MARKETING-PRODUCTION DECISIONS IN AN INDUSTRIAL CHANNEL OF DISTRIBUTION*

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This paper presents a model that considers the interface between marketing and production decisions in a channel of distribution of industrial goods comprised of a manufacturer and a distributor. The key issue investigated is the nature of the coordination within the channel in light of an unstable pattern of seasonal demand. The investigation, which focuses on dynamic issues, considers production activities such as product delivery and inventory policy, and their link to marketing strategies such as pricing policies.

The focus here is on the dynamic nature of the coordinational aspects of the various policies. For example, the following questions are addressed: What is the distributor's pricing strategy throughout the season? What is the nature of the contractual price in the channel? Under what conditions should the distributor carry no inventory throughout the seasonal period? Under what conditions should the manufacturer carry no inventory? What is the impact of the seasonal parameters and differing holding and processing costs upon the various policies of the distribution channel? We propose a mechanism to address these issues, and we provide explicit answers to these questions.

(MARKETING; MARKETING—DISTRIBUTION; PRODUCTION/SCHEDULING)

1. Introduction

Marketing channels can be viewed as "sets of interdependent organizations involved in the process of making a product or service available for use or consumption" (Stern and El-Ansary 1982). Effective distribution system management requires the coordination of the various interdependent marketing and production decisions. As noted by Kotler (1971), one of the major marketing-production problems faced by many distribution systems is the development of efficient production schedules, product delivery, inventory policy, and pricing policies in the face of a predictable but unstable pattern of seasonal demand. This paper will analyze such policies.

In analyzing the above noted policies, it is worth noting that some firms have developed advanced synchronized methodologies for inventory management and control that aid them in improving the efficiency of the system. IBM (1971, 1972), for example, has developed inventory management and control techniques tailored to each level in the marketing channel. Through its IMPACT (Inventory Management Program and Control Techniques) the company has been able to formulate methodologies to solve basic inventory problems. Of course, maintaining a desirable inventory level requires simultaneous control of marketing and production policies. Our model provides formal rationales for such phenomena.

The focus here is on the dynamic nature of the coordinational aspects of the various policies in a channel of distribution. For example, the following questions are addressed: Under what conditions should the distributor operate under a stockless policy throughout the seasonal period? Under what conditions should the manufacturer carry no inventory? Which of the two parties should reach zero inventory earlier as the end of the season approaches? What is the impact of the seasonal parameters and differing holding and processing costs upon the various policies of the channel? We provide explicit answers to these questions.

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As noted, it is quite obvious that an efficient channel coordination can be obtained only by an explicit recognition of the marketing and production considerations and decisions. There have been some related attempts to address this issue. However, to our knowledge, none has appeared in a vertical industrial channel setting. We begin our review with the production literature.

Holt, Modigliani, Muth and Simon (1960) consider an optimal employment, production and inventory policy of a single manufacturer in the face of a given sales forecast. They assume, however, that sales are not influenced by any marketing effort, and hence, the marketing function is treated passively in their model. Tuite (1968), on the other hand, is concerned mainly with the determination of a discount on order placed in slack months for goods that exhibit seasonal sales fluctuations. Thomas (1970) considers the problem of simultaneously making price and production decisions by a monopolist having a fixed plus linear production cost who is facing a known deterministic demand. His major results are in obtaining planning horizons similar to those reported by Wagner and Whitin (1958), and Eppen, Gould and Pashigian (1969). The same problem with convex production costs and constant price has also been treated by Vanthienen (1973) and Richard (1969).

Kunreuther and Richard (1971) investigate the interrelationship between the pricing and inventory decisions of a retailer who orders his outside goods from an outside distributor. The retailer is assumed to incur a fixed cost each time he places an order, so that he finds it desirable to store inventory over time. It is also assumed that the marginal costs of holding and purchasing goods are constant, and hence the retailer maintains the same price throughout the period. Within this framework, comparisons are made between decentralized and centralized decision-making in a firm. Kunreuther and Schrage (1973) later extend this work to the case of a deterministic demand curve that differs from period to period. However, they still assume that the firm wants to maintain the same price for the product throughout the season.

Other attempts worth noting that have been reported in the production/management science literature are by Damon and Schramm¹ (1972), Leitch (1974), and Jorgensen (1985). However, of special relevance to our work is the formulation presented by Pekelman (1974). He considers a single manufacturer with convex production costs who determines simultaneously the price and production over a known horizon when the demand parameters are time-dependent. Pekelman characterizes the production, inventory and pricing policies and shows the relationship between his work and that of Modigliani and Hohn (1955).

Recently the problem of coordination in channels of distribution has received attention by marketing scientists. McGuire and Staelin (1983), for example, investigate the effect of product substitutability on Nash equilibrium distribution structures in a duopoly, where each manufacturer distributes its goods through a single exclusive retailer. They show under what conditions a decentralized distribution system will be desirable. Jeuland and Shugan (1983) also show how the total channel profits can be maximized in a decentralized structure by means of quantity discount. Lal and Staelin (1984) address the problem of why and how a seller should develop a discount pricing structure even if such a pricing structure does not alter ultimate demand. These efforts study pricing schemes that allow the seller to modify the buyer’s behavior and they focus more on the question of decentralization vs. vertical integration. Since they are static analyses they do not focus on within-channel production scheduling and inventory considerations.² By contrast our work begins with a given (decentralized) channel

¹ See Welam (1977) for a discussion of a few shortcomings in Damon and Schramm’s formulation.
² For other related recent work on channel coordination within a static framework, see Moorthy (1984) and Corstjens and Horowitz (1985).
structure, and it provides a dynamic viewpoint of linking the pricing to the production, processing, and inventory policies in the channel.

More specifically, we consider the following problem. A decentralized downstream distribution system, composed of a single manufacturer and a single distributor, who has to process further the product, is facing a seasonal demand condition. Fluctuating demand has been identified as one major characteristic of industrial markets (Kotler 1984). We also note that the distributor is actually acting as another downstream manufacturer. The total quantity of the seasonal product is ordered once, at the beginning of the season, and it is delivered continuously to the distributor throughout the season. The quantity that is physically delivered to (as ordered by) the distributor, at any point in time, is processed immediately. Hence, the type of distribution channel scenario that we wish to consider is especially relevant to industrial marketing, where the product delivered from the manufacturer to the distributor enters the latter’s production facilities completely (e.g., materials and parts) or partly (e.g., capital items). Examples that appear to fit our model well are crude petroleum at the manufacturer level which undergoes additional processing at the distributor level to become refined petroleum, and the distribution of natural gas in the interstate market.

The analytical method employed here is optimal control theory, and the behavior of the system is characterized in terms of the dynamic behavior of the distributor’s pricing, processing, and inventory policies, and the manufacturer’s production and inventory policies. The manufacturer’s pricing policy is handled as follows. Since Kotler (1984, p. 169) points out that “industrial buyers are increasingly responding to long-term contracts with suppliers,” contractual transfer pricing between the two parties is assumed. To account for the “leader” role of many industrial manufacturers (e.g., U.S. Steel Corporation, Standard Oil, Du Pont) the modeling approach we have chosen to take here is that of a sequential differential game. That is, we have formulated our problem as a Stackelberg model (Varian 1978), where it is assumed that the distributor maximizes his profits first, conditioned on any possible manufacturer’s price, and thereby yields a derived demand function, which the manufacturer faces. The latter then maximizes his profits with respect to his decision variables. These results can then be substituted back into the distributor’s policies to provide the unique characterization of the behavior of the system. Our modeling approach is thus similar to McGuire and Staelin’s (1983) leader/follower formulation and it differs from Jeuland and Shugan (1983) and Lal and Staelin’s (1984) joint optimization approach, and from Jorgensen’s (1985) dynamic model which seeks open-loop Nash noncooperative equilibrium strategies. To summarize previous efforts and position our work more clearly, we provide (in Table 1) a summary of various aspects incorporated in the most relevant literature.

The paper is organized as follows. We begin by formally stating the distributor’s and the manufacturer’s problems. We then derive sets of general propositions characterizing the distributor’s policies (§2), and the manufacturer’s policies (§3). Sketches of the proofs are given in an Appendix. More technical details are provided in Eliashberg and Steinberg (1984). From a more technical standpoint, our optimal control solution procedure involves the “indirect adjoining” approach which has not often been used in the literature. Again, we invite the interested readers to review the above reference for additional information. The type of problem examined here is complicated and the derivation of the results involves complex manipulations. Hence, to better illustrate our analyses, the distributor’s and manufacturer’s analytical results are presented in corollaries developed under special circumstances and supplemented with a numerical example (§4). In §5, we summarize the results of the paper and provide suggestions for future research.

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3 See Kotler (1984, p. 467) for a classification of industrial goods.
TABLE 1

Summary of Most Relevant Literature

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2. The Distributor’s Problem Formulation and Policies

2.1. The Distributor’s Problem

The distributor attempts to maximize profits over some known horizon that comprises a season. His profits equal total revenues minus his costs of procurement (which depend, of course, on the price charged by the manufacturer), of processing, and of holding inventory, subject to various constraints. Specifically, the constraints require that: rate of change in inventory is equal to the difference of production and demand, inventory and processing quantity are nonnegative, selling price is at least as great as cost of procurement and no greater than the price which would force demand to zero. To capture the season in its entirety, we let the initial and terminal inventory be equal to zero.

To accommodate the continuous nature of the product flow, and in order to obtain analytical insights, the distributor’s objective function is formulated as a continuous-time problem given by:

$$\max_{P_d(t), Q_d(t)} \int_0^T \{P_d(t)D_d(t) - P_MQ_d(t) - f_d(Q_d(t)) - h_dI_d(t)\} dt$$

where: (2.1)

- $P_d(t)$ denotes the distributor price at time $t$,
- $D_d(P_d(t))$ denotes the quantity demanded by the distributor’s market at time $t$,
- $P_M$ denotes the price that the manufacturer charges the distributor (transfer price)
- $Q_d(t)$ denotes the quantity processed by the distributor at time $t$,
- $f_d(\cdot)$ denotes the distributor's processing cost function,
- $h_d$ denotes the distributor's inventory holding cost per unit,
- $I_d(t)$ denotes the distributor's inventory level at time $t$,
- $T$ denotes the time horizon of concern.

We find it convenient for the control formulation of the problem to observe that for $I_d(0) = I_d(T) = 0$ (this condition characterizes a season):

$$\int_0^T Q_d(t) dt = \int_0^T D_d(t) dt.$$  (2.2)
This follows from the fact that all demand is met and that both initial inventory and terminal inventory are zero. A few additional comments are in order now. First, we note that a constant \( P_M \) is incorporated in the objective function (2.1). This corresponds to situations where contractual pricing exists between the two parties in the distribution channel. Second, we are assuming that there is a fixed cost of playing the Stackelberg game which does not need to be included explicitly in the formulation. This fixed cost can be thought of as the total order cost for the season. Once the relationship between the manufacturer and the distributor has been established, no other fixed order cost is incurred. Third, since the prices are changing continuously over a short horizon, the inventory holding costs are calculated on a per unit basis rather than on a per value of inventory basis. This is a good representation of a firm’s inventory policy in such situations. Finally, no stock-out costs or salvage value are incorporated. The possibility that initial and terminal inventory are equal but non-zero can be accommodated easily in our model and it will not alter our results. The incorporation of stock-out costs and salvage value can also be done; however, it is likely to complicate the analysis and it is not clear that such complexity would be merited.

2.2. The Distributor’s Policies

The following set of propositions demonstrate the distributor’s processing, pricing, and inventory policies. Mathematically, they hold for any assumption made concerning: (1) deterministic distributor demand function, \( D_D(P_D(t)) \), (2) concave seasonality factor (a reversed U-shape which does not necessarily have to be symmetric), and (3) strictly convex nonnegative increasing processing cost function, \( f_D(Q_D(t)) \). We turn now to a discussion of these assumptions.

Our model is deterministic. We acknowledge that demand uncertainty jointly with stockout costs may influence our results. However, this is not always true. For example, if demand uncertainty is modeled via parametric stochasticity (Eliashberg and Chatterjee 1986), it is possible that, for some parameters’ ranges, the amount of uncertainty will lead the distributor to end up the season with some positive inventory level. However, it has been shown (e.g., Eliashberg, Tapiero, and Wind 1985) that in dynamic problems with stochastic parameters, the practically observed values of the model parameters often yield implications which do not deviate greatly from those implied by deterministic models. Hence, the issue is that there are practical situations where deterministic dynamic models do provide a good and reasonable approximation for uncertain phenomena. This, of course, is an empirical issue. Moreover, the proposed procedure can be supplemented with a sensitivity analysis conducted under pessimistic, most likely, and optimistic scenarios to yield policy implications. This practice has been employed extensively by marketing managers.

As far as the convexity of the processing cost function is concerned, according to Nicholson (1978), this is the \textit{a priori} expectation, since the assumption is that many firms are usually operating “near capacity” and that increasing output will raise costs. Nicholson offers that the important question is whether such increasing costs can be brought about by the relatively small fluctuations in output that occur in a firm’s normal experience.

Johnson and Montgomery (1974, p. 208) have also noted in this regard:

\begin{quote}
The convex production cost function can result from situations where there are multiple production (or procurement) sources in a period and it is assumed that production costs are proportional to the quantity produced by a source. By assigning production first to the source with the lowest unit cost until its capacity is reached, then proceeding to use the next cheapest source to capacity, etc., one develops a total production cost that is convex in the amount scheduled for the period.
\end{quote}

Empirically, the traditional econometric efforts have yielded mixed evidence. How-
ever, the convexity hypothesis has received empirical support via an approach called "process analysis," which was taken up by Griffin (1972). Griffin shows that a process analysis approach (Manne 1968), rather than a statistical cost function approach, yields the classical short-run cost function property of rising marginal costs (i.e., convexity). The statistical cost function approach makes use of accounting data and uses sample observations of costs and outputs to estimate the cost function. The process analysis approach describes the cost function from engineering data. Griffin considers why the process analysis approach tends to be overlooked by econometricians. He suggests that the "quantity and relative scarcity of engineering data compared to accounting data may explain the neglect of the process analysis approach. Also, process analysis requires a greater technical knowledge of the industry and greater computational effort than the statistical cost function approach."

Griffin studied the U.S. petroleum refining industry. Emphasizing that his is a short-run analysis since the configurations of capital equipment are fixed, he plots cost vs. output and "confirms the classical textbook shape of marginal cost curves as it rises over a broad range of output." Explains Griffin: "The phenomena producing the upward sloping marginal cost curve can be traced back to the limited capacities of the twelve major process units. . . . When a particular process unit reaches capacity, larger outputs can still be produced through substitution between processes. But such substitution involves a cost." In fact, Griffin finds that the activities of some units are reduced before reaching capacity, because at very high output levels they consume inputs which are more valuable in other uses.

In deriving the various policies, we focus on interior solutions for the distributor's problem. For further treatment of corner solutions, see Eliashberg and Steinberg (1984). All the propositions stated below are valid whenever the distributor's inventory holding cost is sufficiently low. We later provide a precise parametric condition for how low it should be.

**PROPOSITION 1.** The distributor follows a two-part processing strategy. During the first part of his processing schedule, he processes at a constantly increasing rate; during the second part of his processing schedule, which begins at the distributor's stockless point $t_D$, he processes at precisely his market demand rate.

**PROOF.** See Appendix A.

**PROPOSITION 2.** The distributor follows a two-part pricing strategy. He increases his price at a decreasing rate during the first part of the pricing schedule; then, during the second part of the pricing schedule, he decreases his price at an increasing rate.

**PROOF.** See Appendix A.

**PROPOSITION 3.** The distributor follows a three-part inventory strategy. During the first part of his inventory schedule, he builds up inventory at a decreasing rate; during the second part of his inventory schedule, he draws down inventory first at an increasing and then a decreasing rate; during the third part of his inventory schedule, which begins at time $t_2$, he carries no inventory.

**PROOF.** See Appendix A.

We can illustrate graphically the nature of the distributor's policies under the general conditions described above. They are shown in Figure 2.1. As can be seen, $t_D$, the point in time at which the distributor begins carrying no inventory, divides into two parts both the distributor's processing policy ($\mathcal{O}_D^1$ and $\mathcal{O}_D^2$) and his pricing policy ($P_D^1$ and $P_D^2$). Figure 2.1 also illustrates that the distributor's processing policy depends on a function $\psi_D(t)$ which can be constructed from the basic parameters and functions of the problem.
We provide now an intuitive interpretation for the policies shown in Figure 2.1. The distributor, facing a seasonal demand in his market, which first increases and then decreases, can smooth out his operations. If he were to carry no inventory at all throughout the season, i.e., if he did not smooth processing and were to follow a stockless processing policy, he would incur higher costs due to the convexity of his processing cost function. This is the essence of the problem. Instead, under certain circumstances (e.g., sufficiently low inventory holding costs) the distributor can smooth his operation by carrying inventory for some period (at the beginning of the season), and no inventory at all thereafter (right through to the end of the season). This smoothing policy can be obtained by synchronizing the distributor's processing and pricing strategies. We note that the practice of decreasing pricing as the end of the season approaches (Proposition 2) is a phenomenon often observed in many U.S. markets.

To obtain more insight into the distributor's policies we have to invoke a few more specific assumptions concerning $D(P_d(t))$ and $f_d(Q_d(t))$. Three specific assumptions are invoked concerning the distributor's problem. We rationalize and support them now.

1. Linear Demand Function. Linear demand functions have been employed in marketing in various static analyses (McGuire and Staelin 1983) as well as in dynamic analyses (Eliashberg and Jeuland 1986). Here we follow also Pekelman (1974) and adopt the following linear demand equation:

\[
D(P_d(t), t) = D_p(t) = a_p(t) - b_p P_d(t),
\]

where

\[
a_p(t) > 0 \quad \text{for} \quad 0 \leq t \leq T, \quad \text{and} \quad b_p > 0.
\]

Figure 2.1. The Distributor's Policies.
The term \(a_D(t)\) can be interpreted as the time-varying total market potential, i.e., the distributor market that can be captured when price is set equal to zero. The second parameter, \(b_D\), can be interpreted as the coefficient of price sensitivity, which we take to be stationary. Hence, the linear demand function provides an intuitively appealing model which separates conveniently the seasonal and price sensitivity effects.

2. Quadratic Seasonality Effect. In order to capture the seasonality effect we have chosen to model the market potential term, \(a_D(t)\), through a quadratic formulation which provides interesting interpretations. That is,

\[
a_D(t) = -\alpha_1 t^2 + \alpha_2 t + \alpha_3, \quad 0 \leq t \leq T, \quad \text{where} \quad T = \alpha_2/\alpha_1, \quad \alpha_1, \alpha_2, \alpha_3 > 0.
\]

Here, \(\alpha_3\) represents the “nominal” size of the market potential before the season begins. The parameters \(\alpha_1\) and \(\alpha_2\) determine the timing \((\alpha_2/2\alpha_1)\) and the magnitude \((\alpha_3 + (\alpha_2^2/4\alpha_1))\) of the peak sales. It is straightforward to show that larger values of \(\alpha_1\) will move the peak sooner and will lower its magnitude, whereas larger values of \(\alpha_2\) will have opposite effects. Finally, \(T\) is set equal to \(\alpha_2/\alpha_1\) in order to encompass the season in its entirety. Taken together, assumptions (2.3) and (2.3a) provide a reasonable method to model the time varying seasonal distributor’s market.

3. Quadratic Processing Cost Function. The quadratic cost function

\[
f_D(Q_D(t)) = (1/K_D)Q_D^2(t), \quad \text{where} \quad K_D > 0,
\]

has been employed extensively in the production literature (see, for example, Pekelman 1974). In economics, Walters (1963) reports that “the equation relating total costs to output used by most authors is usually a quadratic.”

A quadratic processing cost function may be appropriate under various scenarios. One possibility is the following. Suppose that there are only two input factors of production \(C\) and \(L\) and that the production function is given by the following Cobb-Douglas formulation: \(Q = VC L\), where \(Q\) represents output and \(C\) and \(L\) represent capital and labor input factors, respectively. If one assumes that capital inputs are fixed, then: \(L = Q^2/C\). Now, if the cost of labor is \(w\) dollars per labor and time unit, then the total production variable costs become \(wQ^2/C\). This relationship is captured in our formulation as \(K = C/w\). Thus, \(K\) can be thought of as a measure of capital per labor processing efficiency.

Based on the considerations noted above, and replacing (2.3) and (2.4) in (2.1), the control formulation of the distributor’s problem becomes:

\[
\begin{align*}
\text{Max} \int_0^T \{& (P_D(t) - P_M)(a_D(t) - b_D P_D(t)) - (1/K_D)Q_D^2(t) - h_D I_D(t)\} dt \\
\text{S.T.} \quad & I_D(t) = Q_D(t) - a_D(t) + b_D P_D(t), \\
& I_D(t) \geq 0, \\
& Q_D(t) \geq 0, \\
& P_D(t) \geq P_M, \\
& P_D(t) \leq a_D(t)/b_D, \\
& I_D(0) = I_D(T) = 0.
\end{align*}
\]

\* In a recent paper dealing with competitive issues (Eliashberg and Steinberg 1987), the quadratic formulation of the seasonality effect has been generalized to positive, strictly concave, and increasing-then-decreasing functions. The more general formulation of the seasonality effect will apply to the problem presented here. However, for clarity of exposition, the quadratic form only is presented here.
We first analyze the parametric conditions under which optimal policies to problem (2.5)–(2.11) exist. This has important implications for the more general question: under what circumstances is it worthwhile for the distributor to smooth out his operations? Proposition 4 addresses this issue.

**Proposition 4.** In general, if the distributor’s inventory holding cost per unit is sufficiently low, price sensitivity is low, processing efficiency is low, and the seasonal demand is volatile, he can smooth out his operations. In particular, under assumptions (2.3), (2.3a) and (2.4):

- If (i) \( h_D < \alpha_2/3(b_D + K_D) \), the distributor can smooth out his operations.
- If (ii) \( h_D \geq \alpha_2/3(b_D + K_D) \), the distributor should not smooth out his operations and act according to stockless production policy throughout the season.

**Proof.** See Appendix A.

If the distributor can smooth out his operations, then Corollary 5 shows the appropriate policies he should take.

**Corollary 5.**

\[
Q_D(t) = \begin{cases} 
\left[ K_D/2(b_D + K_D) \right] \left[ \alpha_D(t_D) - h_D(b_D + K_D)(t_D^* - t) - b_D P_M \right], & 0 \leq t \leq t_D^*, \\
\left[ 1/2(b_D + K_D) \right] \left[ (b_D + K_D) a_D(t)/b_D + a_D(t_D) \right], & t_D^* \leq t \leq T.
\end{cases}
\] (2.12)

\[
P_D(t) = \begin{cases} 
\left[ 1/2(b_D + K_D) \right] \left[ (2b_D + K_D) a_D(t)/b_D + K_D P_M \right], & 0 \leq t \leq t_D^*, \\
(\alpha_1/6)(t_D^* - t)^2 t, & 0 \leq t \leq t_D^*, \\
0, & t_D^* \leq t \leq T.
\end{cases}
\] (2.13)

\[
I_D(t) = \begin{cases} 
(\alpha_1/6)(t_D^* - t)^2 t, & 0 \leq t \leq t_D^*, \\
0, & t_D^* \leq t \leq T.
\end{cases}
\] (2.14)

where:

\[ t_D^* = (3/4\alpha_1) \left[ \alpha_2 - (b_D + K_D) h_D \right]. \] (2.15)

**Proof.** See Appendix A.

Some remarks should be in order here. The expression \( a_D(t)/b_D \), which appears in (2.13), represents the maximum possible distributor’s price at time \( t \), and \( P_M \) is the manufacturer’s contractual price over the season. Thus, the second expression in (2.13) states that, after the distributor begins with stockless processing policy (at \( t_D^* \)), his pricing policy at any point in time is a weighted average between his maximum and minimum prices. This result seems intuitively appealing. As far as the distributor’s inventory policy is concerned, equation (2.14) tells us that the inventory builds to a peak at \( t_D^*/3 \), then decreases to zero at \( t_D^* \) and remains at zero until the end of the season. As might be expected, the quantity processed by the distributor decreases as the procurement cost \( (P_M) \) increases. On the other hand, the price he should charge \( (P_D) \) increases as the procurement cost increases.

If the distributor finds it worthwhile to smooth out his operations (Proposition 4), then from (2.15) (Corollary 5) we can learn how early in the season he should stop carrying his inventory. The \( t_D^* \), the time at which the distributor stops carrying inventory, depends on the parameters \( \alpha_1, \alpha_2, b_D, K_D \) and \( h_D \). Hence, for example, relatively larger values of \( K_D \), i.e., greater processing efficiency, lead to the decision to stop carrying inventory sooner (i.e., smaller values of \( t_D^* \)). Also, if inventory holding cost \( h_D \) increases, for example, then again \( t_D^* \) decreases. Finally, since larger values of \( \alpha_1 \) will move the peak sales demand sooner in time and will lower its magnitude, we might expect, under such circumstances, the distributor to begin the stockless production policy earlier. This is also confirmed by (2.15).
3. The Manufacturer’s Problem Formulation and Policies

3.1. The Manufacturer’s Problem

The manufacturer has a profit maximization problem analogous to the distributor’s. The significant differences are as follows. Corresponding to the distributor’s purchase cost per unit of \( P_M \), the manufacturer incurs a raw materials cost of \( C_M \) per unit. Corresponding to the distributor’s demand function of \( \alpha D(t) - \beta DPD(t) \), the manufacturer faces a derived demand function of \( QD(PM, t) \), the distributor’s processing rate which is a function of the manufacturer’s price and time. Also, \( PM \), which is not a function of time, is bounded above by \( P*(t) \) and below by \( C_M \). Otherwise, the formulation carries over directly from the distributor’s problem, where each subscript \( D \) is replaced by a subscript \( M \).

Let \( QM(t), h_M, I_M, \) and \( f_M(\cdot) \) denote the manufacturer’s production rate, inventory holding cost per unit, inventory level, and production cost, respectively. The control formulation of the manufacturer’s problem will thus be:

\[
\text{max}_{Q_M(t), P_M} \int_0^T \left( (P_M - C_M)Q_M(t) - f_M(Q_M(t)) - h_M I_M(t) \right) dt \tag{3.1}
\]

S.T. \( I_M(t) = Q_M(t) - QD(t) \), \( I_M(t) \geq 0 \), \( QM(t) \geq 0 \), \( P_M > C_M \), \( P_M < P*(t) \), \( I_M(0) = I_M(T) = 0 \). \( \tag{3.2-3.7} \)

3.2. The Manufacturer’s Policies

The manufacturer’s production and inventory policies are similar to the distributor’s processing and inventory policies, respectively. They are also valid whenever the manufacturer’s inventory holding cost is sufficiently low. However, in general, the manufacturer’s schedules will divide up the season at different points than the distributor’s schedules. Propositions 6 and 7 formalize it.

PROPOSITION 6. The manufacturer follows a two-part production policy. During the first part, he produces at constantly increasing rate; during the second part of his production schedule, which begins at the manufacturer’s stockless point, \( t_M \), he produces at exactly the distributor’s processing rate.

PROOF. Similar to the proof of Proposition 1.

PROPOSITION 7. The manufacturer follows a three-part inventory policy. During the first part of his inventory schedule, he builds up inventory at decreasing rate; during the second part of his inventory schedule, he draws down inventory at first an increasing and then a decreasing rate; during the third part of his inventory schedule, which begins at time \( t_M \), he holds no inventory.

PROOF. Similar to the proof of Proposition 3.

Figure 3.1 illustrates the general production and inventory policies of the manufacturer and their relationships to the distributor’s processing and inventory policies. The following insights can be obtained.

1. The manufacturer, being the more powerful party in the distribution channel (the “leader” in Stackelberg sense), can smooth out his production even more than the distributor.
(2) The time at which the manufacturer acts according to stockless production policy is later than that of the distributor ($t_M > t_D$). This result is driven mainly by the opportunity given to the manufacturer to smooth out his production activities more.

(3) It is also interesting to note that the function $\Psi_D(t)$ defined earlier at the distributor's level is important at the manufacturer's level. After time $t_M$, the manufacturer's production policy becomes proportional to $\Psi_D(t)$.

As we did in the distributor's problem, we first analyze the parametric conditions under which optimal policies to problem (3.1)-(3.7) under assumptions (2.3), (2.3a), and (2.4) exist. The result is stated in:

**PROPOSITION 8.** In general, if the manufacturer's inventory holding cost per unit is sufficiently low, the distributor's processing efficiency and inventory holding cost per unit are high, the manufacturer can smooth out his operations. In particular, under assumption (2.3), (2.3a), and (2.4):

If (i) $h_M < K_D h_D / K_M$, the manufacturer can smooth out his production.

If (ii) $h_M \geq K_D h_D / K_M$, the manufacturer should not smooth out his operations and act according to stockless production policy throughout the season.

**PROOF.** See Appendix B.

If the manufacturer can smooth out his operations, then we can obtain analytical expressions for his production, pricing and inventory policies. These results are presented in:

**COROLLARY 9.**

$$Q_M^*(t) = \begin{cases} 
K_D / 2(b_D + K_D)[a_D(t_M) - h_M(K_M / K_D)(b_D + K_D)(t_M - t) - b_D P_M] & \text{for } 0 \leq t \leq t_M, \\
K_D / 2(b_D + K_D)[a_D(t) - b_D P_M], & \text{for } t_M \leq t \leq T,
\end{cases}$$

where $t_M = (3/4\alpha_1)[\alpha_2 - (K_M / K_D)(b_D + K_D)h_M]$. 

FIGURE 3.1. The Manufacturer's Policies.
\[ P_M^* = w_1 \left( \frac{1}{T} \int_0^T \frac{(a_D(t)/b_D)dt}{b_D} \right) + w_2 C_M \quad \text{where:} \] (3.10)

\[ w_1 = [1 + (2b_M/K_M)/[2 + (2b_M/K_M)], \quad w_2 = 1/[2 + (2b_M/K_M)], \quad \text{and} \] (3.11)

\[ b_M = b_D K_D/2(b_D + K_D), \] (3.12)

\[ I_M^*(t) = \begin{cases} 
    [K_D/2(b_D + K_D)][(1/3)\alpha_1((t_M^*)^2 - (t_D^*)^2)t - (1/4)[K_D h_D - K_M h_M]t^2, & 0 \leq t \leq t_D^*, \\
    [K_D/2(b_D + K_D)][(1/3)\alpha_1(t_M^* - t)^2], & t_D^* \leq t \leq t_M^*, \\
    0, & t_M^* \leq t \leq T.
\end{cases} \] (3.13)

Some useful insights can be obtained from Corollary 9.

1. The expression: \( (1/T) \int_0^T \frac{(a_D(t)/b_D)dt}{b_D} \), which appears in (3.10) represents the average of the maximum possible distributor's price (hence the average of the maximum possible manufacturer's price) over the season. The parameter \( C_M \) is the per-unit raw materials cost and represents the minimum possible manufacturer's price. Thus (3.10) states that the manufacturer's (contractual) price is a weighted average of his minimum and average maximum prices.

2. Comparing (3.13) with (2.14), we note that the distributor's inventory policy depends only on his parameters \( (\alpha_1, \alpha_2, b_D, K_D, h_D) \), whereas the manufacturer, acting as a leader in the channel, and in some sense, as the channel coordinator, has to consider his own as well as the distributor's parameters in designing his inventory policy.

3. The distributor has a simple strategy with regard to holding inventory once his stockless point is determined: (i) divide the inventory holding period (from time 0 to time \( t_D^* \)) precisely into thirds, (ii) build up inventory at a decreasing rate for the first third, (iii) draw down inventory at an increasing rate for the second third, (iv) draw down inventory at a decreasing rate for the final third. (This follows from (2.14).) The manufacturer, unlike the distributor, cannot simply divide his inventory holding period into thirds to determine the critical points in his inventory policy, since his problem is dependent upon the parameters of the distributor's problem as well as his own (from (3.13)).

To obtain a better feel for the nature of the policies derived, we present in the next section an illustrative example.

4. An Illustrative Example

Suppose that:

\[ h_D = \frac{1}{30}, \quad K_D = 2, \quad a_D(t) = -t^2 + 6t + 12, \]
\[ h_M = \frac{1}{30}, \quad K_M = 2, \quad [\alpha_1 = 1, \alpha_2 = 6, \alpha_3 = 12], \]
\[ b_D = 1, \]
\[ C_M = 3 \frac{5}{16}. \]

Hence, \( T = \alpha_2/\alpha_1 = 6. \) From (3.12) \( b_M = (b_D K_D)/2(b_D + K_D) = \frac{1}{4}. \)

From (3.11), \( w_1 = \frac{9}{10} \) and \( w_2 = \frac{1}{3}. \)

We first verify the parametric conditions: \( h_D < \alpha_2/3(b_D + K_D) \) since \( \frac{1}{6} < \frac{3}{2} \) and \( h_M < K_D h_D/K_M \) since \( \frac{1}{30} < \frac{1}{2}. \)

Since all conditions are satisfied, the derived policies are valid in this example. Since \( T = 6 \) and \( C_M = 3 \frac{5}{16}, \) by (3.10), \( P_M^* = 11.957. \)
Further, from (2.15) and (3.9), $t^*_D = 4.3875$ and $t^*_M = 4.425$. Hence, $a_D(t^*_D) = 19.0748$ and $a_D(t^*_M) = 18.9694$.

\[(2.12) \Rightarrow Q^*_D(t) = \begin{cases} 0.05t + 2.153, & 0 \leq t \leq 4.3875, \\ -0.333t^2 + 2t + 0.014, & 4.3875 \leq t \leq 6. \end{cases} \]

\[(2.13) \Rightarrow P^*_D(t) = \begin{cases} -0.5t^2 + 3.025t + 13.055, & 0 \leq t \leq 4.3875, \\ -0.66t^2 + 4t + 11.986, & 4.3875 \leq t \leq 6. \end{cases} \]
In this paper we have studied the following scenario.

There exist a distributor and a manufacturer where, over a season, the distributor buys only from the manufacturer and the manufacturer sells only to the distributor. The manufacturer, however, is assumed to be the leader (in Stackelberg sense) in this setting. The distributor can affect demand for his product by varying his price over the season. The manufacturer and the distributor have to agree upon a per-unit transfer (contractual) price throughout the season. (Perhaps the unique buyer/seller relationship is predicated on this fixed-price arrangement.) The product is delivered continuously to

5. Summary and Suggestions for Future Research

In this paper we have studied the following scenario.

There exist a distributor and a manufacturer where, over a season, the distributor buys only from the manufacturer and the manufacturer sells only to the distributor. The manufacturer, however, is assumed to be the leader (in Stackelberg sense) in this setting. The distributor can affect demand for his product by varying his price over the season. The manufacturer and the distributor have to agree upon a per-unit transfer (contractual) price throughout the season. (Perhaps the unique buyer/seller relationship is predicated on this fixed-price arrangement.) The product is delivered continuously to
the distributor who can vary his processing rate. The manufacturer can vary his production rate over the season. How should each party behave in order to maximize his total profits over the season? Our analytical results suggest the following guidelines.

**Distributor's Policies.** Whenever the distributor's inventory holding cost per unit is sufficiently low (how low would depend on the distributor's processing efficiency, price sensitivity, and the season volatility), the distributor processes at a constantly increasing rate, where the processing rate is at first greater than the demand rate and then is less than the demand rate. The net effect will be to build up inventory for a while, and then
TABLE 2
Comparison of Profits Under Two Different Scenarios

<table>
<thead>
<tr>
<th></th>
<th>Optimal Manufacturer’s Price</th>
<th>Distributor’s Total Profits</th>
<th>Manufacturer’s Total Profits</th>
<th>Channel’s Total Profits</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Distributor and Manufacturer Both Act Optimally</td>
<td>11.957</td>
<td>45.723</td>
<td>84.315</td>
<td>130.038</td>
</tr>
<tr>
<td>II. Both Distributor and Manufacturer Hold No Inventory Throughout the Season</td>
<td>11.957</td>
<td>43.716</td>
<td>82.804</td>
<td>126.520</td>
</tr>
</tbody>
</table>

to draw down inventory for a while. When inventory reaches zero, the distributor continues with a stockless processing policy, i.e., he processes just enough to meet demand and thus holds no inventory until the end of the season. As to pricing policy, the distributor first increases his price at a decreasing rate, and then decreases his price at an increasing rate.

Manufacturer’s Policies. If the manufacturer’s inventory holding cost per unit is sufficiently low (relative to the distributor’s inventory holding cost, the distributor’s processing efficiency, and the manufacturer’s production efficiency), the manufacturer should produce at a constantly increasing rate, where his production rate is at first greater than the processing rate of the distributor, and then less than the processing rate of the distributor. The net effect will be to build up and then draw down inventory. When inventory reaches zero, the manufacturer continues with a stockless production policy, i.e., producing at the same rate at which the distributor is processing.

Contractual Price within the Channel. The manufacturer’s contractual (constant) price to the distributor lies between the manufacturer’s per-unit raw materials cost and the average of the maximum possible distributor’s price over the season. The weighting scheme depends upon two factors: the manufacturer’s production efficiency and the price sensitivity of the manufacturer’s derived demand function which is determined from the distributor’s processing policy.

The dynamic of coordination in vertical channels of distribution is still a new and, we believe, promising area which deserves further attention. Possible further research avenues include: (1) an explicit analysis of other shapes and types of demand functions such as those incorporating cyclical fluctuations and uncertainty, (2) analysis of other types of contracts such as volume-based contracts, (3) analysis of other channel settings characterized by multiple products/distributors, and (4) analysis of cooperation and bargaining modes of behavior in the distribution channel.

Appendix A

PROOF OF PROPOSITIONS 1–3. In general, the distributor’s problem is:

$$\text{Maximize } \int \left( (P_D(t) - P_M)D_D(P_D(t)) - f_D(Q_D(t)) - h_DI_D(t) \right) dt$$

subject to:

$$\dot{D}_D(t) = Q_D(t) - D_D(P_D(t)).$$

We assume throughout that the demand function exhibits concave seasonality (i.e., reversed U-shape) pattern. For notational simplicity, the time argument will be omitted.

Assuming an interior solution, the Lagrangian is:

$$L_D = \left[ P_D - P_M \right] D_D(P_D) - f_D(Q_D) - h_D I_D + \lambda D_D Q_D - \lambda P_D.$$  

For demand functions concave with respect to price, the necessary and sufficient conditions are:

$$\frac{\partial L_D}{\partial Q_D} = -f’_D(Q_D) + \lambda _p = 0.$$  

Hence,
where \( f'(\cdot) \) denotes the inverse function of the first derivative of \( f_d(\cdot) \). \( \frac{dL_d}{dP_d} = (P^*_d - P_M)D_d(P^*_d) + D_d(P^*_d) - \lambda_d D_d(D_d(P^*_d)) \). Hence,

\[
P^*_d = \left[ D_d(D_d(P^*_d)) / D_d(D_d(P^*_d)) \right] = \lambda_d + \rho_d + P_M.
\]

Equation (A.2) shows that \( P^*_d \) can be expressed as an implicit function of \( \lambda_d + \rho_d \).

Now,

\[
\dot{I}_d = Q_d - D_d(P^*_d) = f'(\lambda_d + \rho_d) - D_d(P^*_d)(\lambda_d + \rho_d))
\]

where \( P^*_d(\lambda_d + \rho_d) \) is the solution to (A.2). Hence, \( \dot{I}_d \equiv 0 \) whenever,

\[
f'(\lambda_d + \rho_d) - D_d(P^*_d)(\lambda_d + \rho_d))) \neq 0.
\]

Equation (A.5) can be viewed as an implicit function of \( \lambda_d + \rho_d \) from which it is possible to define a condition or a function, \( \Psi_d(t) \), which solves (A.5) for \( \lambda_d + \rho_d \).

For example, if \( D_d(P^*_d) = a_d - b_d P_d \) and \( f_d(Q_d) \) is any strictly convex nonnegative increasing function, and if we denote by: \( g_d(\lambda_d + \rho_d) = f_d'(\lambda_d + \rho_d) + (b_d/2)(\lambda_d + \rho_d) \), then: \( \Psi_d(t) = g_d'((a_d(t) - b_d P_M)/2) \).

The behavior of \( Q_d, P_d \) and \( I_d \) can be ascertained from (A.1)-(A.3) based on the behavior and constraints imposed on \( \lambda_d + \rho_d \) and the facts that: (1) \( \Psi_d(t) \) is concave in time because of the seasonality effect, and (2) the time horizon is broken down into two parts: \([0, t_d] \) and \([t_d, T] \), where \( \int_0^t \dot{I}_d(t) = 0 \). Q.E.D. (See the proof of Corollary 5 and Eliashberg and Steinberg 1984 for more detail.)

**Proof of Corollary 5.** For the problem presented by (2.5) through (2.11), the Lagrangian is:

\[
L_d = H_d + Q_d(P_d - a_d + b_d P_d) + \mu_d Q_d + \eta_d(P_d - P_M) + \eta_d a_d/b_d - P_d)
\]

where the Hamiltonian, \( H_d \), is:

\[
H_d = (P_d - P_M)(a_d - b_d P_d) - h_d I_d - (1/K_d)Q_d + a_d + b_d P_d).
\]

**Distributor's Problem: Necessary Conditions**

\[
\frac{dL_d}{dQ_d} = 0 = (-2/K_d)Q_d + \lambda_d + \rho_d + \mu_d.
\]

\[
\frac{dL_d}{dP_d} = 0 = a_d - 2b_d P_d + b_d P_M + \lambda_d b_d + \rho_d b_d + \rho_d - \eta_d b_d.
\]

\[
P^*_d = [a_d + b_d(\lambda_d + \rho_d + P_M) + \eta_d/b_d - \eta_d] / 2b_d.
\]

\[
\frac{dL_d}{d\lambda_d} = 0 = -\frac{dL_d}{dI_d} = h_d.
\]

Observe that \( \partial^2 L_d / \partial Q_d < 0, \partial^2 L_d / \partial P_d < 0 \). Also, since \( Q_d \) and \( P_d \) do not depend on the state variable \( I_d \), the maximized Lagrangian is linear and hence concave in the state variable \( I_d \). Therefore, a solution to the foregoing necessary conditions (A.7)-(A.9) is also a solution to the problem given in (2.5)-(2.11). (See Kamien and Schwartz 1981, p. 207.)

**Distributor's Problem: Complementary and Nonnegativity Conditions**

\[
\rho_d I_d, \rho_d I_d, \mu_d Q_d, \eta_d(P_d - P_M), \eta_d (a_d/b_d - P_d) = 0,
\]

\[
(\rho_d, \mu_d, \eta_d, \eta_d) \geq 0.
\]

The variable \( I_d(t) \) is continuous; \( \lambda_d \) is continuous except possibly at an entry or an exit to the boundary \( I_d(t) = 0; \lambda_d + \rho_d \) is continuous everywhere; and \( \rho_d \leq 0 \) (see McIntyre and Paiewonsky 1967, and Speyer and Bryson 1968).

**Distributor's Problem: Transversality Conditions**

\[
[h_d(T) + \rho_d(T)]I_d(T) = 0,
\]

\[
\lambda_d(T) + \rho_d(T) = 0.
\]

In order to simplify the analysis below, we assume an interior solution. That is, \( a_d/b_d < P_d < P_M \) and \( 0 < Q_d \) for \( 0 \leq t < T \). Rewriting (A.8) it can be shown (see Eliashberg and Steinberg 1984) that the interior solutions for \( P^*_d \) and \( Q^*_d \) are given in terms of \( \lambda_d + \rho_d \) by:

\[
P^*_d = (1/2)(\lambda_d + \rho_d + a_d/b_d + P_M) \quad \text{if} \quad -a_d/b_d + P_M < \lambda_d + \rho_d \leq a_d/b_d - P_M.
\]

\[
Q^*_d = (K_d/2)(\lambda_d + \rho_d) \quad \text{if} \quad \lambda_d + \rho_d \geq 0.
\]
We would like to express \( P_D \) and \( Q_D \) as functions not involving \( XD \) or \( PD \). We do this by examining behavior of \( XD + PD \). We find it convenient to consider two cases: when the distributor’s inventory is positive, and when it is zero. A time interval over which the inventory is positive is called an unconstrained segment. A time interval over which the inventory is zero is called a boundary segment. If inventory is positive, then \( PD \) is zero by (A.10a), and thus \( AD + PD \) equals \( XD \). In order to find \( XD \), we need to know only \( XD(0) \) (the initial value of \( XD \) on that segment) since \( AD \) equals \( hD \) by (A.9). If inventory is zero for a nonzero interval, then we also have that \( ID \) is zero, or equivalently by (2.6) \( QD - (aD - bDPD) = 0 \). It can be shown (Eliashberg and Steinberg 1984) that \( ID = QD - (aD - bDPD) \) can equal zero only whenever \( 0 < XD + PD < \frac{aD}{bD} - PM \). Specifically, \( ID \) will be larger than, equal to, or smaller than zero according to whether \( XD + PD \) is larger than, equal to, or smaller than \( \Psi_D \), where \( \Psi_D \) is defined in terms of the basic parameters and functions of the model such that:

\[
\Psi_D = \left( aD - bDPM \right) / \left( bD + K_D \right).
\]  

(A.15)

Hence, in general, the distributor’s processing, pricing and inventory policies are broken down as follows:

**On an Unconstrained Segment (Inventory is Positive, \( XD(t) = XD(0) + hDt \)):**

\[
(A.10) \quad QD = QD_1 = K_D XD_0 / 2, \\
(A.9) \quad P_D = P_D_1 = (1/2)(\lambda_D + aD/bD + PM), \\
(2.6) \quad I_D = QD_0 = aD + bDPD_1.
\]  

(A.16)

**On a Boundary Segment (Inventory is Zero, \( XD(t) + PD(t) = XD(t) \)):**

\[
(A.10) \quad QD = QD_2 = K_D XD_0 / 2 = K_D(aD - bDPM) / [2(bD + K_D)], \\
(A.9) \quad P_D = P_D_2 = (1/2)(\Psi_D + aD/bD + PM), \\
(2.6) \quad I_D = I_D = 0.
\]  

(A.17)

In order to determine analytically \( T_D \), the time at which entry to the boundary occurs (i.e., inventory is zero for an interval), one needs to solve \( \Psi_D(t_D) = \Psi_D(t_1) \) or, equivalently:

\[
(K_D/2)[\lambda_D(0) + hD \Psi'] = (K_D/2)[\Psi_D(0)](\lambda_D(0) - bDPM) / [bD + K_D].
\]  

(A.18)

Because \( t_D \) is the first point in time after zero at which \( I_D(t) = 0 \) and recalling that \( I_D(0) = 0 \), then:

\[
\int_0^T I_D(t) dt = 0,
\]  

(A.19)

which is a second equation that \( t_D \) must satisfy.

Substituting and solving (A.18) and (A.19) simultaneously for \( XD(0) \) and \( T_D \) yields the distributor’s processing, pricing and inventory policies. Q.E.D.

**Proof of Proposition 4.** For \( t_D \) to exist, and hence, for the smoothing policies (2.12)–(2.15) to be valid, we need \( t_D \) to lie to the right of the point at which \( \Psi_D \) achieves its maximum value \( aD/2 \lambda_D \) (see Figure 2.1). This implies: \( hD < aD/3(bD + K_D) \) which is the required parametric condition. Q.E.D.

**Appendix B**

**Proof of Corollary 9.** The manufacturer’s problem can be written as:

\[
\begin{align*}
\max_{\lambda_D, \lambda_M} \pi_M(P_M) \\
\text{S.T.} \quad C_M < P_M < \min_{t} aM(t)/bD, \\
\pi_M(P_M) = \max_{Q_M} \int_0^T \left[ (P_M - C_M(aM - bMP_M) - (1/K_M)Q_M^2 dt - hM IM \right] dt \\
\text{S.T.} \quad \dot{I}_M = Q_M - Q_E \\
Q_E = \begin{cases} 
Q_1 = aM_1 - bMP_M & \text{for } 0 \leq t \leq t_D, \\
Q_2 = aM_2 - bMP_M & \text{for } t_D \leq t \leq T, \\
I_M(t) \geq 0, \\
Q_M(t) \geq 0, \\
I_M(0) = I_M(T) = 0.
\end{cases}
\end{align*}
\]  

(B.1)  

(B.2)  

(B.3)  

(B.4)  

(B.5)  

(B.6)  

(B.7)  

(B.8)
and where, from (2.12),

\[ a_M(t) = \begin{cases} 
K_D h_D/2 + K_D [a_M(t)] / (b_D + K_D) & \text{for } 0 \leq t \leq t^*_D, \\
K_D a_M(t) / (b_D + K_D) & \text{for } t^*_D \leq t \leq T,
\end{cases} \]

(B.9)

The Lagrangian is:

\[ L_M = H_M + \rho_M (Q_M - a_M + b_M P_M) + \mu_M Q_M. \]

(B.11)

where the Hamiltonian is:

\[ H_M = (P_M - C_M) a_M - b_M P_M - h_M I_M - (1/K_M) Q_M + \lambda_M Q_M - a_M + b_M P_M. \]

Manufacturer's Problem: Necessary Conditions

\[ \frac{dH_M}{dt} = \dot{\lambda}_M = -\frac{\partial L_M}{\partial I_M} = h_M, \quad \frac{dL_M}{\partial Q_M} = 0 = -\frac{2}{K_M} Q_M + \lambda_M + \rho_M + \mu_M. \]

(B.12)

Thus,

\[ Q_M^* = \frac{(K_M/2)(\lambda_M + \rho_M + \mu_M)}{2}. \]

(B.13)

 Sufficiency can be established similarly to the distributor's problem.

Manufacturer's Problem: Complementarity and Nonnegativity Conditions

\[ \rho_M I_M, \rho_M I_M, \mu_M Q_M = 0, \]

(B.14)

\[ \rho_M, \mu_M \geq 0. \]

(B.15)

The variable \( I_M(t) \) is continuous: \( \lambda_M \) is continuous except possibly at an entry or an exit to the boundary \( I_M(t) = 0; \lambda_M + \rho_M \) is continuous everywhere; and \( \rho_M \leq 0 \).

Manufacturer's Problem: Transversality Conditions

\[ [\lambda_M(T) + \rho_M(T)] I_M(T) = 0, \quad \lambda_M(T) + \rho_M(T) \geq 0. \]

(B.16)

Assuming an interior solution for \( Q_M^* \), that is, \( 0 < Q_M^*(t) \) for \( 0 \leq t \leq T \),

\[ Q_M^* = \frac{(K_M/2)(\lambda_M + \rho_M)}{2} \quad \text{if} \quad \lambda_M + \rho_M \geq 0. \]

(B.17)

Employing an analysis similar to that in the proof of Corollary 5, it can be shown that for the manufacturer, \( I_M \) will be larger than, equal to, or smaller than zero according to whether \( \lambda_M + \rho_M \) is larger than, equal to, or smaller than \( \Psi_M \), where

\[ \Psi_M = (2/K_M) (a_M - b_M P_M). \]

To solve for \( \tau_M \), we need to solve simultaneously:

\[ Q_M^* (\tau_M) = Q_M^* (\tau_M) \quad \text{and} \quad \int_0^{\tau_M} I_M(t) dt = 0. \]

(B.18)

(B.19)

The solution implies (3.9). Substituting back, we obtain after some algebraic manipulations (3.8). To derive \( P_M^* \), we first substitute the obtained results in the objective function (B.3). This yields \( \pi_M (P_M^*) \) which is quadratic (see Eliashberg and Steinberg 1984). Simple calculus and some algebraic manipulations yield (3.10)-(3.13).

PROOF OF PROPOSITION 8. For \( \tau_M \) to exist and, hence, for a valid production smoothing policy to hold, we need the slope of \( Q_M^* (t) \) to be smaller than that of \( Q_M (t) \) for \( 0 \leq t \leq \tau_M \). This implies immediately by (3.8) and (2.12): \( K_M h_M < K_D h_D \), which is the required parametric condition. It is a straightforward matter to verify that the condition \( K_M h_M < K_D h_D \) also implies, as needed, \( \tau_M > \tau_M^* \). Q.E.D.

References


