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COMPETITIVE STRATEGIES FOR TWO FIRMS WITH ASYMMETRIC PRODUCTION COST STRUCTURES*

JEHOSHUA ELIASHBERG AND RICHARD STEINBERG
The Wharton School, University of Pennsylvania, Philadelphia, Pennsylvania 19104
Operations Research Department, AT&T Bell Laboratories, HO 33-301,
Crawfords Corner Road, Holmdel, New Jersey 07733

We model joint production-marketing strategies for two firms with asymmetric production cost structures in competition. The first firm, called the “Production-smoother,” faces a convex production cost and a linear inventory holding cost. The second firm, called the “Order-taker,” faces a linear production cost and holds no inventory. Each firm is assumed to vary continuously over time both its production rate and its price in view of an unstable “surge” pattern of demand. The underlying theoretical motivation is to investigate the temporal nature of the equilibrium policies of two competing firms, one operating at or near capacity (the Production-smoother), and one operating significantly below capacity (the Order-taker).

We characterize and compare the equilibrium strategies of the two competing firms. Among our results, we show that if the duopolistic Production-smoother finds it optimal to hold inventory, then he will begin the season by building up inventory, continue by drawing down inventory until it reaches zero, and conclude by following a “zero inventory” policy until the end of the season. This result, which is robust with respect to the market structure, is compared with a monopolistic Production-smoother policy. We also show that due to the “coupling” effect between the two competing firms, the time at which the Production-smoother begins his zero inventory policy is also critical for the Order-taker who divides his production and pricing strategies into two parts determined by the Production-smoother’s zero inventory point. Numerical examples that illustrate certain aspects of our analyses are provided as well.

(PRODUCTION/INVENTORY; MARKETING/COMPETITIVE STRATEGY)

1. Introduction

In this paper, we study the problem of determining production and marketing equilibrium strategies for two competing firms under an unstable “surge” demand condition (e.g., seasonality), where we focus on the asymmetry in the firms’ production cost structures. Demand fluctuation in general, and seasonality in particular, have been shown to be critical issues in aggregate production planning (Hax and Candea 1984). Hax and Candea discuss several methods that managers can use to absorb changing demand patterns. In most practical situations, management can cope with a surge in demand by accumulating inventory over a period of time.

Any cost modeling of a production system must consider the fundamental question of what form the short-term cost function will take. Although convex cost production functions have been used extensively in the literature (Arrow and Karlin 1958, Veinott 1964, Richard 1969, Vanthienen 1973, Pekelman 1974, Eliashberg and Steinberg 1987), linear production cost functions have received an equal share of attention. Many mathematical programming models based on a linear cost assumption have been developed and applied (see, for example, Johnson and Montgomery 1974, §4-2 and Hax and Candea 1984, §3-2). An important question arises: Under what conditions is a linear production cost appropriate, and under what assumptions should convex be used? The conceptual rationales and empirical evidence cited below are similar to that given in Eliashberg and Steinberg (1987).

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1452

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Conceptually, Johnson and Montgomery (1974, p. 208) argue that convexities in manufacturing often result when there are multiple production or procurement sources where production costs are proportional to the quantity produced by a source. The firm will first assign production to the source with the lowest unit cost until its capacity is reached, then use the next cheapest source to capacity, and so on. In this way the total production cost will be convex in the total amount scheduled. If the firm is operating below the capacity of the first source, then since the production costs are proportional to the quantity produced, the total production cost will be linear in the total amount scheduled for that period.

Nicholson (1978) makes a similar argument, and mentions that engineering studies show that optimal operating rates for certain types of machines occur in rather narrow ranges of output. “Attempting to obtain increased output from these machines by using them more intensively in the short run causes marginal costs to rise because of both decreased operating efficiency and an increased frequency of machine malfunctions. . . .” Smith (1969, 1970) argues similarly, claiming that the rate of depreciation of the capital equipment is a convex function of the utilization rate. The traditional econometric efforts have yielded mixed evidence, although Stigler (1966, p. 142) concludes that: “A series of statistical studies have found that short-run marginal cost is approximately constant until ‘capacity’ is reached.”

Perhaps the most convincing of the evidence supporting the convex production cost structure is that of Griffin (1972), who studied the U.S. petroleum refining industry. Griffin applies a process analysis approach (Manne 1958) which uses engineering data to describe the cost function, rather than a statistical cost function approach which uses accounting data and sample observations. Griffin obtains the classical short-run cost function property of rising marginal costs (i.e., convexity).

The objective of this paper is to gain insight into the dynamic nature of the competitive aspects of the various policies of two firms: one operating at or near capacity, facing a convex production cost, and the other operating significantly below capacity, facing a linear cost structure. The firms are assumed to face a demand surge condition. We will refer to the firm operating at or near capacity as the “Production-smoother” and the firm operating below capacity as the “Order-taker.” This is because, in this deterministic setting, the firm with convex costs may find it optimal to smooth production through the use of inventory, while the firm with linear costs will not have an incentive to hold inventory.

This behavior has been described lucidly by Abel (1985): “It is a well-known proposition that a firm producing a storable good under conditions of increasing marginal cost will tend to smooth the time profile of its production relative to the time profile of its sales. The incentive to smooth production arises from the fact that the cost function is a convex function of the level of production. For a given average level of production, average costs can be reduced by reducing the variation in production. Of course, if the cost function is linear in the level of production, then this incentive to smooth production disappears.”

We address the following questions in this paper: What is the temporal nature of each firm’s equilibrium production and pricing policies? For the Production-smoother, what is the temporal nature of the equilibrium inventory policy? We recognize explicitly the importance of synchronizing production and marketing considerations in such settings.

The literature on production-marketing joint decision-making is relatively small. Tuie (1968) is concerned primarily with the determination of a discount on orders placed in slack months for goods that exhibit seasonal sales fluctuations in a noncompetitive context. Thomas (1970) considers the problem of simultaneously making pricing and production decisions by a monopolist having a fixed plus linear production cost who is facing a known deterministic demand. His major results are in obtaining planning horizons. The same problem with convex production cost but constant price has also been treated by
Richard (1969) and Vanthienen (1973), Kunreuther and Richard (1971) and Kunreuther and Schrage (1973) investigate the interrelationship between the pricing and inventory decisions of a retailer who orders his goods from an outside distributor. However, in both of these papers it is assumed that the retailer maintains a constant price throughout the season.

Of special relevance to the present paper are the formulations presented by Pekelman (1974) and Eliashberg and Steinberg (1987). Pekelman (1974) considers a single manufacturer (monopolist) facing a convex production cost and a linear inventory holding cost who determines simultaneously the price and production rate over a known horizon where the demand function is linear in price but time-dependent. Pekelman makes use of a continuous time model and characterizes the solution graphically via optimal control theory. The present paper continues a stream of research on Production-Marketing joint decision-making begun with Eliashberg and Steinberg (1987), which models a manufacturer and a distributor in an industrial channel of distribution where the key issue investigated is the nature of the coordination within the channel over a period. The approach taken by that paper is that of a sequential differential game based on the Stackelberg formulation of the model (Varian 1984). It is shown that, by varying his price and ordering rate over the season, the distributor can maximize his profits with respect to any possible manufacturer's price (called the “contractual price”) held constant throughout the period. This yields a derived demand function to the manufacturer, who can then maximize his profits with respect to his time-varying production rate. These results can then be substituted back into the distributor's equilibrium policies to provide the unique characterization of the behavior of the system. (See also Jorgensen 1986.)

We note that none of the aforementioned papers has considered the dynamic problem in a competitive setting. In fact, in the economic and game-theoretic literature, the usual assumption made is symmetry in costs between the competitors. By contrast, the present paper examines the noncooperative behavior of two firms asymmetric with respect to cost structure. The firms are in competition with each other over a specified period of time during which the price-sensitive demand is increasing-then-decreasing. The modeling approach taken is that of a noncooperative differential game. Nash equilibrium production and pricing strategies are derived. It is assumed that the two firms simultaneously maximize their profits with respect to any possible price set by the competing firm, where each firm can vary its price and production rate over the period. The strategic behavior of the system is characterized in terms of the dynamic behavior of the Production-smoother’s pricing, production and inventory policies, and the Order-taker’s pricing and production policies.

One possible real-world scenario for our model is that of two competing aircraft manufacturers during a period in which there is a surge in demand, where one of the firms is forced to subcontract work due to capacity constraints and thus incurs increased marginal costs. For example, O’Lone (1988) reports that a surge in demand created pressure for Boeing to increase production rates in 1988. This in turn led the aircraft manufacturer to move more work outside the company, incurring higher labor costs, in order to avoid overburdening manufacturing capacity. More specifically, Ramirez (1989) tells how Boeing, in order to deal with this short-run production capacity problem, contracted with Lockheed Corporation to “borrow” 670 workers at premium pay. On the other hand, Boeing’s major domestic competitor, McDonnell Douglas, was apparently operating below capacity during the same time period. The Financial Times of London reports that American Airlines had planned to pick up orders for McDonnell Douglas aircraft

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1 Feichtinger and Hartl (1985) also consider a single manufacturer under a more general demand function and relax the nonnegativity constraint for the inventory by allowing for shortages that are penalized by shortage costs.
that had been placed by British Caledonian and later canceled (Oram 1989).² Oram explains that if American Airlines were indeed to take over the British Caledonian orders, then McDonnell Douglas would still be short of the number of orders it had forecast for the project.

Lockheed is certainly not the only firm in the commercial aircraft industry which can act as a subcontractor. In an August 1989 investment report on the Grumman Corporation, Merrill Lynch (1989) states: “Given that the commercial aerospace industry suffers from a crying need for skilled airplane workers and manufacturing capacity, two assets which Grumman has in abundance, we believe that some steps may be taken to bring demand together with supply.” (Emphasis added.) Thus, the type of scenario we wish to model in this paper is that of an aircraft manufacturer operating at full-capacity (such as Boeing), which finds it necessary due to a surge in demand to subcontract work (e.g., from Lockheed or Grumman) and which competes or is likely to compete with another firm operating below capacity (such as McDonnell Douglas).

The remainder of the paper is organized as follows. We begin by formally stating the problem (§2). We then present the equilibrium policies for the Production-smoother and the Order-taker, as well as an analysis of the strategic implications of these policies (§3). (The mathematical development is presented in the Appendix.) Two numerical examples are then provided (§4). The final section contains concluding remarks which include a summary of the problem and the major results, as well as suggestions for further research (§5).

2. Problem Formulation

The Production-smoother and the Order-taker simultaneously and noncooperatively each attempt to maximize their respective profit over the same known time horizon.

Each firm’s profit equals its revenues minus its unit costs (which may include the cost of raw materials and of shipping), of production, and of holding inventory, subject to various constraints on inventory. (Note that there is no discounting of future profit streams since the period of concern is relatively short.) The constraints require that: rate of change in inventory is equal to the production rate minus the demand rate, inventory level is at all times nonnegative, and initial inventory and terminal inventory are zero. (The possibility that initial inventory and terminal inventory are equal but nonzero can also be accommodated easily in our model.)

Hence, the two objective functions can be written mathematically as:

\[
\Pi_X = \int_0^T \{ P_X(t) \cdot D_X(P_X(t), P_X(t), t) - v_X Q_X(t) \\
- f_X(Q_X(t)) - h_X I_X(t) \} dt
\]

where (2.1)

\[
X \in \{ 1, 2 \mid 1 = \text{Production-smoother}, 2 = \text{Order-taker} \}, \quad X = \{ 1, 2 \} \setminus \{ X \},
\]

\[
P_X(t) \text{ Denotes the price charged at time } t \text{ by firm } X.
\]

\[
Q_X(t) \text{ Denotes firm } X’s \text{ production rate at time } t.
\]

\[
D_X(P_X(t), P_X(t), t) \text{ Denotes the demand rate at time } t \text{ for firm } X’s \text{ product, if firm } X \text{ charges } P_X(t) \text{ and firm } X \text{ charges } P_X(t).
\]

\[
v_X \text{ Denotes firm } X’s \text{ per-unit (variable) cost.}
\]

\[
f_X(Q_X(t)) \text{ Denotes firm } X’s \text{ production cost function exclusive of raw materials cost.}
\]

\[
h_X \text{ Denotes firm } X’s \text{ per-unit inventory holding cost.}
\]

\[
I_X(t) \text{ Denotes firm } X’s \text{ inventory level at time } t.
\]

³ When British Airways took over British Caledonian, it decided not to take delivery of the McDonnell Douglas aircraft.
Consider the following scenario derived from Dixit (1979). Let \( U = x_0 + \alpha'_1 x_1 + \alpha'_2 x_2 - \frac{1}{2} \beta'(x_1^2 + x_2^2) - \gamma' x_1 x_2 \) denote the consumer utility function where \( x_i \) is the quantity purchased of product \( i \) \((i = 1, 2)\) and \( x_0 \) is the quantity purchased from the competitive numéraire sector. Further, let \( P_i \) denote the price of product \( i \), and \( I \) denote the consumer income. The consumer maximization problem is: \( \max_{x_1, x_2, x_0} U(x_1, x_2, x_0) \) subject to \( P_1 x_1 + P_2 x_2 + x_0 = I \). The inverse demand functions are the partial derivatives. That is, \( P_i = \partial U / \partial x_i \) \((i = 1, 2)\). These yield linear inverse demands: \( P_i = \alpha'_i - \beta' x_i - \gamma' x_j \) \((i, j = 1, 2; j \neq i)\). Solving the linear pair of equations for \((x_1, x_2)\) yields the competitive linear demand functions: \( x_i = D_i(P_1, P_2) = a_i - b'_i P_i + \gamma P_j \) \((i, j = 1, 2; j \neq i)\). Letting \( a_i = w a_1, a_2 = (1 - w) a_1 \), and \( b' = (b + \gamma) \), we obtain a structural form equivalent to (2.2a) and (2.2b). A similar approach is taken by Shubik with Levitan (1980, Chapter 7).

The formulation presented in (2.2a) and (2.2b) is, however, more readily interpretable and insightful. There \( a(t) \) denotes the time-varying market potential. The parameters \( w \) and \( 1 - w \) are the market potential fractions for the Production-smoother and Order-taker, respectively. These parameters represent the proportion of the market potential available to each firm as a result of advertising advantage or other price-excluded sources of brand preference. That is, for \( P_1(t) = P_2(t) = P(t) \), \( D_1 > D_2 \) iff \( w > \frac{1}{2} \). In the case where the market is symmetric, \( w = \frac{1}{2} \). The parameter \( b \) can be interpreted as the price-sensitivity coefficient and the parameter \( \gamma \) as the price-differential or competitive effect coefficient. The parameter \( b \) reflects the fact that the product produced by the two firms competes with other goods for the consumer’s wealth. The parameter \( \gamma \) corresponds to the amount of direct price comparison that the consumers make. It captures the degree of substitutability between the two products.

In order to capture the unstable “surge” demand effect in a very general fashion, we have chosen to model the market potential \( a(t) \) with minimal requirements as follows:

\[
\begin{cases}
  a(t) \text{ is a positive, strictly concave function} \\
  \text{increasing—then—decreasing on } [0, T], \\
  \text{where } a(T) = a(0).
\end{cases}
\]  

In (2.3), \( a(0) \) represents the “nominal” size of the market potential at the start of the time horizon. Also, \( a(T) \) is set equal to \( a(0) \) in order to capture the entire seasonal period.

As discussed in the Introduction, the Production-smoother has a convex production cost. This is the essence of his problem. In order to obtain sharp analytical results, we follow Pekelman (1974) and Eliashberg and Steinberg (1987), which consider the case
of a quadratic cost function. Our results are generalizable, however, to any convex cost function. In an exhaustive survey, Walters (1963) reports that “the equation relating total costs to output used by most authors is usually a quadratic.” As discussed in Eliashberg and Steinberg (1987), a quadratic cost function may be appropriate under various scenarios including, for example, the situation where there are only two input factors of production and the production function is given by the Cobb-Douglas formulation (Varian 1984): \( Q = VCL \). Here \( Q \) represents output and \( C \) and \( L \) represent capital and labor factors, respectively. Under the assumption of fixed capital inputs, \( L = Q^2/C \). Further, if the cost of labor is \( W \) dollars per labor and time unit, then the variable production costs, \( f(Q) \), become \( WQ^2/C \). We can write this as \( f(Q) = (1/k)Q^2 \), where \( k = C/W \) is a measure of production efficiency.

Thus, for the Production-smoother and the Order-taker, respectively, we have the following production cost functions:

\[
f_1(Q_1(t)) = \frac{1}{k_1}Q_1^2(t), \quad k_1 > 0, \\
f_2(Q_2(t)) = \frac{1}{k_2}Q_2^2(t), \quad k_2 > 0.
\] (2.4a, 2.4b)

Here \( k_1 \) and \( k_2 \) are parameters corresponding to the production efficiency of the Production-smoother and Order-taker, respectively. In each case, higher values of \( k \) correspond to higher production efficiency and lower production costs.

We assume that each firm meets all of its demand, and thus cumulative production equals cumulative demand over the season:

\[
\int_0^T Q_k(t)dt = \int_0^T D_k(P_k(t), P_{k*}(t), t)dt.
\] (2.5)

Equation (2.5), which is equivalent to saying that initial inventory equals terminal inventory, allows us to rewrite the objective functions of (2.1) in equivalent forms which are more amenable to differential games analysis.

The open-loop Nash equilibrium solution of the two-player game with objective functions

\[
\Pi_1(P_1(t), Q_1(t); P_2(t), Q_2(t)) \quad \text{and} \quad \Pi_2(P_2(t), Q_2(t); P_1(t), Q_1(t))
\] (2.6a, 2.6b)

is the set of strategies \( \{P_1^*, P_2^*(t), Q_1^*(t), Q_2^*(t)\} \) satisfying:

\[
\Pi_1(P_1(t), Q_1(t); P_2^*(t), Q_2^*(t)) \leq \Pi_1(P_1^*(t), Q_1^*(t); P_2^*(t), Q_2^*(t)) \quad \text{and} \quad (2.7a) \\
\Pi_2(P_2(t), Q_2(t); P_1^*(t), Q_1^*(t)) \leq \Pi_2(P_2^*(t), Q_2^*(t); P_1^*(t), Q_1^*(t)). \quad (2.7b)
\]

In other words, the Nash equilibrium strategies are optimal in the sense that neither firm can obtain a better performance for itself while its competitor maintains its own Nash strategy.

Based on the considerations noted above, and from (2.1) through (2.5), we obtain the control formulation of the Production-smoother's problem below. (The time argument will often be omitted for simplicity in the sequel.)

**Problem 1—The Production-Smoother.**

\[
\text{MAX}_{P_1, Q_1} \int_0^T \{ (P_1 - v_1)[w_a - (b + \gamma)P_1 + \gamma P_2] - (1/k_1)Q_1^2 - h_1 I_1 \} dt \quad (2.8)
\]

**S.T.** \( \dot{I}_1 = Q_1 - D_1 = Q_1 - w_a + (b + \gamma)P_1 - \gamma P_2, \) (2.9)

\[ I_1 \geq 0, \quad (2.10) \]
\[ Q_1 \geq 0, \]  
\[ I_1(0) = I_1(T) = 0. \]  
(2.11)  
(2.12)

The analogous control formulation of the Order-taker's problem is:

**Problem 2—The Order-Taker.**

\[ \max_{P_2, Q_2} \int_0^T \left\{ (P_2 - (v_2 + 1/k_2))[(1 - w)a - (b + \gamma)P_2 + \gamma P_1] - h_2 I_2 \right\} dt \]  
\[ \text{S.T.} \quad \dot{I}_2 = Q_2 - D_2 = Q_2 - (1 - w)a + (b + \gamma)P_2 - \gamma P_1, \]  
\[ I_2(0) = I_2(T) = 0. \]  
(2.13)  
(2.14)  
(2.15)  
(2.16)  
(2.17)

**REMARKS.** As easily shown, the Order-taker will never hold inventory, and thus constraints (2.14)--(2.17) in the Order-taker's problem can be replaced by: \( I_2(t) = 0 \) for \( t \in [0, T] \). In addition, his production rate is equal to his demand rate at all times, \( Q^*_2(t) = D^*_2(t) \). (See the Appendix.) Finally, we are looking for interior solutions. Hence, we will be concerned only with strict inequalities for constraint (2.11) of the Productionsmoother's problem and (2.16) of the Order-taker's problem.

### 3. The Equilibrium Policies and Their Strategic Implications

In this section, we present the equilibrium policies resulting from the simultaneous solution of Problems 1 and 2, and discuss the strategic implications of these policies in terms of propositions. The main proofs are presented in the Appendix. Readers interested only in the strategic (qualitative) implications of these results can proceed directly to the propositions.

We will find it useful to define constants \( A, B \) and \( C \) in terms of the basic parameters of the problem:

\[ A = \frac{(b + \gamma)}{\Delta} \left[ \frac{2wb + \gamma(1 + w)}{2} \right], \]  
\[ B = \frac{(b + \gamma)}{\Delta} \left[ \frac{2(b + \gamma)^2 - \gamma^2}{2} \right], \]  
\[ C = \frac{(b + \gamma)}{\Delta} \left[ \frac{2(b + \gamma)^2 - \gamma^2}{2} \right] v_1 - \gamma(b + \gamma)(v_2 + 1/k_2), \]  
where
\[ \Delta = 4(b + \gamma)^2 - \gamma^2 = (2b + \gamma)(2b + 3\gamma). \]  
(3.1a)  
(3.1b)  
(3.1c)  
(3.1d)

**Production-Smoother's Inventory Policy**

\[
I^*_1(t) = \begin{cases} 
A \int_0^t \{ a(t^*) - a(t) \} dt - (B + k_1/2)h_1(t^* - t^2/2) & \text{for } 0 \leq t \leq t^*, \\
0 & \text{for } t^* \leq t \leq T.
\end{cases}
\]  
(3.2)
Here $t^*$ is the zero inventory point which satisfies:

$$A \int_0^{t^*} \{ a(t^*) - a(t) \} \, dt - \frac{1}{2} (B + k_1/2) h_1(t^*)^2 = 0. \tag{3.3}$$

As discussed in the Appendix, in order for the zero inventory point $t^*$ to exist, it must lie to the right of the point at which the market potential $a(t)$ achieves its maximum value.

Based on equation (3.2), we can describe the Production-smoother’s inventory strategy over the entire period. Proposition 3.1 formalizes this aspect.

**PROPOSITION 3.1.** Production-smoother’s Inventory Strategy. If the Production-smoother finds it optimal to hold inventory, then he will begin the period by building up inventory, continue by drawing down inventory until it reaches zero at $t^*$, and conclude by following a “zero inventory” policy from $t^*$ until the end of the seasonal period.

**PROOF.** This follows immediately from (3.2). See also (A.21) in the Appendix. □

The general structure of a Production-smoother’s inventory policy described in Proposition 3.1 is robust with respect to the market structure he is facing. Pekelman (1974) has demonstrated that such an inventory policy holds for a pure monopolist. The analysis presented in Eliashberg and Steinberg (1987), which focuses on manufacturer-distributor relationship in a channel of distribution facing a market with no competition, also shows the optimality of such inventory policies for both the distributor and the manufacturer. Moreover, for quadratic $a(t)$ functions, Eliashberg and Steinberg (1987) provide explicit closed-form solutions (in terms of the basic problem’s parameters) for $t_D^*$ and $t_M^*$, the times at which the distributor and the manufacturer begin their zero inventory policies, respectively.

Proposition 3.1 establishes this property for a firm acting in a competitive market. This result is expected because it captures the essence of production smoothing. However, a question of interest arises: What is the impact of the competitive market effect upon the time at which a duopolistic Production-smoother begins his zero inventory policy vis-à-vis the scenario under which he (the Production-smoother) operates as a monopolist? To investigate this, we examine a specific functional form for $a(t)$. Recall that we denote by $t^*$ the beginning of the zero inventory policy for the Production-smoother in a *duopolistic* market; denote by $t^{**}$ the corresponding variable for a *monopolistic* Production-smoother.

If the market potential $a(t)$ is given by a polynomial expression, then (3.3) can be solved easily by numerical means to obtain $t^*$ to the desired degree of accuracy. In general, if the market potential is given by a polynomial in $t$ of order $n$, the solution for $t^*$ is given by a polynomial in $t$ of order $n - 1$. In particular, for the case of quadratic market potential where $a(t) = -\alpha_1 t^2 + \alpha_2 t + \alpha_3$ ($\alpha_1$, $\alpha_2$, $\alpha_3 > 0$), (3.3) will reduce to a linear equation, yielding immediately

$$t^* = \left[ \frac{3}{4A\alpha_1} \right] \left[ 4A\alpha_2 - h_1(B + k_1/2) \right]. \tag{3.3a}$$

In this case, it can be shown that the condition for the existence of $t^*$ can be interpreted, quite intuitively, viz., that the inventory holding cost, $h_1$, is sufficiently low relative to some combination of the other basic parameters of the problem. Further by (2.3), $a(T) = a(0)$, and thus $\alpha_2/\alpha_1 = T$. Hence

$$t^* = \left( \frac{3}{4} \right) T - \left[ 3 h_1(B + k_1/2) \right] / (4A\alpha_1). \tag{3.3b}$$

For a monopolistic Production-smoother, the corresponding solution, $t^{**}$, can be obtained by setting $\gamma = 0$ and $w = 1$ in Problem 1 (equations (2.8)–(2.12)). This implies:
The above result reproduces the monopolistic Production-smoother distributor result analyzed in Eliashberg and Steinberg (1987, equation (2.15)).

To investigate the impact of some degree of competition on the timing of the start of the zero inventory policy, a Taylor series approximation was developed for \( t^* \) at \( \gamma = 0^+ \) and \( w = 1^- \). The analysis immediately reveals that \( t^* < t^{**} \). We thus conclude that, in the case of quadratic market potential, the competitive pressure faced by the Production-smoother pushes him to begin his zero inventory policy earlier than had he acted as a monopolist, and that in either case he will never hold inventory during the last quarter of the period.

Proposition 3.1 characterizes the essential features of the Production-smoother strategic behavior for any increasing-then-decreasing demand formulation. For the case of linear demand functions with quadratic market potential, the inventory equation (3.2) reduces to the expression:

\[
I_1(t) = \begin{cases} 
(A\alpha_1/3)t(t^* - t)^2 & \text{for } 0 \leq t \leq t^*, \\
0 & \text{for } t^* \leq t \leq T.
\end{cases}
\]

(3.2a)

For \( \gamma = 0 \) and \( w = 1 \) (i.e., the monopoly base case), the above expression reduces to \((A\alpha_1/6)(t^{**} - t)^2t\) for \( 0 \leq t \leq t^{**} \), and zero elsewhere, as in Eliashberg and Steinberg (1987). Note that this implies that the Production-smoother will be building up inventory for precisely the first third of his inventory holding period—as he would do had he been a monopolist, but to a greater maximum inventory level—and drawing inventory down for the remaining two thirds of his inventory holding period. In general, if the market potential is given by a polynomial expression in \( t \) of order \( n \), the nonzero portion of the Production-smoother’s inventory equation will be a polynomial of degree \( n + 1 \).

Returning to the case of general dynamic market potential, \( a(t) \), in (3.4)–(3.7) below we present closed form expressions for the competitor’s equilibrium strategies. Propositions 3.2 through 3.8 characterize these strategies. Note that because of the “coupling” effect between the two competing firms, the time at which the Production-smoother begins his zero inventory policy is also critical for the Order-taker who divides his production and pricing strategies into two parts determined by the Production-smoother’s zero inventory point. As pointed out in the previous section, the Order-taker will never hold inventory.

**Production-Smoother’s Production and Pricing Policies**

\[
Q_1^*(t) = \begin{cases} 
(k_i/2)[Aa(t^*) - C]/[B + k_i/2] - h_i(t^* - t) & \text{for } 0 \leq t \leq t^*, \\
(k_i/2)[Aa(t) - C]/[B + k_i/2] & \text{for } t^* \leq t \leq T,
\end{cases}
\]

(3.4)

\[
P^*_1(t) = [4(b + \gamma)^2/k_1\Delta]Q_1^*(t) + [A/(b + \gamma)]a(t)
+ [(b + \gamma)/\Delta](2(b + \gamma)v_1 + \gamma(v_2 + 1/k_2)).
\]

(3.5)

**Order-Taker’s Production and Pricing Policies**

\[
Q_2^*(t) = [2\gamma(b + \gamma)^2/k_1\Delta]Q_1^*(t) + [(b + \gamma)/(2b + \gamma) - A]a(t)
+ [(b + \gamma)/\Delta]([\gamma(b + \gamma)v_1 - (2b + \gamma)^2 - \gamma^2](v_2 + 1/k_2)),
\]

(3.6)

\[
P^*_2(t) = [2\gamma(b + \gamma)/k_1\Delta]Q_1^*(t) + [1/(2b + \gamma) - A/(b + \gamma)]a(t)
+ [(b + \gamma)/\Delta]([\gamma v_1 + 2(b + \gamma)(v_2 + 1/k_2)].
\]

(3.7)
PROPOSITION 3.2. Production-Smoother’s Production Strategy. During his inventory holding period, the Production-smoother will produce at a constantly increasing rate; during his zero inventory period, he will produce precisely enough to meet his demand rate, and his production rate will be decreasing proportionally to the market potential \( (a(t)) \).

PROOF. This follows from (3.4). See also Eliashberg and Steinberg (1987). □

Like Proposition 3.1, Proposition 3.2 captures another essential aspect of the Production-smoother’s behavior which does not change qualitatively for any general demand function or under other market structures (Pekelman 1974, Eliashberg and Steinberg 1987). The Production-smoother’s strategy always will be characterized by a constantly increasing production rate before \( t^* \) and a production rate identical to the demand rate and proportional to the market potential \( a(t) \) after \( t^* \). A Taylor series-based analysis, similar to the one taken in Proposition 3.1, reveals that the constant production rate for \( 0 \leq t \leq t^* \) remains the same \((dQ^*/dt = k_1h_1/2)\) for the duopolist as it is for the monopolist. The proportionality factor of \( a(t) \), that is, \( A/[2B/k_1 + 1] \), decreases, however, for the duopolistic Production-smoother. This implies that the duopolistic Production-smoother, facing a lower demand rate, decreases his production as he approaches the end of the season at a rate \((|dQ^*/dt|)\) that is smaller than that of a monopolistic Production-smoother.

The next set of propositions focus on the second player, the Order-taker, and thus are unique to the competitive setting discussed in this paper. We also provide a comparison between the two competitors.

PROPOSITION 3.3. Order-Taker’s Production Strategy. During the Production-smoother’s inventory holding period, the Order-taker will be increasing his production rate; during the Production-smoother’s zero inventory period, the Order-taker will be decreasing his production rate, and this decreasing rate will be proportional to the market potential \( (a(t)) \).

PROOF. This follows from (3.6) and (3.4). □

Proposition 3.3 shows that, corresponding to the Order-taker’s different (i.e., linear as opposed to quadratic) production cost structure, the firm has a different production strategy from the Production-smoother. However, the demand functions of the two players are structurally equivalent, and thus our intuition might tell us that the pricing strategies for the two players should be structurally similar. The next proposition confirms this.

PROPOSITION 3.4. Pricing Strategies. Over the season, each player will increase his price past the time at which the market potential \( (a(t)) \) achieves its maximum (which occurs before \( t^* \)) and then decrease his price.

PROOF. This follows from (3.5), (3.7) and (3.4). □

The result presented in Proposition 3.4 is driven by the competitors’ demand formulations; in particular, by the linear fashion in which \( a(t) \) and the competitor’s price are incorporated in the demand functions (2.2a) and (2.2b). It is possible, of course, to posit alternative demand formulations where the pricing strategies differ from those described in the proposition. However, the following pricing rule will always hold for the Order-taker. For any convex downward-sloping demand function (i.e., \( \partial D_2/\partial P_1 < 0 \) and \( \partial^2 D_2/\partial P_2^2 < 0 \)), whenever \( a(t) \) and \( P^*_2(t) \) are simultaneously increasing (decreasing), the Order-taker’s price will also be increasing (decreasing). For more details, see the Appendix (§A.5).

Note that Proposition 3.4 does not imply that \( P_2 \) is in general less than \( P_1 \) or vice versa. However, we do have a result of this kind under a more specific condition. In order to explore further the strategic differences between the prices of the two competing firms, we will assume a symmetric market scenario (i.e., \( w = \frac{1}{2} \)) in Propositions 3.5, 3.6,
and 3.7. Define the price difference at time $t$ as the Production-smoother’s price minus the Order-taker’s price. Note that this difference can be negative. Under this scenario, we have:

**PROPOSITION 3.5. Price Domination.** The Order-taker’s price is strictly less than the Production-smoother’s price over the entire period whenever the sum of the Order-taker’s variable and unit production costs does not exceed the Production-smoother’s variable cost (i.e., $v_2 + 1/k_2 \leq v_1$).

**PROOF.** This follows from (3.5), (3.7), and (3.4). □

Proposition 3.5 implies that, as long as the Order-taker’s total per-unit cost does not exceed the Production-smoother’s per-unit variable cost (excluding the per-unit inventory holding cost), the Order-taker can afford to charge a lower price throughout the seasonal period. This result is somewhat in accordance with another dynamic pricing result (Eliashberg and Jeuland 1986, Proposition 7) obtained for a different competitive demand formulation and for the case where production cost functions are not explicitly considered. It can also be shown that, for the symmetric market case ($\alpha = 1/2$), during the Production-smoother’s inventory holding period, the price difference will be increasing at a constant rate which is strictly smaller than the unit holding cost ($h_1$); and during the Production-smoother’s zero inventory period, the price difference will be decreasing. Note that if the parametric condition $v_2 + 1/k_2 \leq v_1$ is not satisfied, there may be a crossover of price paths.

**PROPOSITION 3.6. Demand Domination.** The Order-taker’s demand is strictly greater than the Production-smoother’s demand over the entire period whenever the sum of the Order-taker’s variable and unit production costs does not exceed the Production-smoother’s variable cost (i.e., $v_2 + 1/k_2 \leq v_1$).

**PROOF.** This follows from equations (A.16a) and (A.16b) in the Appendix. □

Proposition 3.6 follows intuitively from Proposition 3.5 in that, under the specified parametric condition, the Order-taker’s price is lower over the entire period, suggesting his demand will be higher over the entire period.

**PROPOSITION 3.7. Timing of Peak Demands.** The Order-taker reaches his peak demand earlier in the period than the Production-smoother.

**PROOF.** See the Appendix (§A.6). □

*The Zero Inventory Subcase for the Production-Smoother*

Under the parametric condition where $t^*$ does not lie to the right of the point at which $a(t)$ achieves its maximum, the zero inventory case obtains, and the Production-smoother will follow a zero inventory policy over the entire season, as his Order-taker competitor always does. In this case, the mathematical solutions of Problem 1 (2.8)–(2.12) and Problem 2 (2.13)–(2.17) simplify considerably. The double asterisk in the sequel indicates optimal decision variables for the zero inventory case.

\[
Q_1^{**} = k_1(b + \gamma)[P_1^{**}(t) - v_1]/[k_1 + 2(b + \gamma)], \quad \text{where} \quad (3.8)
\]

\[
P_1^{**} = (1/\Delta)[2wb + (1 + w)\gamma]a(t) + 2(b + \gamma)^2v_1 + \gamma(b + \gamma)(v_2 + 1/k_2), \quad \text{and} \quad (3.9)
\]

\[
Q_2^{**} = (b + \gamma)[P_2^{**}(t) - (v_2 + 1/k_2)], \quad \text{where} \quad (3.10)
\]

\[
P_2^{**} = (1/\Delta)[2(1-w)b + (2-w)\gamma]a(t) + \gamma(b + \gamma)v_1 + 2(b + \gamma)^2(v_2 + 1/k_2). \quad (3.11)
\]
We summarize such scenarios formally in the following proposition.

PROPOSITION 3.8. Zero Inventory Subcase. There exist situations where the Production-smoother will follow a zero inventory policy over the entire season, as the Order-taker always does. In this case, the Production-smoother’s equilibrium policies will be similar structurally to the Order-taker, which are as described in Propositions 3.3 and 3.4.

PROOF. This follows from (3.8)–(3.11) and the Appendix. See also Eliashberg and Steinberg (1984).

Of course, sufficiently high inventory holding cost for the Production-smoother is one characteristic of the situation captured by Proposition 3.8.

4. Numerical Examples

In order to make a “fair” comparison between the profitability of the two firms having differing cost structures as described in this paper, we will examine numerical examples where the average marginal production cost over the period for the two firms is identical. This can be expressed by the equation:

$$\frac{2}{k_1} \frac{1}{T} \int_0^T Q^*_t(t) dt = \frac{1}{k_2}. \quad (4.1)$$

EXAMPLE 1. Suppose that the cost parameters are:

$$k_1 = 1, \quad v_1 = 1, \quad h_1 = 0.1, \quad (4.2a)$$
$$k_2 = 0.118, \quad v_2 = 1. \quad (4.2b)$$

Suppose that the demand parameters are given by:

$$a(t) = -t^2 + 6t + 12; \quad \text{hence} \quad \alpha_1 = 1, \quad \alpha_2 = 6, \quad \alpha_3 = 12, \quad (4.3)$$
$$b = 0.1, \quad \gamma = 0.01. \quad (4.4)$$

Thus, from (4.3) and (2.3), $T = 6$.

Further, suppose the market is symmetric, i.e.,

$$w = \frac{1}{2}. \quad (4.5)$$

Then from (3.1):

$$A = 0.262, \quad B = 0.055, \quad C = 0.031, \quad \Delta = 0.048. \quad (4.6)$$

The zero inventory point is $t^* = 4.341 > 3$, the point at which $a(t)$ achieves its maximum. Thus, the optimal policies of §3 hold.

For the Production-smoother, the equilibrium strategies are:

$$P^*_t(t) = \begin{cases} -2.381t^2 + 14.336t + 33.583 & \text{for } 0 \leq t \leq t^*, \\ -2.617t^2 + 15.705t + 32.098 & \text{for } t^* \leq t \leq T, \end{cases} \quad (4.7)$$

$$Q^*_t(t) = \begin{cases} 0.05t + 4.286 & \text{for } 0 \leq t \leq t^*, \\ -0.236t^2 + 1.416t + 2.804 & \text{for } t^* \leq t \leq T, \end{cases} \quad (4.8)$$

$$I^*_t(t) = \begin{cases} 0.087t^3 - 0.758t^2 + 1.645t & \text{for } 0 \leq t \leq t^*, \\ 0 & \text{for } t^* \leq t \leq T. \end{cases} \quad (4.9)$$
For the Order-taker:

\[
P^*_2(t) = \begin{cases} 
-2.381t^2 + 14.288t + 33.537 & \text{for } 0 \leq t \leq t^*, \\
-2.392t^2 + 14.350t + 33.469 & \text{for } t^* \leq t \leq T, 
\end{cases} \tag{4.10}
\]

\[
Q^*_2(t) = \begin{cases} 
-0.262t^2 + 1.572t + 2.467 & \text{for } 0 \leq t \leq t^*, \\
-0.263t^2 + 1.579t + 2.639 & \text{for } t^* \leq t \leq T. 
\end{cases} \tag{4.11}
\]

The relationship of (4.1) can be verified easily. The pricing, production, and inventory policies for Example 1 are illustrated graphically in Figures 4.1–4.3. (Note in Figure 4.1 that there is a crossover of the price paths.) By substituting the equilibrium policies into the objective functions (2.8) and (2.13), we find that the Production-smoother has a larger profit than the Order-taker for both parts of the period (see Table 1).

EXAMPLE 2. Suppose now that, due to higher raw materials or shipping costs, the Production-smoother’s unit variable cost is substantially higher than that of the Order-taker:

\[
k_1 = 1, \quad v_1 = 8, \quad h_1 = 0.1, \tag{4.12a}
\]

\[
k_2 = 0.129, \quad v_2 = 1. \tag{4.12b}
\]

Assume the demand parameters are as in (4.3) and (4.4), and the market is symmetric (4.5). Then from (3.1), the value of \( C \) will change from that obtained in Example 1 (4.6):

\[
A = 0.262, \quad B = 0.055, \quad C = 0.417, \quad \Delta = 0.048. \tag{4.13}
\]

As in Example 1, the zero inventory point is \( t^* = 4.341 \).

The equilibrium policies for the Production-smoother are:

\[
P^*_1(t) = \begin{cases} 
-2.381t^2 + 14.336t + 36.726 & \text{for } 0 \leq t \leq t^*, \\
-2.617t^2 + 15.705t + 35.240 & \text{for } t^* \leq t \leq T, 
\end{cases} \tag{4.14}
\]

\[\text{Figure 4.1. Pricing Policies—Example 1.}\]
COMPETITIVE STRATEGIES FOR TWO FIRMS

Figure 4.2. Production Policies—Example 1.

\[ Q_1^*(t) = \begin{cases} 
0.05t + 3.939 & \text{for } 0 \leq t \leq t^*, \\
-0.236t^2 + 1.416t + 2.456 & \text{for } t^* \leq t \leq T, 
\end{cases} \]  

(4.15)

\[ I_1^*(t) = \begin{cases} 
0.087t^3 - 0.758t^2 + 1.645t & \text{for } 0 \leq t \leq t^*, \\
0 & \text{for } t^* \leq t \leq T. 
\end{cases} \]  

(4.16)

Figure 4.3. Production-Smooth—Example 1.
For the Order-taker:
\[ P^*_2(t) = \begin{cases} -2.381t^2 + 14.288t + 33.318 & \text{for } 0 \leq t \leq t^*, \\ -2.392t^2 + 14.350t + 33.250 & \text{for } t^* \leq t \leq T, \end{cases} \]
\[ Q^*_2(t) = \begin{cases} -0.262t^2 + 1.572t + 2.702 & \text{for } 0 \leq t \leq t^*, \\ -0.263t^2 + 1.579t + 2.695 & \text{for } t^* \leq t \leq T. \end{cases} \]  

As in Example 1, the relationship (4.1) equating average marginal cost can be verified easily. Note that the Production-smoother’s inventory policy in Example 2 is identical (to three decimal places) to the Production-smoother’s inventory policy of Example 1. The equilibrium pricing and production policies are different from those derived in Example 1; they are illustrated graphically in Figures 4.4 and 4.5. By substituting the policies into the objective functions (2.8) and (2.13), we find now that the Order-taker has a larger profit than the Production-smoother for both parts of the period (see Table 2).

5. Concluding Remarks

In this paper, we study the problem of determining production and marketing equilibrium strategies for two competing firms, distinguished as the Production-smoother.
and the Order-taker. We focus on the asymmetry in the firms’ production cost structures and, consequently, on the implications in the handling of inventory for an unstable “surge” demand condition (e.g., seasonality). To our knowledge, no such investigation has been undertaken before where the interface between production and marketing policies in a competitive context is explicitly incorporated.

We have made three major assumptions in our analyses. First, we have invoked the premise of a quadratic cost function for the Production-smoother competitor. Second, we have assumed a linear cost structure for the Order-taker competitor who holds no inventory. Third, we have employed a linear duopolistic demand function. Although each of these assumptions has been supported conceptually (via mathematical axiomatization) or empirically (via statistical estimation), they nonetheless limit the number of scenarios for which the closed-form, analytical implications that we have derived might be applicable. However, qualitatively, the equilibrium strategies we have obtained appear to be generalizable and, for the most part, quite robust with respect to the underlying assumptions.

These assumptions allow us to pose the problem as follows. We seek open-loop Nash equilibrium production and pricing strategies such that neither competitor has any incentive to deviate from his strategies while his rival maintains his own Nash strategies. We address the following questions: What is the temporal nature of each firm’s equilibrium production and pricing policies? For the Production-smoother, what is the temporal

![Figure 4.5. Production Policies—Example 2.](image)

<table>
<thead>
<tr>
<th></th>
<th>Production-smoother’s Inventory Holding Period</th>
<th>Zero Inventory Period</th>
<th>Entire Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production-smoother</td>
<td>734.2</td>
<td>199.2</td>
<td>933.4</td>
</tr>
<tr>
<td>Order-taker</td>
<td>805.0</td>
<td>218.1</td>
<td>1023.1</td>
</tr>
</tbody>
</table>
nature of the equilibrium inventory policy? We recognize explicitly the importance of synchronizing production and marketing considerations in such settings.

Our results fall into two categories, those relating to production and inventory strategies, and those relating to pricing.

We obtain two major, generalizable insights with regard to production and inventory. These are: (1) the Production-smoother's constantly increasing rate of production, which becomes identical to the (decreasing) demand rate throughout the zero inventory period, and (2) the building-up-then-drawing-down Production-smoother's inventory strategy.

The first insight (the Production-smoother's production policy), which captures the essence of production smoothing, is a very robust one. It holds not only for any increasing-then-decreasing competitive demand function, but also for monopolistic Production-smoothers (Pekelman 1974) as well as for the members of industrial channels of distribution (Eliashberg and Steinberg 1987). The constant rate part of this policy will always depend on the inventory holding cost. The (decreasing) demand rate part of the policy will depend, of course, on whether the demand represents a monopolistic, duopolistic, or channel setting. The second insight is driven by the existence of a strictly positive point in time beyond which the Production-smoother carries no inventory.

Given that the Order-taker carries no inventory throughout the season, his production strategy is identical to his demand, which is increasing-then-decreasing. It is interesting to note that the time at which the Production-smoother begins his zero inventory period is also crucial for the Order-taker in planning his production strategy.

We obtain several results with regard to pricing. First, the prices charged by the two firms will each first increase and then decrease. This pricing behavior is sometimes observed in practice where price reductions occur after a peak in demand (e.g., "end of season clearance"). Second, the Order-taker's price over the entire period is strictly lower than the Production-smoother's price whenever the sum of the Order-taker's variable and unit production costs does not exceed the Production-smoother's variable cost. Third, under symmetric market conditions, i.e., when the market potential available to the two firms is equal, the difference between the Production-smoother's price and the Order-taker's price will be increasing at a constant rate until the zero inventory point, after which the price difference will be decreasing. A crossover of price paths is possible.

Under symmetric market conditions, we obtain the following two results with regard to the demands faced by the two players under the optimal policies. First, the Order-taker reaches his peak demand earlier in the period than the Production-smoother. Second, the Order-taker's demand is strictly greater than the Production-smoother's demand over the entire period whenever the Production-smoother variable cost is at least as great as the sum of the Order-taker's variable and unit production costs. (This is the same condition as in the second pricing result above.)

While the "bottom-line" question—which competitor is more profitable?—is interesting to pose, it is very difficult to analyze for the general case due to the myriad number of parametric conditions that need to be considered. Hence, we chose to study numerical examples. In order to make the comparison fair in some sense, however, we held the average marginal production cost over the period for the two competing firms identical. Even under this condition, many possibilities exist for profitability comparison for the most general case. Our numerical examples suggest that no clear profitability pattern seems to emerge. This issue definitely deserves further study.

While we feel that our model is fairly rich, there still exist a number of interesting extensions and directions for further research. One natural extension would be to consider the case of competition among more than two firms. Another possibility would be to incorporate uncertainty into the model and allow for backlogging; this would require the use of stochastic optimal control. A third possibility would be to combine the model presented in Eliashberg and Steinberg (1987) with that of the present paper in order to
gain insight into two distribution channels in competition with each other. Finally, an empirical study of the implications derived in this paper would undoubtedly provide crucial insights into the competitive production-marketing joint decision-making scenarios captured by our model.

Appendix. Developing the Nash Equilibrium Strategies

Our approach is first to derive the necessary conditions for the Order-taker problem which, we will see, leads to simplifications in the more complex Production-smoother’s problem. We then solve them simultaneously. In the sequel, an asterisk will denote the equilibrium policies.

A.1. Optimality Conditions for the Order-taker’s Problem

The objective function (2.13) can be written:

$$\max_{P_2} \int_0^T \{ [P_2 - (v_d + 1/k_2)](1 - w)a - (b + \gamma)P_2 + \gamma P_1 \} dt + \max_{Q_2} \int_0^T \{-h_2 I_2 \} dt.$$  (A.1)

The second integral of (A.1) is maximized at $Q_2^*(t)$ s.t.

$$I_2(t) = 0 \quad \text{for all } t,$$  (A.2a)

which in turn requires

$$I_2(t) = 0 \quad \text{for all } t,$$  (A.2b)

since $I_2(0) = 0$ by (2.17). Hence by (2.14):

$$Q_2^*(t) = D_2^*(P_2^*(t), P_1^*(t), t).$$  (A.3)

By (A.3), the Lagrangian is:

$$L_2 = H_2 + \rho_2(Q_2 - D_2) = H_2.$$  (A.4a)

The Hamiltonian is:

$$H_2 = [P_2 - (v_d + 1/k_2)](1 - w)a - (b + \gamma)P_2 + \gamma P_1 + \lambda_2[q_t - wa + (b + \gamma)P_1 - \gamma P_2] + \lambda_2[Q_2 - (1 - w)a + (b + \gamma)P_2 - \gamma P_1].$$  (A.4b)

By (A.3), this simplifies to:

$$H_2 = [P_2 - (v_d + 1/k_2)](1 - w)a - (b + \gamma)P_2 + \gamma P_1 + \lambda_2[q_t - wa + (b + \gamma)P_1 - \gamma P_2].$$  (A.4c)

Necessary Conditions. Since the Lagrangian for the Order-taker’s problem does not depend on the Production-smoother inventory level, we obtain:

$$(d \lambda_2)/(dt) = \dot{\lambda}_2 = -\partial L_2/\partial I_2 = 0.$$  (A.5)

Also, the terminal condition is:

$$\lambda_2(T) = 0.$$  (A.6)

Thus, (A.5) and (A.6) imply:

$$\lambda_2(t) = 0 \quad \text{for } 0 \leq t \leq T.$$  (A.7)

Making use of (A.2b), the Order-taker’s problem reduces to maximizing the integrand of (2.13), i.e., maximizing instantaneous profit. This yields:

$$P_2^* = \{(1 - w)a + (b + \gamma)(v_d + 1/k_2) + \gamma P_1\}/[2(b + \gamma)].$$  (A.8)

A.2. Optimality Conditions for the Production-Smoother’s Problem

Recall (2.8) through (2.12). This is an optimal control problem with three control variables where the state variable is constrained to be nonnegative. (See Kamien and Schwartz 1981, Part II, §17 for a general discussion of this type of problem, and Eliaishberg and Steinberg 1984 for an application and explanation of the role of the various variables in a monopolistic setting.)
The Lagrangian is:

\[ L_1 = H_1 + \rho_1 [Q_1 - wa + (b + \gamma)P_1 - \gamma P_2]. \] (A.9a)

Since the setting is competitive, the Hamiltonian incorporates the other player's state equation:

\[ H_1 = (P_1 - v_1)[wa - (b + \gamma)P_1 + \gamma P_2] - \frac{1}{k_1}Q_1^2 - h_1 I_1 \]
\[ + \lambda_{12}[Q_2 - wa + (b + \gamma)P_1 - \gamma P_2] - \lambda_{12}[Q_2 - (1 - w)a + (b + \gamma)P_2 - \gamma P_1]. \] (A.9b)

However, by (A.3), the Production-smoother Hamiltonian reduces to (A.9c), given below. For notational simplicity we have replaced \( \lambda_{12} \) by \( \lambda_1 \) in the sequel.

\[ H_1 = (P_1 - v_1)[wa - (b + \gamma)P_1 + \gamma P_2] - \frac{1}{k_1}Q_1^2 - h_1 I_1 + \lambda_1[Q_2 - wa + (b + \gamma)P_1 - \gamma P_2]. \] (A.9c)

**Necessary Conditions.**

(i) \((d\lambda_1)/(dt) = -(dL_1)/(dI_1)\).

Thus, \( \dot{\lambda}_1 = h_1 \) (A.10)

(ii) \((dL_1)/(dQ_1) = 0\).

Thus, \(-2/k_1)Q_1 + \lambda_1 + \rho_1 = 0\).

Hence, \( Q_1^* = (k_1/2)(\lambda_1 + \rho_1) \). (A.11)

(iii) \((dL_1)/(dP_1) = 0\).

Thus, \( wa - 2(b + \gamma)P_1 + \gamma P_2 + \lambda_1(b + \gamma)+(b + \gamma)(\rho_1 + v_1) = 0\).

Hence, \( P_1^* = [wa + (b + \gamma)(\lambda_1 + \rho_1 + v_1) + \gamma P_2]/[2(b + \gamma)] \). (A.12)

\( \rho_1 I_1 = 0, \quad \rho_1 = 0 \). (A.13a)

\( \lambda_1 + \rho_1 \) is continuous except possibly at an entry or an exit to the boundary \( I_1(t) = 0 \);

\( \lambda_1 + \rho_1 \) is continuous everywhere; and \( \rho_1 \leq 0 \). (A.13c)

For discussion of continuity, see Pekelman (1974) and Jacobson et al. (1971). Sufficiency conditions for the control problem (2.8)–(2.12) can be verified in a fashion similar to Eliashberg and Steinberg (1987).

### A.3. The Strategies in Terms of \( \lambda_1 + \rho_1 \)

Solving simultaneously (A.8) and (A.12), we can obtain the Nash equilibrium prices:

\[ P_1^* = (1/\Delta)[2wb + (1 + w)\gamma]a + 2(b + \gamma)^2(\lambda_1 + \rho_1 + v_1), \] (A.14a)

\[ P_2^* = (1/\Delta)[2b(1 - w)b + (2 - w)\gamma]a + \gamma(b + \gamma)(\lambda_1 + \rho_1 + v_1), \] (A.14b)

\[ \Delta = 4(b + \gamma)^2 - \gamma^2 = (2b + 3\gamma)(2b + \gamma). \] (A.14c)

From (A.11), (A.3), and (2.2):

\[ Q_1^* = (k_1/2)(\lambda_1 + \rho_1). \] (A.15a)

\[ Q_2^* = D_2^*(P_1^*(t), P_2^*(t), t) = (1 - w)a - (b + \gamma)P_2^*(t) + \gamma P_1^*(t). \] (A.15b)

From (2.2) and (A.14):

\[ D_1^*(P_1^*(t), P_2^*(t), t) = [(b + \gamma)/\Delta][2wb + (1 + w)a - 2(b + \gamma)^2 - \gamma^2](\lambda_1 + \rho_1), \] (A.16a)

\[ D_2^*(P_2^*(t), P_1^*(t), t) = [(b + \gamma)/\Delta][2b(1 - w)b + (2 - w)\gamma]a [2(b + \gamma)^2 - \gamma^2] = \gamma(b + \gamma)(\lambda_1 + \rho_1), \] (A.16b)
We define $A$, $B$, and $C$ in (3.1). Then from (A.16a) we can write:

$$D_1^a(P_1^a(t), P_2^a(t), t) = Aa(t) - B[\lambda_1(t) + \rho_1(t)] - C.$$  \hfill (A.17)

Equations (A.14), (A.15) and (A.16) show that the equilibrium prices, production rates, and the demands depend on the behavior of $\lambda_1 + \rho_1$.

A.4. Eliminating $\lambda_1 + \rho_1$

Our approach follows Pekelman (1974) and Eliashberg and Steinberg (1987).

A nonzero time interval over which inventory is positive is called an unconstrained segment. A nonzero time interval over which inventory is zero is called a boundary segment. If $I_1 > 0$, then $\rho_1 = 0$ by (A.13a), and thus $\lambda_1 + \rho_1 = \lambda_1$. In order to find $\lambda_1$, we need to know only the initial value of $\lambda_1$ on that segment, since $\lambda_1 = h_1$ by (A.10). If $I_1 = 0$ for a nonzero interval (i.e., on a boundary segment), then we also have $I_1 = 0$ or, equivalently by (2.9),

$$\dot{I}_1 = Q_1 - D_1 = 0.$$  \hfill (A.18)

We will now solve for the value of $\lambda_1 + \rho_1$ which will satisfy (A.18). From (2.9), (A.15a), and (A.17):

$$\dot{I}_1 = (B + k_1/2)(\lambda_1 + \rho_1) - Aa + C.$$  \hfill (A.19)

We conclude:

$$I_1(0) \equiv 0 \iff \lambda_1 + \rho_1 = \gamma \equiv \text{ where } \gamma = (Aa - C)/(B + k_1/2).$$  \hfill (A.20)

We are now in a position to explicitly describe the behavior of $\lambda_1 + \rho_1$ and, hence, the firms’ strategies. The relationship between $\lambda_1 + \rho_1$ and the $\gamma$-curve (A.20) determines the behavior of $I_1(t)$ and $\dot{I}_1(t)$.

We first note that $\gamma(t)$ achieves its maximum where $a(t)$ does. Further, there exist some $t_0$ and $t^*$, where $0 < t_0 < t^* < T$, such that:

- At $t = 0$: $\lambda_1 + \rho_1 = \lambda_1(0) > \gamma(0)$, $I_1(0) > 0$, $I_1(0) = 0$,
- On $(0, t_0)$: $\lambda_1 + \rho_1 = \lambda_1(0) + h_1t > \gamma(t)$, $I_1(t) > 0$, $I_1(t) > 0$,
- At $t = t_0$: $\lambda_1 + \rho_1 = \lambda_1(t_0) + h_1t_0 = \gamma(t_0)$, $I_1(t_0) = 0$, $I_1(t_0) > 0$, \hfill (A.21)
- On $(t_0, t^*)$: $\lambda_1 + \rho_1 = \lambda_1(t_0) + h_1t < \gamma(t)$, $I_1(t) < 0$, $I_1(t) > 0$,
- On $[t^*, T]$: $\lambda_1 + \rho_1 = \gamma(t)$, $I_1(t) = 0$, $I_1(t) = 0$.

The implications of (A.21) for the Production-smoother inventory policy are as follows. The Production-smoother begins building up inventory at $t = 0$ until a point $t_0$ at which time the firm begins drawing down inventory. Inventory is drawn down from $t_0$ until it reaches zero at a point $t^*$, called the zero inventory point. From $t^*$ until the end of the season, $T$, inventory remains at zero. In order for $t^*$ to exist, $t^* > \arg\max \{a(t)\}$. Otherwise, the zero inventory subcase obtains, and the Production-smoother will follow a zero inventory policy over the entire season, and the equilibrium policies are given by (3.8)–(3.11).

In (A.21), $t^*$ determines the behavior of $\lambda_1 + \rho_1$ and consequently the nature of all of the optimal strategies. Here (A.19), (A.20), (A.21) and (3.1) can be used to obtain (3.2) and (3.3).

By (A.21): for $0 \leq t \leq t^*$, we have $\lambda_1 + \rho_1 = \lambda_1(t_0) + h_1t$; for $t^* \leq t \leq T$, we have $\lambda_1 + \rho_1 = \gamma(t)$. Hence, by (A.20) and (A.21):

$$\lambda_1(t) + \rho_1(t) = \begin{cases} [Aa(t^*) - C]/[B + k_1/2] - h_1(t^* - t), & 0 \leq t \leq t^*, \\ [Aa(t) - C]/[B + k_1/2] = \gamma(t), & t^* \leq t \leq T. \end{cases}$$  \hfill (A.22)

A.5. Note on General Demand Functions

For general demand equations $D_i(a(t), P_i(t), P_j(t)) (i \neq j, i = 1, 2)$, the following equations, which generalize (A.8) and (A.12), must be satisfied simultaneously:

$$[P_2 - (v_2 + 1/k_2)](\partial D_i/\partial P_2) + D_i(a(t), P_1, P_2) = 0,$$  \hfill (A.23)

$$[P_1 - (v_1 + \lambda_1 + \rho_1)](\partial D_i/\partial P_1) + D_i(a(t), P_1, P_2) = 0.$$  \hfill (A.24)

The procedure follows along the same lines outlined in §§A.1–A.4. The explicit solution of $\lambda_1(t) + \rho_1(t)$ for $t^* \leq t \leq T$ is, however, not guaranteed; hence neither is the definition of $\gamma(t)$.
A.6. Proof of Proposition 3.7

From (A.16a), (A.16b), and (A.15a), for \( w = \frac{1}{2} \), the demand functions for the Production-smoother and the Order-taker, respectively, can be written:

\[
D^*_1(t) = \begin{cases} 
\tilde{A}a(t) - B_1Q^*_1(t) - \bar{C}_1, & 0 \leq t \leq t^*, \\
Q^*_1(t), & t^* \leq t \leq T, 
\end{cases} \tag{A.25a}
\]

\[
D^*_2(t) = \tilde{A}a(t) + B_2Q^*_1(t) - \bar{C}_2, \quad 0 \leq t \leq T, \tag{A.25b}
\]

where \( \tilde{A}, B_1, B_2 > 0 \).

First note that the maximum in each of \( D^*_1 \) and \( D^*_2 \) is reached before \( t^* \). This is because:

\[
\frac{dD^*_1(t)}{dt} = \frac{dQ^*_1(t)}{dt} < 0, \quad t^* \leq t \leq T, \tag{A.26a}
\]

\[
\frac{dD^*_2(t)}{dt} = \tilde{A} \frac{da(t)}{dt} + B_2 \frac{dQ^*_1(t)}{dt} < 0, \quad t^* \leq t \leq T. \tag{A.26b}
\]

That is, both \( D^*_1(t) \) and \( D^*_2(t) \) are monotonically decreasing after \( t^* \), since \( a(t) \) peaks before \( t^* \) and thus must be decreasing after \( t^* \).

Prior to \( t^* \), that is, for \( 0 < t < t^* \):

\[
\frac{dD^*_1(t)}{dt} = \tilde{A} \frac{da(t)}{dt} - \frac{k_1}{2} h_1B_1, \tag{A.27a}
\]

\[
\frac{dD^*_2(t)}{dt} = \tilde{A} \frac{da(t)}{dt} + B_2 \frac{dQ^*_1(t)}{dt} = \tilde{A} \frac{da(t)}{dt} + \frac{k_1}{2} h_1B_2, \tag{A.27b}
\]

Hence,

\[
\frac{dD^*_1(t)}{dt} > \frac{dD^*_2(t)}{dt}. \tag{A.28}
\]

Consequently, when \( \frac{dD^*_1(t)}{dt} = 0 \), i.e., when \( D^*_1(t) \) reaches its maximum, \( \frac{dD^*_2(t)}{dt} > 0 \), i.e., \( D^*_2(t) \) is still rising to its maximum point. \( \square \)

References


