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AN INVESTIGATION OF COMPETITIVE PREFERENCE STRUCTURES AND POSTERIOR PERFORMANCE THROUGH A BAYESIAN DECISION-THEORETIC APPROACH*

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In this paper we analyze competitive decision-making situations in terms of their preference structures and posterior performance, through a Bayesian decision-theoretic framework. The setting is that of a two-by-two, two-person, non-zero-sum and noncooperative game which is repeated over time. The dynamic behavior of the competitors for different classes of games, as identified by their preference structures, is examined and a classification scheme is proposed for the purpose of unification. The competitors' dynamic behavior and posterior performance for some general classes of games is then derived, and the relationship to the results implied from game-theoretic considerations is discussed. Illustrative examples are given, too.
(GAMES—NONCOOPERATIVE; DECISION ANALYSIS—SEQUENTIAL; UTILITY/PREFERENCE)

1. Introduction

The essence of decision analysis is to provide the decision maker with an optimal decision rule in order to evaluate alternatives under uncertainty, where the outcomes are governed by nature. White [24, p. 17] notes "... when 'optimality' is a valid concept, it is a 'prior' concept and not a 'posterior' one. In other words, a decision is optimal at the point of taking it, in the line of circumstances surrounding it, and not necessarily optimal in retrospect." Consequently, the decision rule that is prescribed is stated in terms of expected value and the focus is on the "goodness" of the decision rather than the "goodness" of the posterior outcomes (performance). Howard [9, p. 86] emphasizes the distinction between a good decision and a good outcome and argues that "Hopefully, by making good decisions in all the situations that we face we shall insure as high a percentage as possible of good outcomes."

Adopting this relative frequency interpretation in repetitive decision-making under uncertainty against nature, one can make only the following statement concerning the relationship between the expected and the posterior performance: the expected performance represents the "average value" of the posterior performance if the decision is to be made repetitively, a large number of times. The concern here is with long-run posterior performance. The analysis of the posterior performance, however, can be more elaborate and insightful when nature is replaced by another decision maker who also behaves according to optimal decision rules and the problem thereby falls within a competitive decision-making framework.

The need for managers to gain better understanding of competitive decision-making and behavior becomes more evident these days since many industries have reached their maturity level and hence, a firm's growth is possible only by attacking the share

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of its competitor. Increased attention by companies to formal strategic planning has highlighted questions such as: What actions are competitors likely to take? What is the best way to respond? How will my industry evolve in the short-run and in the long-run? Porter [11] presents an interesting conceptual framework which attempts to help a firm analyze its industry as a whole, to predict the industry's future evolution, to understand its competitors and its own position, and to translate this analysis into a competitive strategy for a particular business. His approach, however, is qualitative and does not provide answers to the above stated questions in a quantitative sense.

For years, the classical game theory [26] has been regarded as a logical jointly prescriptive quantitative approach for modeling competitive decision-making. A game-theoretic approach is static in its nature, assumes that the competitors do not assign subjective probabilities to each other's choice of a pure action, allows for mixed-strategies as an optimal solution, and emphasizes the existence and stability of competitive equilibrium. Perhaps one of the reasons why game-theoretic ideas have not found more widespread application is that randomization of the decisions seems to have limited appeal in many practical situations. In addition, the evaluation of the industry evolution when mixed-strategies are allowed, is similar to that of games against nature and is stated in expected values terms. Therefore, creating competitive decision-making models where the competitors are assumed to choose only pure strategies at any point of time, and which can still preserve the desired properties of equilibrium in mixed-strategies as developed in game theory, is important in the analysis of competitive industries. This can be done through a Bayesian decision-theoretic approach which is dynamic in its nature. It allows the decision maker to assign subjective probabilities to the opponent's choices of actions and to revise them in light of new information. It prescribes the selection of pure strategies as an optimal behavior, and considers the optimality of the behavior of one competitor.

This paper analyzes competitive situations within a Bayesian decision-theoretic framework. The major objective of the paper is to provide insight and a benchmark for how different industries may evolve over time in terms of posterior performance. The approach presented in this paper can also provide a unifying framework for analyzing industries that are constantly at peace, constantly at war, or cycling between states of peace and war. Using game-theoretic terminology, the paper analyzes competitive situations characterized as two-by-two, two-person, non-zero-sum, repeated and non-cooperative games. The paper is rather conceptual and deals mainly with interpretations and discussions, relying on another paper for the basic mathematical proofs [3]. Only sketches of the proofs will be given here.

In §2, we describe in detail the competitive situations that are studied and the decision-making model employed, and we refer to some of the relevant literature. In §3, we present the major results of the model by examining the behavior of the competitors in various competitive situations and over time, and the relationship between their behavior and that implied from game-theoretic considerations. In §4, illustrative examples are given and discussed. §5 provides a summary and suggestions for further research.

2. The Competitive Situation and the Bayesian Model

The competitive situation studied here involves two competitors (players) I and II, with two actions (strategies) available to each competitor, (α_1, α_2) for Player I and

(β_1, β_2) for Player II, and where the decisions are made simultaneously by the two players. The term "simultaneously" does not refer to the physical flow of time but means "without knowing the decision taken by the other player." Both players know this, and each player knows his own possible returns.

Several possibilities exist as far as the knowledge of the opponent's returns are concerned. Perhaps the most realistic one is the case in which each player does not know his opponent's returns for any of the four possible combinations of the competitors' pairs of strategy choices (outcomes). In this case, in the terminology of game theory, we are dealing with games with incomplete information [8], represented in normal form by a 2×2 matrix. This same competitive situation is repeated many times and allows the competitors to learn about each other's past decisions which are observable. Future decisions of the opponent are not known to each player and can be just inferred from his past behavior. The analysis of this situation can be thought of as taken from the point of view of an industrial analyst or other observer trying to forecast the industry evolution. In the sequel we shall show that the knowledge of the *ordinal* preference of the players over the four possible outcomes, is sufficient to broadly determine how the sequential game will proceed in the future, for certain classes of games. If the outside observer manages to collect or estimate additional and finer information such as the *cardinal* preference of the players, the forecasting of the industry evolution becomes more elaborate. The cardinal preference can be, for example, a von Neumann-Morgenstern utility function over the business goals that are achieved by each player, for each of the four pairs of strategy choices. Other possibilities concerning the knowledge of the opponent's returns, such as complete-information or asymmetric-information, also exist, but will not be considered in this paper.

Although competitive situations depicted as 2×2 games are the simplest two-person games, they have attracted attention of researchers from many disciplines. Rapoport, Guyer and Gordon [16] summarize and interpret what has been learned in the last fifteen years, through experimentation, about social interaction and behavior using this paradigm. Classifications for all 2×2 games have been suggested by Rapoport and Guyer [15] and Harris [6], [7] to aid in combining together games with similar game-theoretic and behavioral aspects. Iterated Prisoners' Dilemma games have also been studied extensively [5], [13]. Sequential games arise in contexts such as economics [22], gaming [23], and stochastic processes [18], [19].

In a business context, the two actions available to each competitor can be thought of as strategic moves such as offensive or defensive moves. Martial language is familiar in many business situations. There are the gasoline price "wars," the "escalating arms budgets" of the soap companies and "invading Coke's markets," to name a few. Porter [11, Chapter 5] describes several conditions that may increase the likelihood of "competitive warfare." Hence, it appears that the language of warfare in business is not just descriptive and bears operational logic to business executives in plotting competitive strategy. At the broadest level, two major strategies can be identified, namely: attack (or defend) vs. keep the status-quo. An attack strategy can then be formulated in terms of product, price, advertising, etc. Strategic decisions concerning these two broad strategies are made periodically and may be interpreted as generating sequential games.

Noting that the results and the conclusions drawn later are constrained of course by the assumptions made, we turn now to a detailed description and explanation of the competitive decision-making model's assumptions. The model developed here assumes that the competitors regard each other's behavior as a stochastic decision process. This assumption is implicit in the "fictitious play" literature [1], [17]. Each player assumes, for lack of other information, that his opponent will behave randomly. Consistent with this assumption, if the pair (p, q) represents the probabilities of Players I and II choosing their first available action, respectively, then q is not known to Player I and p is not known to Player II. However, they can assess some prior probability density functions (p · d · f) over these parameters. The p · d · f's are denoted by $(f^{\mathrm{I}}(q), f^{\mathrm{II}}(p))$. In words, $f^{\mathrm{I}}(q)$ is Player I's prior p · d · f over the event that Player II is choosing his first available action with probability q, and $f^{\mathrm{II}}(p)$ is Player II's prior p · d · f over the event that Player I is choosing his first available action with probability p. It is also assumed in our model that the opponent's behavior can be described by a Bernoulli process. A Bernoulli process is a data-generating process with two possible outcomes on each trial ("success" and "failure"), such that the probabilities of these outcomes are stationary, and the outcomes of the trials are independent. In our context, a trial corresponds to simultaneous decisions and a "success" corresponds to the event that the opponent did choose his first action. After observing each other's decision, the two players learn and revise their prior p · d · f's according to the Bayesian rule and thus obtain their posterior p · d · f's. These serve as prior p · d · f's for the next simultaneous decisions. It has been noted in the statistical decision-theoretic literature [12], [25] that the revision could be difficult to do analytically unless the prior distribution is a member of the family of distributions that is conjugate with respect to the Bernoulli process. The conjugate family in this case is the family of beta distributions.

A beta distribution $f(p \mid r, n)$ for $0 \leqslant p \leqslant 1$ is characterized by two parameters, r and n, where $n > r > 0$ and its mean and variance are:

$$E(p \mid r, n) = r/n \tag{2.1}$$

and

$$V(p \mid r, n) = r(n-r)/n^2(n+1). \tag{2.2}$$

The shape of $f(p \mid r, n)$ depends on r and n, and can accommodate a large number of probabilistic judgments. If the prior parameters at time t are r_t and n_t, and the sample results are r "successes" in n trials, the posterior parameters at time $t+1$, r_{t+1} and n_{t+1}, can be easily computed from:

$$n_{t+1} = n_t + n \tag{2.3}$$

and

$$r_{t+1} = r_t + r. \tag{2.4}$$

In our context, of course, $n = 1$ and $r = 1$ or 0, depending on whether or not the opponent did select his first action. We note, therefore, that n_t and r_t can be viewed as counters such that $n_t - n_0$ counts the number of simultaneous decisions that have been made, and $r_t - r_0$ counts the number of times the opponent has used his first action. Within the Bayesian decision-theoretic framework, the simultaneous decisions amount to the selection of an action which does not influence the subjective probability of the random events (states) associated with this action. This case which is assumed throughout this paper, is called the act-unconditional states case. An alternative

Bayesian decision-making model, act-conditional states [21], allows for the possibility that the selection of an alternative may influence the subjective probability of the random events that will follow the choice. More formally, if $f^{\mathrm{I}}(q \mid \alpha_1)$ and $f^{\mathrm{I}}(q \mid \alpha_2)$ denote Player I's subjective p · d · f over the event that Player II is choosing his first action with probability q, *given* that he (Player I) selects his first and second action respectively, then, under the act-unconditional states assumption, $f^{\mathrm{I}}(q \mid \alpha_1) = f^{\mathrm{I}}(q \mid \alpha_2)$ at any point of time. Symmetrically, $f^{\mathrm{II}}(p \mid \beta_1) = f^{\mathrm{II}}(p \mid \beta_2)$ at any point of time.

After the players revise their prior p · d · f's and obtain the posterior p · d · f's, they use them to compute their expected returns in the next period. It will be assumed that the decision-making rule used by both players is to select their *first* action if and only if its expected return *is strictly greater* than the expected return from the second action. Consequently, the *second* action is chosen by both players if and only if its expected return is *greater than or equal* to the expected return from the first action. No generality is lost by this decision rule since by relabeling the actions, all 2×2 games can be treated this way.

We conclude this section by noting that a model similar somewhat to the sequential game model analyzed in this paper has been briefly discussed by Sanghvi and Sobel [18] as a noncompact game. In their model it is assumed that Player I plays a zero-sum 2×2 game against a programmed opponent who uses a stationary mixed-strategy and never learns about Player I's behavior. Noting the difference between this model and the other models discussed in their paper (compact games), they prove (Theorem 5.1) that this game is ergodic, in the sense that it has a positive probability of being in any state in the long-run.

3. Analysis of Competitive Preference Structures and Posterior Performance

Let Matrix (a) represent a 2×2 game with the following returns to the players:

		Player II's Actions β_1	β_2
Player I's	α_1	$R^{\mathrm{I}}(\sigma_1), R^{\mathrm{II}}(\sigma_1)$	$R^{\mathrm{I}}(\sigma_2), R^{\mathrm{II}}(\sigma_2)$
Actions	α_2	$R^{\mathrm{I}}(\sigma_3), R^{\mathrm{II}}(\sigma_3)$	$R^{\mathrm{I}}(\sigma_4), R^{\mathrm{II}}(\sigma_4)$

Matrix (a)

where σ_i $(i = 1,2,3,4)$ denotes the four possible outcomes (states) of the game, defined by the competitors' pairs of strategy choices, and $R^{\mathrm{I}}(\cdot), R^{\mathrm{II}}(\cdot)$ denote the returns to Players I and II respectively, from each of these four possible states. Next, assume that $R^{\mathrm{I}}(\cdot)$ and $R^{\mathrm{II}}(\cdot)$ are measured on an *ordinal* scale. That is, if $R^{\mathrm{I}}(\cdot)$ is measured, for example, in profit terms, then the information contained in Matrix (a) allows one to make a statement such as: $R^{\mathrm{I}}(\sigma_1) > R^{\mathrm{I}}(\sigma_2)$ means that the profit generated to Player I in state σ_1 is larger than the profit generated to him in state σ_2. Note that in making this statement one does not have to know the exact value of the profit generated in each state. We can also interpret $R(\cdot)$ as an ordinal preference measure and make, for instance, the following statement: $R^{\mathrm{I}}(\sigma_1) > R^{\mathrm{I}}(\sigma_2)$ if and only if Player I prefers state σ_1 to σ_2. Here, several specific goals can be achieved by the player in each state. We shall assume, unless otherwise specified, strict preference ordering of the states. It has been noted by Rapoport and Guyer [15] that cases of indifference between two states can be considered as limiting cases of strict preference.

Our first aim is to investigate whether the ordinal scale is sufficient to determine the course of the sequential game for any possible 2×2 game, and to identify classes of games such that sequential games may evolve in the same pattern within each class. We rely on the taxonomy developed by Rapoport and Guyer [15] for this purpose. They have shown that out of all five hundred and seventy-six possible pairs of preference orderings ($4! \times 4!$), only seventy-eight of these are nonequivalent. Equivalent games can be generated from one another by relabeling actions and/or players. For example, consider the following game, in which the preference ordering is $4 > 3 > 2 > 1$.

	β_1	β_2
α_1	2, 2	4, 1
α_2	1, 4	3, 3

Matrix (b)

By interchanging rows, columns, or both rows and columns, we obtain from Matrix (b) three other matrices representing the same game:

	β_1	β_2
α_2	1, 4	3, 3
α_1	2, 2	4, 1

Matrix (c)

	β_2	β_1
α_1	4, 1	2, 2
α_2	3, 3	1, 4

Matrix (d)

	β_2	β_1
α_2	3, 3	1, 4
α_1	4, 1	2, 2

Matrix (e)

When we interchange the players, however, we obtain a matrix which is identical to Matrix (b). Thus, Matrix (b) can generate only four equivalent games. Other matrices may generate a different number of equivalent games. We shall term a game such as the one described in Matrix (b) as a "competitive preference structure." Rapoport and Guyer classify all 2×2 nonequivalent games into three major classes. Class I: each player has a dominating strategy (games 1–21); Class II: one player has a dominating strategy (games 22–57); and Class III: neither player has a dominating strategy (games 58–78).

To investigate and classify the dynamic competitive behavior implied by our model, we denote the differences in the players' return by S, T, U, and V such that:

$$S = R^{\mathrm{I}}(\sigma_4) - R^{\mathrm{I}}(\sigma_2),$$

$$T = R^{\mathrm{I}}(\sigma_3) - R^{\mathrm{I}}(\sigma_1),$$

$$U = R^{\mathrm{II}}(\sigma_4) - R^{\mathrm{II}}(\sigma_3),$$

$$V = R^{\mathrm{II}}(\sigma_2) - R^{\mathrm{II}}(\sigma_1).$$

Our classification of the 2×2 games is made according to the following relationships:
Class (i): $S > 0$, $T > 0$.
Class (ii): $S < 0, T > 0, U < 0, V > 0$.
Class (iii): $S > 0, T < 0, U < 0, V > 0$.
That is, we shall always rearrange a 2×2 game such that it will fall in one of the above three classes. In our classification, games in Class (i) are such that at least one player has a dominating strategy. This class contains games 1–57 in accordance with Rapoport and Guyer's taxonomy. Our Class (ii) of games contains games 58–65 in their taxonomy by interchanging the columns *or* rows of these games, and in addition, it contains their games 66–69. Finally, our Class (iii) contains Rapoport and Guyer's games 70–78, by interchanging their columns *or* rows.

We turn now to the first proposition which shows that the ordinal information regarding the competitors' returns is sufficient to uniquely determine the course of the game for some classes, and insufficient for other classes. We shall view the generated sequential game as a discrete time Semi-Markov process with a discrete state space $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and which possesses transition probabilities equal to zeros and ones. The state of the process is determined by the play's outcome and we shall characterize the dynamic behavior of the process (game) using terminology from the theory of stochastic processes.

PROPOSITION 1. *Given the state of the process at time t and the Class to which the competitive preference structure belongs, the next* different *state visited by the process is given by the entries of Table* 1.

TABLE 1
Current and Next-Visited Different State for Each Class of Games

		Class of Games		
		(i)	(ii)	(iii)
Current State	σ_1	n/a*	not unique	σ_2
	σ_2	n/a*	none**	σ_4
	σ_3	n/a* or σ_4	none**	σ_1
	σ_4	none**	not unique	σ_3

*not applicable since the game will never enter this state.
**whenever the game is in this state, it remains there on all future plays.

Table 1 shows that for games in Class (i), σ_1 and σ_2 are two states which the process will never enter. This is so because our definition of Class (i) is such that $S > 0$ and $T > 0$. This implies that α_2 is Player I's dominating strategy. Hence, he will always choose this action, regardless of what his subjective probability assessment over the opponent's choice of action is. If Player II also has a dominating strategy, we shall label it as β_2 and in this case state σ_3 will never be visited by the process. If, however, Player II does not have a dominating strategy we shall label as β_2 the strategy that makes $U > 0$, and it is still possible to find the process at some point of time in state σ_3. (Recall that we assume that Player II does not know the returns for Player I.) This state is, however, a transient state and after a finite number of simultaneous decisions the process will leave this state and enter state σ_4. State σ_4 is thus an absorbing state; once the process enters this state, it remains there on all future plays.

An example of a competitive business situation which can be structured as one of Class (i)'s games is shown in the following game matrix:

			Competitor II	
			Keep the Status-Quo β_1	Cut Price β_2
Competitor I	Keep the Status-Quo	α_1	B, F	D, E
	Cut Price	α_2	A, H	C, G

Matrix (f)

Here, $A > B > C > D$ and $E > F > G > H$ are the returns for Player I and Player II, respectively. This is a Prisoner's Dilemma like game matrix where both players have dominating strategies: α_2 and β_2. Of course, due to our incomplete information assumption, neither player knows his opponent's payoffs and when acting as a Bayesian player, our model would then predict that if the game is repeated over time, the decisions made by the competitors each time will always be to cut the price, and hence, the posterior performance will be C and G for Players I and II, respectively, at any point of time. An illustration of this game, in the context of advertising radial tires competition between Sears and Goodyear, is shown and discussed in [4].

Turning to the games of Class (ii), we consider, for example, the following game:

	β_1	β_2
α_1	C, G	B, E
α_2	A, F	D, H

Matrix (g)

where, again, $A > B > C > D$ and $E > F > G > H$. Any competitive situation which has a payoff table with the same properties as game matrix (g) is known as a "Battle of the Sexes". An illustration and discussion of the game in the context of new product introduction is given in [4]. Notice that both pairs of choices (α_1, β_2) (state σ_2) and (α_2, β_1) (state σ_3) are equilibrium pairs in the sense that if the game is in one of these states, it is to neither player's advantage to unilaterally choose a different strategy. Hence, states σ_2 and σ_3 are the absorbing states for games in Class (ii). What will be the course of a Class (ii) game if the current state is σ_1 or σ_4? Unfortunately, the answer to this question is not unique. This will be illustrated by means of an example related to Matrix (g).

Suppose that the game described by Matrix (g) is currently in state σ_4. This is true if and only if, $EV_t^{\mathrm{I}}(\alpha_1) \leqslant EV_t^{\mathrm{I}}(\alpha_2)$ and $EV_t^{\mathrm{II}}(\beta_1) \leqslant EV_t^{\mathrm{II}}(\beta_2)$, where the $EV_t(\ )$'s are the expected returns of the two actions available to the two players, at time t. Of course, in computing the player's expected returns we have to interpret A, B, C, D, E, F, G, H as cardinal returns. However, they can be *any* cardinal numbers that preserve the assumed ordinal relationship. Given that state σ_4 has just occurred, the two players will revise their beta distributions and obtain the following revised expected returns:

$$EV^{\mathrm{I}}_{t+1}(\alpha_1) = \frac{n^{\mathrm{I}}_t EV^{\mathrm{I}}_t(\alpha_1) + B}{n^{\mathrm{I}}_t + 1} \quad \text{and} \quad EV^{\mathrm{I}}_{t+1}(\alpha_2) = \frac{n^{\mathrm{I}}_t EV^{\mathrm{I}}_t(\alpha_2) + D}{n^{\mathrm{I}}_t + 1}$$

$$EV^{\mathrm{II}}_{t+1}(\beta_1) = \frac{n^{\mathrm{II}}_t EV^{\mathrm{II}}_t(\beta_1) + F}{n^{\mathrm{II}}_t + 1} \quad \text{and} \quad EV^{\mathrm{II}}_{t+1}(\beta_2) = \frac{n^{\mathrm{II}}_t EV^{\mathrm{II}}_t(\beta_2) + H}{n^{\mathrm{II}}_t + 1},$$

where n_t denotes one of the two parameters of the beta distribution at time t. For numerical illustration, suppose that $EV^{\mathrm{I}}_t(\alpha_1) = EV^{\mathrm{I}}_t(\alpha_2) = 4$, $EV^{\mathrm{II}}_t(\beta_1) = 3$, $EV^{\mathrm{II}}_t(\beta_2) = 5$, $n^{\mathrm{I}}_t = 3$ and $n^{\mathrm{II}}_t = 4$. It can be readily seen now that the next visited different state will be σ_1 if: $F - H > 8$, or σ_2 if: $F - H \leqslant 8$. Hence, in addition to the ordinal property of the returns, we need to know the difference in the expected returns at time t relative to the difference in the respective returns, in order to determine the future course of the game.

Although dynamic games of Class (ii) are in general nonunique, it is still possible to identify conditions for *some* games, based on their competitive preference structure, where the future course of these games can be uniquely determined. It has been shown by Rapoport and Guyer [15] that some games in Class (ii) contain a single Pareto-optimal outcome and that there are other games in Class (ii) which contain two Pareto-optimal outcomes. A Pareto-optimal outcome of a game is defined as an outcome such that there is no other in which *both* players get larger returns. We turn now to Corollary 1.1 whose proof is straightforward and which formalizes the game-theoretic behavior implied for some of the games in Class (ii). We shall use this property in the sequel, to further investigate the posterior performance of these games.

COROLLARY 1.1. *There exist games in Class* (ii) *with the property that it is possible to identify for them competitive preference structures so that the equilibrium mixed-strategies of the two players will be equal.*

We shall illustrate the corollary by means of an example. Consider the following two game matrices:

	β_1	β_2
α_1	C, H	A, E
α_2	B, F	D, G

Matrix (h)

	β_1	β_2
α_1	B, F	C, E
α_2	A, G	D, H

Matrix (i)

We are still assuming the following ordinal relationship: $A > B > C > D$ and $E > F > G > H$. Matrix (h) contains a single Pareto-optimal outcome (A, E). In Matrix (i), however, the Pareto-optimal outcomes are (C, E) and (A, G).

For non-zero-sum games where the players' returns are measured on a cardinal scale, the prescription provided by game-theory is given in terms of the following equilibrium mixed-strategies (see [14, p. 138] for an example of a specific derivation and the rationale):

	Matrix (h)	Matrix (i)	
	$p^* = \Pr\{\text{I plays } \alpha_1\} = (F - G)/(E + F - G - H)$	$p^* = (G - H)/(E + G - F - H)$	(3.1)
	$q^* = \Pr\{\text{II plays } \beta_1\} = (A - D)/(A + B - C - D)$	$q^* = (C - D)/(A + C - B - D)$	(3.2)

Notice that because of the ordinal relationship that we are assuming, in Matrix (h): $q^* > 1/2$ and $p^* < 1/2$ and hence, $q^* \neq p^*$. However, in Matrix (i) it is possible to identify competitive preference structures such that $p^* = q^*$. The necessary condition for that is: $(A - B)/(C - D) = (E - F)/(G - H)$. We turn now to a proposition concerning the dynamic behavior of the games of Class (ii) discussed in Corollary 1.1.

PROPOSITION 2. *Games in Class* (ii), *where $p^* = q^*$ and where the players' expected values of the beta distribution at $t = 0$ are equal, are developed in cycles, in the sense that the process oscillates between states σ_1 and σ_4.*

It can be seen from Matrix (i), one of the matrices for which Proposition 2 holds, that states σ_2 and σ_3 are two absorbing states from which it is to neither player's advantage to unilaterally move out. Note also that the conditions required by Proposition 2 make these games very symmetric. Consequently, the switch in the players' strategy choice occurs exactly at the same time.

Unlike the games discussed in Proposition 2 in which the process may visit only two states, games of Class (iii) are developed in cycles in which all four states are visited. This is formalized in the following proposition:

PROPOSITION 3. *Games in Class* (iii) *are developed in cycles, in the sense that once the process leaves state σ_i* ($i = 1, 2, 3, 4$), *the probability of returning to this state after a finite number of steps is one.*

The proof of this proposition is based on the preference structure of Class (iii)'s games. Whenever one player switches to a different strategy, his opponent's expected returns are such that it is still optimal for him to keep choosing his old strategy.

In stochastic processes jargon, the process for Class (iii)'s games is a regenerative process. Note from Table 1 that the sequence of the visited states is $\sigma_4 \to \sigma_3 \to \sigma_1 \to \sigma_2$. It is also worth noting that the game described in Matrix (j), for $A > B > C > D$ and $E > F > G > H$,

	β_1	β_2
α_1	B, G	D, E
α_2	C, F	A, H

Matrix (j)

is a pure conflict game and belongs to Class (iii). This competitive preference structure is important because it can capture both non-zero-sum and zero-sum conditions ($E = -D$, $F = -C$, $G = -B$, $H = -A$). The basic mathematical proofs concerning the dynamic behavior of this game are given in [3]. Since games in Class (ii) and (iii) proceed by cycles (Propositions 2, 3), it makes no difference when we start observing them when studying their dynamic behavior. We shall, therefore, assume that at $t = 0$ the process is in state σ_4. A question of interest is: how many plays a game such as the one described in Matrix (j) will be in each of its four possible states? To answer this question we need to know the competitive cardinal preference structure. We denote by i'_t, j'_t, k'_t, l'_t (i'_t, j'_t, k'_t, l'_t are positive integers) the cumulative number of plays that states σ_4, σ_3, σ_1 and σ_2 (in this order), respectively, have been realized for a game in Class (iii), during the first t cycles. We also denote by $E_0^{\mathrm{I}}(q)$ and $E_0^{\mathrm{II}}(p)$ the expected values, at $t = 0$, of the prior beta distributions of Players I and II, respectively. Proposition 4 presents the necessary and sufficient condition for this game to be realized, and sets lower bounds for i_t', j_t', k_t', and l_t'. Its proof is based on the computation of the expected returns of the competitors' actions, at times when transitions occur, and on some algebraic manipulations.

PROPOSITION 4. *For games of Class* (iii) *and under the Bayesian model, if at* $t = 0$, $E_0^I(q) \leqslant q^*$ *and* $E_0^{II}(p) \geqslant p^*$, *then the process starts in state* σ_4 *and the following inequalities hold*:

$$i_t' > [l_{t-1}' + k_{t-1}']\left[\frac{V}{-U}\right] - j_{t-1}' + \frac{\delta}{-U}, \tag{3.3}$$

$$j_t' > [i_t' + l_{t-1}']\left[\frac{S}{-T}\right] - k_{t-1}' + \frac{\epsilon}{-T}, \tag{3.4}$$

$$k_t' \geqslant [j_t' + i_t']\left[\frac{-U}{V}\right] - l_{t-1}' - \frac{\delta}{V}, \tag{3.5}$$

$$l_t' \geqslant [k_t' + j_t']\left[\frac{-T}{S}\right] - i_t' - \frac{\epsilon}{S}, \tag{3.6}$$

where $\delta = n_0^{II}[EV_0^{II}(\beta_2) - EV_0^{II}(\beta_1)]$ *and* $\epsilon = n_0^I[EV_0^I(\alpha_2) - EV_0^I(\alpha_1)]$.

Series for i_t', j_t', k_t' and l_t' can be formed recursively, under certain conditions, from (3.3)–(3.6). Let $S/(-T) = K$, $(-U)/V = L$, $\delta/(-U) = M$ and $\epsilon/(-T) = N$. We can turn now to Proposition 5 which provides insight on the cumulative number of times that each state has been visited during the first t cycles, and we show that under certain conditions, it can be expressed by a second-order power series. The required conditions are stated in the proposition and the proof is based on Proposition 4 and mathematical induction.

PROPOSITION 5. *For games of Class* (iii), *where* $L = 1$; $M, N \geqslant 0$ *and integers*; $K \geqslant N + 3$ *and integer*, *and under the Bayesian model, if at* $t = 0$, $E_0^I(q) \leqslant q^*$ *and* $E_0^{II}(p) \geqslant p^*$, *then the following equations hold*:

$$i_t' = t^2 + Mt \qquad \textit{for } t = 1, 2, 3, \ldots, \tag{3.7}$$

$$j_t' = \begin{cases} Kt^2 + \left[MK + \dfrac{K-1}{2}\right]t - \dfrac{K-3}{2} + N & \textit{for } t = 1, 3, 5, \ldots, \\ Kt^2 + \left[MK + \dfrac{K-1}{2}\right]t & \textit{for } t = 2, 4, 6, \ldots, \end{cases} \tag{3.8}$$

$$k_t' = \begin{cases} Kt^2 + \left[MK + \dfrac{K+1}{2}\right]t - \dfrac{K-3}{2} + N & \textit{for } t = 1, 3, 5, \ldots, \\ Kt^2 + \left[MK + \dfrac{K+1}{2}\right]t & \textit{for } t = 2, 4, 6, \ldots, \end{cases} \tag{3.9}$$

$$l_t' = t^2 + (M+1)t \qquad \textit{for } t = 1, 2, 3, \ldots. \tag{3.10}$$

REMARK. Although we require in the proposition that $K \geqslant N + 3$, proofs for $2 \leqslant K < N + 3$ are similar but have to be considered separately.

To compare the dynamic behavior of Class (iii)'s games with that of games of Class (ii) (the ones discussed in Proposition 2), we present now Proposition 6 whose proof is similar to Proposition 5's proof. We let now $(-S)/T = K = (-U)/V = L$, $\delta/(-U) = M$ and $\epsilon/T = N$.

PROPOSITION 6. *For games of Class* (ii), *where* $K \geqslant 1$ *and integer*; $M, N \geqslant 0$ *and integers*; $p^* = q^*$ *and* $E_0^I(q) = E_0^{II}(p)$, *and under the Bayesian model, if at* $t = 0$, $E_0^I(q) \geqslant q^*$ $(E_0^{II}(p) \geqslant p^*)$, *then the following equations hold*:

$$i_0 = M + 1, \tag{3.11}$$

$$k_t' = \left[(M+1)K - N\right]t \quad \textit{for } t = 1, 2, 3, \ldots, \tag{3.12}$$

$$i_t' = t \quad \textit{for } t = 1, 2, 3, \ldots . \tag{3.13}$$

We can now comment on some of the major differences and similarities in the dynamic behavior of games of Class (ii) and Class (iii). First, games of Class (ii) proceed by *fixed* cycles; that is, the number of simultaneous decisions made in each cycle, remains constant. On the other hand, the cycles are variable and becoming longer for games of Class (iii). Second, for games of Class (ii) there exists a transient period of time (i_0) before the process starts cycling. This transient period of time gets longer, the larger the difference between the players' initial expectations, $E_0(\cdot)$, and the game-theoretic equilibrium mixed-strategies. Games of Class (iii), on the other hand, start cycling from their beginning. Finally, it can be shown that for both classes of games, the empirical relative frequencies of strategy choices converge to the game-theoretic equilibrium strategies. This interesting result has been conjectured by Brown [1] and proved by Robinson [17] for finite two-person zero-sum games. Our competitive model extends this result to non-zero-sum games. It should be noted, however, that the convergence is not the same in the two classes of games. The games of Class (ii) converge immediately and with the same rate to their equilibrium mixed-strategies because of their symmetric structure ($p^* = q^*$ and $E_0^{\mathrm{I}}(q) = E_0^{\mathrm{II}}(p)$). The convergence, however, may take different forms for games of Class (iii). It can come from above or from below, and with different convergence rates. For example, for games such as the ones described in Proposition 5, Player II converges more rapidly than Player I to his game-theoretic equilibrium strategy. This may be due to the assumption that the initial state is σ_4 which is the least desirable one for Player II.

This concludes our analysis of competitive preference structures and posterior performance and the presentation of the results implied by our competitive decision-making model. The next section illustrates some of our findings with examples.

4. Illustrative Examples

EXAMPLE 1. In this example we consider two industries (1) and (2), where in each industry Player I can be thought of as a possible entrant and Player II can be viewed as the market leader. The competitive preference structures of the two industries are represented by Matrices (4.1.1) and (4.1.2), where the preferences are measured, for instance, on a 0–100 cardinal scale.

Entrant \ Leader		Defend β_1	Keep the Status-Quo β_2
Keep the Status-Quo	α_1	40, 20	20, 40
Attack	α_2	30, 30	80, 10

Matrix (4.1.1)—Industry (1)

Entrant \ Leader		Defend β_1	Keep the Status-Quo β_2
Keep the Status-Quo	α_1	60, 20	20, 40
Attack	α_2	40, 30	100, 10

Matrix (4.1.2)—Industry (2)

Several arguments may support these competitive preference structures which are basically pure conflict games. From the leader's point of view, his most preferred outcome is that both sides keep the status-quo (i.e., peace prevails). His second most preferred outcome is a combination of an offensive move on the part of the competitor and a defensive move on his part. This may be justified since the disciplining action can lead any aggressor to expect that retaliation will always occur. Porter [11, p. 99] refers to this strategy as "discipline as a form of defense." The leader's third most preferred outcome is a defensive move on his part while the competitor is keeping the status-quo. This may mean unjustified warfare costs for the leader. Finally, the leader's least preferred outcome is to keep the status-quo while the competitor is attacking. For the entrant, the preference ordering of the four states is exactly reversed due to similar arguments.

We shall use now the notations developed in §3 to investigate the evolution of the two industries.

Industry (1)	Industry (2)
$S = 60, T = -10, U = -20, V = 20, K = 6$	$S = 80, T = -20, U = -20, V = 20, K = 4,$
$L = 1, p^* = 1/2, q^* = 6/7, E_0^{\mathrm{I}}(q) = 4/5,$	$L = 1, p^* = 1/2, q^* = 4/5, E_0^{\mathrm{I}}(q) = 8/11,$
$E_0^{\mathrm{II}}(p) = 2/3, n_0^{\mathrm{I}} = 2.5, n_0^{\mathrm{II}} = 3, N = 1,$	$E_0^{\mathrm{II}}(p) = 2/3, n_0^{\mathrm{I}} = 2.75, n_0^{\mathrm{II}} = 3, N = 1,$
$M = 1.$	$M = 1.$

Note that the two industries are similar in every respect except for their K-ratios. Given the competitive preference structures, we identify the two games as belonging to Class (iii). Hence, from the data and due to Propositions 3 and 5 we know that currently both industries are in state σ_4, that is, the entrant is attacking while the leader is keeping the status-quo. How many plays they will stay in each state can be determined from equations (3.7)–(3.10). For example, the first cycle will last as follows:

Industry (1)	Industry (2)
$i_1 = 2, j_1 = 14, k_1 = 15, l_1 = 3$	$i_1 = 2, j_1 = 10, k_1 = 11, l_1 = 3$
$i_1 + j_1 + k_1 + l_1 = 34$	$i_1 + j_1 + k_1 + l_1 = 26$

Clearly, the cycles in Industry (1) are longer because of its larger K-ratio. An insight as to the asymptotic dynamic behavior of the two industries can be obtained from the limits of the empirical relative frequencies which we denote by $pp_t'(\alpha_1)$ and $pp_t'(\beta_1)$ for strategies α_1 and β_1, respectively.

$$pp_t'(\alpha_1) = \frac{k_t' + l_t'}{i_t' + j_t' + k_t' + l_t'} \tag{4.1}$$

and

$$pp_t'(\beta_1) = \frac{j_t' + k_t'}{i_t' + j_t' + k_t' + l_t'}\,. \tag{4.2}$$

Substituting (3.7)–(3.10) for some t odd we obtain:

$$pp_t'(\alpha_1) = \frac{(K+1)t^2 + \left[MK + (K+3)/2 + M\right]t - (K-3)/2 + N}{2(K+1)t^2 + (2M + 2MK + K + 1)t - K + 3 + 2N} \tag{4.3}$$

and

$$pp_t'(\beta_1) = \frac{2Kt^2 + K(2M+1)t - K + 3 + 2N}{2(K+1)t^2 + (2M + 2MK + K + 1)t - K + 3 + 2N}. \tag{4.4}$$

For the two industries being investigated we obtain:

Industry (1)	Industry (2)
$pp_t'(\alpha_1) = \dfrac{7t^2 + 11.5t - 0.5}{14t^2 + 21t - 1}$	$pp_t'(\alpha_1) = \dfrac{5t^2 + 8.5t + 0.5}{10t^2 + 15t + 1}$
$pp_t'(\beta_1) = \dfrac{12t^2 + 18t - 1}{14t^2 + 21t - 1}$	$pp_t'(\beta_1) = \dfrac{8t^2 + 12t + 1}{10t^2 + 15t + 1}$

It turns out that $pp_t'(\alpha_1)$ converges to p^* from above in both industries. The convergence of $pp_t'(\beta_1)$ to q^* is from below in Industry (1) and from above in Industry (2). We also note that $pp_t'(\beta_1)$ converges to its equilibrium more rapidly than $pp_t'(\alpha_1)$ in both industries.

EXAMPLE 2. Consider the industry with a competitive preference structure represented by Matrix (4.2.1) and where the preferences are measured, say, on a 0–20 cardinal scale.

			Competitor II: Keep the Status-Quo β_1	Competitor II: Attack β_2
Competitor I	Keep the Status-Quo	α_1	10, 12	8, 15
	Attack	α_2	12, 10	2, 1

Matrix (4.2.1)

This industry is symmetric with respect to the competitors' preference ordering of the four states. However, the strength of the preference is different. The competitive preference structure described in Matrix (4.2.1) may represent an industry where mutual war could be a disaster for both competitors and hence, it is their least preferred state. The second least preferred state for each competitor is the case where he is keeping the status-quo while his opponent is attacking. The second most preferred state is the case where both competitors are in peace. Finally, the most preferred outcome for each competitor is to attack while his opponent is keeping the status-quo.

What would be the evolution of the industry for some given expectations over the competitor's likelihood of choosing his strategies? We shall determine it from the industry's parameters, using the notations presented in §3: $S = -6, T = 2, U = -9, V = 3, K = 3, L = 3$, $p^* = 3/4, q^* = 3/4, E_0^I(q) = 3/4, E_0^{II}(p) = 3/4, M = 0, N = 0$. We note that the competitive preference structure belongs to Class (ii) and meets the conditions specified in Proposition 6. Hence, the industry will cycle through two states:

σ_1 and σ_4, namely, keeping peace simultaneously or mutual war. From the data and Proposition 6 we can determine the number of plays that these states will be visited. The process will start with one simultaneous decision of war ($i_0 = 1$). This will be the transient period. Then, cycles will develop and always be composed of three simultaneous peace decisions ($k = 3$) and a single simultaneous war decision ($i = 1$).

EXAMPLE 3. In this example we shall illustrate the effect of the competitors' attitudes toward risk upon their posterior performance. Consider the competitive preference structures represented by Matrices (4.3.1) and (4.3.2).

	β_1	β_2
α_1	40, 20	20, 40
α_2	30, 30	80, 10

Matrix (4.3.1)

	β_1	β_2
α_1	3.50, 20	1.87, 40
α_2	2.56, 30	12.23, 10

Matrix (4.3.2)

Here, Matrix (4.3.1) is identical to Matrix (4.1.1) presented in Example 1, and we can use the results illustrated there. However, we shall interpret now the numerical returns as *monetary payoffs*. Matrix (4.3.2) is derived from Matrix (4.3.1) in some particular manner. It is assumed that Player II is still risk neutral and hence, his returns are the same as in Matrix (4.3.1). For Player I, though, Matrix (4.3.2) is constructed from the monetary payoffs in Matrix (4.3.1) and under the assumption that he is a risk taker with an exponential utility function: $U(x) = e^{0.0313x}$. Exponential utility is viewed as a reasonable approximation to the preferences of many decision makers [10], [20]. Note that the preference ordering of the four states does not change but the parameters of the industry represented by Matrix (4.3.2) now become: $S = 10.36, T = -0.94, U = -20, V = 20, K = 11, L = 1, p^* = 1/2, q^* = 11/12, E_0^{\mathrm{I}}(q) = 4/5, E_0^{\mathrm{II}}(p) = 2/3, n_0^{\mathrm{I}} = 2.5, n_0^{\mathrm{II}} = 3, N = 3, M = 1$. Consequently, the posterior performance of the competitors during the first cycle will be: $i_1 = 2, j_1 = 26, k_1 = 27$ and $l_1 = 3$. The length of the first cycle is now composed of 58 simultaneous decisions compared with 34 decisions for the two risk-neutrals case, i.e., the cycle is longer. The same implication holds for any other cycle. A more general discussion of the effect of the competitors' attitudes toward risk is given in [3]. It is based on the results that have been reported in [2].

5. Summary

This paper has investigated competitive preference structures which can be represented by 2×2 game matrices, and examined their implied posterior performances. The approach taken was Bayesian decision-theoretic, where the decision maker regards his opponent's behavior as a stochastic decision process. Each decision maker is assumed to assess subjective probability distributions over the likelihoods of his opponent's choices of strategies. After observing each other's decision, the two competitors learn and revise their probability distributions. At each play, each player acts in such a way as to maximize his expected return and selects an optimal pure strategy, based on the players' mutual past history.

It was first shown that different competitive preference structures may generate different sequential games. A classification of the games was then presented such that the same pattern of dynamic behavior is to be observed within each class of games. For some general classes of games, bounds and expressions on the number of times in a sequence of plays that certain strategy pairs will be employed, were obtained. The convergence of the dynamic competitive behavior implied by our model, to the one implied from game-theoretic considerations, was noted and characterized, too. Examples which illustrate the implications of the results were also presented.

The analysis of real competitive situations can be quite complicated. However, it is felt that the approach presented in this paper could provide some insight and a benchmark as to how different industries may evolve over time, in terms of the competitors' posterior performance. This understanding becomes more important these days since many industries have reached their maturity level and a firm's growth is possible only by attacking the share of its competitor. In addition, the results reported in this paper can be used for generating hypotheses regarding actual dynamic behavior in competitive situations. These hypotheses can then be tested in an experimental gaming setting or with industry data and may provide stepping stones to developing theory in the direction of greater relevance to the "real world."

Several possibilities exist for future work in the same spirit as the work reported here. Perhaps, the two most interesting changes in the details of the game that should be considered are the following. First, a dynamic preference theory can be incorporated to account for the temporal aspects of the problem, and the results could be compared with the stable preference assumed in this paper. Second, a relaxation of the act—unconditional states assumption may provide an insight on the effect of the players' beliefs concerning possible "information leaks" upon their competitive behavior.[1]

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