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Pricing and Capacity Rationing for Rentals with Uncertain Durations

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We consider a rental firm with two types of customers. Contract customers pay fixed, prenegotiated rental fees and expect a high quality of service. Walk-in customers have no contractual relations with the firm and are "shopping for price." Given multiple contract and walk-in classes, the rental firm has to decide when to offer service to contract customers and what fees to charge walk-in customers for service.

We formulate this rental management problem as a problem in stochastic control and characterize optimal policies for managing contract and walk-in customers. We also consider static, myopic controls that are simpler to implement, and we analytically establish conditions under which these policies perform optimally. Complementary numerical tests provide a sense of the range of systems for which myopic policies are effective.

Key words: dynamic programming; services; rentals; revenue management
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1. Introduction

Rental businesses provide a cost-effective alternative to ownership in many branches of the economy. The range of products that are offered for rent extends from relatively inexpensive items such as videotapes, DVDs, and home electronics to expensive ones such as cars, trucks, real estate properties, and construction equipment. Despite many obvious differences, all rental businesses possess some common characteristics: A rental company acquires and maintains an inventory of items, which are used by the customers for a limited period of time. Typically, rental durations are short compared to the items' useful lifetime, and items become available for future rentals shortly after they are returned to inventory.

Heterogeneity of customer preferences is often an important factor in determining rental management practices, and in many industries rental companies separate their customer base into two groups. The first, *contract* group consists of customers whose rentals are regulated by prenegotiated contracts, which usually specify a fixed rental fee as well as certain service obligations. The second customer group consists of *walk-in* customers, to which a rental company has no long-term contractual obligations. Typically, these customers "shop for price" and do not expect a high degree of service.

The nature of contract and walk-in profiles can depend on the type of rental business. For example in car rentals, business customers play the role of the contract group, and leisure customers are best described as walk-ins. On the other hand, in truck-trailer rentals there are no leisure customers, and the role of the walk-ins is assumed by the small business customers, whereas contract customers are large corporations that typically bring in significant rental volumes. In these systems, short-term capacity imbalances—either congestion or idleness—can be managed through a number of mechanisms. Demand from walk-in customers can be controlled by the raising or lowering of prices. Similarly, rental requests from contract customers can be honored or denied, but at a price.

Customer heterogeneity, combined with demand and supply uncertainty, makes the determination of how best to control capacity a difficult problem in these systems. At what point should contract customers be turned away? And by how much should walk-in prices be raised as the system becomes congested? Furthermore, which classes should be considered "better" than others and be given priority or better prices? In this paper, we provide some initial answers to these questions.

We consider a setting in which there are multiple classes of contract and walk-in customers. Arriving contract customers can be offered a rental unit and charged their contract's class-specific rental fee, or they may be rejected, in which case the rental system pays a class-specific penalty. Arriving walk-in customers are quoted prices that may dynamically vary with the system's level of congestion. Given a quote, these customers either accept or balk, with class-specific probabilities.

This model is related to two different streams of research in the literature on capacity management. The first one considers the use of admission controls to allocate fixed capacity among different classes of customers. For example, see Littlewood (1972), Alstrup et al. (1986), Brumelle and McGill (1993), Belobaba (1996), Kleywegt and Papastavrou (1998), Rothstein (1974), Ladany (1977), Williams (1977), Liberman and Yechiali (1978), Bitran and Gilbert (1996), Ross and Tsang (1989), Ross and Yao (1990), Altman et al. (2001), Örmeci et al. (2001, 2002), Örmeci and van der Wal (2006), Örmeci and Burnetas (2004, 2005), and Savin et al. (2005). These allocation papers assume that all prices are fixed and that capacity is managed solely via admission controls. The second group analyzes models in which capacity is controlled via dynamic pricing. For example, see Low (1974), Kelley (1997), Kelley et al. (1998), Zhao and Zheng (2000), Paschalidis and Tsitsiklis (2000), Yoon and Lewis (2004), and Gayon et al. (2004). These pricing papers assume that service fees can be changed as often as needed for all customer classes.

The present paper brings together these two streams of research. We model a more general setting, in which some customer classes (the contract-customer group) have fixed rental fees and are controlled through admission control, whereas others (the walk-in group) may tolerate dynamic price setting. Our analysis characterizes the nature of effective controls in these "hybrid" systems.

The contract and walk-in customer groups of our model are also similar in spirit to the *guaranteed* and *best-effort* customers in Maglaras and Zeevi (2005), a paper that models information services. Its processor-sharing model, for the service of best-effort customers, does not fit well the dynamics of traditional rental businesses, however. In addition, it does not consider the dynamic admission and pricing controls, which are central to our analysis. Rather, it concentrates on the asymptotic optimality of simpler, static controls.

Our model is Markovian—arrivals of different customer classes are independent Poisson processes and service times are exponentially distributed—and we analyze the system as an infinite-horizon Markov decision process (MDP). Arrival rates are class-specific.

We first model the various customer classes as sharing a common service rate. This assumption, although restrictive, places our initial formulation and analysis as direct parallels with more traditional, finite-horizon revenue-management problems. For example, in airline revenue-management problems, all customers are assumed to take one (equally sized) seat, no matter what their ticket prices are. As such, this "special case" is of interest in and of itself.

The assumption also allows us to clearly and fully characterize the structure of optimal policies, as well as the sensitivity of optimal policy parameters to primitive system parameters. Specifically, our analysis yields the following insights:

1. We show that threshold-based policies are optimal for managing the admission of contract customers and that the optimal fees charged to walk-in customers grow with the degree of system congestion. We also establish that the optimal rental fees to be charged to a particular walk-in customer class are at least as high as the fee that maximizes the expected revenue from a walk-in rental.

2. In turn, we demonstrate that optimal policy parameters are monotone with respect to system parameters. Each contract class's optimal threshold is decreasing in customer arrival intensities and rental durations, as well as in the rental fees paid by other contract customers, and it is increasing in the rental fees paid by the customers of that class. Similarly, the optimal rental fee for a class of walk-in customer is increasing in arrival rates and rental durations, as well as the fixed fees paid by contract customers.

3. We also show how the problem's revenue structure establishes hierarchies among the various customer classes: For contract customers, higher fees guarantee higher level of access; and for walk-ins, a specific form of price sensitivity guarantees systematically lower rental fees.

4. We then introduce the related notions of preferred customer classes and myopic revenue-management policies. We establish sufficient conditions for a particular class to be preferred, as well as a range of problem parameters under which myopic policies are optimal.

We next consider the more general model in which the expected duration of contract rentals differs from that of walk-ins. Although this additional problem complexity prevents us from demonstrating the monotonicity properties outlined (in point 1) above, we are able to prove that variants of all of our other analytical characterizations carry over. In particular, we note that limiting results for the asymptotic optimality of myopic policies (for large systems), trivially hold for systems with non-Markovian rental durations as well.

Our paper concludes with two sets of numerical tests that complement the analytical results described above. In the first, we consider the effectiveness of myopic policies as simple, heuristic solutions to the revenue-management problem. Here we find that, given capacity levels that roughly match some (pricing-dependent) measure of the offered load, as well as contract fees that fall sufficiently close to those offered to walk-in customers, myopic policies can be optimal. Conversely, when these two conditions are not met, more complex control policies can perform significantly better.

In the second, we consider how the use of a given control, myopic or optimal, affects the choice of rental fleet size. Here, we find that the fleet size that maximizes expected profits, when using a myopic policy, can be smaller or larger than the one that maximizes profits under the optimal control. Furthermore, the relative price sensitivity of walk-in customers helps to drive this effect: Lower walk-in price sensitivity drives optimal capacity under the myopic policy to exceed that under the optimal control.

Finally, we note that our MDP results hold for both infinite-horizon, discounted, and average-reward objectives. Our numerical results are based on average-reward formulations. Because average-reward results are independent of a system's starting state, they facilitate the interpretation of the numerical tests.

The remainder of the paper is organized as follows. In the next section we formulate our model and its associated dynamic programming value function. Then in §3, we derive structural and sensitivity properties of optimal capacity-management policies. In §4, we focus on myopic capacity-management policies, as well as the related issue of preferred customer classes, and we establish sufficient conditions for the preferred status of a particular customer class. Section 5 extends our basic model to the case in which contract and walk-in customers rent equipment for different durations. Section 6 reports the results of our numerical study. We conclude with a brief discussion of the results, as well as directions for future research.

2. The Model and Associated Value Function

In this section we formulate the rental profit management problem, along with its associated dynamic programming value function.

2.1. Model Description

We consider a firm operating a fleet of c identical units of rental equipment. The fleet is accessed by two groups of customers, contract and walk-in, each represented by several customer classes. The various classes of contract customers are indexed $i = 1, \ldots, N$, and those for walk-ins, $j = 1, \ldots, M$.

The system's dynamics are Markovian. Requests for rental services arrive according to independent Poisson processes with parameters $\lambda_i^{\mathscr{C}}$ for contract customers and $\lambda_j^{\mathscr{W}}$ for walk-ins. Each rental request is for one unit of capacity, and a request that is fulfilled rents the unit of capacity for an exponentially distributed quantity of time. Rental durations are independent of each other and of the arrival processes, and we assume that the expected duration of all requests is the same, $1/\mu$. (In §5 we consider the more general model in which contract and walk-in customers' expected rental durations may differ.)

When a contract customer arrives to the system and demands a unit of capacity, the firm must decide whether to grant or to deny the rental request. If the request is accepted, then a contract customer from class i pays a prenegotiated revenue of ϱ_i per unit of time for the duration of the rental. If it is rejected, then the customer departs immediately and the firm incurs a lump-sum service penalty π_i. Such a penalty may be interpreted as a direct rebate that the rental firm pays to customers that it cannot accommodate or as a more indirect good-will cost. (For more on service penalties, as well as their relationship to service-level constraints, see Savin et al. 2005.) If an arrival occurs when all equipment units have already been rented out, then the service request is likewise denied and a lump-sum service penalty of π_i is paid.

If the service request comes from a walk-in customer of class j and at least one unit of capacity is available, the firm decides on the spot what rental fee to charge. The fee that is chosen from a finite (but arbitrarily large) set of possible walk-in rates $W = (\omega_1, \ldots, \omega_L)$ consisting of L ordered elements: $\omega_1 < \omega_2 < \cdots < \omega_L$. If a fee of ω_l is offered to a walk-in customer of class j, she accepts it with probability p_{jl} and pays ω_l per unit of time for the duration of the rental. This price-response scheme is analogous to that in Low (1974).

Alternatively, a walk-in customer of class j rejects the offer ω_l with probability $1 - p_{jl}$ and is immediately lost. In this case, no penalty is imposed on the rental firm. The absence of a penalty reflects our assumption that walk-in customers do not have high service expectations and are mostly interested in obtaining a "good" price. Again, if no equipment is available, then the request is simply denied and, in this case, there is no service penalty.

We make two mild assumptions concerning acceptance probabilities:

Assumption 1. *For any walk-in class j*: (1) $p_{j1} > p_{j2} > \cdots > p_{jL}$ *is a monotone sequence*; *and* (2) *the maximum price offered*, w_L, *has an acceptance probability* $p_{jL} = 0$.

Part (1) is quite natural: customers that are offered higher prices are less likely to buy. Part (2) is a simply a mechanism by which the rental firm may deny access to an arriving walk-in customer, if warranted.

We formulate the problem as a Markov decision process. We define a continuous-time discount rate of

$\gamma > 0$ and define the firm's objective as the maximization of expected discounted reward, the difference between the expected discounted revenues it receives from rentals and the expected discounted penalties it must pay due to contract-customer rejections. Below we will show that these results also extend directly to the limiting case, $\gamma \to 0$, in which the firm maximizes expected average rewards.

2.2. MDP Formulation and Value Function

As stated, our problem evolves over continuous time. More difficultly, the rate at which rental revenues accrue depends on the numbers of each class of customer in the system, a representation that leads to an $N + M$-dimensional state space. To overcome these complications, we make a simple transformation that will allow us to analyze the problem using a lower-dimensional state space in discrete time.

In particular, we note that the expected duration—hence expected discounted revenue—of a given rental is independent of the system state. Therefore, rather than tracking revenues in continuous time, we track expected discounted revenues on arrival. In effect, we treat rental revenues as lump sums that are paid on arrival, and because the rate at which a given class of customer pays does not affect system evolution, it does not need to be tracked in the system state.

Given an expected rental duration of $1/\mu$, we denote the expected discounted revenue of a class i contract customer as $r_i = \varrho_i/(\mu + \gamma)$, and that of a walk-in customer that accepts the offered price l as $w_l = \omega_l/(\mu + \gamma)$. As before, we label the set of expected walk-in revenues $W = (w_1, \ldots, w_L)$, order them $w_1 < \cdots < w_N$, and assume that the corresponding class j acceptance probabilities are ordered $p_{j1} > \cdots > p_{jL}$.

The fact that interarrival times and rental durations are exponentially distributed means that the system evolves as a continuous-time Markov chain. This, in turn, allows us to make three common simplifications. First, it implies that, at times between events, the state of the rental system can be completely described by the numbers of various types of customers in service. Furthermore, because all customer classes have the same mean service time, μ, the system state may be described by the total number of rental units in service, k. Formally, we define the state space for the rental problem as a set $S = (k \mid 0 \le k \le c)$.

Second, it implies that system controls—in the form of the acceptance or rejection of an arriving class i customer, or the price offered to a class j customer—needs to be exercised only at arrival epochs. That is, when determining the form of effective system controls, it is sufficient to consider only the discrete-time process embedded at arrival and departure epochs (see Puterman 1994, Chapter 11).

Third, Markovian dynamics allow us to "uniformize" the system, so that the distribution of times between events, such as arrivals and service completions, occurs at a constant (uniform) rate. Uniformization ensures that the expected discounted rewards earned at these discrete event epochs equal the expected discounted rewards that would have accrued in continuous time (Lippman 1975).

We define the aggregate event rate to be $\Gamma = \sum_{i=1}^{N} \lambda_i^{\mathscr{C}} + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} + \mu c + \gamma$, so that the system evolves with exponentially distributed interevent times of uniform rate Γ. Without loss of generality, we define the time scale so that $\Gamma = 1$. This implies that, in any state, k, the probability that the next event is a class i or class j arrival equals $\lambda_i^{\mathscr{C}}$ or $\lambda_j^{\mathscr{W}}$, respectively. Analogously, given there are k customers in the system, the probability that the next event is a service completion equals $k\mu$, and the probability that it is a nonevent—which sends the system back into the same state—equals $(c - k)\mu$.

The discount rate, γ, can be interpreted as the probability that the next event is terminating, in which case the system "stops" and no more profits are earned. The use of such a termination probability is equivalent to discounting (see Puterman 1994, §5.3) Furthermore, the formulation allows us to extend our analysis and results straightforwardly to case of average rewards: As $\gamma \to 0$, $r_i = \varrho_i/(\mu + \gamma)$ and $w_l/(\mu + \gamma)$ converge to their average-reward counterparts, and the interpretation of event rates as probabilities remains unchanged.

An equivalent, alternative representation defines the aggregate event rate to be $\sum_{i=1}^{N} \lambda_i^{\mathscr{C}} + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} + \mu c$ and explicitly discounts future rewards. This alternative makes the relationship between the discounted and average-reward problems less transparent, however. In §EC.1 in the online appendix (provided in the e-companion)[1] we explain the relationship between the two uniformization schemes.

Finally, rather than formally stating and analyzing the objective of maximizing discounted expected cash flows, we will instead analyze the MDP value function associated with the problem. Formally, we note that the problem's state and action spaces are both finite, as are all one-period rewards. Therefore, there exists a stationary, deterministic allocation/pricing policy that is optimal, and such value function exists (see Puterman 1994, §6.2).

The value function associated with our control problem is defined as follows:

[1] An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

$$v(k) = \sum_{i=1}^{N} \lambda_i^{\mathscr{C}} H_i^{\mathscr{C}}[v(k)] + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} H_j^{\mathscr{W}}[v(k)] + \mu k v(k-1) + \mu(c-k)v(k), \tag{1}$$

where the operators

$$H_i^{\mathscr{C}}[f(k)] = \begin{cases} \max[f(k)-\pi_i, f(k+1)+r_i] & \text{when } k<c \\ f(k)-\pi_i & \text{when } k=c \end{cases} \tag{2}$$

and

$$H_j^{\mathscr{W}}[f(k)] = \begin{cases} \max_l [p_{jl}(f(k+1)+w_l)+(1-p_{jl})f(k)] \\ \qquad \text{when } k<c \\ f(k) \quad \text{when } k=c \end{cases} \tag{3}$$

are defined for any arbitrary function f defined on S. The operator $H_i^{\mathscr{C}}$ in (2) reflects the capacity allocation choice between accepting or rejecting a rental request coming from class i contract customer. The pricing operator $H_j^{\mathscr{W}}$, introduced in (3), represents the choice of the best fee for rental requests from walk-in customers of class j.

Therefore, $v(k)$ is the expected discounted stream of future rewards and penalties, given the system is now in state k and is operated optimally. In the next section we analyze the value function (1) to derive structural properties of optimal profit management policies.

3. The Structure of Optimal Profit Management Policies

Given a discounted problem ($\gamma > 0$) with time units selected so that $\Gamma = 1$, the sum of the arrival and departure rates, $\sum_{i=1}^{N} \lambda_i^{\mathscr{C}} + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} + \mu c$, is strictly less than 1. In this case, the so-called value iteration operator, T, defined as

$$Tf(k) = \sum_{i=1}^{N} \lambda_i^{\mathscr{C}} H_i^{\mathscr{C}}[f(k)] + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} H_j^{\mathscr{W}}[f(k)] + \mu k f(k-1) + \mu(c-k)f(k), \tag{4}$$

is a contraction operator and can be repeatedly applied to find the value function through successive approximation. If the c-vector v_0 represents an initial estimate of v, then one pass of the value-iteration procedure produces, $v_1 = Tv_0$, n applications of T produce v_n, and $\lim_{n\to\infty} v_n = v$ (see Puterman 1994, §6.3).

This fact is important for establishing structural properties of optimal profit management policies. In particular, when v_0 has a specific property, such as monotonicity or concavity, and the application of T can be shown to maintain the property, then we can inductively prove that the value function, $v(k)$, itself has the property as well (Porteus 1982).

When $\gamma = 0$, our problem's structure is also sufficient to ensure that we can continue to use value iteration to approximate the so-called gain, the optimum expected reward per transition. Specifically, we observe that, given the use of any (arbitrary) stationary policy, there exists a state, $k = 0$, to which there is a positive probability of returning within $c < \infty$ transitions, no matter what the starting state. This implies that the problem is unichain.

In this case, T can be shown to be a so-called J-step contraction operator, so that the value-iteration procedure can be used to successively approximate the average reward per transition—the gain—with any precision (see Puterman 1994, §8.5). By demonstrating that the operator T maintains a desirable property, such as concavity, we can again prove that value iteration identifies an average-reward-optimal policy with that property.

Thus, the MDP results we report hold for both discounted and average-reward problems. To avoid confusion regarding the definition of $v(k)$, we state all MDP-related analytical results in the context of discounted problems. In contrast, we perform all of the paper's numerical tests using an average-reward criterion. Because average reward per transition (the gain) does not depend on the starting state, k, average-reward results are more straightforward to interpret numerically.

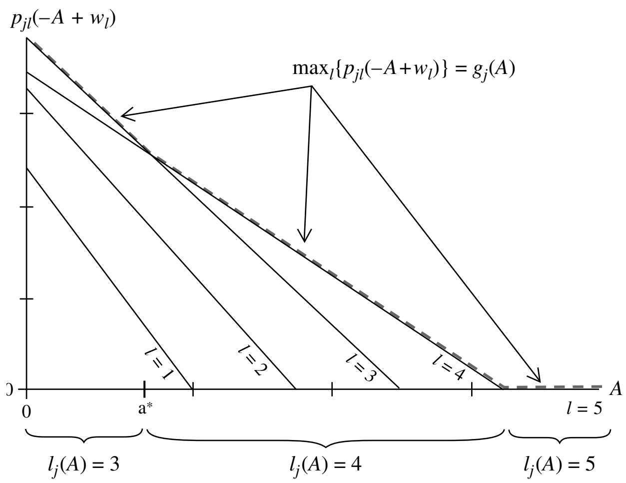
3.1. Structural Properties of the Value Function

The first maximization operator, $H_i^{\mathscr{C}}$, is quite similar to those found in a variety of capacity-allocation studies. (For an early example see Miller 1969; for a recent one see Lewis et al. 1999.) These papers often prove that the associated value function is nonincreasing and concave and, in turn, that so-called "trunk-reservation" (or threshold) policies are optimal. Our first task is to check for the existence of this type of structure.

In our setting, however, the addition of walk-in customers and controls, $H_j^{\mathscr{W}}$, complicates the analysis. Although we have imposed little structure on the form of the relationship between walk-in prices and acceptance probabilities, the restrictions imposed by Assumption 1 are sufficient to ensure that optimal walk-in prices also propagate the desired properties (see also Low 1974).

To demonstrate this, we first note that, formally, we denote a function $f(k)$ as nonincreasing in k whenever $f(k+1) \le f(k)$ for $k = 0, \ldots, c-1$. Similarly, we write that $f(k)$ is concave in k whenever $f(k+1) - f(k) \le f(k) - f(k-1)$ for $k = 1, \ldots, c-1$. We then define three quantities that are intimately related to $H_j^{\mathscr{W}}$.

Figure 1 **Example of the Functions $g_j(A)$ and $l_j(A)$ Defined in (5) and (7)**



First, for $A \geq 0$ let

$$g_j(A) = \max_l [p_{jl}(-A + w_l)], \tag{5}$$

and note that, when $k < c$,

$$H_j^{\mathcal{W}}[f(k)] = g_j(f(k) - f(k+1)) + f(k). \tag{6}$$

If $v(k) - v(k+1)$ is the opportunity cost associated with the acceptance of an arriving walk-in customer when in state k, then $g_j(v(k) - v(k+1))$ is the expected net gain from offering the optimal price, should that customer be of class j. From Assumption 1, we know that $p_{jL} = 0$, so $g_j(A) \geq 0$ for all j and any $A \geq 0$.

Second, define

$$l_j(A) = \arg\max_l [p_{jl}(-A + w_l)], \tag{7}$$

to be the index of the optimal price. If there is more than one maximizer, then define $l_j(A)$ to be the largest of them. Then from (6) we see that $l_j(v(k) - v(k+1))$ is also the maximizer of $H_j^{\mathcal{W}}[v(k)]$.

Figure 1 illustrates the functions $g_j(A)$ and $l_j(A)$ for a walk-in pricing function that includes five price-probability pairs, $l = 1, 2, 3, 4, 5$. The figure's horizontal axis tracks A, and the solid lines display $p_{jl}(-A + w_l)$ for each of the five different price points. The dashed line shows $g_j(A)$, the point-wise maximum of the solid lines. We use the brackets below the horizontal axis to denote ranges of A for which various l's are optimal. Recall that, when more than one index maximizes $g_j(A)$, we define $l_j(A)$ to be the greatest such index. For example, at a^*, the point at which both $l = 3$ and $l = 4$ maximize $g_j(A)$, we let $l_j(A) = 4$.

Several of the figure's features are worth noting. First, note that the lines' slopes are the $-p_{jl}$'s, which increase with increasing l. Furthermore, in this example, the lines for first two price points, $l = 1$ and $l = 2$, are dominated; it is never optimal to use them for any $A \geq 0$. Most important are following two monotonicity properties shown in the graph.

One is that $l_j(A)$ cannot decrease as A grows larger. That this is always true is a fact that follows from the assumption that lower indices (and lower prices) are associated with higher p_{jl}'s. That is, lines with higher slopes ($-p_{jl}$'s) dominate lower-sloped lines as A grows. In turn, the higher the opportunity cost of the capacity (the greater is A), the higher the index of the optimal walk-in price.

The other is the fact that $g_j(A)$ is nonincreasing in A. It is not difficult to see that this is always the case as well, because the first derivative of $g_j(A)$ is some $-p_{jl}$, a negative of probability. Note that for $k < c - 1$ we have

$$\begin{aligned} &H_j^{\mathcal{W}}[f(k+1)] - H_j^{\mathcal{W}}[f(k)] \\ &\quad = (g_j(f(k+1) - f(k+2)) - g_j(f(k) - f(k+1))) \\ &\qquad + (f(k+1) - f(k)). \end{aligned}$$

Therefore, because $g_j(A)$ is nonincreasing in A, we can use $g_j(A)$ to show that $H_j^{\mathcal{W}}[f(k+1)]$ is nonincreasing whenever $f(k)$ is nonincreasing and concave.

The third quantity that is closely related to $H_j^{\mathcal{W}}[f(k)]$ is

$$h_j(A) = -A - g_j(A).$$

That $h_j(A)$ is nonincreasing in A is also easily demonstrated: Its first derivative is $(-1 + p_{jl})$ for some $p_{jl} \in [0, 1]$. In turn, for $k < c - 2$, we have

$$\begin{aligned} &(H_j^{\mathcal{W}}[f(k+2)] - H_j^{\mathcal{W}}[f(k+1)]) \\ &\quad - (H_j^{\mathcal{W}}[f(k+1)] - H_j^{\mathcal{W}}[f(k)]) \\ &= (h_j(f(k+1) - f(k+2)) - h_j(f(k) - f(k+1))) \\ &\quad + (g_j(f(k+2) - f(k+3)) - g_j(f(k+1) - f(k+2))), \end{aligned} \tag{8}$$

and the fact that both $g_j(A)$ and $h_j(A)$ are nonincreasing implies that $H_j^{\mathcal{W}}[f(k)]$ propagates the concavity of $f(k)$.

The monotonicity of $g_j(A)$, $l_j(A)$, and $h_j(A)$ suggests that $H_j^{\mathcal{W}}[f(k)]$ itself propagates the monotonicity and concavity of $f(k)$, and this is in fact the case:

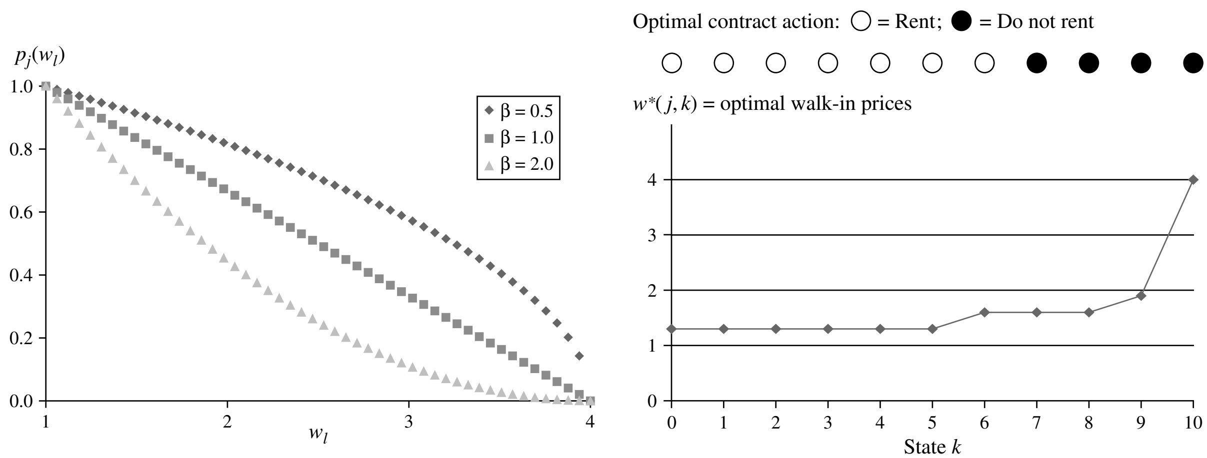
Theorem 1. (a) *If $f(k)$ is a nonincreasing concave function of k, then $Tf(k)$ is also a nonincreasing concave function of k. Consequently, the optimal value function $v(k)$ is a nonincreasing concave function of k.*

(b) *For contract class i, let*

$$k_i^* = \begin{cases} c, & \text{if } v(c) - v(c-1) \geq -(\pi_i + r_i), \\ \min(k \mid v(k+1) - v(k) < -(\pi_i + r_i)), & \\ \quad \text{if } v(c) - v(c-1) < -(\pi_i + r_i). \end{cases} \tag{9}$$

Then, the optimal policy provides a rental unit to an arriving class i customer if and only if the number already in service is $k < k_i^*$.

Figure 2 Example Walk-in Acceptance Probability Function (Left); Example Optimal Policy (Right)



(c) *Let $w^*(j, k)$ be the optimal price to offer a walk-in customer of class j that arrives to find k rental units in service. Then $w^*(j, k+1) \geq w^*(j, k)$, $k = 0, \ldots, c-1$.*

A formal statement of the monotonicity properties of $g_j(A)$, $l_j(A)$, and $h_j(A)$, as well as all proofs, can be found in the online appendix.

Given the concavity result of Part (a), Part (b) states that, as can be expected, the optimal allocation of rental capacity between contract classes is achieved by threshold policies. Part (c) further verifies that, as in Low (1974), optimal walk-in pricing is congestion dependent: for a bigger k, the rental of a unit implies greater loss $(v(k) - v(k+1))$ and induces a higher price to be charged.

Conversely, a loss of $v(k) - v(k+1) = 0$ would induce the lowest possible price. We observe that, in fact, the case in which there is no congestion cost corresponds to one of (effectively) unlimited residual capacity. Furthermore, the fact that $l_j(A)$ is nondecreasing in A implies that this minimum price maximizes expected discounted revenue from the rental itself:

Corollary 1. *Define the index of the price that maximizes the expected (discounted) rental revenue from a class j customer as*

$$e_j = l_j(0) = \arg\max_l [w_l \cdot p_{jl}]. \tag{10}$$

Then $w^*(j, k) \geq w_{e_j}$, $k = 0, \ldots, c-1$.

Thus, the myopic, revenue-maximizing walk-in price, w_{e_j}, provides a lower bound on the price offered to any class j customer. In turn, it is the implied cost of incremental congestion that drives the rental firm to increase its price above w_{e_j}. A price increase above w_{e_j} lowers immediate expected (discounted) revenue, and it effectively reserves capacity by lowering the associated acceptance probability. Furthermore, it increases the actual (conditional) revenue that is paid, should the offer be accepted and additional congestion be incurred.

We next provide a brief example that illustrates the results of Theorem 1 for the case of $N = 1$ customer class and $M = 1$ walk-in class. In this and all of the paper's numerical examples and tests, we use the following form for "walk-in" acceptance probability functions

$$p_j(w_l) = \left(\frac{w_{\max} - w_l}{w_{\max} - w_{\min}}\right)^{\beta_j}, \tag{11}$$

where $w_{\max}$, $w_{\min}$ are constants $p_j(w_{\min}) = 1$ and $p_j(w_{\max}) = 0$, and β_j is the parameter that determines the "curvature" of p_j. The values of the walk-in prices are set to uniformly cover the interval $[w_{\min}, w_{\max}]$:

$$w_l = w_{\min} + (w_{\max} - w_{\min})\frac{l-1}{L-1}, \quad l = 1, \ldots, L, \tag{12}$$

so that $w_1 = w_{\min}$, $w_L = w_{\max}$.

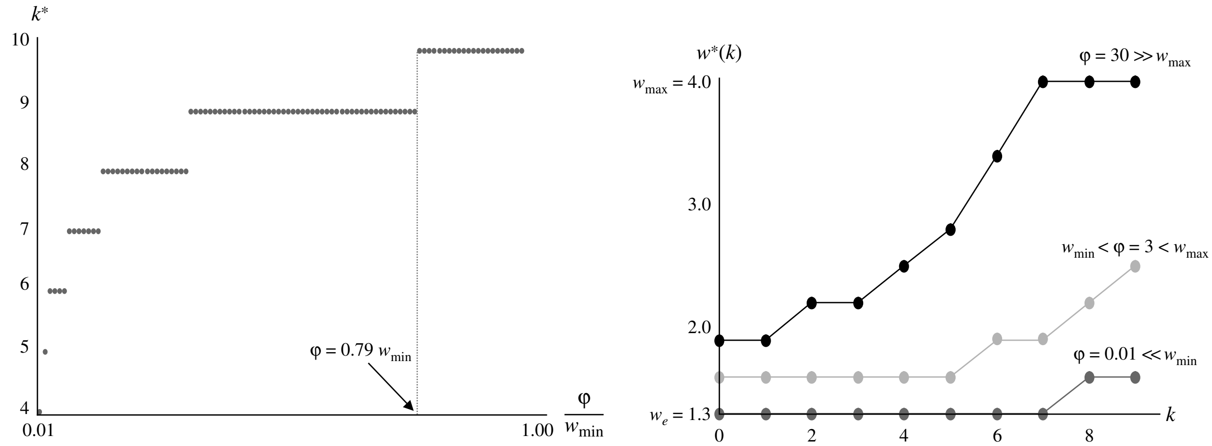
The left panel of Figure 2 illustrates the shape of the acceptance probability function (11)–(12) for three different values of the price sensitivity factor β. In each of the three curves, $w_{\min} = w_1 = 1$, $w_{\max} = w_L = 4$, and $L = 50$.

We note that the class of functions that can be defined by (11) is quite broad. For larger β, the acceptance probability declines more quickly with price increases, reflecting increased sensitivity to higher prices. As β drops from 1, toward 0, the decline weakens and becomes more concave, and as β increases beyond 1, the decline becomes more strongly convex.

At the same time we emphasize that this class is just one example of a much wider range of pricing functions addressed in the paper: The only restrictions we place on the relationship among the p_{jl}'s and w_l's are those found in Assumption 1.

The right panel of Figure 2 depicts an example of the results of Theorem 1. To generate it, we let $\lambda^{\mathscr{C}} = \lambda^{\mathscr{W}} = 7$, $\mu = 1$, and $c = 10$. For contract customers, we let $r = 0.2$ and $\pi = 0$, and for walk-ins

Figure 3 Optimal Contract Thresholds k^* (Left) and Walk-in Prices (Right) as Functions of φ



we use the pricing function (11)–(12) with $w_{\min} = 1$, $w_{\max} = 4$, $L = 10$, and $\beta = 2$. The panel then displays the optimal rental policy under the average-reward objective: The top shows that the rental requests from contract customers are accepted as long as the number of rented units is less than $k^* = 7$; the bottom shows that optimal walk-in fees form a monotone sequence, increasing with the degree of congestion in the rental system. Furthermore, when the system is far from being congested ($k \leq 5$), it is optimal to charge the fee, which maximizes the expected revenue from a walk-in rental, $w_e = 1.3$.

Although Theorem 1 characterizes the structure of optimal policies—for example, thresholds for contract customers—it does not provide direct insight into the nature of optimal policy parameters. The concavity of the value function allows us to characterize aspects of the parameters themselves, however. In fact, optimal capacity allocation and pricing policies are sensitive to the choice of the demand and service parameters, as well as to revenue and penalty values.

For contract customers it is the incremental benefit of accepting—rather than rejecting—a request that drives the choice of control parameters, and to capture this fact we define the *penalty-adjusted* contract rental fee to be $\varphi_i = r_i + \pi_i$. Then given this definition, the nature of the relationship is as follows:

Theorem 2. (a) *The optimal threshold level for class i contract customers, k_i^*, is a nonincreasing function of the arrival rates $\lambda_n^{\mathscr{C}}$, $n = 1, \ldots, N$, and $\lambda_j^{\mathscr{W}}$, $j = 1, \ldots, M$, as well as a nondecreasing function of the service rate μ. Also k_i^* is a nondecreasing function of the class i penalty-adjusted rental fee, φ_i, and a nonincreasing function of penalty-adjusted rental fees φ_n, $n \neq i$, of the other contract classes.*

(b) *For every state of the system, k, the optimal price for class j walk-in customers, $w^*(j, k)$, is a nondecreasing function of the arrival rates $\lambda_i^{\mathscr{C}}$, $i = 1, \ldots, N$, and $\lambda_j^{\mathscr{W}}$, $j = 1, \ldots, M$, as well as a nonincreasing function of the service rate μ. Also, $w^*(j, k)$ is a nondecreasing function of the penalty-adjusted rental fees of contract customers: φ_i, $i = 1, \ldots, N$.*

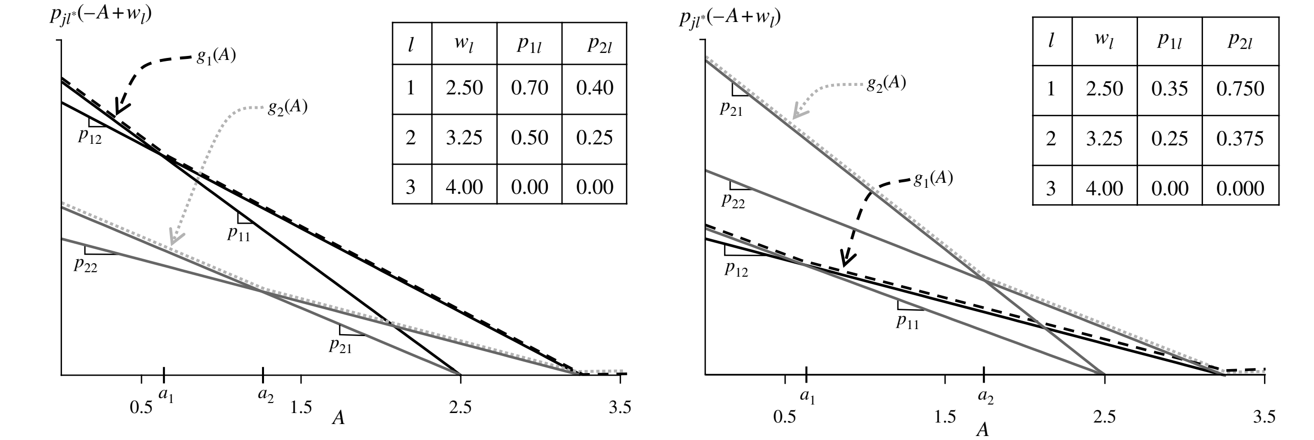
Theorem 2 indicates that, as expected, capacity rationing intensifies as the offered service load from any customer class is increased. Such an increase in service load raises the "value" of each unit of the available capacity, forcing the system manager to reduce thresholds for competing classes of contract customers and to charge higher fees for walk-in services. An increase in the penalty-adjusted fee for any contract class has a similar effect: Although access increases for the class with the increased penalty-adjusted fee, it is further restricted for all other classes.

We illustrate the results of Theorem 2 in Figure 3. For this example there are $N = 1$ contract and $M = 1$ walk-in classes, $\lambda^{\mathscr{C}} = \lambda^{\mathscr{W}} = 5$, $\mu = 1$, $c = 10$, and the walk-in probability function is (11)–(12) with $w_{\min} = 1$, $w_{\max} = 4$, $\beta = 2$, and $L = 10$. Again, the maximization criterion is average reward.

The left panel of Figure 3 shows the optimal contract threshold k^* as a function of the penalty-adjusted contract fee, φ. Note that the smallest value of the fee that ensures that contract customers are served whenever possible, 0.79, is equal to only a fraction of the smallest walk-in fee, $w_{\min} = 1$.

The figure's right panel shows how optimal walk-in pricing changes as a function of φ. For a low value of φ, optimal walk-in pricing is virtually static, set at or near the level that maximizes immediate expected walk-in revenue, $w_e = 1.3$. As φ increases, so does the value of each unit of rental capacity, however, and walk-in prices increase with greater levels of congestion. Finally, when the contract fee is high enough, it becomes optimal to deny walk-in customers access to rentals in highly congested states by charging a price that is rejected with certainty.

Figure 4 Two Examples of Two Fee-Acceptance Probability Functions for Which p_{1l}/p_{2l} Is Increasing in l



l	w_l	p_{1l}	p_{2l}
1	2.50	0.70	0.40
2	3.25	0.50	0.25
3	4.00	0.00	0.00

l	w_l	p_{1l}	p_{2l}
1	2.50	0.35	0.750
2	3.25	0.25	0.375
3	4.00	0.00	0.000

The monotonicity properties described in Theorem 2 suggest that the value of the penalty-adjusted fee is an important characteristic determining the capacity allocated to serving a particular contract class. Analogously, we expect the status of a walk-in class j to be determined by the shape of its fee-acceptance probability curve:

Theorem 3. *For contract classes i_1 and i_2, $\varphi_{i_1} \geq \varphi_{i_2} \Rightarrow k^*_{i_1} \geq k^*_{i_2}$. For walk-in classes j_1 and j_2, if $p_{j_1 l}/p_{j_2 l}$ is an increasing function of l for $l = 1, \ldots, L-1$, then $w^*(j_1, k) \geq w^*(j_2, k)$ for all $k = 0, \ldots, c-1$.*

Theorem 3 states that a higher penalty-adjusted fee affords greater access to the rental units. Similarly, the rate at which the fee-acceptance probability declines with l determines the price sensitivity of a given walk-in class, and more price-sensitive classes should be offered systematically lower prices.

The fact that one class is more price-sensitive than another does *not*, however, imply that the first is more or less profitable than the second. For instance, Figure 4 shows two examples in which class 1 is less price sensitive than class 2; in both, class 2 customers are offered a lower price over the range $[a_2, a_1]$. In the left panel $g_1(A) \geq g_2(A)$ for all $A \geq 0$, whereas in the right the reverse is true.

4. Myopic Profit Management and Preferred Customer Classes

The value of the penalty-adjusted rental fee for a contract class and the degree of price sensitivity for a walk-in class serve as indicators of the relative "importance" of a particular class. High values of these indicators may result in a preferential treatment of rental requests coming from customers of these classes. Formally, we make the following definition:

Definition 1. Contract class i is called preferred if $k^*_i = c$. Walk-in class j is called preferred if $w^*(j, k) = w_{e_j}$, for all $k = 0, \ldots, c-1$.

Preferred classes receive treatment that can be characterized as myopic: A preferred contract customer receives service as long as there is available capacity; a preferred walk-in customer is offered a fee that maximizes the expected (discounted) revenue immediately obtained from his or her rental, as well as the probability the he will accept the offer in the first place. Clearly, such profit management policies would be optimal in the case of unlimited rental capacity.

4.1. Asymptotic Optimality of Myopic Policies

More interestingly, myopic capacity management is also asymptotically optimal in highly utilized systems, as both the offered load and system capacity become large. To establish this result, we note that, under a myopic policy, the rental system behaves like an Erlang loss system with arrival rate

$$\lambda_{\mathcal{M}} = \left(\sum_{i=1}^{n} \lambda_i^{\mathscr{C}} + \sum_{j=1}^{m} \lambda_j^{\mathscr{W}} p_{e_j} \right), \tag{13}$$

and offered load $\rho_{\mathcal{M}} = \lambda_{\mathcal{M}}/\mu$. Here, the subscript, $\mathcal{M}$, denotes statistics related to the myopic policy. Note that the effective arrival rate includes only walk-in customers that accept the myopic offer of w_{e_j}.

For a system with an offered load of ρ and c rental units, the Erlang loss function allows us to calculate the probability that an arriving customer is blocked,

$$B(\rho, c) = \frac{\rho^c / c!}{\sum_{l=0}^{c} \rho^l / l!}, \tag{14}$$

and given $B(\rho, c)$ it is straightforward to calculate expected reward per unit of time. Assuming no blocking, the potential expected revenue on customer arrival is

$$r_{\mathcal{M}} = \frac{\sum_{i=1}^{n} \lambda_i^{\mathscr{C}} r_i + \sum_{j=1}^{m} \lambda_j^{\mathscr{W}} p_{e_j} w_{e_j}}{\sum_{i=1}^{n} \lambda_i^{\mathscr{C}} + \sum_{j=1}^{m} \lambda_j^{\mathscr{W}} p_{e_j}}, \tag{15}$$

and, given blocking, the expected penalty on a loss is

$$\Omega_{\mathcal{M}} = \frac{\sum_{i=1}^{n} \lambda_i^{\mathscr{C}} \pi_i}{\sum_{i=1}^{n} \lambda_i^{\mathscr{C}} + \sum_{j=1}^{m} \lambda_j^{\mathscr{W}} p_{e_j}}, \tag{16}$$

Then, under the myopic policy, the average reward per unit of time can be written as

$$r_{\mathcal{M}}\lambda_{\mathcal{M}}(1-B(\rho_{\mathcal{M}},c))-\Omega_{\mathcal{M}}\lambda_{\mathcal{M}}B(\rho_{\mathcal{M}},c)$$
$$=r_{\mathcal{M}}\lambda_{\mathcal{M}}-(r_{\mathcal{M}}+\Omega_{\mathcal{M}})\lambda_{\mathcal{M}}B(\rho_{\mathcal{M}},c). \quad (17)$$

Clearly, $r_{\mathcal{M}}\lambda_{\mathcal{M}}$ represents an upper bound on the profit earned under any capacity-management policy, including the optimal profit, and the loss due to a myopic policy is bounded above by

$$(r_{\mathcal{M}}+\Omega_{\mathcal{M}})\lambda_{\mathcal{M}}B(\rho_{\mathcal{M}},c). \quad (18)$$

In turn, it is not difficult to show that, for large c, this loss can be small, even for $\rho_{\mathcal{M}}$ "close" to c. In particular, we have the following results:

Lemma 1. (a) *Suppose* $\rho_{\mathcal{M}}=c+\alpha\sqrt{c}$ *for some fixed, real* α. *Then* $\lim_{c\to\infty}\rho_{\mathcal{M}}/c=1$ *and*

$$\lim_{c\to\infty}\sqrt{c}\cdot B(\rho_{\mathcal{M}},c)=\beta+o(1/\sqrt{c}), \quad (19)$$

where β *is a function of* α.

(b) *Suppose* $\rho_{\mathcal{M}}=\alpha c$ *for some fixed* $0<\alpha<1$. *Then*

$$B(\rho_{\mathcal{M}},c)\le\frac{\sqrt{1/2\pi c}(\alpha e^{(1-\alpha)})^{c}e^{-1/(12c+1)}}{1-(\alpha e^{(1-\alpha)})^{c}}. \quad (20)$$

Part (a) of Lemma 1 is due to Jagerman (1974), and the proof of Part (b) can be found in the online appendix. These two results are complementary.

The limiting result in Part (a) shows that, when $\rho_{\mathcal{M}}$ and c are nearly balanced—so that the difference can be measured in units of $\sqrt{c}$—the rate at which the blocking probability drops is of order $1/\sqrt{c}$ as $c\to\infty$. Therefore, the fraction of customers that is blocked vanishes, and from (18) we see that the absolute rate at which revenue is lost, $(r_{\mathcal{M}}+\Omega_{\mathcal{M}})\lambda_{\mathcal{M}}B(\rho_{\mathcal{M}},c)$, is approximately

$$(r_{\mathcal{M}}+\Omega_{\mathcal{M}})(c+\alpha\sqrt{c})\mu\frac{\beta}{\sqrt{c}}$$
$$=(r_{\mathcal{M}}+\Omega_{\mathcal{M}})\mu\cdot(\sqrt{c}+\alpha)\beta=O(\sqrt{c}); \quad (21)$$

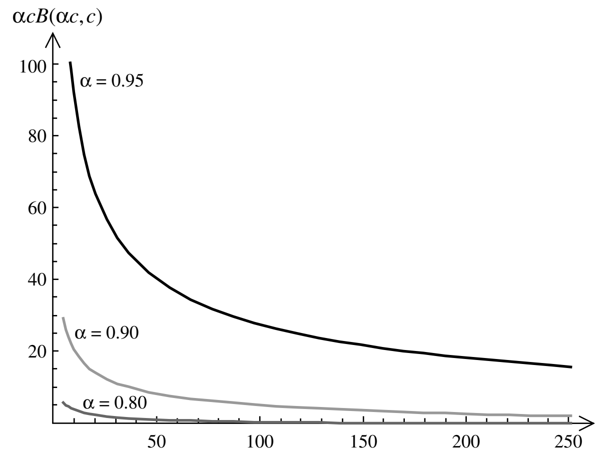
a rate which may be large in absolute terms but vanishes (is $O(\sqrt{c}/c)=O(1/\sqrt{c})$) as a percentage of the total reward rate.

The result of Part (b) is an upper bound on the blocking probability given a fixed $\alpha<1$, so that excess capacity remains of order c. The term $\alpha e^{1-\alpha}<1$, so the bound declines exponentially quickly in c. In contrast to the result of Part (a), the absolute loss rate is itself bounded by

$$(r_{\mathcal{M}}+\Omega_{\mathcal{M}})\lambda_{\mathcal{M}}B(\rho_{\mathcal{M}},c)$$
$$\le(r_{\mathcal{M}}+\Omega_{\mathcal{M}})\mu\cdot\alpha c\frac{\sqrt{1/2\pi c}(\alpha e^{(1-\alpha)})^{c}e^{-1/(12c+1)}}{1-(\alpha e^{(1-\alpha)})^{c}}$$
$$=O(\sqrt{c}\cdot(\alpha e^{1-\alpha})^{c}), \quad (22)$$

and absolute losses also vanish as c becomes large.

Figure 5 **Upper Bound on the Loss Rate Calculated as in (22), with** $r_{\mathcal{M}}+\Omega_{\mathcal{M}}=1$**, and** $\mu=1$



For large systems with some excess capacity, myopic policies will afford nearly optimal average rewards. Indeed, when α is not too close to one, the bound declines quickly with c. But for a given c, the bound grows rapidly as $\alpha\uparrow 1$. Figure 5 illustrates these effects for various α's and c's.

At the same time, longer-run capacity-sizing decisions must account for capacity costs as well, and when accounting for these costs, it may not be optimal to operate with as much as $O(c)$ units of excess capacity. In particular, (21) implies that excess capacity on the order $O(\sqrt{c})$ generates losses of that order. Therefore, to the extent that capacity costs, revenues, and loss penalties are of the same order of magnitude, lower orders of excess capacity will generate higher long-run average profits for very large systems.

Remark. The comparison among scaling results presented in Lemma 1, as well as the resulting cost and revenue trade-offs, loosely follows along the lines of Borst et al. (2004). This paper systematically analyzes various scaling regimes for $M/M/c$ queueing systems, describes systems with excess capacity of order $O(c)$ as "quality driven" and those with excess capacity of order $O(\sqrt{c})$ "rationalized." For more on differences among scaling regimes, see also Gans et al. (2003, §4).

4.2. Sufficient Conditions for Classes to Be Preferred

The notion of a preferred class introduced in Definition 1 also naturally leads to the following questions: In the presence of constraining rental capacity, does there always exist such a class? If so, what makes a class preferred? We address these questions below.

We first establish conditions that ensure "preferred" status for a contract or a walk-in class. Without loss of generality we assume that $\varphi_1\ge\varphi_2\ge\cdots\ge\varphi_N$, and to ease the notational burden, we define $\bar{w}_j(A)=w_{l_j(A)}$ and $\bar{p}_j(A)=p_{jl_j(A)}$.

Thus, $\bar{w}_j(A)$ represents the optimal walk-in fee to be charged to class j customers when the expected opportunity cost of capacity is $A \geq 0$, and $\bar{p}_j(A)$ reflects the probability that such fee will be accepted. We will show below that $\bar{w}_j(\varphi_i)$ and $\bar{p}_j(\varphi_i)$ are important characteristics when determining the preferred status of walk-in class j.

The analogous quantity for class j walk-in customers is

$$\hat{r}_j = \inf(A \geq 0 \mid l_j(A) > e_j). \tag{23}$$

That is, $\hat{r}_j$ is the minimum opportunity cost such that the optimal price index $l_j(A)$ falls above that for myopic pricing. For example, in Figure 1, $\hat{r} = a^*$, and in both panels of Figure 4, $\hat{r}_1 = a_1$ and $\hat{r}_2 = a_2$.

Furthermore at $\hat{r}_j$, the myopic price is also optimal, $\bar{p}_j(\hat{r}_j)(-\hat{r}_j + \bar{w}_j(\hat{r}_j)) = p_{e_j}(-\hat{r}_j + w_{e_j})$, so

$$\hat{r}_j = \frac{w_{e_j} p_{e_j} - \bar{w}_j(\hat{r}_j)\bar{p}_j(\hat{r}_j)}{p_{e_j} - \bar{p}_j(\hat{r}_j)}. \tag{24}$$

Note that the value of $\hat{r}_j$ is completely determined by the set of walk-in fees w_l and by the shape of the fee-sensitivity function p_{jl}. In particular, it does not depend on any demand or service characteristics.

Using the definitions above, we can characterize conditions that ensure a preferred status for a particular customer class:

Theorem 4. (a) *Define*

$$G_i^{\mathscr{C}}(A) = \frac{\sum_{n\neq i} \lambda_n^{\mathscr{C}} \max(A, \varphi_n) + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} \bar{w}_j(A)\bar{p}_j(A)}{\sum_{n\neq i} \lambda_n^{\mathscr{C}} + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} \bar{p}_j(A) + \mu}. \tag{25}$$

Then there exists a unique A_i^* such that $A_i^* = G_i^{\mathscr{C}}(A_i^*)$, and $A > G_i^{\mathscr{C}}(A)$ for all $A > A_i^*$. Furthermore, contract class i is preferred whenever $\varphi_i \geq A_i^*$.

(b) *Define*

$$G^{\mathscr{W}}(A) = \frac{\sum_{i=1}^{N} \lambda_i^{\mathscr{C}} \max(A, \varphi_i) + \sum_{m=1}^{M} \lambda_m^{\mathscr{W}} \bar{w}_m(A)\bar{p}_m(A)}{\sum_{i=1}^{N} \lambda_i^{\mathscr{C}} + \sum_{m=1}^{M} \lambda_m^{\mathscr{W}} \bar{p}_m(A) + \mu}. \tag{26}$$

Then there exists a unique A^* such that $A^* = G^{\mathscr{W}}(A^*)$, and $A > G^{\mathscr{W}}(A)$ for all $A > A^*$. Furthermore, walk-in class j is preferred whenever $\hat{r}_j \geq A^*$.

Thus, A_i^* and A^* are lower bounds on the benefit that an arriving class i or j customer must bring to the system to be preferred. For contract customers, the lower bound is on the penalty-adjusted fee, φ_i, whereas for walk-in customers the bound is on the maximum loss for which a myopic price offer of w_{e_j} is still optimal.

In addition, if we recall that arrival and service rates correspond to event probabilities, then the bounds associated with (25)–(26) can be interpreted probabilistically. To more clearly illustrate these results, we consider three special cases.

Our first example is one in which the entire customer base consists of contract customers: $N > 1$ and $M = 0$. In this case, the revenue-management problem reduces to a simplified version of a stochastic knapsack problem (Ross and Tsang 1989). Recalling that $\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_N$, the sufficient condition (25) can be expressed as

$$\varphi_i \geq \frac{\sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} \varphi_n}{\sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} + \mu}. \tag{27}$$

For class $i = 1$, the sufficient condition (27) becomes $\varphi_1 \geq 0$, and we see that contract class 1 always has unrestricted access to available rental units. Rental requests of class 2 customers, however, are always granted only if the class 2 penalty-adjusted fee $\varphi_2 \geq (\lambda_1^{\mathscr{C}}/(\lambda_1^{\mathscr{C}} + \mu))\varphi_1$, a fraction of the corresponding fee for class 1 customers. This fraction represents the probability that a class 1 customer will arrive before a class 2 customer returns the rental equipment to the rental pool, and it is an upper bound on the probability that a class 1 customer will be lost because a class 2 customer is put into service.

A similar argument applies for any contract class. The benefit of renting to a class i customer, φ_i, should outweigh the expected revenue lost due to blocking of higher-paying customers, and the expression on the right-hand side of (27) is an upper bound on this opportunity cost. We note that the expression only includes the contribution from the customers who pay higher penalty-adjusted fees because, as indicated in Theorem 3, the denial of access to class i customers implies that all customers of lower-paying classes are also rejected.

The second example is one in which the entire customer base consists of a single walk-in class: $N = 0$ and $M = 1$. Here, our problem reduces to a simplified version of a pricing problem considered in Paschalidis and Tsitsiklis (2000). Because there is only one class of customer, we drop the subscript j, and (26) reduces to

$$A^* = \frac{\lambda^{\mathscr{W}} \bar{w}(A^*)\bar{p}(A^*)}{\lambda^{\mathscr{W}} \bar{p}(A^*) + \mu}, \tag{28}$$

for A^*.

Now suppose $\hat{r} > A^*$. Then it follows from (23) that $\bar{w}(A^*) = w_e$ and $\bar{p}(A^*) = p_e$, where w_e is the fee that maximizes the immediate expected revenue from a given walk-in customer. In turn, we have

$$\hat{r} > A^* = \frac{\lambda^{\mathscr{W}} p_e}{\lambda^{\mathscr{W}} p_e + \mu} w_e. \tag{29}$$

Note that the expression on the right-hand side is an upper bound on the expected revenue lost due to the blocking of a walk-in customer.

Next define $l_e = l(0)$ to be the index of the myopic price. Then (23) also implies that $p_e(-\hat{r} + w_e) \geq p_l(-\hat{r} + w_l)$ for $l = l_e + 1, \dots, L$ or equivalently

$$\frac{p_e w_e - p_l w_l}{p_e - p_l} \geq \hat{r} \quad l = l_e + 1, \dots, L.$$

Together with (29) this implies

$$\frac{p_e w_e - p_l w_l}{p_e - p_l} > \frac{\lambda^{\mathcal{W}} p_e}{\lambda^{\mathcal{W}} p_e + \mu} w_e, \quad l = l_e + 1, \dots, L,$$

and the above inequality can be re-expressed as

$$p_e\left(w_e - \frac{\lambda^{\mathcal{W}} p_e}{\lambda^{\mathcal{W}} p_e + \mu} w_e\right) > p_l\left(w_l - \frac{\lambda^{\mathcal{W}} p_e}{\lambda^{\mathcal{W}} p_e + \mu} w_e\right), \quad l = l_e + 1, \dots, L. \tag{30}$$

Recalling (5), we see that (30) states that myopic pricing is optimal for an expected loss of $(\lambda^{\mathcal{W}} p_e / (\lambda^{\mathcal{W}} p_e + \mu)) w_e$. But this quantity is an upper bound on the expected loss when admitting a customer, and we find that, for $\hat{r} > A^*$, a price of w_e, maximizes expected profit in any state.

Comparing these first two examples, we note that there exists an important difference between contract and walk-in classes. The first example showed that, in the presence of only contract classes, there always exists at least one preferred class: the one with the highest value of penalty-adjusted fee. The second analysis was predicated on $\hat{r} > A^*$, however. Indeed, even if the customer base is uniform and consists of a single walk-in class, the preferred status of such a class is not guaranteed.

In the third example, there exists a single contract and a single walk-in class: $N = 1$ and $M = 1$, and again for simplicity we drop the indices i and j. Here, (25) and (26) become

$$\varphi \geq \frac{\lambda^{\mathcal{W}} \bar{w}(\varphi) \bar{p}(\varphi)}{\lambda^{\mathcal{W}} \bar{p}(\varphi) + \mu}, \quad \text{and}$$
$$\hat{r} \geq \frac{\lambda^{\mathcal{C}} \max(\varphi, \hat{r}) + \lambda^{\mathcal{W}} \bar{w}(\hat{r}) \bar{p}(\hat{r})}{\lambda^{\mathcal{C}} + \lambda^{\mathcal{W}} \bar{p}(\hat{r}) + \mu}. \tag{31}$$

The first inequality in (31) stipulates that contract customers are preferred whenever the gain from their admittance exceeds the potential losses due to reduced walk-in rental capacity. The expression on the inequality's right-hand side represents an estimate of these losses and, similar to (27), it includes only the contribution from the customers who pay a fee of at least φ. (Note that $\bar{w}(\varphi) \geq \varphi$ provided that $\bar{p}(\varphi) > 0$.) This expression can be interpreted as an unrealized revenue $\bar{w}(\varphi)$ multiplied by the probability $\lambda^{\mathcal{W}} \bar{p}(\varphi) / (\lambda^{\mathcal{W}} \bar{p}(\varphi) + \mu)$ that a walk-in customer would arrive and accept an offer of $\bar{w}(\varphi)$ before the admitted contract customer returns her rental. The second inequality in (31) is similar to (28) and reduces to it when $\hat{r} \geq \varphi$. It has a closely related interpretation.

5. Different Rental Durations for Contract and Walk-in Classes

The analysis of §§3 and 4 assumes that all customers' rental durations have the same distribution, irrespective of the group to which they belong. In this section we extend this analysis to the more general case in which rental durations of contract customers differ from those of walk-in customers. Despite this additional complexity, we show that many of the properties of the optimal capacity-management policies established in the simpler setting can be extended to the case of group-dependent rental times.

We assume that rental durations are independent, exponentially distributed random variables with expectations $1/\mu_{\mathcal{C}}$ and $1/\mu_{\mathcal{W}}$ for contract and walk-in customers, respectively. Given this assumption, at any time t we must distinguish between the numbers of contract and walk-in customers already in service. Thus, the state of the rental system is now described by the two-dimensional vector $(k_{\mathcal{C}}, k_{\mathcal{W}})$ representing the numbers of contract and walk-in customers in service. The new state space for the rental problem is defined as $\tilde{S} = ((k_{\mathcal{C}}, k_{\mathcal{W}}) \mid 0 \leq k_{\mathcal{C}} \leq c,\ 0 \leq k_{\mathcal{W}} \leq c,\ 0 \leq k_{\mathcal{C}} + k_{\mathcal{W}} \leq c)$.

As before, both the state space and the number of available actions are finite, and it is not difficult to show that, under any stationary, deterministic policy, there is a positive probability of returning to the idle state $(0, 0)$ within $c < \infty$ transitions. Therefore, for both the discounted and average-reward problems, there exists a stationary, deterministic policy that is optimal, and it can identified via value-iteration.

The difference between service rates $\mu_{\mathcal{C}}$ and $\mu_{\mathcal{W}}$ leads us to uniformize the system at rate $\Gamma = \sum_{i=1}^{N} \lambda_i^{\mathcal{C}} + \sum_{j=1}^{M} \lambda_j^{\mathcal{W}} + (\mu_{\mathcal{C}} + \mu_{\mathcal{W}})c + \gamma$. Again we choose the time scale so that $\Gamma = 1$ and its component transition rates may be viewed as probabilities. In turn, the optimality equation (1) is replaced by

$$\begin{aligned} v(k_{\mathcal{C}}, k_{\mathcal{W}}) &= \sum_{i=1}^{N} \lambda_i^{\mathcal{C}} H_i^{\mathcal{C}}[v(k_{\mathcal{C}}, k_{\mathcal{W}})] + \sum_{j=1}^{M} \lambda_j^{\mathcal{W}} H_j^{\mathcal{W}}[v(k_{\mathcal{C}}, k_{\mathcal{W}})] \\ &\quad + (\mu_{\mathcal{C}} k_{\mathcal{C}} v(k_{\mathcal{C}} - 1, k_{\mathcal{W}}) + \mu_{\mathcal{W}} k_{\mathcal{W}} v(k_{\mathcal{C}}, k_{\mathcal{W}} - 1)) \\ &\quad + (\mu_{\mathcal{C}}(c - k_{\mathcal{C}}) + \mu_{\mathcal{W}}(c - k_{\mathcal{W}})) v(k_{\mathcal{C}}, k_{\mathcal{W}}), \end{aligned} \tag{32}$$

where the maximization operators are extended to become

$$H_i^{\mathcal{C}}[f(k_{\mathcal{C}}, k_{\mathcal{W}})] = \begin{cases} \max[f(k_{\mathcal{C}}, k_{\mathcal{W}}) - \pi_i,\ f(k_{\mathcal{C}} + 1, k_{\mathcal{W}}) + r_i] & \text{when } k_{\mathcal{C}} + k_{\mathcal{W}} < c, \\ f(k_{\mathcal{C}}, k_{\mathcal{W}}) - \pi_i & \text{when } k_{\mathcal{C}} + k_{\mathcal{W}} = c. \end{cases} \tag{33}$$

and

$$H_j^{\mathcal{W}}[f(k_{\mathcal{C}}, k_{\mathcal{W}})] = \begin{cases} \max_l [p_{jl}(f(k_{\mathcal{C}}, k_{\mathcal{W}}+1)+w_l)+(1-p_{jl})f(k_{\mathcal{C}}, k_{\mathcal{W}})] \\ \qquad\qquad \text{when } k_{\mathcal{C}}+k_{\mathcal{W}}<c, \\ f(k_{\mathcal{C}}, k_{\mathcal{W}}) \quad \text{when } k_{\mathcal{C}}+k_{\mathcal{W}}=c. \end{cases} \tag{34}$$

for any arbitrary function f defined on $\tilde{S}$.

The nonincreasing and concave properties of optimal profit management policies that were established in §3 were defined for functions with one dimensional domains. To extend them, we provide the following definitions. First, a function $f(k_{\mathcal{C}}, k_{\mathcal{W}})$ defined on $\tilde{S}$ is nonincreasing in $k_{\mathcal{C}}$ and $k_{\mathcal{W}}$ if $f(k_{\mathcal{C}}+1, k_{\mathcal{W}}) \le f(k_{\mathcal{C}}, k_{\mathcal{W}})$ and $f(k_{\mathcal{C}}, k_{\mathcal{W}}+1) \le f(k_{\mathcal{C}}, k_{\mathcal{W}})$, respectively. Similarly, we say that $f(k_{\mathcal{C}}, k_{\mathcal{W}})$ is submodular on $\tilde{S}$ if $f(k_{\mathcal{C}}+1, k_{\mathcal{W}}+1) - f(k_{\mathcal{C}}+1, k_{\mathcal{W}}) \le f(k_{\mathcal{C}}, k_{\mathcal{W}}+1) - f(k_{\mathcal{C}}, k_{\mathcal{W}})$ whenever $0 \le k_{\mathcal{C}}+k_{\mathcal{W}} \le c-2$.

Then given the definitions above, we can state our first results for systems with $\mu_{\mathcal{C}} \ne \mu_{\mathcal{W}}$:

THEOREM 5. (a) *The optimal profit function* $v(k_{\mathcal{C}}, k_{\mathcal{W}})$ *is a nonincreasing, submodular function on* $\tilde{S}$.

(b) *Let* $(k_{\mathcal{C}}, k_{\mathcal{W}}) \in \tilde{S}$ *be the state of the rental system at the time of a class i arrival. The rental request will be granted if and only if* $k_{\mathcal{W}}$ *is less than a certain threshold value,* $k_i^*(k_{\mathcal{C}})$.

(c) *Let* $w^*(j, k_{\mathcal{C}}, k_{\mathcal{W}})$ *be the optimal fee to charge for the walk-in service of an class j customer when that arrives to a system in state* $(k_{\mathcal{C}}, k_{\mathcal{W}})$. *Then,* $w^*(j, k_{\mathcal{C}}+1, k_{\mathcal{W}}) \ge w^*(j, k_{\mathcal{C}}, k_{\mathcal{W}}) \ge w_{e_j}$, *for* $(k_{\mathcal{C}}, k_{\mathcal{W}}), (k_{\mathcal{C}}+1, k_{\mathcal{W}}) \in \tilde{S}$.

(d) *For contract classes* i_1 *and* i_2, $\varphi_{i_1} \ge \varphi_{i_2} \Rightarrow k_{i_1}^*(k_{\mathcal{C}}) \ge k_{i_2}^*(k_{\mathcal{C}})$ *for all* $k_{\mathcal{C}} = 0, \ldots, c-1$. *For walk-in classes* j_1 *and* j_2, *if* $p_{j_1 r}/p_{j_2 r}$ *is an increasing function of r, then* $w^*(j_1, k_{\mathcal{C}}, k_{\mathcal{W}}) \ge w^*(j_2, k_{\mathcal{C}}, k_{\mathcal{W}})$ *for all* $(k_{\mathcal{C}}, k_{\mathcal{W}}) \in \tilde{S}$.

Parts (a)–(c) of the theorem are analogues to the statements of Theorem 1, and Part (a) directly implies Parts (b) and (c). Part (b) states that optimal admission of contract-class customers can be achieved by "switching curve" policies (Altman et al. 2001, Savin et al. 2005). These policies place thresholds on the number of *walk-in* customers that are renting units. This differs both from Theorem 1, which considers the total number of rental units in service, as well as from controls on the numbers of contract customers in servers. Similarly, Part (c) shows that walk-in prices are increasing in the number of units currently rented to *contract* customers. In particular, it is the "decreasing differences" property of $v(k_{\mathcal{C}}, k_{\mathcal{W}})$ that makes the control of one customer group depend on the system state of the other.

Part (d) of the theorem is a direct analog to Theorem 3. As before, class i customers with higher penalty-adjusted fees and class j customer that are more price sensitive continue to be granted greater access to rental units. Furthermore, the characterization of preferred customer classes in Theorem 4 also has a direct analog:

THEOREM 6. (a) *For contract class i define*

$$\tilde{G}_i^{\mathcal{C}}(A) = \frac{\sum_{n \ne i} \lambda_n^{\mathcal{C}} \max(A, \varphi_n) + \sum_{j=1}^{M} \lambda_j^{\mathcal{W}} w_{e_j} p_{j, e_j}}{\sum_{n \ne i} \lambda_n^{\mathcal{C}} + \mu_{\mathcal{C}}}. \tag{35}$$

Then there exists a unique $\tilde{A}_i^*$ *such that* $\tilde{A}_i^* = \tilde{G}_i^{\mathcal{C}}(\tilde{A}_i^*)$, *and* $A > \tilde{G}_i^{\mathcal{C}}(A)$ *for all* $A > \tilde{A}_i^*$. *Furthermore, contract class i is preferred whenever* $\varphi_i \ge A_i^*$.

(b) *Define*

$$\tilde{G}^{\mathcal{W}}(A) = \frac{\sum_{i=1}^{N} \lambda_i^{\mathcal{C}} \varphi_i + \sum_{m=1}^{M} \lambda_m^{\mathcal{W}} \bar{w}_m(A) \bar{p}_m(A)}{\sum_{m=1}^{M} \lambda_m^{\mathcal{W}} \bar{p}_m(A) + \mu_{\mathcal{W}}}. \tag{36}$$

Then there exists a unique $\tilde{A}^*$ *such that* $\tilde{A}^* = \tilde{G}^{\mathcal{W}}(\tilde{A}^*)$, *and* $A > \tilde{G}^{\mathcal{W}}(A)$ *for all* $A > \tilde{A}^*$. *Furthermore, walk-in class j is preferred whenever* $\hat{r}_j \ge A^*$.

Finally, we note that the well-known insensitivity property of the Erlang loss function implies that the results of Lemma 1 also hold for a much broader class of problems. (For example, see Ross 1996.)

COROLLARY 2. *Suppose class i customers have generally-distributed service times with mean* $1/\mu_i < \infty$, $i = 1, 2, \ldots, N$. *Similarly, suppose class j customers have generally-distributed service times with mean* $1/\mu_j < \infty$, $j = 1, 2, \ldots, M$. *Then the results of Lemma* 1 *hold without modification.*

Therefore, our limiting results, which show that myopic policies are asymptotically (average-reward) optimal in large, heavily-loaded systems, hold for generally-distributed, class-specific service times as well.

Thus, many of the essential characterizations that we developed for systems with $\mu_{\mathcal{C}} = \mu_{\mathcal{W}}$ also hold for the more general case of $\mu_{\mathcal{C}} \ne \mu_{\mathcal{W}}$. In particular, the form of our characterization of preferred customer classes holds for the wider range of systems as well.

6. Effectiveness of Myopic Profit Management: Numerical Study

Myopic profit management policies have several important advantages. They are simple to justify and implement; they depend neither on the values of demand and service parameters nor on the state of the rental system. They also turn out to be optimal for a range of problem parameters, as indicated in Lemma 1 and Theorem 4. It is clear, however, that the inflexibility of myopic policies may be a disadvantage in instances in which demand for rentals significantly exceeds capacity, leading to the loss of profits compared to more flexible state-dependent policies. To better understand the effectiveness of myopic policies, we conducted two sets of numerical studies.

The first examines the situation in which overall rental capacity is fixed. Here, we focus on the revenue performance of the myopic heuristic as we vary the overall intensity and the composition of the rental demand, as well as price parameters of contract and walk-in customers.

The second considers the case in which capacity can be changed to maximize the expected value of rental profit. In this setting, we fix the rental demand intensity and vary the composition of the demand, the value of the walk-in fee sensitivity factor as well as the unit holding cost for the rental capacity. For each example, we then compare the optimal number of rental units, c, given the use of the optimal and the myopic policy.

In both sets of tests, we consider examples with $N = 1$ contract class and $M = 1$ walk-in classes, a simple setting which highlights the competition between contract and walk-in classes. (We also report results of preliminary tests with $N = 2$ and $M = 2$ in §EC.10 in the online appendix.) Given the results of Lemma 1, we also consider primarily examples with small numbers of rental units—c in the ones or low teens—because these are the cases in which myopic policies are most likely to perform poorly. As before, we report results for average-reward problems.

6.1. Performance of Myopic Control Given Fixed Rental Capacity

First, we focus on the case in which the value of the rental capacity is fixed. Our test suite is constructed as follows. The rental capacity level is set at $c = 10$ units, and of the three demand and service parameters ($\lambda^{\mathscr{C}}$, $\lambda^{\mathscr{W}}$, and μ), we fix the service rate μ at 1. For the walk-in class, we use a walk-in pricing function (11)–(12) with fixed $w_{\min} = 1$, $w_{\max} = 4$, and $L = 10$. We then conduct numerical studies for different combinations of penalty-adjusted contract fees, walk-in customer price sensitivities, and demand intensities.

Let $R(\Delta, c)$ denote the expected net contribution (expected revenues, less expected penalties) per unit of time achieved under policy Δ for rental capacity c. Then Table 1 reports the average and the maximum (in parentheses) percentage deviation between the optimal (OPT) and myopic (MYO) policies: $(R(OPT, c) \div R(MYO, c) - 1) \times 100\%$. In figures we refer to this quantity as the *myopic % shortfall*.

All tests shown in Table 1 use $\beta = 1$, and each cell in the table reports the results of 9 numerical examples. In each cell, we fixed values of φ $(0.1, 0.5, 1, 2, 3, 10)$ and $\rho = (\lambda^{\mathscr{C}} + \lambda^{\mathscr{W}})/(\mu c)$ $(0.1, 0.5, 1, 1.5, 2)$, and for these fixed values we run 9 test cases: $\lambda^{\mathscr{C}}/(\lambda^{\mathscr{C}} + \lambda^{\mathscr{W}}) = 0.1, \ldots, 0.9$.

The table's results indicate that myopic profit management policies can be effective as long as two general conditions are satisfied. First, as already indicated in Lemma 1, the overall demand load should be adequately matched by the rental capacity. Conversely, the importance of careful capacity-management becomes apparent in cases in which the offered load, ρ, significantly exceeds one. Second, the contract penalty-adjusted fee φ should be in the neighborhood of $w_{\min}$ and $w_{\max}$. Indeed, when $\varphi \ll w_{\min}$, the policy severely restricts the service of the low-paying contract customers, and the performance of the myopic policy, which freely admits those customers, understandably deteriorates. For $\varphi \gg w_{\max}$ the situation is reversed: Walk-in customers have very little to offer and are, therefore, virtually priced out by the optimal policy. In this case, the performance of the myopic policy is compromised because it continues to charge the low, myopically optimal walk-in fee, w_e.

The nonmonotonicity of the performance of the myopic policy as a function of the penalty-adjusted contract fee is further illustrated in Figure 6, which fixes $\lambda^{\mathscr{C}} = \lambda^{\mathscr{W}} = 5$ (so that $\rho = 1$) and systematically varies φ. (The rest of the example's parameters are fixed as in Table 1.) The figure's left panel shows the myopic policy's percentage revenue shortfall over the full range of φ, whereas its right panel shows differences in policy actions for three representative φ's.

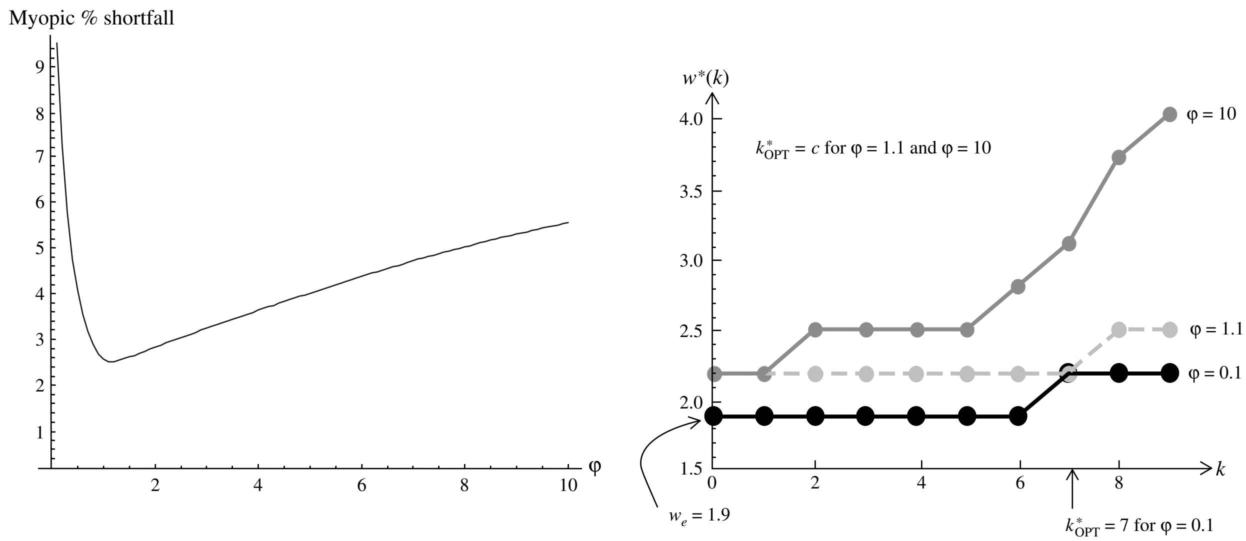
Note that the myopic policy performs best when the penalty-adjusted fee is near (slightly below) the myopic walk-in fee of $w_e = 1.9$. In this case, the loss of one type of customer—contract or walk-in—is not significantly different from that of another, and the reservation of capacity for a particular class of customer does not have a significant effect.

Table 1 Performance of the Myopic Profit Management Policies for One Contract and One Walk-in Class

	$\varphi = 0.1$	$\varphi = 0.5$	$\varphi = 1$	$\varphi = 2$	$\varphi = 3$	$\varphi = 10$
$\rho = 0.1$	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
$\rho = 0.5$	2.9 (3.1)	0.05 (0.08)	0.06 (0.08)	0.08 (0.09)	0.09 (0.11)	0.18 (0.25)
$\rho = 1.0$	9.2 (11.8)	3.9 (5.1)	2.4 (3.6)	2.6 (3.6)	3.1 (3.9)	5.3 (6.7)
$\rho = 1.5$	24.2 (28.6)	12.1 (15.7)	7.5 (11.0)	6.3 (10.3)	7.4 (10.7)	13.3 (17.3)
$\rho = 2.0$	35.5 (41.6)	21.6 (26.0)	13.0 (18.0)	8.9 (16.0)	10.5 (16.9)	19.1 (26.3)

Note. The average (and maximum in parentheses) percentage deviations from the optimal profit.

Figure 6 Average Performance Gap (Left) and Policy Differences (Right) for Different Values of φ



It is plausible that both of these general conditions are satisfied in many practical settings. On the one hand, whereas short-term mismatches between rental demand and the supply of rental capacity are almost unavoidable in stochastic settings, in the long run one should expect that demand and supply should be properly matched. (In the next subsection, we analyze this issue in more detail.) On the other hand, given the fact that contract and walk-in customers derive the same type of economic benefit from the use of a rental, it is reasonable to expect that φ is in the neighborhood of $[w_{\min}, w_{\max}]$.

The arguments above indicate that, among the combinations of parameters we studied, the ones with $\lambda^{\mathscr{C}} + \lambda^{\mathscr{W}} \approx \mu c$ and $w_{\min} \leq \varphi \leq w_{\max}$ are of particular importance. For these settings, the maximum relative difference between the performances of myopic and optimal policies, 3.9%, is observed for high value of penalty-adjusted contract fee φ ($\varphi = 3$).

Figure 7 highlights differences between the myopic and the optimal capacity-management policies as a function of the walk-in fee sensitivity factor, β. In these examples, $\lambda^{\mathscr{C}} = \lambda^{\mathscr{W}} = 5$, $c = 10$, $\mu = 1$, and the walk-in pricing function (11)–(12) has $w_{\min} = 1$, $w_{\max} = 4$, and $L = 10$. Then we fix $\varphi = 3$ and systematically vary β between 0.1 and 10. The figure's left panel shows the myopic's percentage revenue shortfall over the full range of β, and the right shows differences in policy actions for three representative β's.

The left panel shows that, as with φ, percentage revenue shortfall is not necessarily a monotonic function of β, and the right panel provides some insight into why this is the case. When β is near 0, walk-in customers are willing to accept high rental fees and both myopic pricing and myopic capacity allocation are close to optimal. For high values of β, walk-in customers are relatively inflexible, and the myopic price values are driven down toward $w_{\min}$. However, even

Figure 7 Average Performance Gap (Left) And Policy Differences (Right) for Different Values of β

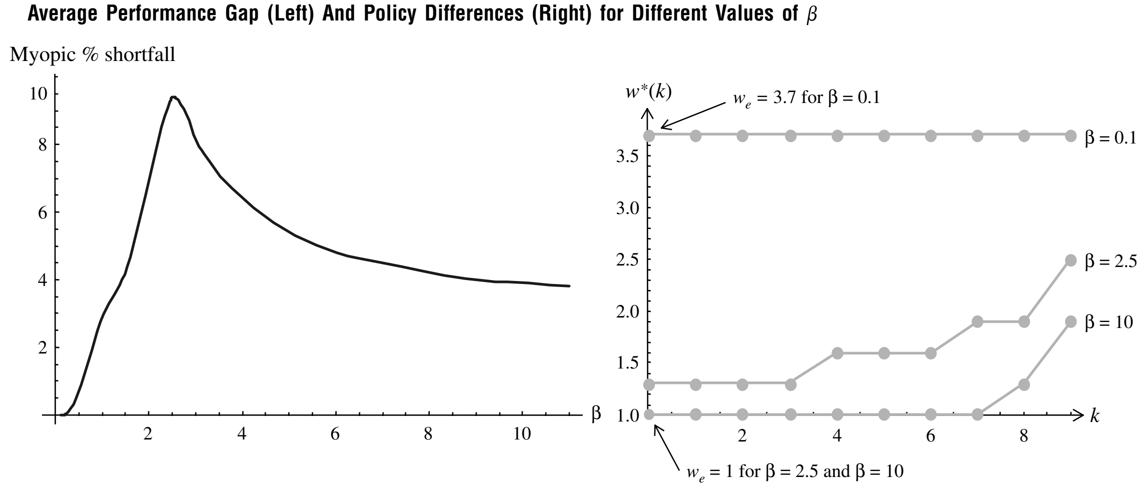


Table 2 Optimal Fleet Sizes and Expected Profits for the Optimal (OPT) and Myopic (MYO) Policies

		Optimal fleet size, $c^*(\Delta)$						Profit, $\Pi(\Delta, c^*(\Delta))$					
$\lambda^{\mathscr{C}}/(\lambda^{\mathscr{C}}+\lambda^{\mathscr{W}})$		0.1		0.5		0.9		0.1		0.5		0.9	
h/φ	β	OPT	MYO	OPT	MYO	OPT	MYO	OPT	MYO	OPT	MYO	OPT	MYO
0.3	0.2	11	12	12	12	13	13	16.1	16.06	13.58	13.57	11.12	11.12
	1	9	10	11	11	13	13	7.33	6.71	8.73	8.39	10.15	10.08
	5	8	10	11	12	13	13	3.03	2.68	6.36	6.02	9.68	9.6
0.5	0.2	10	10	10	11	11	11	12.03	11.92	9.18	9.14	6.55	6.53
	1	6	8	8	9	10	10	4.44	3.38	5.06	4.5	5.74	5.62
	5	4	3	7	8	10	10	0.85	0.05	2.99	2.09	5.32	5.09
0.7	0.2	8	9	8	8	8	8	8.3	8.05	5.3	5.18	2.62	2.61
	1	5	5	6	6	7	7	2.12	0.81	2.06	1.34	2.03	1.89
	5	0	0	4	0	7	7	0	0	0.69	0	1.75	1.43

Note. In all cases the following parameters are fixed: $\varphi = 2$, $w_{\min} = 1$, $w_{\max} = 4$, $L = 10$, $\mu = 1$, $\lambda^{\mathscr{C}} + \lambda^{\mathscr{W}} = 10$.

in these cases higher fees are demanded from walk-in customers as the rental capacity becomes tight. That is, to successfully compete for capacity with contract customers, whose penalty adjusted fees may be greater than $w_{\min}$, walk-in customers must also pay more. Thus, the optimal capacity-management policy is to treat contract customers as a preferred class, while raising walk-in fees above the myopic levels when occupancy is nearing the system's capacity.

Remark. Results of numerical tests for systems with $N = 2$ classes of contract customers and $M = 2$ classes of walk-in customers, whereas more complex, are generally consistent with those described above. First, as penalty-adjusted contract fees, φ_i, move outside of the interval $[w_{\min}, w_{\max}]$, the myopic policy's performance suffers. Second, myopic policies appear to perform well for very price-sensitive and very price-insensitive walk-in customers. Myopic pricing performs less well for walk-in customers with intermediate β's, however. Here, dynamic pricing appears to more effectively modulate walk-in demand. Section EC.10 in the online appendix provides a detailed description of the experiments and results.

6.2. The Effect of Myopic Controls on Capacity Choice and Overall Performance

In the previous subsection we tested the myopic pricing heuristic in the setting in which rental capacity is fixed. Below, we focus on the setting in which the overall rental demand is fixed and the rental company can adjust its rental capacity to maximize expected profits.

To account for the expenses of capacity investment and maintenance, we introduce a holding cost of h per unit of capacity per unit of time, and we let $\Pi(\Delta, c) = R(\Delta, c) - hc$ for the overall expected profit per unit of time. Then given a set of problem parameters and chosen capacity-management policy, Δ, we search for the rental capacity, c, that maximizes $\Pi(\Delta, c)$, and we denote the optimal capacity as $c^*(\Delta)$.

The problem parameters used in the numerical study are as follows. Overall demand for rental services is fixed at $\lambda^{\mathscr{C}} + \lambda^{\mathscr{W}} = 10$. The service rate for rental customers and the contract penalty-adjusted fee are fixed at $\mu = 1$ and $\varphi = 2$. The walk-in pricing function (11)–(12) uses parameters $w_{\min} = 1$, $w_{\max} = 4$, and $L = 10$. We then systematically vary the customer mix, $\lambda^{\mathscr{C}}/(\lambda^{\mathscr{C}} + \lambda^{\mathscr{W}}) = (0.1, 0.5, 0.9)$, the price sensitivity of walk-in customers, $\beta = (0.2, 1.0, 5.0)$, and the relative cost of capacity, $h = (0.3\varphi, 0.5\varphi, 0.7\varphi)$.

Table 2 illustrates the results of these 27 examples for both the OPT and MYO policies. The left side of the table reports optimal rental fleet sizes, $c^*(\Delta)$. The right lists optimal profits $\Pi(\Delta, c^*(\Delta))$. The tests' results prompt several observations. First, the performances of both the optimal and the myopic controls change in a similar, intuitive fashion as the problem parameters are varied. All other parameters being fixed, an increase in the capacity holding cost, h, results in the decrease in the optimal size of the rental fleet, as well as the values of profits (and profit margins). An increase in the value of the walk-in fee sensitivity factor, β, implies more price-sensitive walk-in customers. Lower prices, in turn, result in decreased profits and fleet sizes. At the same time, sensitivity to β is naturally moderated by fleet composition: When $\lambda^{\mathscr{C}}$ comprises 90% of the offered load, the price sensitivity of walk-in customers matters little.

Second, we note that the optimal capacity under a myopic policy can be either higher or lower than the one under optimal revenue control. When capacity costs are low enough, $h = 0.3\varphi$ in our examples, then it is profitable to add capacity to satisfy the additional load induced by myopic pricing. As capacity costs climb, however, this added load is only profitable if walk-in customers are not too price sensitive. In particular, for $h = 0.5\varphi$, there is a reversal. For $\beta \in \{0.2, 1.0\}$, optimal capacity for the MYO policy exceeds that for OPT, and for $\beta = 5.0$ it falls short.

In fact, this type of reversal is an example of a more general phenomenon. Our myopic policy, by definition, restricts the offered load less than more dynamic controls. Therefore, when capacity costs are low, it is worth adding capacity so that marginal (lower-value) customers—which would have been excluded under the optimal control—do not "crowd out" more profitable customers. As capacity costs grow, however, the rental firm cannot afford to serve them. An extreme instance of the latter can be seen in the example in which $h = 0.7\varphi$, $\beta = 5$, and $\lambda^{\mathscr{C}}$ makes up 50% of total demand. (For a more general discussion and analysis, see Savin et al. 2005.)

Finally, we note that the myopic policy's profits (and profit margins) improve, relative to those of the optimal policy, as the fraction of contract customers increases and as capacity costs decrease. Thus, in the examples shown in Table 2, it appears to be the optimal policy's ability to control walk-in demand—rather than contact customers—that boosts its performance.

7. Conclusion

In this paper, we analyze a model of rental operations in which customers are heterogeneous and both rental requests and durations are stochastic. An important, novel feature of our model is the explicit treatment of the interaction between customer groups with fundamentally different attitudes toward rental fees and quality of service: Contract customers expect stable rental fees and a high quality of service; in contrast, walk-in customers can be dynamically quoted prices and have no expectations regarding equipment availability.

We derive structural properties of optimal capacity-management policies and show how policy parameters are affected by changes in problem parameters. We also characterize preferred customer classes and related myopic policies. We demonstrate that myopic policies can perform well in a wide variety of circumstances. On the one hand, we develop sufficient conditions for the preferred status of a particular customer class that do not depend on the number of rental units, c, only on offered load and price attributes. When satisfied by all customer classes, these conditions identify a range of problem's parameters, independent of c, for which myopic policies are optimal. On the other, we provide limiting results that imply that, in large systems, myopic policies are (asymptotically) optimal for any set of pricing attributes, as long as c roughly matches the offered load.

Numerical results indicate that myopic management can be effective even in smaller systems, given capacity is roughly balanced with demand. Conversely, in cases in which rental capacity is severely constrained, the inflexibility of the myopic policies may lead to significant profit losses.

Although our analysis captures some of the important characteristics of rental businesses, more work remains to be done. One immediate question concerns the directness with which our numerical results translate to more complex systems that have multiple classes of contract and walk-in customers. The numerical tests in §EC.10 of the online appendix suggest that, with $N = 2$ and $M = 2$, the relative performance of myopic policies is not inconsistent with the results depicted in Figures 6 and 7. Still, we do not yet thoroughly understand system behavior, even in this relatively simple setting, and a more systematic analysis is warranted.

In addition, two potential extensions to the paper's model look especially relevant and interesting to us. The first addresses the fact that, in many cases, rental units can be heterogeneous, and customers may be inclined to substitute one type of unit for another. The second explicitly models arrival rates, and perhaps rental durations, as being affected by price and availability.

8. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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Electronic Companion—"Pricing and Capacity Rationing for Rentals with Uncertain Durations" by Noah Gans and Sergei Savin, *Management Science* 2007, 53(3) 390–407.

Online Appendix

EC.1. Uniformization

An alternative representation of uniformization defines the aggregate event rate to be $\Upsilon = \sum_{i=1}^{N} \lambda_i^{\mathscr{C}} + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} + \mu c$, so that the system evolves with exponentially distributed interevent times of Υ. Again the normalized version of each event rate, for example $\lambda_i^{\mathscr{C}}/\Upsilon$, becomes the conditional probability that the next event is of that rate's type. In this case, one explicitly defines a discrete-time discount factor associated with the expected interevent time, $\int_0^\infty e^{-\gamma t}\Upsilon e^{-\Upsilon t}\,dt = \Upsilon/(\Upsilon+\gamma)$.

It is not difficult to see that this formulation is equivalent to the one that we use. For instance, suppose that the next event is a type i arrival and that the resulting expected discounted value at that arrival epoch is X. Then, under this scheme, the marginal contribution to expected discounted value (now) is $\lambda_i^{\mathscr{C}}/\Upsilon \cdot \Upsilon/(\Upsilon+\gamma)\cdot X = \lambda_i^{\mathscr{C}}/\Gamma \cdot X$, precisely the same contribution as under our scheme.

EC.2. Properties of $g_j(A)$, $l_j(A)$, and $h_j(A)$

We demonstrate the properties of $g_j(A)$, $l_j(A)$, and $h_j(A)$:

Lemma 2.
(a) *$l_j(A)$ is nondecreasing in A;*
(b) *$g_j(A)$ is nonincreasing in A; and*
(c) *$h_j(A)$ is nonincreasing in A.*

Proof.
Part (a)—By definition,

$$p_{jl_j(A)}(-B+w_{l_j(A)}) \le p_{jl_j(B)}(-B+w_{l_j(B)}).$$

Now let $A > B$, and by contradiction, suppose that $l_j(B) > l_j(A)$. Then $p_{jl_j(B)} < p_{jl_j(A)}$, so

$$\begin{aligned} 0 &> p_{jl_j(A)}(-B+w_{l_j(A)}) - p_{jl_j(B)}(-B+w_{l_j(B)}) \\ &= p_{jl_j(A)}(-A+w_{l_j(A)}) - p_{jl_j(B)}(-A+w_{l_j(B)}) + (B-A)(p_{jl_j(B)} - p_{jl_j(A)}). \end{aligned}$$

But the expression on the right-hand side is positive, which is a contradiction.

Part (b)—$B > A$ implies

$$g_j(B) = p_{j,\,l(B)}(-B+w_{l(B)}) < p_{j,\,l(B)}(-A+w_{l(B)}) \le \max_l [p_{jl}(-A+w_l)] = g_j(A).$$

Part (c)—$B > A$ implies

$$\begin{aligned} h(B)-h(A) &= -B+A-p_{j,\,l(B)}(-B+w_{l(B)}) + p_{j,\,l(A)}(-A+w_{l(A)}) \\ &\ge B-A-p_{j,\,l(B)}(B+w_{l(B)}) + p_{j,\,l(B)}(A+w_{l(B)}) = (1-p_{j,\,l(B)})(B-A) \ge 0. \quad \square \end{aligned}$$

EC.3. Proof of Theorem 1

Part (a)—Consider a nonincreasing concave function $f(k)$. The image of this function under T can be expressed as

$$Tf(k) = \sum_{i=1}^{N} \lambda_i^{\mathscr{C}} H_i^{\mathscr{C}}[f(k)] + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} H_j^{\mathscr{W}}[f(k)] + \mu H(k), \tag{EC1}$$

where $H(k) = kf(k-1) + (c-k)f(k)$. Because $\lambda_i^{\mathscr{C}}$, $\lambda_j^{\mathscr{W}}$, and μ are all positive, it is sufficient to show that $H_i^{\mathscr{C}}[f(k)]$, $H_j^{\mathscr{W}}[f(k)]$, and $H(k)$ are all nonincreasing concave functions of k.

We start with $H(k)$. For any $0 \le k \le c-1$,

$$H(k+1) - H(k) = k(f(k) - f(k-1)) + (c-k-1)(f(k+1) - f(k)) \le 0. \tag{EC2}$$

Also, for any $0 \le k \le c-2$, we have

$$\begin{aligned} &H(k+2) + H(k) - 2H(k+1) \\ &\quad = (c-k-2)(f(k+2) + f(k) - 2f(k+1)) + k(f(k+1) + f(k-1) - 2f(k)) \le 0. \end{aligned} \tag{EC3}$$

Next, consider $H_i^{\mathscr{C}}[f(k)]$ for any $i = 1, \ldots, N$. When $k = c-1$, $H_i^{\mathscr{C}}[f(k+1)] = f(c) - \pi_i$ so that

$$H_i^{\mathscr{C}}[f(k+1)] - H_i^{\mathscr{C}}[f(k)] = f(k+1) - \max[f(k+1) + r_i + \pi_i, f(k)] \le 0; \tag{EC4}$$

for $k < c-1$,

$$H_i^{\mathscr{C}}[f(k+1)] - H_i^{\mathscr{C}}[f(k)] = \max[f(k+2) + r_i + \pi_i, f(k+1)] - \max[f(k+1) + r_i + \pi_i, f(k)], \tag{EC5}$$

and three cases are possible. If $f(k+2) - f(k+1) \ge -(r_i + \pi_i)$, then the fact that $f(k)$ is nonincreasing implies that $H_i^{\mathscr{C}}[f(k+1)] - H_i^{\mathscr{C}}[f(k)] = f(k+2) - f(k+1) \le 0$. If $f(k+2) - f(k+1) < -(r_i + \pi_i) \le f(k+1) - f(k)$ then $H_i^{\mathscr{C}}[f(k+1)] - H_i^{\mathscr{C}}[f(k)] = -(r_i + \pi_i) \le 0$. Finally, if $f(k+1) - f(k) < -(r_i + \pi_i)$, then we have $H_i^{\mathscr{C}}[f(k+1)] - H_i^{\mathscr{C}}[f(k)] = f(k+1) - f(k) \le 0$.

For the proof of concavity of $H_i^{\mathscr{C}}[f(k)]$ we again start with the boundary case of $k = c-2$:

$$\begin{aligned} H_i^{\mathscr{C}}[f(k+2)] + H_i^{\mathscr{C}}[f(k)] - 2H_i^{\mathscr{C}}[f(k+1)] &= f(k+2) + \max[f(k+1) + r_i + \pi_i, f(k)] \\ &\quad - 2\max[f(k+2) + r_i + \pi_i, f(k+1)]. \end{aligned} \tag{EC6}$$

We use the monotonicity and concavity of $f(k)$ to evaluate each of the 3 cases that must be considered. If $f(k+2) - f(k+1) \ge -(r_i + \pi_i)$, (EC6) becomes $f(k+1) - f(k+2) - r_i - \pi_i \le 0$. If $f(k+2) - f(k+1) < -(r_i + \pi_i) \le f(k+1) - f(k)$, then (EC6) reduces to $f(k+2) - f(k+1) + r_i + \pi_i \le 0$. Finally, if $f(k+1) - f(k) < -(r_i + \pi_i)$, we obtain $H_i^{\mathscr{C}}[f(k+2)] + H_i^{\mathscr{C}}[f(k)] - 2H_i^{\mathscr{C}}[f(k+1)] = f(k+2) + f(k) - 2f(k+1) \le 0$.

In considering nonboundary states, with $k < c-2$, we note that we only have to prove concavity for the case in which $f(k+3) - f(k+2) \ge -(r_i + \pi_i)$, because the other two cases were effectively considered above. Here, the monotonicity of $H_i^{\mathscr{C}}[f(k)]$ implies that $H_i^{\mathscr{C}}[f(k+2)] + H_i^{\mathscr{C}}[f(k)] - 2H_i^{\mathscr{C}}[f(k+1)]$ reduces directly to $f(k+3) + f(k+1) - 2f(k+2)$, which is less than or equal to zero by the concavity of $f(k)$.

Now, consider $H_j^{\mathscr{W}}[f(k)]$ for any $j = 1, \ldots, M$. First, we demonstrate that $H_j^{\mathscr{W}}[f(k)]$ is nonincreasing. Let $l_1 = \arg\max_l [p_{jl}(f(k+1) + w_l) + (1 - p_{jl})f(k)]$. Then for $k = c-1$

$$\begin{aligned} H_j^{\mathscr{W}}[f(k+1)] - H_j^{\mathscr{W}}[f(k)] &= f(k+1) - p_{jl_1}(f(k+1) + w_{l_1}) - (1 - p_{jl_1})f(k) \\ &= (1 - p_{jl_1})(f(k+1) - f(k)) - p_{jl_1} w_{l_1} \le 0, \end{aligned} \tag{EC7}$$

given nonincreasing $f(k)$. Similarly, let $l_2 = \arg\max_l [p_{jl}(f(k+2) + w_l) + (1 - p_{jl})f(k+1)]$. Then for $k < c-1$, we have

$$H_j^{\mathscr{W}}[f(k+1)] - H_j^{\mathscr{W}}[f(k)] \le H_j^{\mathscr{W}}[f(k+1)] - (p_{jl_2}(f(k+1) + w_{l_2}) - (1 - p_{jl_2})f(k))$$

because (p_{jl_1}, w_{l_1}) maximizes $H_j^{\mathscr{W}}[f(k)]$. In turn, we have

$$\begin{aligned} H_j^{\mathscr{W}}[f(k+1)] - H_j^{\mathscr{W}}[f(k)] &\le p_{jl_2}(f(k+2) + w_{l_2}) + (1 - p_{jl_2})f(k+1) - (p_{jl_2}(f(k+1) + w_{l_2}) - (1 - p_{jl_2})f(k)) \\ &= f(k+1) - f(k) + p_{jl_2}(f(k+2) + f(k) - 2f(k+1)) \le 0, \end{aligned} \tag{EC8}$$

which follows from the monotonicity and concavity of $f(k)$.

Finally, we show that $H_j^{\mathscr{W}}[f(k)]$ is concave. For $k = c-2$, we have

$$\begin{aligned} H_j^{\mathscr{W}}[f(k+2)] + H_j^{\mathscr{W}}[f(k)] - 2H_j^{\mathscr{W}}[f(k+1)] &= f(k+2) + H_j^{\mathscr{W}}[f(k)] - 2H_j^{\mathscr{W}}[f(k+1)] \\ &= f(k+2) + f(k) - 2f(k+1) + p_{jl_1}(f(k+1) - f(k) + w_{l_1}) \\ &\quad - 2p_{jl_2}(f(k+2) - f(k+1) + w_{l_2}), \end{aligned} \tag{EC9}$$

where l_1 and l_2 are defined as before. Again, because (p_{jl_1}, w_{l_1}) maximizes $H_j^{\mathscr{W}}[f(k)]$, we have

$$\begin{aligned}H_j^{\mathscr{W}}&[f(k+2)]+H_j^{\mathscr{W}}[f(k)]-2H_j^{\mathscr{W}}[f(k+1)]\\&\le f(k+2)+f(k)-2f(k+1)+p_{jl_1}(f(k+1)-f(k)+w_{l_1})-2p_{jl_1}(f(k+2)-f(k+1)+w_{l_1})\\&=(1-p_{jl_1})(f(k+1)+f(k)-2f(k+1))+p_{jl_1}(f(k+1)-f(k+2)-w_{l_1})\\&\le 0+p_{jl_1}(f(k)-f(k+1)-w_{l_1}),\end{aligned}\tag{EC10}$$

given the concavity of $f(k)$. By definition (3), $p_{jl_1}(f(k+1)+w_{l_1})+(1-p_{jl_1})f(k)\ge f(k)$, which implies $p_{jl_1}(f(k)-f(k+1)-w_{l_1})\le 0$ and completes the argument for $k=c-2$. For $k<c-2$, we define $l_3=\arg\max_l[p_{jl}(f(k+3)+w_l)+(1-p_{jl})f(k+2)]$, so that

$$\begin{aligned}H_j^{\mathscr{W}}&[f(k+2)]+H_j^{\mathscr{W}}[f(k)]-2H_j^{\mathscr{W}}[f(k+1)]\\&=f(k+2)+f(k)-2f(k+1)+p_{jl_3}(f(k+3)-f(k+2)+w_{l_3})\\&\quad+p_{jl_1}(f(k+1)-f(k)+w_{l_1})-2p_{jl_2}(f(k+2)-f(k+1)+w_{l_2}).\end{aligned}\tag{EC11}$$

Then because (p_{jl_2}, w_{l_2}) maximizes $H_j^{\mathscr{W}}[f(k+1)]$, we have

$$\begin{aligned}H_j^{\mathscr{W}}&[f(k+2)]+H_j^{\mathscr{W}}[f(k)]-2H_j^{\mathscr{W}}[f(k+1)]\\&\le f(k+2)+f(k)-2f(k+1)+p_{jl_3}(f(k+3)-f(k+2)+w_{l_3})\\&\quad+p_{jl_1}(f(k+1)-f(k)+w_{l_1})-p_{jl_3}(f(k+2)-f(k+1)+w_{l_3})-p_{jl_1}(f(k+2)-f(k+1)+w_{l_1})\\&=(1-p_{jl_1})(f(k+2)+f(k)-2f(k+1))+p_{jl_3}(f(k+3)+f(k+1)-2f(k+2))\le 0.\end{aligned}\tag{EC12}$$

The fact that the class of concave functions is closed under the action T implies that the optimal profit function $v(k)$ is also concave.

Part (b)—Using the definition of k_i^* (9), for $i=1,\dots,N$ we observe that $H_i^{\mathscr{C}}[v(k)]=v(k+1)+r_i$ for $k<k_i^*$, and $H_i^{\mathscr{C}}[v(k)]=v(k)-\pi_i$ for $k\ge k_i^*$. Thus, class i contract customers are admitted into service if and only if $k<k_i^*$.

Part (c)—Recall that $w_1<w_2<\cdots<w_L$ and that w_L is offered whenever a walk-in arrival finds $k\ge c$ rental units busy. Then Lemma 2(a) and the concavity of $v(k)$ imply that the fee indices form a monotone sequence: $l_j(v(k+1)-v(k+2))\ge l_j(v(k)-v(k+1))$, so that $w^*(j,k+1)\ge w^*(j,k)$, $k=0,\dots,c-1$.

EC.4. Proof of Theorem 2

Part (a)—Recall that the optimality equation for the profit function can be expressed as

$$v(k)=Tv(k),\tag{EC13}$$

where

$$Tv(k)=\sum_{i=1}^{N}\lambda_i^{\mathscr{C}}H_i^{\mathscr{C}}[v(k)]+\sum_{j=1}^{M}\lambda_j^{\mathscr{W}}H_j^{\mathscr{W}}[v(k)]+\mu k v(k-1)+\mu(c-k)v(k).\tag{EC14}$$

We start by proving monotonicity of the optimal thresholds for contract classes with respect to the contract demand intensities. Suppose that a demand intensity for contract class m is changed from $\lambda_m^{\mathscr{C}}$ to $\hat\lambda_m^{\mathscr{C}}\ge\lambda_m^{\mathscr{C}}$. Define by $T(\lambda_m^{\mathscr{C}})$ and $T(\hat\lambda_m^{\mathscr{C}})$ the respective dynamic programming operators in (EC14) and by $v(k,\lambda_m^{\mathscr{C}})$ and $v(k,\hat\lambda_m^{\mathscr{C}})$ the corresponding solutions to the Markov decision process (MDP) optimality equation (EC13), so that $v(k,\lambda_m^{\mathscr{C}})=T(\lambda_m^{\mathscr{C}})v(k,\lambda_m^{\mathscr{C}})$, $v(k,\hat\lambda_m^{\mathscr{C}})=T(\hat\lambda_m^{\mathscr{C}})v(k,\hat\lambda_m^{\mathscr{C}})$, $k=0,\dots,c$. We will show that

$$v(k+1,\hat\lambda_m^{\mathscr{C}})-v(k,\hat\lambda_m^{\mathscr{C}})\le v(k+1,\lambda_m^{\mathscr{C}})-v(k,\lambda_m^{\mathscr{C}})\tag{EC15}$$

for every $k=0,\dots,c-1$, which, according to (9) would imply that the optimal threshold under demand intensity $\lambda_m^{\mathscr{C}}$ is at least as big as the one under $\hat\lambda_m^{\mathscr{C}}$.

To establish (EC15), we proceed as follows. First, we define the sequence of approximations for the optimal discounted profit function $v_n(k,\hat\lambda_m^{\mathscr{C}})$, $n=0,1,\dots,$ where $v_0(k,\hat\lambda_m^{\mathscr{C}})\equiv v(k,\lambda_m^{\mathscr{C}})$, and $v_n(k,\hat\lambda_m^{\mathscr{C}})=T(\hat\lambda_m^{\mathscr{C}})v_{n-1}(k,\hat\lambda_m^{\mathscr{C}})$ for $n\ge 1$. Given the contracting nature of $T(\hat\lambda_m^{\mathscr{C}})$, $v(k,\hat\lambda_m^{\mathscr{C}})=\lim_{n\to\infty}v_n(k,\hat\lambda_m^{\mathscr{C}})$. Thus, (EC15) will be proven if we show that

$$v_{n+1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{n+1}(k, \hat{\lambda}_m^{\mathscr{C}}) \le v_n(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_n(k, \hat{\lambda}_m^{\mathscr{C}}), \quad n \ge 0. \tag{EC16}$$

Note that v_n is the analog of the arbitrary concave, decreasing, function, f, that was the argument of T in the proof of Theorem 1. Here, our initial approximation uses a value function, $v_0(k, \hat{\lambda}_m^{\mathscr{C}}) \equiv v(k, \lambda_m^{\mathscr{C}})$, so in this proof we use the name v_0, rather than f or f_n.

We conduct the proof of (EC16) by induction.

For $n = 1$ we have

$$\begin{aligned} v_1(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_1(k, \hat{\lambda}_m^{\mathscr{C}}) &= T(\hat{\lambda}_m^{\mathscr{C}})v(k+1, \lambda_m^{\mathscr{C}}) - T(\hat{\lambda}_m^{\mathscr{C}})v(k, \lambda_m^{\mathscr{C}}) \\ &= T(\lambda_m^{\mathscr{C}})v(k+1, \lambda_m^{\mathscr{C}}) - T(\lambda_m^{\mathscr{C}})v(k, \lambda_m^{\mathscr{C}}) + (T(\hat{\lambda}_m^{\mathscr{C}})v(k+1, \lambda_m^{\mathscr{C}}) - T(\lambda_m^{\mathscr{C}})v(k+1, \lambda_m^{\mathscr{C}})) \\ &\quad - (T(\hat{\lambda}_m^{\mathscr{C}})v(k, \lambda_m^{\mathscr{C}}) - T(\lambda_m^{\mathscr{C}})v(k, \lambda_m^{\mathscr{C}})) \\ &= v(k+1, \lambda_m^{\mathscr{C}}) - v(k, \lambda_m^{\mathscr{C}}) + (\hat{\lambda}_m^{\mathscr{C}} - \lambda_m^{\mathscr{C}})(H_m^{\mathscr{C}}[v(k+1, \lambda_m^{\mathscr{C}})] - v(k+1, \lambda_m^{\mathscr{C}}) \\ &\qquad - H_m^{\mathscr{C}}[v(k, \lambda_m^{\mathscr{C}})] + v(k, \lambda_m^{\mathscr{C}})). \end{aligned} \tag{EC17}$$

Now we focus on the last expression in (EC17). For $k+1 = c$, we obtain

$$\begin{aligned} &H_m^{\mathscr{C}}[v(k+1, \lambda_m^{\mathscr{C}})] - v(k+1, \lambda_m^{\mathscr{C}}) - H_m^{\mathscr{C}}[v(k, \lambda_m^{\mathscr{C}})] + v(k, \lambda_m^{\mathscr{C}}) \\ &\quad = -\max[0, v(k+1, \lambda_m^{\mathscr{C}}) - v(k, \lambda_m^{\mathscr{C}}) + \varphi_m] \le 0. \end{aligned} \tag{EC18}$$

For $k+1 < c$ we have

$$\begin{aligned} &H_m^{\mathscr{C}}[v(k+1, \lambda_m^{\mathscr{C}})] - H_m^{\mathscr{C}}[v(k, \lambda_m^{\mathscr{C}})] - v(k+1, \lambda_m^{\mathscr{C}}) + v(k, \lambda_m^{\mathscr{C}}) \\ &\quad = \max[0, v(k+2, \lambda_m^{\mathscr{C}}) - v(k+1, \lambda_m^{\mathscr{C}}) + \varphi_m] - \max[0, v(k+1, \lambda_m^{\mathscr{C}}) - v(k, \lambda_m^{\mathscr{C}}) + \varphi_m] \le 0, \end{aligned} \tag{EC19}$$

given the concavity of $v(k, \lambda_m^{\mathscr{C}})$. Thus, for any $k = 0, \ldots, c-1$, $v_1(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_1(k, \hat{\lambda}_m^{\mathscr{C}}) \le v(k+1, \lambda_m^{\mathscr{C}}) - v(k, \lambda_m^{\mathscr{C}})$.

To proceed with the induction, we need the following two intermediate results:

Lemma 3. *For any $k = 0, \ldots, c-1$,*

$$H_i^{\mathscr{C}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})] \le H_i^{\mathscr{C}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})]. \tag{EC20}$$

Proof. For $k = c-1$, consider the difference

$$\begin{aligned} &(H_i^{\mathscr{C}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})]) - (H_i^{\mathscr{C}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})]) \\ &\quad = (v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}})) - (v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})) \\ &\qquad - \max[0, v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i] + \max[0, v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i]. \end{aligned} \tag{EC21}$$

The induction assumption $v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}) \le v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})$ implies that there are three possible cases to consider in (EC21). First, if $v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i \ge v_K(k, \hat{\lambda}_m^{\mathscr{C}})$, then $v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i \ge v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})$ as well, and

$$(H_i^{\mathscr{C}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})]) - (H_i^{\mathscr{C}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})]) = 0. \tag{EC22}$$

Second, if $v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i \le v_K(k, \hat{\lambda}_m^{\mathscr{C}})$ and $v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i \ge v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})$, then

$$\begin{aligned} &(H_i^{\mathscr{C}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})]) - (H_i^{\mathscr{C}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})]) \\ &\quad = v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i \le 0. \end{aligned} \tag{EC23}$$

Finally, if $v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i \le v_K(k, \hat{\lambda}_m^{\mathscr{C}})$ and $v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i \le v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})$, we obtain

$$\begin{aligned} &(H_i^{\mathscr{C}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})]) - (H_i^{\mathscr{C}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})]) \\ &\quad = (v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}})) - (v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})) \le 0. \end{aligned} \tag{EC24}$$

For $k < c-1$, the right-hand side of (EC21) contains an extra term

$$\max[0, v_K(k+2, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i] - \max[0, v_{K-1}(k+2, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) + \varphi_i],$$

which is clearly nonpositive due to induction assumption. □

Lemma 4.

$$H_j^{\mathscr{W}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})] \le H_j^{\mathscr{W}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})] \quad \text{(EC25)}$$

for any $k = 0, \ldots, c-1$.

Proof. For $k = c-1$, we have

$$\begin{aligned}
&(H_j^{\mathscr{W}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})]) - (H_j^{\mathscr{W}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})])\\
&\quad = \big(v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}) - \max_l[p_{jl}(v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}) + w_l)]\big)\\
&\qquad - \big(v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}) - \max_l[p_{jl}(v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}) + w_l)]\big) \le 0, \quad \text{(EC26)}
\end{aligned}$$

where the final inequality follows from the induction assumption and the result of Lemma 2(b).
For $k < c-1$,

$$\begin{aligned}
&(H_j^{\mathscr{W}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})]) - (H_j^{\mathscr{W}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})])\\
&\quad = (v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}})) - (v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) + v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}))\\
&\qquad + \max_l[p_{jl}(v_K(k+2, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) + w_l)] - \max_l[p_{jl}(v_{K-1}(k+2, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) + w_l)]\\
&\qquad - \max_l[p_{jl}(v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}) + w_l)] + \max_l[p_{jl}(v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}) + w_l)]\\
&\quad \le (v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}})) - (v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}))\\
&\qquad - \max_l[p_{jl}(v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}) + w_l)] + \max_l[p_{jl}(v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}) + w_l)] \le 0. \quad \text{(EC27)}
\end{aligned}$$

where the first inequality follows from the induction assumption and Lemma 2(c) and the second is the same as (EC26). □

Now we proceed with the induction step. Assume that (EC16) is valid for all $n \le K$. Then,

$$\begin{aligned}
&v_{K+1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K+1}(k, \hat{\lambda}_m^{\mathscr{C}})\\
&\quad = T(\hat{\lambda}_m^{\mathscr{C}}) v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - T(\hat{\lambda}_m^{\mathscr{C}}) v_K(k, \hat{\lambda}_m^{\mathscr{C}})\\
&\quad = \sum_{i=1}^{N} \tilde{\lambda}_i^{\mathscr{C}} (H_i^{\mathscr{C}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})]) + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} (H_j^{\mathscr{W}}[v_K(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_K(k, \hat{\lambda}_m^{\mathscr{C}})])\\
&\qquad + \mu k(v_K(k, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k-1, \hat{\lambda}_m^{\mathscr{C}})) + \mu(c-(k+1))(v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}})), \quad \text{(EC28)}
\end{aligned}$$

where we denote $\tilde{\lambda}_i^{\mathscr{C}} = \lambda_i^{\mathscr{C}}$, $i \ne m$, $\tilde{\lambda}_m^{\mathscr{C}} = \hat{\lambda}_m^{\mathscr{C}}$. Then using (EC28), (EC20), and (EC25), we obtain

$$\begin{aligned}
&v_{K+1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K+1}(k, \hat{\lambda}_m^{\mathscr{C}})\\
&\quad \le \sum_{i=1}^{N} \tilde{\lambda}_i^{\mathscr{C}} (H_i^{\mathscr{C}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_i^{\mathscr{C}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})])\\
&\qquad + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} (H_j^{\mathscr{W}}[v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}})] - H_j^{\mathscr{W}}[v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})])\\
&\qquad + \mu k(v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k-1, \hat{\lambda}_m^{\mathscr{C}})) + \mu(c-(k+1))(v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}}))\\
&\quad = T(\hat{\lambda}_m^{\mathscr{C}}) v_{K-1}(k+1, \hat{\lambda}_m^{\mathscr{C}}) - T(\hat{\lambda}_m^{\mathscr{C}}) v_{K-1}(k, \hat{\lambda}_m^{\mathscr{C}})\\
&\quad = v_K(k+1, \hat{\lambda}_m^{\mathscr{C}}) - v_K(k, \hat{\lambda}_m^{\mathscr{C}}), \quad \text{(EC29)}
\end{aligned}$$

which completes the induction argument. Thus, the optimal thresholds for the contract classes are nonincreasing functions of contract demand intensities.

The proof of the monotonicity of contract thresholds with respect to walk-in demand intensities is analogous to the proof for contract demand intensities outlined above. The only difference lies in the proof of the first induction step, which we present below.

Suppose that a demand intensity for walk-in class s is changed from $\lambda_s^{\mathcal{W}}$ to $\hat{\lambda}_s^{\mathcal{W}} \geq \lambda_s^{\mathcal{W}}$. As before, we define by $T(\lambda_s^{\mathcal{W}})$ and $T(\hat{\lambda}_s^{\mathcal{W}})$ the dynamic programming operators in (EC14) and by $v(k, \lambda_s^{\mathcal{W}})$ and $v(k, \hat{\lambda}_s^{\mathcal{W}})$ the corresponding solutions to the MDP optimality equations. We also define the sequence of approximations for the optimal discounted profit function $v_n(k, \hat{\lambda}_s^{\mathcal{W}})$, $n \geq 0$, such that $v_n(k, \hat{\lambda}_s^{\mathcal{W}}) = T(\hat{\lambda}_s^{\mathcal{W}}) v_{n-1}(k, \hat{\lambda}_s^{\mathcal{W}})$, $n \geq 1$, $v_0(k, \hat{\lambda}_s^{\mathcal{W}}) = v(k, \lambda_s^{\mathcal{W}})$. For the first induction step, we obtain

$$v_1(k+1, \hat{\lambda}_s^{\mathcal{W}}) - v_1(k, \hat{\lambda}_s^{\mathcal{W}}) = v(k+1, \lambda_s^{\mathcal{W}}) - v(k, \lambda_s^{\mathcal{W}}) + (\hat{\lambda}_s^{\mathcal{W}} - \lambda_s^{\mathcal{W}})(H_s^{\mathcal{W}}[v(k+1, \lambda_s^{\mathcal{W}})] - H_s^{\mathcal{W}}[v(k, \lambda_s^{\mathcal{W}})] - v(k+1, \lambda_s^{\mathcal{W}}) + v(k, \lambda_s^{\mathcal{W}})). \quad \text{(EC30)}$$

For $k+1 = c$, we have

$$(H_s^{\mathcal{W}}[v(k+1, \lambda_s^{\mathcal{W}})] - H_s^{\mathcal{W}}[v(k, \lambda_s^{\mathcal{W}})]) - (v(k+1, \lambda_s^{\mathcal{W}}) - v(k, \lambda_s^{\mathcal{W}})) = -\max_l [p_{jl}(v(k+1, \lambda_s^{\mathcal{W}}) - v(k, \lambda_s^{\mathcal{W}}) + w_l)] \leq 0, \quad \text{(EC31)}$$

because $\max_l [p_{jl}(v(k+1, \lambda_s^{\mathcal{W}}) - v(k, \lambda_s^{\mathcal{W}}) + w_l)] \geq 0$. Similarly, for $k+1 < c$, we have

$$(H_s^{\mathcal{W}}[v(k+1, \lambda_s^{\mathcal{W}})] - H_s^{\mathcal{W}}[v(k, \lambda_s^{\mathcal{W}})]) - (v(k+1, \lambda_s^{\mathcal{W}}) - v(k, \lambda_s^{\mathcal{W}})) = \max_l [p_{jl}(v(k+2, \lambda_s^{\mathcal{W}}) - v(k+1, \lambda_s^{\mathcal{W}}) + w_l)] - \max_l [p_{jl}(v(k+1, \lambda_s^{\mathcal{W}}) - v(k, \lambda_s^{\mathcal{W}}) + w_l)] \leq 0. \quad \text{(EC32)}$$

Thus, $v_1(k+1, \hat{\lambda}_s^{\mathcal{W}}) - v_1(k, \hat{\lambda}_s^{\mathcal{W}}) \leq v(k+1, \lambda_s^{\mathcal{W}}) - v(k, \lambda_s^{\mathcal{W}})$.

To obtain the monotonicity result with respect to changes in the rental rate μ, we note that the induction result summarized in (EC29) can be reversed, i.e., using the same arguments, we can show that $v_K(k+1) - v_K(k) \geq v_{K-1}(k+1) - v_{K-1}(k)$ implies that $v_{K+1}(k+1) - v_{K+1}(k) \geq v_K(k+1) - v_K(k)$. (For simplicity, we have omitted the second argument in the profit functions here.) In particular, the results of Lemmas 3 and 4 will also be reversed. Replacing μ by $\hat{\mu} > \mu$, we define operators $T(\hat{\mu})$ and $T(\mu)$ and optimal profit functions $v(k, \hat{\mu})$ and $v(k, \mu)$. We then define a sequence of approximations for the optimal discounted profit function $v_n(k, \hat{\mu})$, $n \geq 0$ as $v_n(k, \hat{\mu}) = T(\hat{\mu}) v_{n-1}(k, \hat{\mu})$, $n \geq 1$, $v_0(k, \hat{\mu}) = v(k, \mu)$. Then,

$$v_1(k+1, \hat{\mu}) - v_1(k, \hat{\mu}) = v(k+1, \mu) - v(k, \mu) + (\hat{\mu} - \mu)((k+1)(v(k, \mu) - v(k+1, \mu)) - k(v(k-1, \mu) - v(k, \mu))). \quad \text{(EC33)}$$

Rearranging terms in the last expression on the right-hand side of (EC33), we obtain

$$(k+1)(v(k, \mu) - v(k+1, \mu)) - k(v(k-1, \mu) - v(k, \mu)) = k(2v(k, \mu) - v(k-1, \mu) - v(k+1, \mu)) + v(k, \mu) - v(k+1, \mu) \geq 0, \quad \text{(EC34)}$$

because $v(k)$ in nonincreasing and concave. Thus, $v_1(k+1, \hat{\mu}) - v_1(k, \hat{\mu}) \geq v_0(k+1, \hat{\mu}) - v_0(k, \hat{\mu})$, implying the monotonicity result in question.

We now focus on the effects of changes in contract fees and penalties on the optimal thresholds. First, we note that, without loss of generality, we can focus on a single penalty-adjusted fee, $\varphi_i = r_i + \pi_i$. For example, the accrual of penalty, π_i, when a customer is lost and revenue, r_i, when a customer is accepted is equivalent to "losing" π_i at every class i arrival and "gaining" $\varphi_i = r_i + \pi_i$ only on acceptance.

Next, we show that an increase in the penalty-adjusted fee for class i leads to more stringent control of admissions for other contract customers. As before, we denote by $T(\varphi_i)$ and $T(\hat{\varphi}_i)$ the operators in (EC14) for the penalty-adjusted fees φ_i and $\hat{\varphi}_i$ and by $v(k, \varphi_i)$ and $v(k, \hat{\varphi}_i)$ the corresponding solutions to the MDP optimality equations. Below we will show that $v(k+1, \hat{\varphi}_i) - v(k, \hat{\varphi}_i) \leq v(k+1, \varphi_i) - v(k, \varphi_i)$ for all $k = 0, \ldots, c-1$, which, in turn, will imply that the optimal thresholds for all contract classes other than i will either decrease or remain the same when class i fee increases.

We define the sequence of approximations for the optimal discounted profit function $v_n(k, \hat{\varphi}_i)$, $n \geq 0$, as $v_n(k, \hat{\varphi}_i) = T(\hat{\varphi}_i)v_{n-1}(k, \hat{\varphi}_i)$, $n \geq 1$, $v_0(k, \hat{\varphi}_i) = v(k, \varphi_i)$. Then using an induction argument that is an analogous to that in (EC29) we prove the result. In this case, the induction step is the same for all $n = 1, 2, \ldots,$ and here we show only $v_1(k+1, \hat{\varphi}_i) - v_1(k, \hat{\varphi}_i) \leq v_0(k+1, \hat{\varphi}_i) - v_0(k, \hat{\varphi}_i)$:

$$\begin{aligned}&(v_1(k+1, \hat{\varphi}_i) - v_1(k, \hat{\varphi}_i)) - (v(k+1, \varphi_i) - v(k, \varphi_i))\\ &\quad = (\lambda_i^{\mathscr{C}})(\max[0, v(k+2, \varphi_i) - v(k+1, \varphi_i) + \hat{\varphi}_i] - \max[0, v(k+1, \varphi_i) - v(k, \varphi_i) + \hat{\varphi}_i]) \leq 0, \end{aligned} \quad \text{(EC35)}$$

where the inequality follows from the concavity of $v(k)$.

Finally, it is intuitive that the acceptance threshold for class i customers would not decrease with an increase of φ_i. That is, given that all other problem parameters are fixed, the acceptance of class i customer paying φ_i while in state k implies the acceptance of the same customer if he or she pays $\hat{\varphi}_i > \varphi_i$.

Formally, we employ the same type of inductive argument used above to show that $v_n(k+1, \hat{\varphi}_i) - v_n(k, \hat{\varphi}_i) + \hat{\varphi}_i \geq v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_i$ for all $n \geq 1$ and all $0 \leq k \leq c-1$. Beginning with $v_0(k, \hat{\varphi}_i) \equiv v(k, \varphi_i)$, we have for $0 \leq k \leq c-2$

$$\begin{aligned} v_1(k+1, \hat{\varphi}_i) - v_1(k, \hat{\varphi}_i) + \hat{\varphi}_i &= T(\hat{\varphi}_i)v(k+1, \varphi_i) - T(\hat{\varphi}_i)v(k, \varphi_i) + \hat{\varphi}_i \\ &= v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_i + (\hat{\varphi}_i - \varphi_i) + \lambda_i^{\mathscr{C}}(\max[v(k+2, \varphi_i) + \hat{\varphi}_i, v(k+1, \varphi_i)] \\ &\qquad - \max[v(k+2, \varphi_i) + \varphi_i, v(k+1, \varphi_i)] - \max[v(k+1, \varphi_i) + \hat{\varphi}_i, v(k, \varphi_i)] \\ &\qquad + \max[v(k+1, \varphi_i) + \varphi_i, v(k, \varphi_i)]) \end{aligned}$$

Noting that $0 \leq \max[A + \epsilon, B] - \max[A, B] \leq \epsilon$ for $\epsilon > 0$, this further reduces to

$$\begin{aligned} v_1(k+1, \hat{\varphi}_i) - v_1(k, \hat{\varphi}_i) + \hat{\varphi}_i &\geq v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_i + \lambda_i^{\mathscr{C}}(0 - (\hat{\varphi}_i - \varphi_i)) + (\hat{\varphi}_i - \varphi_i) \\ &\geq v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_i. \end{aligned}$$

The same result for $k = c-1$ is established in a similar fashion.

Now we assume, by induction, that $v_n(k+1, \hat{\varphi}_i) - v_n(k, \hat{\varphi}_i) + \hat{\varphi}_i \geq v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_i$ for all $n \leq K$ and all $0 \leq k \leq c-1$. Then, by definition, we have

$$\begin{aligned} &v_{K+1}(k+1, \hat{\varphi}_i) - v_{K+1}(k, \hat{\varphi}_i) + \hat{\varphi}_i \\ &\quad = T(\hat{\varphi}_i)v_K(k+1, \hat{\varphi}_i) - T(\hat{\varphi}_i)v_K(k, \hat{\varphi}_i) + \hat{\varphi}_i \\ &\quad = \sum_{l=1}^{N} \lambda_l^{\mathscr{C}}(H_l^{\mathscr{C}}(\hat{\varphi}_i)[v_K(k+1, \hat{\varphi}_i)] - H_l^{\mathscr{C}}(\hat{\varphi}_i)[v_K(k, \hat{\varphi}_i)]) + \sum_{j=1}^{M} \lambda_l^{\mathscr{W}}(H_j^{\mathscr{W}}[v_K(k+1, \hat{\varphi}_i)] - H_j^{\mathscr{W}}[v_K(k, \hat{\varphi}_i)]) \\ &\qquad + \mu k(v_K(k, \hat{\varphi}_i) - v_K(k-1, \hat{\varphi}_i)) + \mu(c - (k+1))(v_K(k+1, \hat{\varphi}_i) - v_K(k, \hat{\varphi}_i)) + \hat{\varphi}_i. \end{aligned} \quad \text{(EC36)}$$

Here the argument to the operator, $H_l^{\mathscr{C}}$, serves to distinguish between operators that use different type i penalty-adjusted fees.

Next, we bound the differences of the operators, $H_l^{\mathscr{C}}(\hat{\varphi}_i)[\cdot]$ and $H_j^{\mathscr{W}}[\cdot]$ in terms of their analogs for systems with type i penalty-adjusted revenue, φ_i. In the deriving the bounds, we consider only cases in which $0 \leq k \leq c-2$, because proofs for the boundary case $k = c-1$ are similar.

For $l \neq i$, we have

$$\begin{aligned} &H_l^{\mathscr{C}}(\hat{\varphi}_i)[v_K(k+1, \hat{\varphi}_i)] - H_l^{\mathscr{C}}(\hat{\varphi}_i)[v_K(k, \hat{\varphi}_i)] \\ &\quad = v_K(k+1, \hat{\varphi}_i) - v_K(k, \hat{\varphi}_i) + \max[v_K(k+2, \hat{\varphi}_i) - v_K(k+1, \hat{\varphi}_i) + \varphi_l, 0] \\ &\qquad - \max[v_K(k+1, \hat{\varphi}_i) - v_K(k, \hat{\varphi}_i) + \varphi_l, 0] \\ &\quad \geq v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_i - \hat{\varphi}_i + \max[v(k+2, \varphi_i) - v(k+1, \varphi_i) + \varphi_i - \hat{\varphi}_i + \varphi_l, 0] \\ &\qquad - \max[v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_i - \hat{\varphi}_i + \varphi_l, 0] \\ &\quad \geq v(k+1, \varphi_i) - v(k, \varphi_i) + \max[v(k+2, \varphi_i) - v(k+1, \varphi_i) + \varphi_l, 0] \\ &\qquad - \max[v(k+1, \varphi_i) - v(k, \varphi_i) + \varphi_l, 0] + (\varphi_i - \hat{\varphi}_i) \\ &\quad = H_l^{\mathscr{C}}(\hat{\varphi}_i)[v_K(k+1, \hat{\varphi}_i)] - H_l^{\mathscr{C}}(\hat{\varphi}_i)[v_K(k, \hat{\varphi}_i)] + (\varphi_i - \hat{\varphi}_i). \end{aligned} \quad \text{(EC37)}$$

Here, the first inequality follows from the induction assumption and the fact that $A-\max[A+\varphi_l,0]$ is a nondecreasing function of A. The second inequality follows from the concavity of $v(k)$ and the fact that, for $A\ge B$, $\max[A+\varepsilon,0]-\max[B+\varepsilon,0]$ is increasing in ε.

Similarly, for $l=i$, we can directly obtain

$$\begin{aligned}&H_i^{\mathscr{C}}(\hat\varphi_i)[v_K(k+1,\hat\varphi_i)]-H_i^{\mathscr{C}}(\hat\varphi_i)[v_K(k,\hat\varphi_i)]\\&\quad\ge v(k+1,\varphi_i)-v(k,\varphi_i)+\varphi_i-\hat\varphi_i+\max[v(k+2,\varphi_i)-v(k+1,\varphi_i)+\varphi_i,0]\\&\qquad-\max[v(k+1,\varphi_i)-v(k,\varphi_i)+\varphi_i,0]\\&\quad=H_i^{\mathscr{C}}(\hat\varphi_i)[v_K(k+1,\hat\varphi_i)]-H_i^{\mathscr{C}}(\hat\varphi_i)[v_K(k,\hat\varphi_i)]+(\varphi_i-\hat\varphi_i).\end{aligned}\tag{EC38}$$

For $H_j^{\mathscr{W}}$ we have

$$\begin{aligned}&H_j^{\mathscr{W}}[v_K(k+1,\hat\varphi_i)]-H_j^{\mathscr{W}}[v_K(k,\hat\varphi_i)]\\&\quad=v_K(k+1,\hat\varphi_i)-v_K(k,\hat\varphi_i)+g_j(v_K(k+1,\hat\varphi_i)-v_K(k+2,\hat\varphi_i))-g_j(v_K(k,\hat\varphi_i)-v_K(k+1,\hat\varphi_i))\\&\quad\ge v(k+1,\varphi_i)-v(k,\varphi_i)+\varphi_i-\hat\varphi_i+g_j(v(k+1,\varphi_i)-v(k+2,\varphi_i)+\hat\varphi_i-\varphi_i)\\&\qquad-g_j(v(k,\varphi_i)-v(k+1,\varphi_i)+\hat\varphi_i-\varphi_i),\end{aligned}\tag{EC39}$$

where the inequality follows from the induction assumption and the fact that $g_j(A)$ and $h_j(A)$ are nonincreasing functions of A (see Lemma 2).

To complete the derivation of the bound, we need to demonstrate the convexity of $g_j(A)$, which we do as follows. Letting $l^*=l_j(\alpha A+(1-\alpha)B)$, we have

$$\begin{aligned}g_j(\alpha A+(1-\alpha)B)&=p_{jl^*}(-\alpha A-(1-\alpha)B+w_{l^*})\\&=\alpha p_{jl^*}(-A+w_{l^*})+(1-\alpha)p_{jl^*}(-B+w_{l^*})\le\alpha g_j(A)+(1-\alpha)g_j(B).\end{aligned}$$

Then from (EC39) and the convexity of $g_j(A)$, we have

$$\begin{aligned}&H_l^{\mathscr{W}}[v_K(k+1,\hat\varphi_i)]-H_l^{\mathscr{W}}[v_K(k,\hat\varphi_i)]\\&\quad\ge v(k+1,\varphi_i)-v(k,\varphi_i)+(\varphi_i-\hat\varphi_i)+g_l(v(k+1,\varphi_i)-v(k+2,\varphi_i))-g_l(v(k,\varphi_i)-v(k+1,\varphi_i))\\&\quad=H_l^{\mathscr{W}}[v_K(k+1,\varphi_i)]-H_l^{\mathscr{W}}[v_K(k,\varphi_i)]+(\varphi_i-\hat\varphi_i).\end{aligned}\tag{EC40}$$

Finally, combining (EC36), (EC37), (EC38), and (EC40), we obtain

$$\begin{aligned}&v_{K+1}(k+1,\hat\varphi_i)-v_{K+1}(k,\hat\varphi_i)+\hat\varphi_i\\&\quad\ge\sum_{l=1}^N\lambda_l^{\mathscr{C}}(H_l^{\mathscr{C}}(\varphi_i)[v(k+1,\varphi_i)]-H_l^{\mathscr{C}}(\varphi_i)[v(k,\varphi_i)]+(\varphi_i-\hat\varphi_i))\\&\qquad+\sum_{j=1}^M\lambda_j^{\mathscr{W}}(H_j^{\mathscr{W}}[v(k+1,\varphi_i)]-H_j^{\mathscr{W}}[v_K(k,\varphi_i)]+(\varphi_i-\hat\varphi_i))+\mu k(v_K(k,\hat\varphi_i)-v_K(k-1,\hat\varphi_i))\\&\qquad+\mu(c-(k+1))(v_K(k+1,\hat\varphi_i)-v_K(k,\hat\varphi_i))+\hat\varphi_i\\&\quad\ge\sum_{l=1}^N\lambda_l^{\mathscr{C}}(H_l^{\mathscr{C}}(\varphi_i)[v(k+1,\varphi_i)]-H_l^{\mathscr{C}}(\varphi_i)[v(k,\varphi_i)]+(\varphi_i-\hat\varphi_i))\\&\qquad+\sum_{j=1}^M\lambda_j^{\mathscr{W}}(H_j^{\mathscr{W}}[v(k+1,\varphi_i)]-H_j^{\mathscr{W}}[v_K(k,\varphi_i)]+(\varphi_i-\hat\varphi_i))+\mu k(v(k,\varphi_i)-v(k-1,\varphi_i)+(\varphi_i-\hat\varphi_i))\\&\qquad+\mu(c-(k+1))(v(k+1,\varphi_i)-v(k,\varphi_i)+(\varphi_i-\hat\varphi_i))+\hat\varphi_i.\end{aligned}\tag{EC41}$$

Here the second inequality follows from the induction assumption. Collecting terms and defining $\lambda=\sum_{l=1}^N\lambda_l^{\mathscr{C}}+\sum_{j=1}^M\lambda_j^{\mathscr{W}}$, we see that the right-hand side of (EC41) equals

$$\begin{aligned}&T(\varphi_i)v(k+1,\varphi_i)-T(\varphi_i)v(k,\varphi_i)+(\lambda+\mu c-\mu)(\varphi_i-\hat\varphi_i)+\hat\varphi_i\\&\quad=v(k+1,\varphi_i)-v(k,\varphi_i)+\Lambda\varphi_i+(1-\Lambda)\hat\varphi_i\ge v(k+1,\varphi_i)-v(k,\varphi_i)+\varphi_i,\end{aligned}$$

where $0<\Lambda=\lambda+\mu(c-1)<1$. This completes the proof.

Part (b)—From (7) we recall that

$$l_j(v(k+1)-v(k)) = \arg\max_l [p_{jl}(v(k+1)-v(k)+w_l)]$$

denotes the index of the optimal fee for class j walk-in customers for a given state of the system, k. In Part (a) of the theorem, we have shown that $v(k+1)-v(k)$ is a nonincreasing function of the contract and walk-in demand intensities as well as of the contract penalty-adjusted rental fees, and a non-decreasing function of the service rate μ. Combining this with the result of Lemma 2(a), we obtain the required monotonicity result.

EC.5. Proof of Theorem 3

The statement on the ordering of the optimal contract thresholds follows directly from the definition (9). The "walk-in" result is obtained by contraction, as follows. Suppose, by contradiction, that for walk-in classes j_1 and j_2, $w^*(j_1,k) < w^*(j_2,k)$ for some k, and define

$$l_1 = \arg\max_l (p_{j_1 l}(v(k+1)-v(k)+w_l)), \qquad l_2 = \arg\max_l (p_{j_2 l}(v(k+1)-v(k)+w_l)), \tag{EC42}$$

so that $w_{l_1} = w^*(j_1,k) < w^*(j_2,k) = w_{l_2}$. Then,

$$\begin{aligned} p_{j_1 l_1}(v(k+1)-v(k)+w_{l_1}) &\geq p_{j_1 l_2}(v(k+1)-v(k)+w_{l_2}),\\ p_{j_2 l_2}(v(k+1)-v(k)+w_{l_2}) &\geq p_{j_2 l_1}(v(k+1)-v(k)+w_{l_1}), \end{aligned} \tag{EC43}$$

and, because $p_{jl_1} > p_{jl_2}$ for $j = j_1, j_2$,

$$\frac{w_{l_1}p_{j_2 l_1} - w_{l_2}p_{j_2 l_2}}{p_{j_2 l_1} - p_{j_2 l_2}} \leq v(k) - v(k+1) \leq \frac{w_{l_1}p_{j_1 l_1} - w_{l_2}p_{j_1 l_2}}{p_{j_1 l_1} - p_{j_1 l_2}}, \tag{EC44}$$

or, equivalently,

$$\frac{p_{j_1 l_2}}{p_{j_2 l_2}} \leq \frac{p_{j_1 l_1}}{p_{j_2 l_1}}, \tag{EC45}$$

a contradiction. Hence, $w^*(j_1,k) \geq w^*(j_2,k)$ for all k.

EC.6. Proof of Lemma 1(b)

$$B(\alpha c, c) = \frac{(\alpha c)^c/c!}{\sum_{k=0}^{c}(\alpha c)^k/k!} = \frac{e^{-\alpha c}(\alpha c)^c}{c!P(X_{\alpha c} \leq c)}, \tag{EC46}$$

where $X_{\alpha c}$ is a Poisson random variable with parameter αc. Feller (1968) provides a bound for the factorial expression in the denominator of (EC46):

$$c! \geq \sqrt{2\pi c}\left(\frac{c}{e}\right)^c e^{1/(12c+1)}. \tag{EC47}$$

Also, using Chernoff bound on the Poisson probability (Ross 1996)

$$P(X_{\alpha c} \leq c) \geq 1 - (\alpha e^{1-\alpha})^c, \tag{EC48}$$

valid for $\alpha < 1$, we get the statement of the lemma.

EC.7. Proof of Theorem 4

Part (a)—Define

$$f_i(A) = A\mu + \sum_{n\neq i} \lambda_n^{\mathscr{C}}(A - \max(A, \varphi_n)) + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}}(A - \bar{w}_j(A))\bar{p}_j(A). \tag{EC49}$$

Then $A = G_i^{\mathscr{C}}(A)$ is equivalent to $f_i(A) = 0$, and we can show that $f_i(A)$ is strictly increasing in A. The first term of (EC49) is strictly increasing in A, and the second is nondecreasing in A. To see that the third is nondecreasing as well, note that, it follows from Lemma 2(b) that, for $A < B$,

$$(A - \overline{w}_j(A))\bar{p}_j(A) = -g_j(A) \le -g_j(B) = (B - \overline{w}_j(B))\bar{p}_j(B).$$

Using this fact, we can then demonstrate the desired properties of $G_i^{\mathscr{C}}(A)$.

First, there exists a solution to $A = G_i^{\mathscr{C}}(A)$. For $A = 0$, we have $f_i(A) < 0$, and for $A = \max(\max_n(\varphi_n), w_L)$, we have $f_i(A) > 0$. Together with the strictly increasing nature of $f_i(\cdot)$, this implies that $A = G_i^{\mathscr{C}}(A)$ has a unique solution, which we call A_i^*.

Second, the fact that $f_i(A)$ is strictly increasing in A also implies that $f_i(A) < 0$ for all $A < A_i^*$, and $f_i(A) > 0$ for all $A > A_i^*$. Algebraic manipulation shows that this is equivalent, $A < G_i^{\mathscr{C}}(A)$ for $A < A_i^*$, and $A > G_i^{\mathscr{C}}(A)$ for $A > A_i^*$.

For $\varphi_i \ge A_i^*$ we therefore have $\varphi_i \ge G_i^{\mathscr{C}}(\varphi_i)$, or

$$\varphi_i \ge \frac{\sum_{n\neq i} \lambda_n^{\mathscr{C}} \max(\varphi_i, \varphi_n) + \sum_{j=1}^M \lambda_j^{\mathscr{W}} \overline{w}_j(\varphi_i)\bar{p}_j(\varphi_i)}{\sum_{n\neq i} \lambda_n^{\mathscr{C}} + \sum_{j=1}^M \lambda_j^{\mathscr{W}} \bar{p}_j(\varphi_i) + \mu}. \tag{EC50}$$

Finally, we show that any φ_i that satisfies (EC50) is large enough for class i to be preferred. To do so, we consider the class of functions F_i defined on a set $k = 0, \ldots, c$, such that for every $f \in F_i$, $f(k+1) - f(k) \le f(k) - f(k-1)$, $k = 1, \ldots, c-1$, and $f(k) - f(k+1) \le \varphi_i$, $k = 0, \ldots, c-1$. We will show that F_i is closed under the operator T defined in (4). That is, for every $f \in F_i$, Tf also belongs to F_i. This will imply that the optimal profit function $v(k)$ also belongs to F_i, so that $v(k+1) - v(k) \ge -\varphi_i$, for $\forall\, k = 0, \ldots, c-1$ and $k_i^* = c$.

The fact that concavity of f is preserved under the action of T was proved in Theorem 1. Thus, we only need to show that $Tf(k) - Tf(k+1) \le \varphi_i$ for $k = c-1$. For any contract class n, we have

$$H_n^{\mathscr{C}}[f(c-1)] - H_n^{\mathscr{C}}[f(c)] = \max[f(c-1) - f(c), \varphi_n] \le \max[\varphi_i, \varphi_n] \tag{EC51}$$

Further, using the result of Lemma 2(b), we obtain:

$$\begin{aligned} H_j^{\mathscr{W}}[f(c-1)] - H_j^{\mathscr{W}}[f(c)] &= f(c-1) - f(c) + \max_l [p_{jl}(f(c) - f(c-1) + w_l)] \\ &\le (1 - \bar{p}_j(\varphi_i))\varphi_i + \overline{w}_j(\varphi_i)\bar{p}_j(\varphi_i). \end{aligned} \tag{EC52}$$

Finally,

$$\mu(c-1)f(c-2) + \mu f(c-1) - \mu c f(c-1) = \mu(c-1)(f(c-2) - f(c-1)) \le \mu(c-1)\varphi_i^{\mathscr{C}}. \tag{EC53}$$

Recall that the N contract classes are labelled so that $\varphi_1 \ge \varphi_2 \cdots \ge \varphi_N$. Then combining (EC51), (EC52), and (EC53), we obtain

$$\begin{aligned} Tf(c-1) - Tf(c) &\le \sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} \varphi_n + \sum_{n=i}^{N} \lambda_n^{\mathscr{C}} \varphi_i + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} ((1 - \bar{p}_j(\varphi_i))\varphi_i + \overline{w}_j(\varphi_i)\bar{p}_j(\varphi_i)) + \mu(c-1)\varphi_i \\ &\le \varphi_i + \sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} \varphi_n + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} \overline{w}_j(\varphi_i)\bar{p}_j(\varphi_i) - \varphi_i \left(\sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} \bar{p}_j(\varphi_i) + \mu \right) \\ &\le \varphi_i, \end{aligned} \tag{EC54}$$

where the last inequality follows from the theorem's assumption, (25).

Part (b)—The proof follows that of Part (a). We use the same argument to establish the existence of the unique solution to $A = G^{\mathscr{W}}(A)$, A^*, as well as the other required properties of $G^{\mathscr{W}}(A)$.

We then demonstrate that

$$\hat{r}_j \ge G_j(\hat{r}_j) = \frac{\sum_{i=1}^N \lambda_i^{\mathscr{C}} \max(\hat{r}_j, \varphi_i) + \sum_{m=1}^M \lambda_m^{\mathscr{W}} \overline{w}_m(\hat{r}_j)\bar{p}_m(\hat{r}_j)}{\sum_{i=1}^N \lambda_i^{\mathscr{C}} + \sum_{m=1}^M \lambda_m^{\mathscr{W}} \bar{p}_m(\hat{r}_j) + \mu} \tag{EC55}$$

is sufficient for class j to be preferred.

Here we consider a class of functions F_j defined on a set $k=0,\ldots,c$, such that for every $f\in F_j$, $f(k+1)-f(k)\le f(k)-f(k-1)$, $k=1,\ldots,c-1$, and $f(k)-f(k+1)\le \hat{r}_j$, $k=0,\ldots,c-1$. Then, for a contract class i, we obtain

$$H_i^{\mathscr{C}}[f(c-1)]-H_i^{\mathscr{C}}[f(c)]=\max[f(c-1)-f(c),\varphi_i]\le\max[\varphi_i,\hat{r}_j]. \tag{EC56}$$

For a walk-in class m, we have

$$\begin{aligned}H_m^{\mathscr{W}}[f(c-1)]-H_m^{\mathscr{W}}[f(c)] &= f(c-1)-f(c)+\max_l[p_{ml}(f(c)-f(c-1)+w_l)]\\ &\le (1-\bar{p}_m(\hat{r}_j))\hat{r}_j+\overline{w}_m(\hat{r}_j)\bar{p}_m(\hat{r}_j),\end{aligned} \tag{EC57}$$

and the analog to (EC53) is

$$\mu(c-1)f(c-2)+\mu f(c-1)-\mu c f(c-1)\le\mu(c-1)\hat{r}_j. \tag{EC58}$$

Then inequalities that are analogs to those leading to (EC54) imply that $Tf(c-1)-Tf(c)\le\hat{r}_j$.

EC.8. Proof of Theorem 5

Part (a)—Similar to the proof of Theorem 1, we look at a class $\widetilde{F}$ of nonincreasing submodular functions defined on $\tilde{S}$. Consider a function $f\in\widetilde{F}$. The MDP transformation operator $\widetilde{T}$ can be defined as

$$\widetilde{T}f(k_{\mathscr{C}},k_{\mathscr{W}})=\sum_{i=1}^{N}\lambda_i^{\mathscr{C}}H_i^{\mathscr{C}}[f(k_{\mathscr{C}},k_{\mathscr{W}})]+\sum_{j=1}^{M}\lambda_j^{\mathscr{W}}H_j^{\mathscr{W}}[f(k_{\mathscr{C}},k_{\mathscr{W}})]+H(k_{\mathscr{C}},k_{\mathscr{W}}), \tag{EC59}$$

where

$$H(k_{\mathscr{C}},k_{\mathscr{W}})=\mu_{\mathscr{C}}k_{\mathscr{C}}f(k_{\mathscr{C}}-1,k_{\mathscr{W}})+\mu_{\mathscr{W}}k_{\mathscr{W}}f(k_{\mathscr{C}},k_{\mathscr{W}}-1)+(\mu_{\mathscr{C}}(c-k_{\mathscr{C}})+\mu_{\mathscr{W}}(c-k_{\mathscr{W}}))v(k_{\mathscr{C}},k_{\mathscr{W}}).$$

Below we will show that $H_i^{\mathscr{C}}[f(k_{\mathscr{C}},k_{\mathscr{W}})]$, $H_j^{\mathscr{W}}[f(k_{\mathscr{C}},k_{\mathscr{W}})]$, and $H(k_{\mathscr{C}},k_{\mathscr{W}})$ all belong to $\widetilde{F}$.

We start with $H(k_{\mathscr{C}},k_{\mathscr{W}})$. For any $(k_{\mathscr{C}},k_{\mathscr{W}}),(k_{\mathscr{C}}+1,k_{\mathscr{W}})\in\tilde{S}$,

$$\begin{aligned}H(k_{\mathscr{C}}+1,k_{\mathscr{W}})-H(k_{\mathscr{C}},k_{\mathscr{W}}) &= \mu_{\mathscr{C}}k_{\mathscr{C}}(f(k_{\mathscr{C}},k_{\mathscr{W}})-f(k_{\mathscr{C}}-1,k_{\mathscr{W}}))+\mu_{\mathscr{W}}k_{\mathscr{W}}(f(k_{\mathscr{C}}+1,k_{\mathscr{W}}-1)-f(k_{\mathscr{C}},k_{\mathscr{W}}-1))\\ &\quad+(\mu_{\mathscr{C}}(c-k_{\mathscr{C}}-1)+\mu_{\mathscr{W}}(c-k_{\mathscr{W}}))(f(k_{\mathscr{C}}+1,k_{\mathscr{W}})-f(k_{\mathscr{C}},k_{\mathscr{W}}))\le 0.\end{aligned} \tag{EC60}$$

Thus $H(k_{\mathscr{C}},k_{\mathscr{W}})$ is nonincreasing in $k_{\mathscr{C}}$. Similarly, $H(k_{\mathscr{C}},k_{\mathscr{W}})$ is nonincreasing in $k_{\mathscr{W}}$:

$$\begin{aligned}H(k_{\mathscr{C}},k_{\mathscr{W}}+1)-H(k_{\mathscr{C}},k_{\mathscr{W}}) &= \mu_{\mathscr{C}}k_{\mathscr{C}}(f(k_{\mathscr{C}}-1,k_{\mathscr{W}}+1)-f(k_{\mathscr{C}}-1,k_{\mathscr{W}}))\\ &\quad+\mu_{\mathscr{W}}k_{\mathscr{W}}(f(k_{\mathscr{C}},k_{\mathscr{W}})-f(k_{\mathscr{C}},k_{\mathscr{W}}-1))\\ &\quad+(\mu_{\mathscr{C}}(c-k_{\mathscr{C}})+\mu_{\mathscr{W}}(c-k_{\mathscr{W}}-1))(f(k_{\mathscr{C}},k_{\mathscr{W}}+1)-f(k_{\mathscr{C}},k_{\mathscr{W}}))\le 0.\end{aligned} \tag{EC61}$$

Finally, for submodularity we have

$$\begin{aligned}&(H(k_{\mathscr{C}}+1,k_{\mathscr{W}}+1)-H(k_{\mathscr{C}},k_{\mathscr{W}}+1))-(H(k_{\mathscr{C}}+1,k_{\mathscr{W}})+H(k_{\mathscr{C}},k_{\mathscr{W}}))\\ &\quad=\mu_{\mathscr{C}}k_{\mathscr{C}}((f(k_{\mathscr{C}},k_{\mathscr{W}}+1)-f(k_{\mathscr{C}}-1,k_{\mathscr{W}}+1))-(f(k_{\mathscr{C}},k_{\mathscr{W}})-f(k_{\mathscr{C}}-1,k_{\mathscr{W}})))\\ &\qquad+\mu_{\mathscr{W}}k_{\mathscr{W}}((f(k_{\mathscr{C}}+1,k_{\mathscr{W}})-f(k_{\mathscr{C}},k_{\mathscr{W}}))-(f(k_{\mathscr{C}}+1,k_{\mathscr{W}}-1)-f(k_{\mathscr{C}},k_{\mathscr{W}}-1)))\\ &\qquad+(\mu_{\mathscr{C}}(c-k_{\mathscr{C}}-1)+\mu_{\mathscr{W}}(c-k_{\mathscr{W}}-1))\times((f(k_{\mathscr{C}}+1,k_{\mathscr{W}}+1)\\ &\qquad-f(k_{\mathscr{C}},k_{\mathscr{W}}+1))-(f(k_{\mathscr{C}}+1,k_{\mathscr{W}})-f(k_{\mathscr{C}},k_{\mathscr{W}})))\le 0.\end{aligned} \tag{EC62}$$

Next, we consider $H_i^{\mathscr{C}}[f(k_{\mathscr{C}},k_{\mathscr{W}})]$ for any $i=1,\ldots,N$. For increases in $k_{\mathscr{C}}$, we have two cases. Given $k_{\mathscr{C}}+k_{\mathscr{W}}=c-1$,

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1,k_{\mathscr{W}})]-H_i^{\mathscr{C}}[f(k_{\mathscr{C}},k_{\mathscr{W}})]=f(k_{\mathscr{C}}+1,k_{\mathscr{W}})-\max[f(k_{\mathscr{C}}+1,k_{\mathscr{W}})+\varphi_i,f(k_{\mathscr{C}},k_{\mathscr{W}})]\le 0, \tag{EC63}$$

where $\varphi_i = r_i + \pi_i$. If $k_{\mathscr{C}} + k_{\mathscr{W}} < c-1$ then

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] = \max[f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] \\ - \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}})]. \qquad \text{(EC64)}$$

The case of $f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) < -\varphi_i$ reduces to (EC63), and for $f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) \geq -\varphi_i$ we have

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] = f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) + \varphi_i - \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}})] \\ \leq f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) \leq 0. \qquad \text{(EC65)}$$

The cases are the same for increases in $k_{\mathscr{W}}$. For $k_{\mathscr{C}} + k_{\mathscr{W}} = c-1$,

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] = f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}})] \\ \leq f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) \leq 0, \qquad \text{(EC66)}$$

and, for $k_{\mathscr{C}} + k_{\mathscr{W}} < c-1$,

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] = \max[f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ - \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}})] \leq 0, \qquad \text{(EC67)}$$

no matter which of $f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) + \varphi_i$ or $f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)$ is greater in the first maximization operator.

Now we turn to the proof of submodularity of $H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})]$. As before, we first look at the boundary case of $k_{\mathscr{C}} + k_{\mathscr{W}} = c-2$:

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)] + H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ = (f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - \max[f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}+1, k_{\mathscr{W}})]) \\ + (\max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}})] - \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)]). \qquad \text{(EC68)}$$

Because $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ is nonincreasing in $k_{\mathscr{C}}$, the first difference must be less than or equal to zero. Similarly, the fact that $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ is nonincreasing in both $k_{\mathscr{C}}$ and $k_{\mathscr{W}}$ implies that the second is as well.

For states with $k_{\mathscr{C}} + k_{\mathscr{W}} < c-2$, we have

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)] + H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ = \max[f(k_{\mathscr{C}}+2, k_{\mathscr{W}}+1) + \varphi_i, f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)] + \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}})] \\ - \max[f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)]. \qquad \text{(EC69)}$$

Note that in the case of $f(k_{\mathscr{C}}+2, k_{\mathscr{W}}+1) + \varphi_i \leq f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)$ (EC69) reduces to (EC68). On the other hand, $f(k_{\mathscr{C}}+2, k_{\mathscr{W}}+1) + \varphi_i > f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)$ implies, due to submodularity of f, that $f(k_{\mathscr{C}}+2, k_{\mathscr{W}}) + \varphi_i > f(k_{\mathscr{C}}+1, k_{\mathscr{W}})$. Then, (EC69) becomes

$$H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)] + H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ = (f(k_{\mathscr{C}}+2, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+2, k_{\mathscr{W}})) - (\max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ - \max[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \varphi_i, f(k_{\mathscr{C}}, k_{\mathscr{W}})]). \qquad \text{(EC70)}$$

Again, the fact that $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ is nonincreasing in both arguments implies that each of the two differences is less than or equal to zero.

Now, consider $H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})]$ for any $j = 1, \ldots, M$. First, we show that $H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})]$ is nonincreasing in $k_{\mathscr{C}}$. For $k_{\mathscr{C}} + k_{\mathscr{W}} = c-1$, we have

$$H_j^{\mathscr{W}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] = f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) \\ - \max_l[p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)] \leq 0, \qquad \text{(EC71)}$$

because both expressions on the right-hand side are nonpositive. For $k_{\mathscr{C}} + k_{\mathscr{W}} < c-1$, we have

$$\begin{aligned} &H_j^{\mathscr{W}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] \\ &\quad = (f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}})) + \Big(\max_l [p_{jl}(f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + w_l)] \\ &\qquad - \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)]\Big) \le 0 \end{aligned} \tag{EC72}$$

because, once again, the fact that $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ in nonincreasing in both arguments implies that both differences on the right-hand side are nonpositive. The result for the second difference follows, in particular, from Lemma 2(c).

Similarly, we show that $H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})]$ is nonincreasing in $k_{\mathscr{W}}$. For $k_{\mathscr{C}} + k_{\mathscr{W}} = c-1$, the monotonicity of $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ with respect to $k_{\mathscr{W}}$ implies

$$\begin{aligned} H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] &= f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) - \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)] \\ &\le 0. \end{aligned} \tag{EC73}$$

For $k_{\mathscr{C}} + k_{\mathscr{W}} < c-1$, we let

$$p_{jl^*} = \arg\max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) + w_l) + (1-p_{jl}) f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)].$$

Then

$$\begin{aligned} H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] &= \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) + w_l) + (1-p_{jl}) f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ &= p_{jl^*}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) + w_{l^*}) + (1-p_{jl^*}) f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) \\ &\le p_{jl^*}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) + w_{l^*}) + (1-p_{jl^*}) f(k_{\mathscr{C}}, k_{\mathscr{W}}) \\ &\le \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) + w_l) + (1-p_{jl}) f(k_{\mathscr{C}}, k_{\mathscr{W}})] \\ &= H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})], \end{aligned} \tag{EC74}$$

where the first inequality is implied by the fact that $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ is nonincreasing in $k_{\mathscr{W}}$.

Now we demonstrate the submodularity of $H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})]$. For $k_{\mathscr{C}} + k_{\mathscr{W}} = c-2$,

$$\begin{aligned} &H_j^{\mathscr{W}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)] + H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ &\quad = f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) + f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) \\ &\qquad + \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)] - \max_l [p_{jl}(f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + w_l)] \\ &\qquad - \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) + w_l)] \\ &\quad \le (f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - \max_l [p_{jl}(f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + w_l)]) \\ &\qquad - (f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) - \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)]) \le 0. \end{aligned} \tag{EC75}$$

Here the first inequality follows from the fact that $p_{jL} = 0$, so that

$$\max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) + w_l)] \ge 0.$$

The second inequality follows from the submodularity of $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ and Lemma 2(b). Finally, for $k_{\mathscr{C}} + k_{\mathscr{W}} < c-2$, we have

$$\begin{aligned} &H_j^{\mathscr{W}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1)] + H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] \\ &\quad = f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) + f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) \\ &\qquad + \max_l [p_{jl}(f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+2) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) + w_l)] - \max_l [p_{jl}(f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + w_l)] \\ &\qquad - \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) + w_l)] + \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)] \\ &\quad \le f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - \max_l [p_{jl}(f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + w_l)] \\ &\qquad - (f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) - \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)]) \le 0. \end{aligned} \tag{EC76}$$

Here, the first inequality is implied by the fact that

$$\max_l [p_{jl}(f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+2) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) + w_l)] \le \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+2) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) + w_l)],$$

which in turn follows from the submodularity of $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ and Lemma 2(b). The second inequality follows from the submodularity of $f(k_{\mathscr{C}}, k_{\mathscr{W}})$ and Lemma 2(c).

Part (b)—The fact that the class of submodular functions is closed under the action $\widetilde{T}$ implies that the optimal profit function $v(k_{\mathscr{C}}, k_{\mathscr{W}})$ is also submodular (Porteus 1982). Then, defining

$$k_i^*(k_{\mathscr{C}}) = \begin{cases} c - k_{\mathscr{C}}, & \text{if } v(k_{\mathscr{C}}, c-k_{\mathscr{C}}) - v(k_{\mathscr{C}}, c-k_{\mathscr{C}}-1) \ge -\varphi_i, \\ \min(k_{\mathscr{W}} \mid v(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) < -\varphi_i), & \text{if } v(k_{\mathscr{C}}, c-k_{\mathscr{C}}) - v(k_{\mathscr{C}}, c-k_{\mathscr{C}}-1) < -\varphi_i, \end{cases} \tag{EC77}$$

for $i = 1, \ldots, N$ and $k_{\mathscr{C}} = 0, \ldots, c-1$, we observe that $H_i^{\mathscr{C}}[v(k_{\mathscr{C}}, k_{\mathscr{W}})] = v(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + r_i$ for $k_{\mathscr{W}} < k_i^*(k_{\mathscr{C}})$, and $H_i^{\mathscr{C}}[v(k_{\mathscr{C}}, k_{\mathscr{W}})] = v(k_{\mathscr{C}}, k_{\mathscr{W}}) - \pi_i$ for $k_{\mathscr{W}} \ge k_i^*(k_{\mathscr{C}})$. Thus, class i contract customers are admitted into service if and only if $k_{\mathscr{W}} < k_i^*(k_{\mathscr{C}})$.

Part (c)—For given state of the rental system $(k_{\mathscr{C}}, k_{\mathscr{W}})$, the optimal fee to charge for a walk-in service of class j customer corresponds to a price index that is analogous to (7):

$$l_j(k_{\mathscr{C}}, k_{\mathscr{W}}) = \arg\max_l [p_{jl}(v(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)]. \tag{EC78}$$

From Lemma 2(a), we know that these indices form a monotone sequence: $l_j(k_{\mathscr{C}}+1, k_{\mathscr{W}}) \ge l_j(k_{\mathscr{C}}, k_{\mathscr{W}})$, so that $w^*(j, k_{\mathscr{C}}+1, k_{\mathscr{W}}) = w_{l_j(k_{\mathscr{C}}, k_{\mathscr{W}})} \ge w_{l_j(k_{\mathscr{C}}, k_{\mathscr{W}})} = w^*(j, k_{\mathscr{C}}, k_{\mathscr{W}})$, for $(k_{\mathscr{C}}, k_{\mathscr{W}}), (k_{\mathscr{C}}+1, k_{\mathscr{W}}) \in \tilde{S}$.

Now, suppose that there exists a state $(k_{\mathscr{C}}, k_{\mathscr{W}})$ and an index j such that $l_j(k_{\mathscr{C}}, k_{\mathscr{W}}) < e_j$. From the optimality of $l_j(k_{\mathscr{C}}, k_{\mathscr{W}})$, we have

$$p_{j, l_j(k_{\mathscr{C}}, k_{\mathscr{W}})}(v(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_{l_j(k_{\mathscr{C}}, k_{\mathscr{W}})}) \ge p_{j, e_j}(v(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_{e_j}), \tag{EC79}$$

so that

$$v(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) \ge \frac{w_{e_j} p_{j, e_j} - w_{l_j(k_{\mathscr{C}}, k_{\mathscr{W}})} p_{j, l_j(k_{\mathscr{C}}, k_{\mathscr{W}})}}{p_{j, l_j(k_{\mathscr{C}}, k_{\mathscr{W}})} - p_{j, e_j}} > 0, \tag{EC80}$$

a contradiction with the monotonicity results in Part (a). Thus, $l_j(k_{\mathscr{C}}, k_{\mathscr{W}}) \ge e_j$ for all $(k_{\mathscr{C}}, k_{\mathscr{W}}) \in \tilde{S}$ and j.

Part (d)—The statement on the ordering of the optimal contract thresholds follows from the definition (EC77). The walk-in result is obtained as follows. Suppose that for some $(k_{\mathscr{C}}, k_{\mathscr{W}}) \in \tilde{S}$, $w^*(j_1, k_{\mathscr{C}}, k_{\mathscr{W}}) < w^*(j_2, k_{\mathscr{C}}, k_{\mathscr{W}})$. Define

$$\begin{aligned} l_1 &= \arg\max_l (p_{j_1 l}(v(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)), \\ l_2 &= \arg\max_l (p_{j_2 l}(v(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)), \end{aligned} \tag{EC81}$$

so that $w^*(j_1, k_{\mathscr{C}}, k_{\mathscr{W}}) = w_{l_1}$, $w^*(j_2, k_{\mathscr{C}}, k_{\mathscr{W}}) = w_{l_2}$. Then, the same argument used to prove Theorem 3 holds here as well.

EC.9. Proof of Theorem 6

The proof follows along the lines of that of Theorem 4. Arguments for the uniqueness of the solutions, as well as the inequalities $\varphi_i \ge \widetilde{G}_i^{\mathscr{C}}(\varphi_i)$ and $\hat{r}_j \ge \widetilde{G}^{\mathscr{W}}(\hat{r}_j)$ are direct analogues. Here we concentrate on sufficient conditions for preferred customer classes.

Part (a)—We consider a class of functions F_i defined on $\tilde{S}$, such that for every $f \in F_i$,

$$f(k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) \le f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}), \tag{EC82}$$

and

$$f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) \le \varphi_i \tag{EC83}$$

for $(k_{\mathscr{C}}, k_{\mathscr{W}}), (k_{\mathscr{C}}+1, k_{\mathscr{W}}), (k_{\mathscr{C}}, k_{\mathscr{W}}+1), (k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) \in \tilde{S}$. Below we will show that, given (35) holds, F_i is closed under $\widetilde{T}$ defined in (EC59). That is, for every $f \in F_i$, $\widetilde{T} f$ also belongs to F_i. This will imply that the optimal profit function $v(k_{\mathscr{C}}, k_{\mathscr{W}})$ also belongs to F_i, so that $v(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - v(k_{\mathscr{C}}, k_{\mathscr{W}}) \geq -\varphi_i$, and $k_i^*(k_{\mathscr{C}}) + k_{\mathscr{C}} = c$.

The fact that the submodularity of f is preserved under the action of $\widetilde{T}$ was proved in Theorem 5. Thus, we only need to show that $\widetilde{T}f(k_{\mathscr{C}}, k_{\mathscr{W}}) - \widetilde{T}f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) \leq \varphi_i$ for all $(k_{\mathscr{C}}, k_{\mathscr{W}})$ such that $k_{\mathscr{C}} + k_{\mathscr{W}} = c-1$. If the property holds in the "border" states, then submodularity implies that it will hold in states with $k_{\mathscr{C}} + k_{\mathscr{W}} < c-1$ as well.

Therefore, $k_{\mathscr{C}} + k_{\mathscr{W}} = c-1$ and for any contract class n, we have

$$\begin{aligned} H_n^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_n^{\mathscr{C}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] &= \max[f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}), \varphi_n] \\ &\leq \max[\varphi_i, \varphi_n] \end{aligned} \tag{EC84}$$

by (EC83). Similarly,

$$\begin{aligned} H_j^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_j^{\mathscr{W}}[f(k_{\mathscr{C}}+1, k_{\mathscr{W}})] &= f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) + \max_l [p_{jl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)] \\ &\leq \varphi_i + w_{e_j} p_{j,e_j}. \end{aligned} \tag{EC85}$$

Finally, we have

$$\begin{aligned} &H(k_{\mathscr{C}}, k_{\mathscr{W}}) - H(k_{\mathscr{C}}+1, k_{\mathscr{W}}) \\ &\quad = \mu_{\mathscr{C}} k_{\mathscr{C}} f(k_{\mathscr{C}}-1, k_{\mathscr{W}}) + \mu_{\mathscr{W}} k_{\mathscr{W}} f(k_{\mathscr{C}}, k_{\mathscr{W}}-1) + ((\mu_{\mathscr{C}} + \mu_{\mathscr{W}})c - \mu_{\mathscr{C}} k_{\mathscr{C}} - \mu_{\mathscr{W}} k_{\mathscr{W}}) f(k_{\mathscr{C}}, k_{\mathscr{W}}) \\ &\qquad - \mu_{\mathscr{C}}(k_{\mathscr{C}}+1) f(k_{\mathscr{C}}, k_{\mathscr{W}}) - \mu_{\mathscr{W}} k_{\mathscr{W}} f(k_{\mathscr{C}}+1, k_{\mathscr{W}}-1) - ((\mu_{\mathscr{C}} + \mu_{\mathscr{W}})c - \mu_{\mathscr{C}}(k_{\mathscr{C}}+1) - \mu_{\mathscr{W}} k_{\mathscr{W}}) f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) \\ &\quad = \mu_{\mathscr{C}} k_{\mathscr{C}} (f(k_{\mathscr{C}}-1, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}})) + \mu_{\mathscr{W}} k_{\mathscr{W}} (f(k_{\mathscr{C}}, k_{\mathscr{W}}-1) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}}-1)) \\ &\qquad + ((\mu_{\mathscr{C}} + \mu_{\mathscr{W}})c - \mu_{\mathscr{C}}(k_{\mathscr{C}}+1) - \mu_{\mathscr{W}} k_{\mathscr{W}})(f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}+1, k_{\mathscr{W}})) \\ &\quad \leq (\mu_{\mathscr{C}} + \mu_{\mathscr{W}}) c \varphi_i - \mu_{\mathscr{C}} \varphi_i. \end{aligned} \tag{EC86}$$

Combining (EC84), (EC85), and (EC86), we obtain

$$\begin{aligned} Tf(k_{\mathscr{C}}, k_{\mathscr{W}}) - Tf(k_{\mathscr{C}}+1, k_{\mathscr{W}}) &\leq \sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} \varphi_n + \sum_{n=i}^{N} \lambda_n^{\mathscr{C}} \varphi_i + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} \varphi_i + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} w_{e_j} p_{j,e_j} + (\mu_{\mathscr{C}} + \mu_{\mathscr{W}}) c \varphi_i - \mu_{\mathscr{C}} \varphi_i \\ &\leq \varphi_i + \sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} \varphi_n + \sum_{j=1}^{M} \lambda_j^{\mathscr{W}} w_{e_j} p_{j,e_j} - \varphi_i \left(\sum_{n=1}^{i-1} \lambda_n^{\mathscr{C}} + \mu_{\mathscr{C}} \right) \\ &\leq \varphi_i, \end{aligned} \tag{EC87}$$

where the last inequality follows from (35).

Part (b)—As in Part (a), we consider a class of functions F_j defined on the set $\tilde{S}$, such that for every $f \in F_j$, (EC82) and

$$f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) \leq \hat{r}_j \tag{EC88}$$

hold for $(k_{\mathscr{C}}, k_{\mathscr{W}}), (k_{\mathscr{C}}+1, k_{\mathscr{W}}), (k_{\mathscr{C}}, k_{\mathscr{W}}+1), (k_{\mathscr{C}}+1, k_{\mathscr{W}}+1) \in \tilde{S}$. Then, for a contract class i and boundary states $(k_{\mathscr{C}}, k_{\mathscr{W}})$ such that $k_{\mathscr{C}} + k_{\mathscr{W}} = c-1$ we have

$$\begin{aligned} H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_i^{\mathscr{C}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] &= \max[0, f(k_{\mathscr{C}}+1, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + \varphi_i] + f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) \\ &\leq \varphi_i + \hat{r}_j. \end{aligned} \tag{EC89}$$

For walk-in class n, we have

$$\begin{aligned} H_n^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}})] - H_n^{\mathscr{W}}[f(k_{\mathscr{C}}, k_{\mathscr{W}}+1)] &= f(k_{\mathscr{C}}, k_{\mathscr{W}}) - f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) + \max_l [p_{nl}(f(k_{\mathscr{C}}, k_{\mathscr{W}}+1) - f(k_{\mathscr{C}}, k_{\mathscr{W}}) + w_l)] \\ &\leq (1 - \bar{p}_n(\hat{r}_j)) \hat{r}_j + \overline{w}_n(\hat{r}_j) \bar{p}_n(\hat{r}_j), \end{aligned} \tag{EC90}$$

where we have used the result of Lemma 2(b). As in (EC86), we also have

$$H(k_{\mathscr{C}}, k_{\mathscr{W}}) - H(k_{\mathscr{C}}, k_{\mathscr{W}}+1) \leq (\mu_{\mathscr{C}} + \mu_{\mathscr{W}}) c \hat{r}_j - \mu_{\mathscr{W}} \hat{r}_j. \tag{EC91}$$

Then, together (EC89), (EC90), and (EC91) imply

$$\begin{aligned} Tf(k_{\mathscr{C}},k_{\mathscr{W}})-Tf(k_{\mathscr{C}},k_{\mathscr{W}}+1) &\le \sum_{i=1}^{N}\lambda_i^{\mathscr{C}}\varphi_i+\sum_{i=1}^{N}\lambda_i^{\mathscr{C}}\hat{r}_j+\sum_{n=1}^{M}\lambda_n^{\mathscr{W}}\hat{r}_j+\sum_{n=1}^{M}\lambda_n^{\mathscr{W}}(\overline{w}_n(\hat{r}_j)-\hat{r}_j)\bar{p}_n(\hat{r}_j)+(\mu_{\mathscr{C}}+\mu_{\mathscr{W}})c-\mu_{\mathscr{W}}\hat{r}_j \\ &\le \hat{r}_j+\sum_{i=1}^{N}\lambda_i^{\mathscr{C}}\varphi_i+\sum_{n=1}^{M}\lambda_n^{\mathscr{W}}\overline{w}_n(\hat{r}_j)\bar{p}_n(\hat{r}_j)-\hat{r}_j\left(\sum_{n=1}^{M}\lambda_n^{\mathscr{W}}\bar{p}_n(\hat{r}_j)+\mu_{\mathscr{W}}\right) \\ &\le \hat{r}_j, \end{aligned} \tag{EC92}$$

where the last inequality follows from (36).

EC.10. Numerical Study of Systems with More Than One Contract and Walk-in Class

In this numerical study we consider a system with $N=2$ contract and $M=2$ walk-in classes. The purpose of the study is to see how the observations made in §6, which were based on single contract and single walk-in class, carry over to a setting with multiple classes.

EC.10.1. Setup

We construct the test suite as follows. The rental capacity and service rate are fixed at $c=10$ and $\mu=1$, respectively, and as before we vary $\lambda=\lambda^{\mathscr{C}}+\lambda^{\mathscr{W}}$ so that the arrival rates cover a wide range of offered loads: $\rho=\lambda/(c\mu)\in\{0.1, 0.5, 1.0, 1.5, 2.0\}$. For each of the two contract and walk-in classes, we then evenly split arrivals: $\lambda_1^{\mathscr{C}}=\lambda_2^{\mathscr{C}}=0.5\lambda^{\mathscr{C}}$ and $\lambda_1^{\mathscr{W}}=\lambda_2^{\mathscr{W}}=0.5\lambda^{\mathscr{W}}$. The walk-in price parameters are set at $w_{\min}=1$, $w_{\max}=4$, $L=10$, just as in Table 1 in §6.

In two sets of numerical experiments, shown in Tables EC.1 and EC.2 below, we then systematically vary the difference between the penalty-adjusted fees for the two contract classes, as well as the difference between the price sensitivities of the two walk-in classes. Specifically, in Table EC.1 we set

- $\varphi=3$ and let $\varphi_1=\varphi-\delta$ and $\varphi_2=\varphi+\delta$ for $\delta\in\{0, 0.5, 1.5\}$, and
- $\beta=1$ and let $\beta_1=\beta-\Delta$ and $\beta_2=\beta+\Delta$ for $\Delta\in\{0, 0.5, 0.9\}$.

Similarly, in Table EC.2 we set

- $\varphi=3$ and let $\varphi_1=\varphi-\delta$ and $\varphi_2=\varphi+\delta$ for $\delta\in\{0, 0.5, 1.5\}$, and
- $\beta=3$ and let $\beta_1=\beta-\Delta$ and $\beta_2=\beta+\Delta$ for $\Delta\in\{0, 0.5, 0.9\}$.

Thus, the walk-in customers in Table EC.1 are, one the whole less price sensitive—both in absolute terms and relative to contract customers—than those in Table EC.2. Within each table, the examples with $\delta=0$ and $\Delta=0$ conform to systems with one class of contract customer and one class of walk-in, and as δ (and Δ) increases, the price sensitivities of the two contract (and the two walk-in) classes diverge.

For each cell in the tables we fix the values of $\rho=(\lambda^{\mathscr{C}}+\lambda^{\mathscr{W}})/(c\mu)$, δ, and Δ, and we vary the ratio $\lambda^{\mathscr{C}}/(\lambda^{\mathscr{C}}+\lambda^{\mathscr{W}})$ across 9 test cases, $\{0.1, \ldots, 0.9\}$, just as in Table 1. The values in the cells represent the average and the maximum (in parentheses) percentage shortfall in the performance of the myopic heuristic, as compared to the optimal profit management policy.

EC.10.2. Results

We begin by recalling that the first column of each table has $\delta=\Delta=0$ and corresponds to a setting with a single contract and a single walk-in class. Thus, the performance-gap values in the first column of Table EC.1 coincide with the values for $\varphi=3$ in Table 1.

Table EC.1 Average (Maximum) Percentage Profit Shortfall of the Myopic Policy as Compared to the Optimal Policy: Examples with $\beta=1$

	$\delta=0$			$\delta=1.0$			$\delta=1.5$		
ρ	$\Delta=0$	$\Delta=0.5$	$\Delta=0.9$	$\Delta=0$	$\Delta=0.5$	$\Delta=0.9$	$\Delta=0$	$\Delta=0.5$	$\Delta=0.9$
0.1	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)
0.5	0.09 (0.11)	0.04 (0.05)	0.13 (0.15)	0.09 (0.12)	0.04 (0.05)	0.12 (0.15)	0.09 (0.12)	0.04 (0.05)	0.13 (0.15)
1.0	3.1 (3.9)	2.0 (2.5)	3.5 (5.1)	3.1 (3.9)	2.0 (2.5)	3.5 (5.1)	4.2 (4.5)	3.2 (4.3)	5.0 (5.2)
1.5	7.4 (10.7)	5.7 (8.0)	7.6 (12.6)	7.6 (10.7)	5.9 (8.0)	8.1 (12.6)	12.2 (13.7)	10.7 (13.4)	13.4 (13.9)
2.0	10.5 (16.9)	8.5 (13.6)	10.0 (17.8)	11.4 (17.0)	9.5 (13.6)	11.6 (18.0)	19.3 (21.3)	17.7 (20.9)	20.1 (21.2)

Table EC.2 Average (Maximum) Percentage Profit Shortfall of the Myopic Policy as Compared to the Optimal Policy: Examples with $\beta = 3$

	$\delta = 0$			$\delta = 1.0$			$\delta = 1.5$		
ρ	$\Delta = 0$	$\Delta = 0.5$	$\Delta = 0.9$	$\Delta = 0$	$\Delta = 0.5$	$\Delta = 0.9$	$\Delta = 0$	$\Delta = 0.5$	$\Delta = 0.9$
0.1	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)	0.0 (0.0)
0.5	0.4 (0.5)	0.6 (0.9)	0.1 (0.2)	0.4 (0.5)	0.6 (0.9)	0.1 (0.2)	0.4 (0.5)	0.6 (0.9)	0.1 (0.2)
1.0	8.1 (10.5)	8.3 (10.8)	5.6 (6.9)	8.1 (10.5)	8.3 (10.8)	5.6 (6.9)	9.0 (10.5)	9.2 (10.8)	6.5 (7.1)
1.5	16.3 (22.0)	6.4 (22.3)	13.0 (17.1)	16.4 (22.0)	16.5 (22.3)	13.1 (17.1)	20.0 (22.1)	20.1 (22.5)	16.8 (17.8)
2.0	21.2 (30.0)	21.3 (30.2)	17.8 (24.8)	21.8 (30.0)	22.0 (30.2)	18.4 (24.8)	28.1 (30.4)	28.2 (30.7)	24.9 (26.0)

Next, we note that, though more complex than before, the tables' results are largely consistent with those for the case of a single contract and a single walk-in class: The myopic policy remains effective when the overall demand load is not too high. In addition, we make the following two observations concerning the more complex behavior of the tables' multi-class examples.

First, when the aggregate penalty-adjusted contract fee is $\varphi = 3$, the performance of the myopic policy appears to decrease as the heterogeneity of the two contract classes, δ, increases. This effect can be readily explained: as δ increases, both $\varphi_1 = \varphi - \delta$ and $\varphi_2 = \varphi + \delta$ rapidly move outside of the interval $[w_{\min}, w_{\max}]$ in which contract fees are "similar" to walk-in fees and the myopic performs well.

This behavior is consistent with the performance declines shown in Figure 6. A similar pattern in the "multiple classes" case is demonstrated in the more detailed set of examples whose results are reported in Figure EC.1, below. The figure's results are for a system with $\lambda_1^{\mathscr{C}} = \lambda_2^{\mathscr{C}} = \lambda_1^{\mathscr{W}} = \lambda_2^{\mathscr{W}} = 2.5$, $\mu = 1$, $c = 10$, and $\beta_1 = \beta_2 = 1$.

Second, heterogeneity among walk-in customers appears to have a mixed effect on the performance of myopic policies. For $\beta = 1$, the relative performance gap decreases slightly for $\Delta = 0.5$ and then increases for $\Delta = 0.9$. In contrast, for $\beta = 3$, the effect is reversed: The relative performance gap increases slightly for $\Delta = 0.5$ and then decreases for $\Delta = 0.9$.

Figure 7 helps to understand this phenomenon. When β is either very large or very small, then a myopic policy is optimal. In the former case, customers are very price sensitive, and it is optimal to (myopically) maintain a fixed, low price, and in the latter customers are price insensitive, and it pays to myopically maintain a high price. For intermediate levels of price sensitivity dynamic policies are valuable and for myopic policies they are "dangerous."

When walk-in heterogeneity is large, then both walk-in classes are price (in)sensitive enough that both fall out of the above "danger zone" in the middle of Figure 7's graph, and the performance of the myopic policy is good. This is likely to be the case for $\beta = 3$ in Table EC.2. If, however, the heterogeneity pushes one of the β's into that danger zone, while pushing the other one out of the zone, then the impact on myopic performance is hard to predict. This is the case for $\beta = 1$ in Table EC.2.

Figure EC.2 provides a two-class analog to Figure 7. In the figure, we set $\lambda_1^{\mathscr{C}} = \lambda_2^{\mathscr{C}} = \lambda_1^{\mathscr{W}} = \lambda_2^{\mathscr{W}} = 2.5$, $\mu = 1$, $c = 10$, and $\varphi_1 - \varphi_2 = 3$. Then each curve represents the results for a different system, one with

Figure EC.1 Percentage Performance Shortfall for the Myopic Policy as a Function of the Penalty-Adjusted Fee for Contract Class 2, φ_2

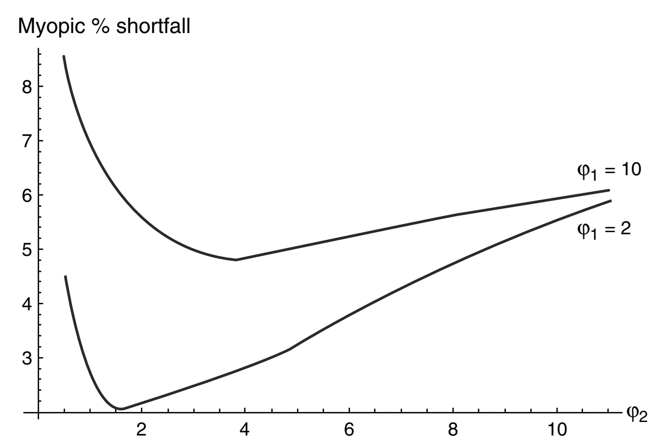
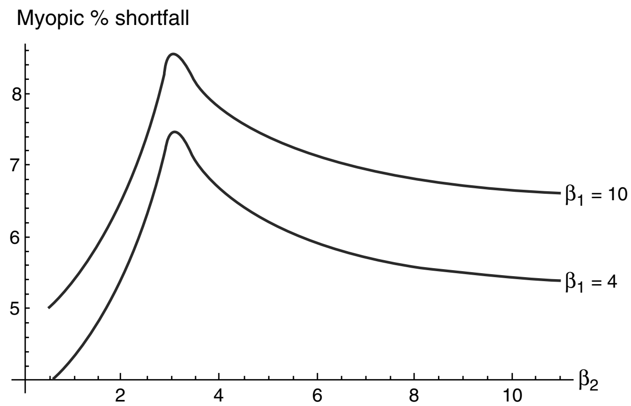


Figure EC.2 Percentage Performance Shortfall for the Myopic Policy as a Function of the Price Sensitivity of Walk-in Class 2, β_2



$\beta_1 = 4$, and the other with $\beta_1 = 10$. In both cases we systematically vary β_2 and record the myopic policy's percentage shortfall from optimality.

Note that both curves display peaks at $\beta_2 = 3$. In both cases, the myopic policy remains the same for all $\beta_2 \geq 3$.

References

See references list in the main paper.

Feller, W. 1968. *An Introduction to Probability Theory and Its Applications*, Vol. 1. John Wiley and Sons, New York, 50–53.