Supplement for

Joint Stocking and Sourcing Policies for a Single–Depot, Single–Base, Two–Echelon Environments with Repairable Parts: The Role of Flexibility

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Appendix A: Formulation of the availability constraint

Proposition 1: For any stocking solution (S_1, S_0) and base sourcing fraction, r_1, the expected number of backorders at the depot is,

$$BO_0(S_0) = \lambda * (1 - r_1) * L_0 - S_0 + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda * (1 - r_1) * L_0)^n}{n!} * (S_0 - n)$$

Proof:

By Palm's Theorem, $R_{0,}$ is distributed as Poisson with mean equal to $\lambda * r_0 * L_0$, i.e $P(R_o = n) = e^{-\lambda*r_0*L_0} * \frac{(\lambda*r_0*L_0)^n}{n!}$. Let $BO(S)$ be the expected number of backorders at a location when its target stocking level is S, $BO(S) = \sum_{n=S+1}^{\infty}(n - S) * P(R = n)$ (when appropriate we suppress the location subscript). The expected number of backorders at a location for any S is:

$$BO(S) = \sum_{n=S+1}^{\infty} (n - S) * P(R = n) = e^{-BO(0)} \sum_{n=S+1}^{\infty} (n - S) * \frac{BO(0)^n}{n!}$$

The incremental reduction in expected backorders is equal to,

$$\delta(S) = BO(S + 1) - BO(S) = \sum_{n=S+2}^{\infty}(n - S) * P(R = n) - \sum_{n=S+2}^{\infty} P(R = n) - \sum_{n=S+2}^{\infty}(n - S) * P(R = n) - P(R = S + 1) = -\sum_{n=S+1}^{\infty} P(R = n) = \sum_{n=0}^{S} P(R = n) - 1 = P(R \leq S) - 1$$

When the central depot inventory is 0 then the expected number of backorders equals the number of systems in repair, $BO_0(0) = \lambda * r_0 * L_0 = \lambda * (1 - r_1) * L_0$ and

$BO_0(S_0 + 1) = BO_0(S_0) + \left(\sum_{n=0}^{S_0} P(R_o = n) - 1\right) = BO_0(S_0) + \left(e^{-BO_0(0)} \sum_{n=0}^{S_0} \frac{BO_0(0)^n}{n!} - 1\right)$,

$S_0 = 0,1,2, \ldots$. The expected number of backorders at the depot for any S_0 is: $BO_0(S_0) = \lambda * (1 - r_1) * L_0 - S_0 + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n)$

Proposition 2: For any stocking solution (S_1, S_0) and base sourcing fraction, r_1, the expected number of backorders at the base is, $BO_1(S_0,S_1) = \lambda * r_1 * L_1 + \lambda * (1 - r_1) * TT + \lambda * (1 - r_1) * L_0 - (S_1 + S_0) + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n) +$

$e^{-\left(\lambda*r_1*L_1+\lambda*(1-r_1)*TT+\lambda*(1-r_1)*L_0-S_0+e^{-\lambda*(1-r_1)*L_0}\sum_{n=0}^{S_0-1}\frac{(\lambda*(1-r_1)*L_0)^n}{n!}*(S_0-n)\right)} *$

$\sum_{k=0}^{S_1-1} \frac{\left(\lambda*r_1*L_1+\lambda*(1-r_1)*TT+\lambda*(1-r_1)*L_0-S_0+e^{-\lambda*(1-r_1)*L_0}\sum_{n=0}^{S_0-1}\frac{(\lambda*(1-r_1)*L_0)^n}{n!}*(S_0-n)\right)^k}{k!} * (S_1 - k)$

Proof:

The expected delay for replenishment orders at the depot is defined as Backorder delay time=$BODT = \frac{BO_0(S_0)}{\lambda*r_0}$. The "effective" lead time at the base is $ELT_1 = r_1 * L_1 + r_0 * TT + r_0 * BODT$

It follows then that, $BO_1(S_0, 0) = \lambda * ELT_1$, where $BO_1(S_0, 0)$ refers to $S_1 = 0$. The expected backorders at the base can be computed as, $BO_1(S_0,S_1) = BO_1(S_0, 0) + \sum_{n=0}^{S_1-1} \delta(n) = \lambda * ELT_1 + \sum_{n=0}^{S_1-1}(P(R_1 \leq n) - 1) = \lambda * (r_1 * L_1 + r_0 * TT + r_0 * BODT) - S_1 + e^{-\lambda*ELT_1} \sum_{n=0}^{S_1-1} \sum_{k=0}^{n} \frac{(\lambda*ELT_1)^k}{k!}$

After placing in the previous equation $BODT$, ELT_1 and $r_0 = 1 - r_1$we find $BO_1(S_1, S_0)$ to be:

$BO_1(S_0,S_1) = \lambda * r_1 * L_1 + \lambda * (1 - r_1) * TT + \lambda * (1 - r_1) * L_0 - (S_1 + S_0) +$

$e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n) +$

$e^{-\left(\lambda*r_1*L_1+\lambda*(1-r_1)*TT+\lambda*(1-r_1)*L_0-S_0+e^{-\lambda*(1-r_1)*L_0}\sum_{n=0}^{S_0-1}\frac{(\lambda*(1-r_1)*L_0)^n}{n!}*(S_0-n)\right)} *$

$\sum_{k=0}^{S_1-1} \frac{\left(\lambda*r_1*L_1+\lambda*(1-r_1)*TT+\lambda*(1-r_1)*L_0-S_0+e^{-\lambda*(1-r_1)*L_0}\sum_{n=0}^{S_0-1}\frac{(\lambda*(1-r_1)*L_0)^n}{n!}*(S_0-n)\right)^k}{k!} * (S_1 - k)$

Appendix B: Derivatives of BO_1 by the decision variables

Proposition 3: For any stocking solution (S_1, S_0), the partial derivative of expected backorders at the base, with respect to r_1 is,

$$\frac{\partial BO_1(S_0,S_1,r_1)}{\partial r_1} = \lambda * \Big(L_1 - \big(TT + L_0 * (1 - P(R_0 \le S_0 - 1))\big)\Big) * \big(1 - P(R_1 \le S_1 - 1)\big)$$

Proof:

We differentiate BO_1 by r_1 and get,

$$\frac{\partial BO_1(S_0,S_1,r_1)}{\partial r_1} = \lambda * (L_1 - L_0 - TT) + \lambda * L_0 * e^{-\lambda * L_0 *(1-r_1)} \sum_{n=0}^{S_0-1} \frac{(\lambda * L_0 * (1-r_1))^n}{n!} * (S_0 - n) +$$

$$e^{-\lambda * L_0 *(1-r_1)} * \sum_{n=0}^{S_0-1} - \frac{(\lambda * L_0 * (1-r_1))^n * n}{(1-r_1) * n!} * (S_0 - n) + \Big(-\lambda * (L_1 - L_0 - TT) - \lambda * L_0 *$$

$$e^{-\lambda * L_0 *(1-r_1)} \sum_{n=0}^{S_0-1} \frac{(\lambda * L_0 * (1-r_1))^n}{n!} * (S_0 - n) - e^{-\lambda * L_0 *(1-r_1)} * \sum_{n=0}^{S_0-1} - \frac{(\lambda * L_0 * (1-r_1))^n * n}{(1-r_1) * n!} * (S_0 - n)\Big) *$$

$$e^{-\left(\lambda * r_1 * L_1 + \lambda *(1-r_1) * TT + \lambda * (1-r_1) * L_0 - S_0 + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n)\right)} *$$

$$\sum_{k=0}^{S_1-1} \frac{\left(\lambda * r_1 * (L_1 - L_0 - TT) + \lambda * (L_0 + TT) - S_0 + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n)\right)^k}{k!} * (S_1 - k) +$$

$$e^{-\left(\lambda * r_1 * L_1 + \lambda *(1-r_1) * TT + \lambda * (1-r_1) * L_0 - S_0 + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n)\right)} *$$

$$\sum_{k=0}^{S_1-1} \frac{\left(\lambda * r_1 * (L_1 - L_0 - TT) + \lambda * (L_0 + TT) - S_0 + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n)\right)^k}{\left(\lambda * r_1 * (L_1 - L_0 - TT) + \lambda * (L_0 + TT) - S_0 + e^{-\lambda*(1-r_1)*L_0} \sum_{n=0}^{S_0-1} \frac{(\lambda*(1-r_1)*L_0)^n}{n!} * (S_0 - n)\right) * k!} * k * \Big(\lambda * (L_1 - L_0 -$$

$$TT) + \lambda * L_0 * e^{-\lambda * L_0 *(1-r_1)} \sum_{n=0}^{S_0-1} \frac{(\lambda * L_0 * (1-r_1))^n}{n!} * (S_0 - n) + e^{-\lambda * L_0 *(1-r_1)} *$$

$$\sum_{n=0}^{S_0-1} - \frac{(\lambda * L_0 * (1-r_1))^n * n}{(1-r_1) * n!} * (S_0 - n)\Big) * (S_1 - k)$$

Let us open up the following expressions:

(B1) $$\sum_{n=0}^{S_0-1} \frac{(\lambda * L_0 * (1-r_1))^n}{n!} * (S_0 - n) = S_0 * \sum_{n=0}^{S_0-1} \frac{(\lambda * L_0 * (1-r_1))^n}{n!} - (\lambda * L_0 * (1 - r_1)) * \sum_{n=0}^{S_0-2} \frac{(\lambda * L_0 * (1-r_1))^n}{n!}$$

and

(B2) $$\sum_{n=0}^{S_0-1} - \frac{(\lambda * L_0 * (1-r_1))^n * n}{(1-r_1) * n!} * (S_0 - n) = -S_0 * \lambda * L_0 * \sum_{n=0}^{S_0-2} \frac{(\lambda * L_0 * (1-r_1))^n}{n!} + \lambda * L_0 * \sum_{n=0}^{S_0-2} \frac{(\lambda * L_0 * (1-r_1))^n}{n!} + \lambda^2 * {L_0}^2 * (1 - r_1) \sum_{n=0}^{S_0-3} \frac{(\lambda * L_0 * (1-r_1))^n}{n!}$$

Let us define $A = \lambda * (L_1 - L_0 - TT) + \lambda * L_0 * e^{-\lambda * L_0 * (1-r_1)} \sum_{n=0}^{S_0-1} \frac{\left(\lambda * L_0 * (1-r_1)\right)^n}{n!} * (S_0 - n) + e^{-\lambda * L_0 * (1-r_1)} * \sum_{n=0}^{S_0-1} - \frac{\left(\lambda * L_0 * (1-r_1)\right)^n * n}{(1-r_1) * n!} * (S_0 - n)$. Now we place into A Equations (B1) and (B2) to get

$$A = \lambda * (L_1 - L_0 - TT) + \lambda * L_0 * e^{-\lambda * L_0 * (1-r_1)} * \left(S_0 * \sum_{n=0}^{S_0-1} \frac{\left(\lambda * L_0 * (1-r_1)\right)^n}{n!} - \left(\lambda * L_0 * (1-r_1)\right) * \sum_{n=0}^{S_0-2} \frac{\left(\lambda * L_0 * (1-r_1)\right)^n}{n!}\right) + \lambda * L_0 * e^{-\lambda * L_0 * (1-r_1)} * \left(-S_0 * \sum_{n=0}^{S_0-2} \frac{\left(\lambda * L_0 * (1-r_1)\right)^n}{n!} + \sum_{n=0}^{S_0-2} \frac{\left(\lambda * L_0 * (1-r_1)\right)^n}{n!} + \lambda * L_0 * (1-r_1) \sum_{n=0}^{S_0-3} \frac{\left(\lambda * L_0 * (1-r_1)\right)^n}{n!}\right)$$

We notice that some of the expressions cancel, place probabilities instead of their corresponding expressions and re-order the equation which results in:

$$A = \lambda = \lambda * \left(L_1 - \left(TT + L_0 * P(R_0 > S_0 - 1)\right)\right)$$

Now we place A and $BO_1(S_0, 0)$ into the derivative to get a more compact expression:

$$\frac{\partial BO_1(S_0, S_1, r_1)}{\partial r_1} = A - A * e^{-BO_1(0,S_0)} * \sum_{k=0}^{S_1-1} \frac{BO_1(0,S_0)^k}{k!} * (S_1 - k) + e^{-BO_1(0,S_0)} * \sum_{k=0}^{S_1-1} \frac{BO_1(0,S_0)^k}{BO_1(0,S_0) * k!} * k * A * (S_1 - k)$$

Let us open up the following expressions:

(B3) $\sum_{k=0}^{S_1-1} \frac{BO_1(0,S_0)^k}{k!} * (S_1 - k) = S_1 * \sum_{k=0}^{S_1-1} \frac{BO_1(0,S_0)^k}{k!} - BO_1(0, S_0) * \sum_{k=0}^{S_1-2} \frac{BO_1(0,S_0)^k}{k!}$

and

(B4) $\sum_{k=0}^{S_1-1} \frac{BO_1(0,S_0)^k}{BO_1(0,S_0) * k!} * k * A * (S_1 - k) = A * S_1 * \sum_{k=0}^{S_1-2} \frac{BO_1(0,S_0)^k}{k!} - A * \sum_{k=0}^{S_1-2} \frac{BO_1(0,S_0)^k}{k!} - A * BO_1(0, S_0) * \sum_{k=0}^{S_1-3} \frac{BO_1(0,S_0)^k}{k!}$

After placing (B3) and (B4) in the derivative, ordering and placing probabilities instead of their corresponding expressions we receive:

$$\frac{\partial BO_1(S_0, S_1, r_1)}{\partial r_1} == A * P(R_1 > S_1 - 1)$$

Placing A leads to the final expression, $\frac{\partial BO_1(S_0, S_1, r_1)}{\partial r_1} = \lambda * \left(L_1 - \left(TT + L_0 * P(R_0 > S_0 - 1)\right)\right) * P(R_1 > S_1 - 1) = \lambda * \left(L_1 - \left(TT + L_0 * \left(1 - P(R_0 \le S_0 - 1)\right)\right)\right) * \left(1 - P(R_1 \le S_1 - 1)\right)$

Proposition 4: BO_1 is convex and decreasing with S_1.

Proof:

Since the stock values are integers use forward differences equations as an approximation of the derivatives. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and if h is fixed (non-zero) then the approximation of the derivative is $\frac{\Delta_h f(x)}{h} = \frac{f(x+h)-f(x)}{h}$. The approximation for second derivative is: $f''(x) \approx \frac{\Delta_h^2 f(x)}{h^2} = \frac{\Delta_h f(x+h)-\Delta_h f(x)}{h^2} = \frac{f(x+2h)-f(x+h)-f(x+h)+f(x)}{h^2} = \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$. In our case h is simply 1 and we shall show the first and second differences of the base backorders by S_1. Note that

$\frac{\partial BO_1(S_0,S_1)}{\partial S_1} \approx \delta_{1,S_0}(S_1) = P(R_1 \leq S_1) - 1 = \sum_{k=0}^{S_1} \frac{(\lambda * ELT_1)^k}{k!} * e^{-\lambda * ELT_1} - 1 \leq 0$ and $\frac{\partial^2 BO_1(S_0,S_1)}{\partial S_1^2} \approx BO_1(S_0,S_1+2) - 2BO_1(S_0,S_1+1) + BO_1(S_0,S_1) = \delta_{1,S_0}(S_1+1) - \delta_{1,S_0}(S_1) = P(R_1 \leq S_1 + 1) - P(R_1 \leq S_1) = P(R_1 = S_1 + 1) \geq 0$ which completes the proof.

Proposition 5: BO_1 is convex and decreasing with S_0.

Proof:

$\frac{\partial BO_1(S_0,S_1)}{\partial S_0} \approx BO_1(S_0+1,S_1) - BO_1(S_0,S_1) = BO_1(S_0+1,0) + \sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) - \left(BO_1(S_0,0) + \sum_{n=0}^{S_1-1} \delta_{1,S_0}(n)\right) = \delta_0(S_0) + \sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0}(n) = \delta_0(S_0) - \left(\sum_{n=0}^{S_1-1} P(R_1 > n|S_0+1) - \sum_{n=0}^{S_1-1} P(R_1 > n|S_0)\right) = \delta_0(S_0) - \left(\sum_{n=0}^{S_1-1} \sum_{k=n+1}^{\infty} P(R_1 = k|S_0+1) - \sum_{n=0}^{S_1-1} \sum_{k=n+1}^{\infty} P(R_1 = k|S_0)\right) = \delta_0(S_0) - \left(\sum_{n=1}^{S_1} n * P(R_1 = n|S_0+1) + S_1 * P(R_1 > S_1|S_0+1) - \sum_{n=1}^{S_1} n * P(R_1 = n|S_0) - S_1 * P(R_1 > S_1|S_0)\right)$

Note that the term $\sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0}(n) = -\left(\sum_{n=0}^{S_1-1} P(R_1 > n|S_0+1) - \sum_{n=0}^{S_1-1} P(R_1 > n|S_0)\right)$ is always positive and increasing with S_1. Its maximal value is achieved when $S_1 \to \infty$ and then its value is $-\delta_0(S_0)$ and $\frac{\partial BO_1(S_0,S_1)}{\partial S_0} = 0$. It follows immediately that for lower values of S_1, $\frac{\partial BO_1(S_0,S_1)}{\partial S_0} < 0$. As $S_0 \to \infty$ then $\delta_0(S_0) \to 0$ and the derivative $\frac{dBO_1(S_1,S_0)}{dS_0} \to 0$.

We formulate the second derivative.

$\frac{\partial^2 BO_1(S_0,S_1)}{\partial S_0^2} \approx BO_1(S_0+2,S_1) - 2BO_1(S_0+1,S_1) + BO_1(S_0,S_1) = \left(\delta_0(S_0+1) - \delta_0(S_0)\right) + \left(\sum_{n=0}^{S_1-1} \delta_{1,S_0+2}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n)\right) - \left(\sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0}(n)\right) = \left(\delta_0(S_0+1) - \delta_0(S_0)\right) - \left(\sum_{n=0}^{S_1-1} P(R_1 > n|S_0+2) - \sum_{n=0}^{S_1-1} P(R_1 > n|S_0+1)\right) + \left(\sum_{n=0}^{S_1-1} P(R_1 > n|S_0+1) - \sum_{n=0}^{S_1-1} P(R_1 > n|S_0)\right)$. Note that each one of the three terms: $\delta_0(S_0+1) - \delta_0(S_0) = -\left(P(R_0 > S_0+1) - P(R_0 > S_0)\right)$,$\sum_{n=0}^{S_1-1} \delta_{1,S_0+2}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) = -\left(\sum_{n=0}^{S_1-1} P(R_1 > n|S_0+2) - \sum_{n=0}^{S_1-1} P(R_1 > n|S_0+1)\right)$ and $\sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0}(n) = -\left(\sum_{n=0}^{S_1-1} P(R_1 > n|S_0+1) - \sum_{n=0}^{S_1-1} P(R_1 > n|S_0)\right)$ is always positive and the last two terms are increasing with S_1. We notice also that $\sum_{n=0}^{S_1-1} \delta_{1,S_0+2}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) < \sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0}(n)$. So we conclude that the expression $\left(\sum_{n=0}^{S_1-1} \delta_{1,S_0+2}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n)\right) - \left(\sum_{n=0}^{S_1-1} \delta_{1,S_0+1}(n) - \sum_{n=0}^{S_1-1} \delta_{1,S_0}(n)\right)$ is negative and decreasing with S_1. Its minimal value, that is achieved when $S_1 \to \infty$, is $-\delta_0(S_0+1) + \delta_0(S_0)$. Thus, $\frac{\partial^2 BO_1(S_0,S_1\to\infty)}{\partial S_0^2} = \left(\delta_0(S_0+1) - \delta_0(S_0)\right) - \delta_0(S_0+1) + \delta_0(S_0) = 0$. It follows directly that for all smaller values of S_1, $\frac{\partial^2 BO_1(S_0,S_1\to\infty)}{\partial S_0^2} > 0$ which completes the proof.

Appendix C: Proof of Theorem 2

Case 1: Both the repair lead time and the repair cost are higher at the base then at the central depot, i.e. $L_1 > L_0 + TT$ and $c_1 \geq c_0$.

Proof for Case 1:

From Proposition 3 we know that $\frac{\partial BO_1(S_0,S_1,r_1)}{\partial r_1} > 0$ so we are achieving a minimal value of backorders by setting $r_1 = 0$ (for any given values of S_0, S_1). When $r_1 = 0$ the repair costs are also minimized since we are repairing everything at the cheaper depot.

Case 2: Repair lead time is again smaller at the central depot but the repair cost is higher there, i.e. $L_1 > L_0 + TT$ and $c_1 < c_0$.

Proof for Case 2:

From Proposition 3 we know that $\frac{\partial BO_1(S_0,S_1,r_1)}{\partial r_1} > 0$. So if we didn't have to consider $c_1 < c_0$ the lower repair costs at the base we would have chosen $r_1 = 0$. Let us define a relative value of $\alpha = \lambda * \frac{c_1 - c_0}{p} < 0$ and the objective function can be written as $\min_{S_0,S_1,r_1} \lambda * \frac{c_0}{p} + \alpha * r_1 + (S_1 + S_0)$. Thus for a small increase in r_1, say to $r_1 + \Delta r_1$ we decrease the objective by $\Delta r_1 * \alpha$ and "indirectly" we increase $(S_1 + S_0)$. $(S_1 + S_0)$ increase since by increasing r_1 we increase the effective lead time and the backorders. We have to increase $(S_1 + S_0)$ to decrease the backorders to satisfy the availability constraint (see Propositions 4 and 5). The maximal reduction of the repair cost if we increase r_1 from 0 to 1 is α. When $\alpha \to 0$, the change in the objective function value is very small and the optimal solution will be attained when $r_1 \to 0$. When $\alpha \to -\infty$ then $r_1 \to 1$ and we have to increase $(S_1 + S_0)$. So we have to conduct a tradeoff analysis to find r_1.

Case 3: Both the repair lead time and the repair cost are lower at the base than at the central depot, i.e. $L_1 \leq L_0 + TT$ and $c_1 < c_0$.

Proof for Case 3:

Assume that $L_1 \leq TT$ then from Proposition 3 we know that $\frac{dBO_1(S_0,S_1,r_1)}{dr_1} < 0$ always and it is optimal to set $r_1 = 1$ and $S_0 = 0$. This follows because with $r_1 = 1$ there are no repairs sourced from the depot hence no value in putting inventory at the depot to reduce its delay time. Then we find the optimal value of S_1 for the single site (base) problem. For a given S_0, S_1, when $TT \leq L_1 \leq L_0 + TT$ then by Proposition 3, $\frac{dBO_1(S_0,S_1,r_1)}{dr_1}$ may be positive or negative dependent upon the value of S_0 and r_1 (but not S_1). However, for any fixed level of $S_0 + S_1$ the shortest effective lead time and respectively minimal number of backorders is when $r_1 = 1$ and $S_0 = 0$ and this is the dominant case. Since for $r_1 = 1$ the repair costs are the lowest, this is the optimal solution. So, for this case the optimal solution is: $r_1 = 1$ and $S_0 = 0$.

Case 4: Repair lead time is shorter at the base, but the repair cost there is higher, i.e. $L_1 < L_0 + TT$ and $c_1 > c_0$.

Proof for Case 4:

This case is the opposite of Case 2. We know by Case 3 that it is beneficial from an availability standpoint to repair everything at the base. But the base repair cost is higher there so there is a tradeoff that may dictate satisfying the availability constraint by increasing the stock level, and sometimes it may be better to increase the fraction of repairs at the depot (i.e., to repair more at the more expensive place, the depot). Let us use the definition of Case 2 so the objective is $\min_{S_0,S_1,r_1} \lambda * \frac{c_0}{p} + \alpha * r_1 + (S_1 + S_0)$ and $\alpha = \lambda * \frac{c_1 - c_0}{p} > 0$. Thus for a small decrease in r_1, say to $r_1 - \Delta r_1$ we decrease the objective by $\Delta r_1 * \alpha$ and we increase $(S_1 + S_0)$ through increasing the effective lead time and the backorders. Thus, we have to increase $(S_1 + S_0)$ to satisfy the availability constraint (see Propositions 4 and 5).

The maximal reduction of the repair cost if we reduce r_1 from 1 to 0 is α. When $\alpha \to 0$, the change in the objective function value is very small and the optimal solution will be attained when $r_1 = 1$. When $\alpha \to \infty$ then $r_1 \to 0$ and we have to increase $(S_1 + S_0)$. So for the general case we have to conduct a tradeoff analysis to find the optimal r_1.

Appendix D: Test data based on aerospace and defense program

(S_0, S_1, r_1) marks the algorithm's solution assuming a single LRU system and $S_0 = 0$.

Part #	p	λ	L_0	TT	L_1	c_0	c_1	Optimal policy	(S_0, S_1, r_1)	Objective cost
1	102058.5	29.6	0.25	0.02	0.40	15308.8	20962.4	Central	(0,10,0)	1473146.9
2	84406.0	24.5	0.14	0.02	0.18	20560.2	21786.2	Central	(0,5,0)	926098.5
3	13014.4	21.3	0.13	0.02	0.15	1921.2	1588.8	Mixed	(0,4,1)	85858.8
4	39089.8	21.1	0.19	0.02	0.26	11068.9	16422.8	Central	(0,6,0)	468004.9
5	4959.3	18.6	0.24	0.02	0.33	899.5	459.2	Mixed	(0,6,0.1)	45636.5
6	25204.3	18.0	0.18	0.02	0.13	3214.4	2990.4	Base	(0,3,1)	129488.4
7	177458.8	15.4	0.18	0.02	0.21	54885.1	76533.5	Central	(0,4,0)	1557689.1
8	12666.8	15.4	0.24	0.02	0.28	4225.7	4926.3	Central	(0,5,0)	128419.8
9	71589.3	13.8	0.21	0.02	0.20	14532.6	8933.2	Base	(0,4,1)	409212.7
10	18946.0	12.5	0.19	0.02	0.30	7909.2	7625.4	Mixed	(0,3,0)	155508.5
11	42740.8	11.8	0.43	0.02	0.57	1938.3	2562.1	Central	(0,7,0)	322086.4
12	24057.0	11.4	0.18	0.02	0.20	4615.6	6342.7	Central	(0,3,0)	124658.3
13	6997.2	10.4	0.28	0.02	0.22	3192.4	3331.5	Mixed	(0,3,0.52)	55040.9
14	25530.6	10.4	0.23	0.02	0.21	10837.5	10621.6	Base	(0,3,1)	187296.8
15	110046.5	10.3	0.15	0.02	0.25	46585.5	38144.0	Mixed	(0,2,0.26)	677531.0
16	25138.714	10.3	0.18	0.02	0.25	6585.3	9710.7	Central	(0,3,0)	143072.7
17	8278.3	10.0	0.19	0.02	0.19	3473.5	5170.6	Mixed	(0,3,0)	59527.3
18	81316.2	9.9	0.21	0.02	0.19	20303.4	12505.0	Base	(0,2,1)	286343.1
19	20271.0	9.9	0.18	0.02	0.14	4497.5	3710.9	Base	(0,2,1)	77217.4
20	27474.4	9.5	0.19	0.02	0.12	10790.3	8174.9	Base	(0,1,1)	105294.7
21	4640.9	9.4	0.18	0.02	0.27	2160.3	1509.0	Mixed	(0,3,1)	28057.2
22	4773.2	9.2	0.13	0.02	0.23	2159.7	1660.3	Mixed	(0,2,0.78)	25854.2
23	9495.0	8.6	0.20	0.02	0.22	2550.2	2985.0	Central	(0,2,0)	40906.0
24	16617.8	8.0	0.25	0.02	0.19	2610.8	3871.1	Mixed	(0,2,0.34)	57593.9
25	5966.1	8.0	0.17	0.02	0.13	2441.2	3415.9	Mixed	(0,1,0.63)	30328.4
26	7109.0	7.5	0.22	0.02	0.19	2618.8	2973.2	Mixed	(0,2,0)	40906.0
27	125335.5	7.1	0.19	0.02	0.32	43315.1	44677.2	Central	(0,2,0)	557912.3
28	120781.8	7.0	0.15	0.02	0.24	34494.6	17985.7	Mixed	(0,2,1)	367835.2
29	14276.9	6.6	0.38	0.02	0.54	5980.8	8197.5	Central	(0,3,0)	82092.4
30	76191.7	5.2	0.11	0.02	0.16	3883.8	2357.8	Mixed	(0,1,1)	88567.9
31	70079.3	5.1	0.12	0.02	0.13	21491.2	21847.1	Mixed	(0,1,0)	180052.1