}{x} \ = \ 0$, where $\overline{M} \in \mathbb{R}^+ \cup \{\infty\}$. Then,*

$$\lim_{\beta \to \infty} \left(\frac{L(h, \beta \cdot b', X)}{L(h, \beta \cdot b, X)} \right) \ = \ 1 \ \ \textit{for all} \ \ (h, b', b) \ .$$

We know from (1) that

$$\begin{aligned} C^*(h, \operatorname{avg}(\mathbf{b})) &= N \cdot L(h, \operatorname{avg}(\mathbf{b}), V_{\infty}^{\mathbf{1},1} + D^1) \ \text{ and} \\ C^*(h, \min(\mathbf{b})) &= N \cdot L(h, \min(\mathbf{b}), V_{\infty}^{\mathbf{1},1} + D^1) \ . \end{aligned}$$

Therefore,

$$\left(\frac{C^{\mathbf{1}*}(h, \mathbf{b})}{C^*(h, \mathbf{b})} \right) \ \leq \ \left(\frac{L(h, \operatorname{avg}(\mathbf{b}), V_{\infty}^{\mathbf{1},1} + D^1)}{L(h, \min(\mathbf{b}), V_{\infty}^{\mathbf{1},1} + D^1)} \right) \ .$$

The desired asymptotic result now follows directly from Lemma 13 and Assumption 2 (c). This completes the proof of Theorem 7.

Proof of Lemma 8

We begin our proof with some preliminaries from Esary et al. (1967) who state and prove the following properties of associated random variables that we will use. (We reproduce these properties verbatim.) *(P1) Any subset of associated random variables are associated. (P2) If two sets of associated random variables are independent of one another, then their union is a set of associated random variables. (P3) The set consisting of a single random variable is associated. (P4) Non-decreasing functions of associated random variables are associated. (P5) If $T_1^{(k)}, T_2^{(k)}, \ldots, T_n^{(k)}$ are associated for each k, and $\mathbf{T}^{(k)} \to \mathbf{T}$ in distribution, then $T_1, T_2, \ldots, T_n$ are associated.*

We will first show that the random variables $W_\infty^{\mathbf{1},1}, W_\infty^{\mathbf{1},2}, \ldots, W_\infty^{\mathbf{1},N}$ are associated.

Recall that for every j, $W_\infty^{\mathbf{1},j} = V_\infty^{\mathbf{1},j} + D^j$. Let $\mathbf{V}$ denote the vector $(V_\infty^{\mathbf{1},1}, V_\infty^{\mathbf{1},2}, \ldots, V_\infty^{\mathbf{1},N})$ and let $\mathbf{D}$ denote the vector $(D^1, D^2, \ldots, D^N)$. Since these two vectors are independent of each other and the demands are associated, properties (P2) and (P4) imply that the random variables $W_\infty^{\mathbf{1},1}, W_\infty^{\mathbf{1},2}, \ldots, W_\infty^{\mathbf{1},N}$ are associated if the random variables $V_\infty^{\mathbf{1},1}$, $V_\infty^{\mathbf{1},2}$, …, $V_\infty^{\mathbf{1},N}$ are associated. Since these steady state shortfall random variables are the limits (in the sense of convergence in distribution) of the corresponding shortfalls in period $t \geq 1$, we again know from property (P5) that it is sufficient to show that the random variables $V_t^{\mathbf{1},1}$, $V_t^{\mathbf{1},2}$, …, $V_t^{\mathbf{1},N}$ are associated for every $t \geq 1$. Recall that the evolution of the shortfall vector under the symmetric allocation policy can be expressed as a recursion of the form

$$\mathbf{V}_{t+1}^{\mathbf{1}} = \Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t) \ ,$$

for a specific componentwise increasing mapping Λ (please see the discussion following the definition of weighted balancing policies in Section 3). We claim that $\mathbf{V}_t^{\mathbf{1}}$ is associated by induction.

Claim 1. *For all $t \geq 1$, the vector $\mathbf{V}_t^{\mathbf{1}}$ is associated.*

Proof of Claim: The proof is by induction. Since $\mathbf{V}_1^{\mathbf{1}} = \mathbf{0}$, it is trivially associated. We proceed by assuming $\mathbf{V}_t^{\mathbf{1}}$ is associated. We are required to show that $\mathbf{V}_{t+1}^{\mathbf{1}} = \Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t)$ is associated. In other words, for any two non-decreasing functions f and g such that $E[f(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]$, $E[g(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]$ and $E[f(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))g(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]$ exist, it remains to show that

$$E[f(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))g(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))] \geq E[f(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]E[g(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]. \quad (5)$$

Since $\mathbf{V}_t^{\mathbf{1}}$ and $\mathbf{D}_t$ are independent of each other, we can rewrite this equation as

$$E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[f(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))g(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]\right] \geq E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[f(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]\right] E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[g(\Lambda(\mathbf{V}_t^{\mathbf{1}} + \mathbf{D}_t))]\right]. \quad (6)$$

It suffices to prove (6). To do this, we need to define some functions: For every $\mathbf{d} \in \mathbb{R}^N$ and every $\mathbf{v} \in \mathbb{R}^N$, let

$$\hat{f}_{\mathbf{d}}(\mathbf{v}) := f(\Lambda(\mathbf{v}+\mathbf{d})) \text{ and } \hat{g}_{\mathbf{d}}(\mathbf{v}) := g(\Lambda(\mathbf{v}+\mathbf{d})).$$

Then, (6) can be rewritten as

$$E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{f}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})\hat{g}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right] \geq E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{f}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right] E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{g}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right]. \quad (7)$$

We proceed to prove (7). Since Λ is an non-decreasing function, we observe that, for every $\mathbf{d}$, the functions $\hat{f}_{\mathbf{d}}$ and $\hat{g}_{\mathbf{d}}$ are also non-decreasing functions. Therefore, since $\mathbf{V}_t^{\mathbf{1}}$ is an associated random vector, we know that the L.H.S. of (7) can be bounded from below as follows:

$$E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{f}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})\hat{g}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right] \geq E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{f}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{g}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right]. \quad (8)$$

For every $\mathbf{d} \in \mathbb{R}^N$, let

$$F(\mathbf{d}) := E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{f}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})] \text{ and } G(\mathbf{d}) := E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{g}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})].$$

Using these definitions, we can rewrite (8) as

$$E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{f}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})\hat{g}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right] \geq E_{\mathbf{D}_t}\left[F(\mathbf{D}_t)G(\mathbf{D}_t)\right]. \quad (9)$$

Since f, g and Λ are all non-decreasing functions, it follows from the definitions of $\hat{f}_{\mathbf{d}}$, $\hat{g}_{\mathbf{d}}$, F and G that the functions F and G are also non-decreasing. This, along with the fact that $\mathbf{D}_t$ is an associated random vector, implies

$$E_{\mathbf{D}_t}\left[F(\mathbf{D}_t)G(\mathbf{D}_t)\right] \geq E_{\mathbf{D}_t}\left[F(\mathbf{D}_t)\right] E_{\mathbf{D}_t}\left[G(\mathbf{D}_t)\right]. \quad (10)$$

Using the definitions of F and G, we see that the R.H.S. of this equation can be expressed as follows:

$$E_{\mathbf{D}_t}\left[F(\mathbf{D}_t)\right] E_{\mathbf{D}_t}\left[G(\mathbf{D}_t)\right] = E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{f}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right] E_{\mathbf{D}_t}\left[E_{\mathbf{V}_t^{\mathbf{1}}}[\hat{g}_{\mathbf{d}}(\mathbf{V}_t^{\mathbf{1}})]|\mathbf{D}_t=\mathbf{d}\right]. \quad (11)$$

Combining (9)-(11) yields (7), thus completing the proof of Claim 1. $\diamond$

We are now ready to show that the random variable $W_\infty^{\mathbf{1},1}$ is light-tailed. We will first show that the aggregate shortfall is light tailed and use that fact to show that $W_\infty^{\mathbf{1},1}$ is light-tailed.

Let $W_\infty^{agg} = W_\infty^{\mathbf{1},1} + W_\infty^{\mathbf{1},2} + \ldots + W_\infty^{\mathbf{1},N}$ denote the steady-state aggregate end-of-period shortfall random variable. Then, we know that W_∞^{agg} is the convolution of V_∞^{agg} and D^{agg}, where $V_\infty^{agg} = V_\infty^{\mathbf{1},1} + V_\infty^{\mathbf{1},2} + \ldots + V_\infty^{\mathbf{1},N}$ and $D^{agg} = D^1 + D^2 + \ldots + D^N$. We also know that V_∞^{agg}, the steady state aggregate shortfall (at the beginning of a period), is the limit (in the sense of convergence in distribution) of the recursion $V_{t+1}^{agg} = (V_t^{agg} + D_t^{agg} - \kappa)^+$, where $V_0^{agg} = 0$ and D_t^{agg} is the aggregate demand in period t. As mentioned earlier, when D^{agg} is light-tailed, the *Cramér – Lundberg* approximation (please see Asmussen, 2000, Glasserman, 1997) can be used to show that as x becomes large, $P(V_\infty^{agg} + D^{agg} > x)$ approaches an exponential function of $(-x)$. More precisely, there exist constants θ_1 and θ_2 such that

$$P(W_\infty^{agg} > x) \sim \theta_1 e^{-\theta_2 x} \ ,$$

where the notation $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$.

Next, since all products are symmetric, we know that $W_\infty^{\mathbf{1},1}$, $W_\infty^{\mathbf{1},2}$, ..., $W_\infty^{\mathbf{1},N}$ all have the same distribution. Thus,

$$\begin{aligned}
P(W_\infty^{\mathbf{1},1} > x) &= \left(P(W_\infty^{\mathbf{1},1} > x)P(W_\infty^{\mathbf{1},2} > x)\ldots P(W_\infty^{\mathbf{1},N} > x)\right)^{1/N} \\
&\leq \left(P(W_\infty^{\mathbf{1},1} > x, W_\infty^{\mathbf{1},2} > x, \ldots, W_\infty^{\mathbf{1},N} > x)\right)^{1/N} \because \ W_\infty^{\mathbf{1},1},\ W_\infty^{\mathbf{1},2}, \ldots, W_\infty^{\mathbf{1},N} \text{ are associated} \\
&\leq P(W_\infty^{\mathbf{1},1} + W_\infty^{\mathbf{1},2} + \ldots + W_\infty^{\mathbf{1},N} > Nx)^{1/N} = P(W_\infty^{agg} > Nx)^{1/N} \\
&\sim (\theta_1 e^{-\theta_2 N x})^{1/N} = (\theta_1)^{1/N} e^{-\theta_2 x}.
\end{aligned}$$

This implies the desired result that $W_\infty^{\mathbf{1},1}$ is light-tailed.

Proof of Lemma 9

We will establish the result by proving the following more general claim.

Claim 2. *Consider a non-negative random variable X with distribution function F and a strictly positive density function f in $(0,\infty)$. Assume that X is light-tailed; that is, there exist non-negative constants α and β such that $P(X \geq x) \leq \alpha e^{-\beta x}$ for all $x \geq 0$. Furthermore, assume that* $\lim_{x\to\infty} \overline{F}(x)/f(x)$ *exists. Then,* $\lim_{x\to\infty} m(x)/x = 0$, *where* $m(x) = E[X - x|X > x]$.

Proof. Notice that $m(x)$ can also be written as $\int_x^\infty \overline{F}(u)du/\overline{F}(x)$, where $\overline{F}(x) = 1 - F(x)$. Then,

$$\lim_{x\to\infty} m(x) = \lim_{x\to\infty} \overline{F}(x)/f(x)$$

by L'Hopital's rule. Also,

$$\lim_{x\to\infty} -x/ln(\overline{F}(x)) = \lim_{x\to\infty} \overline{F}(x)/f(x)$$

by L'Hopital's rule. Thus,

$$\lim_{x\to\infty} m(x) = \lim_{x\to\infty} -x/ln(\overline{F}(x)).$$

But

$$\overline{F}(x) \leq \alpha e^{-\beta x} \ \forall\ x \geq\ 0.$$

So,

$$\ln(\overline{F}(x)) \leq \ln(\alpha) - \beta x.$$

Thus,

$$-\ln(\overline{F}(x)) \geq \beta x - \ln(\alpha) > 0$$

for sufficiently large x. This implies that

$$-x/\ln(\overline{F}(x)) \leq x/(\beta x - \ln(\alpha))$$

for sufficiently large x. Therefore,

$$\lim_{x\to\infty} -x/\ln(\overline{F}(x)) \leq \lim_{x\to\infty} x/(\beta x - \ln(\alpha)) = 1/\beta < \infty \ .$$

Thus,

$$\lim_{x\to\infty} m(x) = \lim_{x\to\infty} -x/\ln(\overline{F}(x)) = 1/\beta < \infty \ .$$

Therefore, $\lim_{x\to\infty} m(x)/x = 0$. □

Proof of Lemma 10

Proof. First, we observe that

$$C^*(\mathbf{h}, \mathbf{b}, \kappa) \geq C^*(h, b, \kappa) \ , \text{ if } \ 0 \ < \ h \ \leq \ h^j \ \forall \ j \ \text{ and } \ 0 \ < \ b \ < \ b^j \ \forall \ j \ , \tag{12}$$

where $C^*(h, b, \kappa)$ is the optimal cost of a system in which all products have the same holding cost h and the same backorder cost b. Thus, it suffices to show that, for any $h > 0$ and $b > 0$,

$$\lim_{\kappa\downarrow\mu} C^*(h, b, \kappa) = \infty \ .$$

Next, let us define $V_\infty^{agg}(\kappa)$ as the steady state version of the aggregate shortfall process $\{V_t^{agg}(\kappa)\}$ defined by the recursion $V_{t+1}^{agg}(\kappa) = (V_t^{agg}(\kappa) + D^{agg} - \kappa)^+$ (recall that $D^{agg} = \sum_{j=1}^N D^j$). We claim that

$$C^*(h, b, \kappa) \geq \min_S h \cdot E[(S - V_\infty^{agg}(\kappa) - D^{agg})^+] + b \cdot E[(D^{agg} + V_\infty^{agg}(\kappa) - S)^+] \ . \tag{13}$$

The proof of the claim is the following: Consider any feasible policy in the multi-product system. We can use this policy to construct a feasible policy in the "aggregate system" whose optimal long run average cost is represented on the right side of (13) such that the cost in the latter system (and therefore, the long run average cost) is smaller than that in the former system every period. This is done by ordering, in the latter system, the sum of the quantities ordered for all the products in the former system – the fact that the cost in the latter system is smaller in every period follows from the inequalities

$$\sum_{j=1}^{N}(x^j - d^j)^+ \geq \left(\sum_{j=1}^{N}(x^j - d^j)\right)^+ \text{ and } \sum_{j=1}^{N}(d^j - x^j)^+ \geq \left(\sum_{j=1}^{N}(d^j - x^j)\right)^+ .$$

This proves the claim.

Thus, it only remains to show that

$$\lim_{\kappa\downarrow\mu}\min_{S} h \cdot E[(S - V_\infty^{agg}(\kappa) - D^{agg})^+] + b \cdot E[(D^{agg} + V_\infty^{agg}(\kappa) - S)^+] = \infty.$$

To show this, we first note that replacing D^{agg} by its expectation, μ, in the expression within the limit above we obtain a lower bound on that expression (this is a consequence of Jensen's inequality and the convexity of the function $(x)^+$). Letting $\tilde{S} = S - \mu$, it is sufficient to show that

$$\lim_{\kappa\downarrow\mu}\min_{\tilde{S}} h \cdot E[(\tilde{S} - V_\infty^{agg}(\kappa))^+] + b \cdot E[(V_\infty^{agg}(\kappa) - \tilde{S})^+] = \infty. \tag{14}$$

Next, observe that the recursion for $\{V_t^{agg}(\kappa)\}$ is the same as that for the waiting time process for a $G/G/1$ queue in which the inter-arrival times are deterministic and equal to κ and the service time for the t^{th} customer is D_t^{agg}. We know from Kingman (1962) that the distribution of the random variable $[\frac{(\kappa-\mu)}{\sigma^2}] \cdot V_\infty^{agg}(\kappa)$ converges to an exponential distribution with mean 1/2, i.e.,

$$\lim_{\kappa\downarrow\mu} P\left(\frac{(\kappa-\mu)}{\sigma^2} \cdot V_\infty^{agg}(\kappa) \geq z\right) = e^{-2z} \ , \quad \text{for all} \quad z \ \geq \ 0 \ ,$$

where σ^2 is the variance of the aggregate demand D^{agg}. We can verify using straight forward calculus that this implies that

$$\begin{aligned} &\lim_{\kappa\downarrow\mu}\min_{S'} h \cdot E\left[(S' - \frac{(\kappa-\mu)}{\sigma^2} \cdot V_\infty^{agg}(\kappa))^+\right] + b \cdot E\left[(\frac{(\kappa-\mu)}{\sigma^2} \cdot V_\infty^{agg}(\kappa) - S')^+\right] \\ = \ &(h/2) \cdot \ln\left((b+h)/h\right). \end{aligned} \tag{15}$$

It is easy to verify that the desired equality in (14) follows directly from (15). □

Proof of Theorem 11

Proof. The second statement follows directly from the first statement and Lemma 10. We proceed to show the first statement. Our plan is to find an upper bound on $C^P(\mathbf{h}, \mathbf{b}, \kappa)$ and a lower bound on $C^*(\mathbf{h}, \mathbf{b}, \kappa)$ and show that the difference between these bounds is finite for all κ.

Let $S(\kappa)$ be defined as $\arg\min_S h^N \cdot E[(S - D^{agg} - V_\infty^{agg}(\kappa))^+] + b^N \cdot E[(D^{agg} + V_\infty^{agg}(\kappa) - S)^+]$. Now, consider a policy π which uses the same priority rule as P but uses the following non-optimal base-stock levels:

$$S^j = 0 \text{ for all } j < N \text{ and } S^N = S(\kappa) \ .$$

Let $C^{\pi}(\mathbf{h},\mathbf{b},\kappa)$ ($C^{\pi,N}(h^N,b^N,\kappa)$) denote the long run average cost for the system (product N) under π given the respective parameters. Since P uses the optimal base-stock levels under the given priority allocation rule and π does not, we obtain the following relations:

$$\begin{aligned} C^P(\mathbf{h},\mathbf{b},\kappa) &\leq C^{\pi}(\mathbf{h},\mathbf{b},\kappa) \\ &= \sum_{j=1}^{N-1} b^j \cdot E[D^j + V_{\infty}^{P,j}] + C^{\pi,N}(h^N,b^N,\kappa)\ . \end{aligned} \tag{16}$$

The equality above follows from the fact that under π, there is never any inventory of products 1 through $N-1$ on hand and from the fact that the shortfall process under π is the same as that under P. Let $b_{max} := \max\{b^1, b^2, \ldots, b^N\}$. It follows from (16) and the definition of π that

$$\begin{aligned} C^P(\mathbf{h},\mathbf{b},\kappa) &\leq b_{max} \cdot \sum_{j=1}^{N-1} \left(E[D^j + V_{\infty}^{P,j}]\right) + C^{\pi,N}(h^N,b^N,\kappa) \\ &= b_{max} \cdot \sum_{j=1}^{N-1} \left(E[D^j + V_{\infty}^{P,j}]\right) \\ &\quad + h^N \cdot E[(S(\kappa) - D^N - V_{\infty}^{P,N}(\kappa))^+] + b^N \cdot E[(D^N + V_{\infty}^{P,N}(\kappa) - S(\kappa))^+]\ . \end{aligned} \tag{17}$$

The inequality above provides an upper bound on $C^P(\mathbf{h},\mathbf{b},\kappa)$.

Next, we proceed to identify a lower bound on $C^*(\mathbf{h},\mathbf{b},\kappa)$. By Assumption 5, we have

$$C^*(\mathbf{h},\mathbf{b},\kappa) \geq C^*(h^N,b^N,\kappa)\ . \tag{18}$$

Now, observe that $C^*(h^N,b^N,\kappa)$ is the optimal cost of a multi-product inventory system in which all products have identical holding and shortage costs. We have shown in the proof of Lemma 10 that this quantity exceeds the optimal cost of a single product inventory system with a holding cost h^N, backorder cost b^N, capacity κ and demand distribution D^{agg}. That is,

$$\begin{aligned} C^*(h^N,b^N,\kappa) &\geq \min_S h^N \cdot E[(S - D^{agg} - V_{\infty}^{agg}(\kappa))^+] + b^N \cdot E[(D^{agg} + V_{\infty}^{agg}(\kappa) - S)^+]\ , \\ &= h^N \cdot E[(S(\kappa) - D^{agg} - V_{\infty}^{agg}(\kappa))^+] + b^N \cdot E[(D^{agg} + V_{\infty}^{agg}(\kappa) - S(\kappa))^+]\ . \end{aligned} \tag{19}$$

Let us define $V_{\infty}^{P,[1,N-1]}$ as $\sum_{j=1}^{N-1} V_{\infty}^{P,j}$ and $D^{[1,N-1]}$ as $\sum_{j=1}^{N-1} D^j$. Now, comparing (17) and (19) and using (18), we can write

$$\begin{aligned} & C^P(\mathbf{h},\mathbf{b},\kappa) - C^*(\mathbf{h},\mathbf{b},\kappa) \\ \leq\ & b_{max} \cdot \sum_{j=1}^{N-1} \left(E[D^j + V_{\infty}^{P,j}]\right) + h^N \cdot E[D^{[1,N-1]} + V_{\infty}^{P,[1,N-1]}(\kappa)] \\ =\ & (b_{max} + h^N) \cdot E[D^{[1,N-1]} + V_{\infty}^{P,[1,N-1]}(\kappa)]\ , \end{aligned} \tag{20}$$

where the inequality follows from the facts that

$$(S(\kappa) - D^N - V_\infty^{P,N}(\kappa))^+ - (S(\kappa) - D^{agg} - V_\infty^{agg}(\kappa))^+ \leq D^{[1,N-1]} + V_\infty^{P,[1,N-1]}$$

and that $(D^N + V_\infty^{P,N}(\kappa) - S(\kappa))^+ \leq (D^{agg} + V_\infty^{agg}(\kappa) - S(\kappa))^+$. Notice that $V_\infty^{P,[1,N-1]}(\kappa)$ is the steady state distribution of the stochastic process $\{V_t^{P,[1,N-1]}(\kappa)\}$ which evolves according to the recursion

$$V_{t+1}^{P,[1,N-1]}(\kappa) = \left(V_t^{P,[1,N-1]}(\kappa) + D_t^{[1,N-1]} - \kappa\right)^+ . \tag{21}$$

The reason for this recursion is that under the priority policy P, the inventory and shortfall dynamics of product 1 are the same as that in a system with only product 1 present and with κ units of production capacity. So, product 1's shortfall process follows the recursion $V_{t+1}^{P,1}(\kappa) = \left(V_t^{P,1}(\kappa) + D_t^1 - \kappa\right)^+$. By the same reasoning, for any $n \leq N$, we have the more general recursion $V_{t+1}^{P,[1,n]}(\kappa) = \left(V_t^{P,[1,n]}(\kappa) + D_t^{[1,n]} - \kappa\right)^+$.

Returning to (21), we observe that since $\mu > E[D^{[1,N-1]}]$, $\overline{V} := \lim_{\kappa\downarrow\mu} E[V_\infty^{P,[1,N-1]}(\kappa)]$ exists (in fact, $\overline{V} := E[V_\infty^{P,[1,N-1]}(\mu)]$) and is finite. Using this observation in (20), we obtain $C^P(\mathbf{h}, \mathbf{b}, \kappa) - C^*(\mathbf{h}, \mathbf{b}, \kappa) \leq \overline{M} := (b_{max} + h^N) \cdot E[D^{[1,N-1]} + \overline{V}] < \infty$ for all $\kappa > \mu$. This completes the proof of the theorem. □

Proof of Lemma 12

Consider any non-anticipatory policy π. Let y_t^π denote the aggregate inventory level after ordering in period t, when this policy is followed. Similarly let $\mathbf{y}_t^\pi$ ($\mathbf{x}_t^\pi$) denote the vector of inventory levels after (before) ordering in period t and let C_t^π be the cost incurred in that period. Thus, $E[C_t^\pi] = \sum_{n=1}^N G^n(y_t^{\pi,n})$. Therefore, we know from the definition of F_1 that

$$E[C_t^\pi] \geq F_1(y_t^\pi) .$$

$$\Rightarrow \inf_{\pi\in\Pi} \limsup_{T\to\infty} \frac{E\left[\sum_{t=1}^T C_t^\pi\right]}{T} \geq \inf_{\pi\in\Pi} \limsup_{T\to\infty} \frac{E\left[\sum_{t=1}^T F_1(y_t^\pi)\right]}{T} .$$

Note that Π is the class of non-anticipatory policies satisfying the constraints $\mathbf{y}_t^\pi \geq \mathbf{x}_t^\pi$ and $\sum_{n=1}^N y_t^{\pi,n} \leq \sum_{n=1}^N x_t^{\pi,n} + \kappa$, in every period. Let Π' denote the larger class of policies which are non-anticipatory and require that only the second constraint, i.e. the capacity constraint, is satisfied in every period. This implies that

$$\inf_{\pi\in\Pi} \limsup_{T\to\infty} \frac{E\left[\sum_{t=1}^T F_1(y_t^\pi)\right]}{T} \geq \inf_{\pi\in\Pi'} \limsup_{T\to\infty} \frac{E\left[\sum_{t=1}^T F_1(y_t^\pi)\right]}{T} .$$

The quantity on the right side of the above inequality is nothing but the long run average optimal cost for a single product inventory problem with a capacity limit of κ and an expected single period cost $F_1(\cdot)$, which is a convex function. We know from Federgruen and Zipkin (1986) and Huh et al. (2011) that a base-stock policy is optimal for this problem. Thus, we obtain

$$\inf_{\pi \in \Pi'} \limsup_{T \to \infty} \frac{E\left[\sum_{t=1}^{T} F_1(y_t^\pi)\right]}{T} = \min_S E[F_1(S - V_\infty^{agg})]$$

using the strong law of large numbers for Markov Chains (see Resnick (1992) for details). This leads to the desired result that

$$\inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{E\left[\sum_{t=1}^{T} C_t^\pi\right]}{T} \geq \min_S E[F_1(S - V_\infty^{agg})] \ .$$