OPTIMAL CONTROL APPLICATIONS & METHODS, VOL. 10, 313–331 (1989)

OPTIMAL ADVERTISING AND PRICING FOR A THREE-STAGE TIME-LAGGED MONOPOLISTIC DIFFUSION MODEL INCORPORATING INCOME

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SUMMARY

A three-stage time-lagged diffusion model that incorporates consumers' income, advertising and price effects is proposed. The derivation of the model synthesizes and relies upon a number of important arguments made in the diffusion and economic literature. Optimal control theory is used to derive normative advertising and pricing strategic implications for a monopolist introducing a new durable product.

KEY WORDS Innovations diffusion Advertising Pricing Economic models

1. INTRODUCTION

The temporal diffusion of social phenomena has been studied in depth by researchers in a number of disciplines. Economists and marketing scientists, for example, have studied the diffusion of new products and services within markets (e.g. References 1–3), while sociologists have studied the spread of ideas and practices within different societies; Rogers,[4] for instance, defining diffusion as the process by which an innovation is adopted by a society over time, provides an important behavioural foundation for the process.

Mathematically, the diffusion process has frequently been modelled via a two-stage single differential equation approach, representing the epidemic manner in which the penetration and adoption of the innovation are influenced simultaneously by external and internal sources.[2,5,6] The price and advertising variables have been typically incorporated in these models to determine the basic parameters of the differential equation.[7–10] A similar approach has been taken in other dynamic models.[11,12]

In this paper we take a three-stage time-lagged modelling approach to the process and explicitly incorporate income effect in the innovation diffusion process. Although consumers' purchasing power (e.g. income) has long been recognized by economists as one of the major variables influencing demand for durable products and many services, it has not been incorporated explicitly in most diffusion models. We assume that consumers evolve through the

0143–2087/89/040313–19$09.50

Received 22 February 1988
Revised 15 February 1989

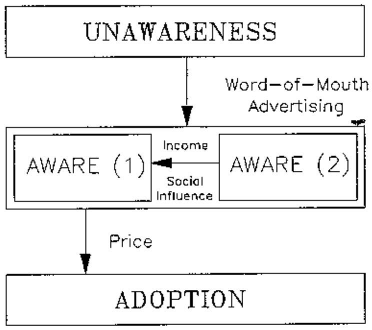


Figure 1

following three stages in their adoption process: unawareness, awareness and adoption (see Figure 1). Awareness diffusion, which is modelled as a simple epidemic model, is generated by internal word-of-mouth communication and is also controlled externally by advertising. Adoption, which is the ability to purchase the new product, depends upon the current product price. Consumers' ability to purchase the innovation is determined through a 'critical income' which is varying over time because of social influence. Only aware consumers whose income is greater than the 'critical income' are thus considered as potential adopters of the innovation. Another limitation of previous efforts relaxed here is the absence of word-of-mouth communication time lag. That is, the possibility that the activity of spreading information about the innovation is delayed and limited in time has not been modelled explicitly to our knowledge. We model that via a parameter τ that captures the time period over which the word-of-mouth communication is effective.

2. THE DIFFUSION MODEL

2.1. *Conceptual framework*

We define three states in the diffusion process: unawareness, awareness and adoption (Figure 1). At any point in time a consumer is in one and only one state. The flow of consumers from unawareness to awareness is generated by word-of-mouth communication and advertising. However, the flow from awareness to adoption is conditional on income and depends on the price level of the innovation. It is not sufficient that consumers know about the innovation; they need to have the financial resources (i.e. income) in order to become potential adopters. An aware customer is characterized by a reservation price he or she assigns to the new product. The level of such a reservation price depends on income considerations, the nature and strength of the social influence and the spending priorities of the consumer. Higher-income people are postulated to have higher reservation prices.[13] Positive social influence, however, is supposed to enhance consumers' valuation of the innovation[4] and as a result make them revise their spending plans to allow for actual adoption at higher reservation prices. We thus consider a potential adopter as an aware consumer whose current reservation price exceeds the minimum possible price the manufacturer of the innovation could ever charge. We call such a minimum price the floor price F. The potential market is therefore comprised of all those customers who are potential adopters of the innovation. The potential market is represented by the substate 'Aware (1)' in Figure 1. 'Aware (2)', in contrast, represents those consumers whose current reservation prices do not exceed the floor price. The transition from the substate

'Aware (2)' to the substate 'Aware (1)' depends on economic and social factors (i.e. rise in income, social influence). Finally, as shown in Figure 1, the timing of adoption (i.e. the transition from 'Aware (1)' to 'Adoption') depends on the current price of the innovation. A high (low) price is supposed to impede (speed) this transition. This conceptual framework is the basis of the development of our proposed model which we discuss next.

2.2. *Determination of the market potential*

To quantify the level of 'Aware (1)' we have related our modelling approach to the economic approach proposed by Duesenberry[13] and which is discussed below.

2.2.1. Static considerations. The indifference map shown in Figure 2 is drawn on the basis of the assumption that the individual buys one unit of the product or none.[13] It suggests that as an individual's income increases, his reservation price becomes higher, i.e. he is willing to give up more of the 'other goods' to get one unit of the new product. Also, the figure shows that an individual with income W_0 is indifferent between having all his income spent on 'other goods' (alternative (1) in the figure) and having one unit of the new product with the rest of his or her income spent on other goods (alternative (2)).

Assuming that all individuals behave accordingly, Duesenberry postulates a linear relationship between reservation price R and income W:[13]

$$R = cW + d \quad (c > 0)$$

or equivalently,

$$W = aR + b \tag{1}$$

where $a = 1/c$ and $b = -d/c$ are parameters to be estimated.

Equation (1) underlies the identification and quantification of the *static* market potential. In our formulation we only consider aware individuals whose reservation price (R) is greater than or equal to the floor price (F) as potential adopters of the innovation. This amounts to considering only those aware individuals whose income (W) is greater than or equal to some 'critical income' since there is a one-to-one relationship between income and reservation price. Let us denote by W_F the 'critical income' that corresonds to a reservation price equal to the floor price (F), i.e. $W_F = aF + b$. Therefore we can write the (static) market potential at the

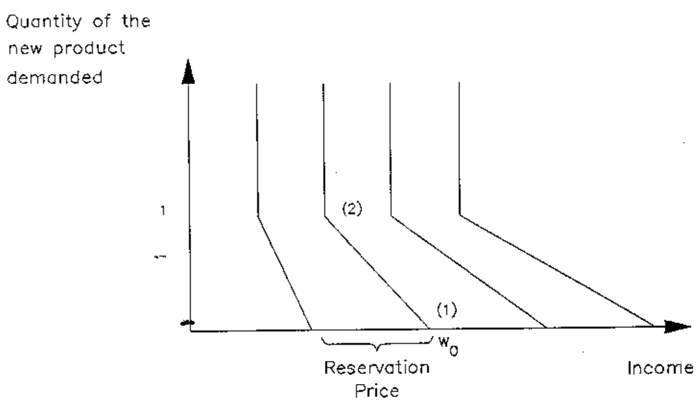


Figure 2

time of the launch of the new product, t_0, as

$$N(I(t_0), t_0) = I(t_0)\Pr[W \geqslant W_F] = I(t_0)\Pr[W \geqslant aF + b] \quad (2)$$

Equation (2), which is static, states that the market potential at t_0, $N(I, t_0)$, is equal to the number of people aware of the innovation, $I(t_0)$, times the proportion of people whose income is greater than or equal to the critical income W_F.

2.2.2. Dynamic considerations. To incorporate dynamic effects into equation (1) we assume that the consumer's valuation of the innovation (i.e. his or her reservation price) is subject to social influence in the sense that it may vary with respect to the cumulative number of adopters, $X(t)$. This valuation obviously increases if the innovation is successful and decreases if the innovation does not perform well. Some reasons why the valuation of new products generally increases with increasing adoption (penetration) include:

(1) Individual comparisons — because of social status, an individual generally makes individual comparisons between the quality of his living standards and that of others of the same or higher status. For example, when an individual is asked about his or her reaction to a friend's VCR which he or she does not have, a possible reaction will be a feeling of dissatisfaction with one's own situation. If this feeling is produced often enough, it may lead to creating a need for the new product and thus eventual adoption.
(2) Increasing adoption (penetration) generates more word-of-mouth information about the product and thus reduces the uncertainties about the innovation.
(3) Penetration can enhance preference for the new product by creating a service infrastructure or networking (e.g. a telephone becomes more useful as more people have one).

This increase (decrease) in the valuation of the new product is assumed to be manifested by a flattening (steepening) of the indifference curves at the individual level. This means that internal social influence has induced the individual, who is aware of the innovation, to revise his or her plan to allow for actual adoption at a higher reservation price. At the aggregate level this effect can be captured via the slope term a in equation (1) by allowing it to become an increasing (decreasing) function of $X(t)$, the cumulative number of adopters, i.e. $a = a(X(t))$. The dynamics of $a(X(t))$ implies the dynamics of W_F, the critical income, since $W_F(t) = a(X(t))F + b$. Thus, if the innovation is successful then the social influence is positive, $\mathrm{d}a/\mathrm{d}X < 0$, and W_F will be decreasing over time; that is, consumers from lower-income classes will exhibit a higher tendency to purchase the new product.

From the above development we can write the *dynamic* market potential as follows:

$$N(X(t), I(t), t) = I(t)\Pr(W \geqslant W_F(t)) = I(t)\Pr(W \geqslant a(X(t))F + b) \quad (3)$$

Equation (3) simply states that the market potential at any time t is equal to the cumulative number of knowers of the innovation, $I(t)$, times the proportion of people whose income is greater than or equal to the critical income at time t, $W_F(t)$. Note that $N(X(t), I(t), t)$ represents 'Aware (1)' in Figure 1.

2.3. Mathematical formulation of the diffusion model

Let $I(t)$ be the cumulative number of knowers of the innovation by time t, $X(t)$ be the cumulative number of adopters by time t and N_o be the size of the population of interest (e.g. number of households in the U.S.A.). It is assumed that all individuals have the same likelihood of getting the information about the new product and that there is a fixed period of time (τ) an adopter is active in passing the information about the innovation. It is also assumed that the marketer of the new product is controlling externally the information diffusion by advertising. Therefore, the rate of growth in awareness is proportional to the number of active 'transmitters', $X(t)-X(t-\tau)$, and the advertising effectiveness $f(A(t))$, where $A(t)$ is the advertising expenditure rate. We assume, naturally, that the advertising response function $f(A)$ satisfies the requirements of monotonicity and diminishing return ($f'(A)>0$; $f''(A)<0$). The awareness rate can thus be stated by the following differential equation:

$$\dot{I}(t)=[f(A(t))+\beta(X(t)-X(t-\tau))][N_o-I(t)] \tag{4}$$

Equation (4) represents the flow between the unawareness and awareness states in Figure 1.

Once an individual becomes aware of the innovation, if his or her income is greater than or equal to the critical income, then he or she becomes a potential adopter, that is, a member of $N(X(t), I(t), t)$. This is modelled via equation (3). We postulate that the rate of adoption is proportional to the number of people who have not adopted yet, i.e. $N(X(t), I(t), t)-X(t)$, multiplied by the price effect, $\exp(-kP(t))$, where k is a price sensitivity parameter and $P(t)$ is the price at time t. The price effect we postulate here is akin to that of Robinson and Lakhani.[14] In sum, the general innovation diffusion model we propose can be stated by the following system of differential equations:

$$\dot{I}(t)=[f(A(t))+\beta(X(t)-X(t-\tau))][N_o-I(t)], \qquad I(t_0)=I_0 \tag{5}$$

$$\dot{X}(t)=[N(X(t), I(t), t)-X(t)]\ \exp(-kP(t)), \qquad X(t_0)=X_0 \tag{6}$$

For a general income density function $g(W)$,

$$N(X(t), I(t), t)=I(t)\int_{a(X(t))F+b}^{\infty} g(W)\,\mathrm{d}W \tag{7}$$

In what follows we drop the arguments of the various functions whenever there is no confusion.

Readers can easily verify from equation (7) that $\partial N/\partial X \geqslant 0$ whenever $\mathrm{d}a(X)/\mathrm{d}X<0$ (see Appendix, equation (26)). That is, positive *social influence* can be captured via either one of the two derivatives. In the sequel, however, we will focus on the former derivative to characterize the nature of the social influence.

We also note that the model structure is compatible with some behavioural findings in diffusion research reviewed by Rogers[4] and by Gatignon and Robertson,[15] namely that:

(1) Internal influence (the totality of word-of-mouth effect and social influence), whenever operative, is interdependent with mass media and its effect is most pronounced at later stages of the adoption process.
(2) The speed of awareness and adoption increases with the average time of active information dissemination (i.e. τ) within the social system.
(3) Many empirically observed diffusion curves are sigmoid or exponential. The new model can produce a variety of diffusion curves. All depends on the income distribution of the population, the speed of information diffusion and the strength of the social influence.

Having described the diffusion model employed in the paper, it is worthwhile noting its relation to some other models that appear relevant to ours.[16–20] We first note that none of these models incorporates the time lag effect. Dodson and Muller[16] have also proposed a three-stage diffusion model, but with a diffusion structure different from ours and incorporating only advertising. In addition, their focus was not on deriving optimal strategies. Horsky,[17] taking an individual level approach, has developed a one-stage diffusion model incorporating information, price and income effects. While he does not derive optimal pricing strategies analytically, some conjectures are made. Jeuland[18] has also proposed a one-stage diffusion model incorporating income without derivation of optimal strategies. Kalish[19] has developed a three-stage diffusion model incorporating uncertainty, price and advertising. He has also derived optimal advertising and pricing strategies for a monopolist. Kalish's work differs from ours, however, mainly in terms of the following characteristics: (1) treatment of the price and market potential effect, (2) treatment of the word-of-mouth dissemination effect and (3) treatment of the social influence effect. Finally, Seidman *et al.*[20] have constructed a very general model capable of incorporating various types of communication that may take place, for instance, between two distinctly distributed subgroups of the population, such as 'Aware (1)' and 'Aware (2)' in our income-distributed formulation. They also consider issues related to the existence of optimal advertising policies. Their formulation of the social influence effect is different from ours, however, and they do not consider the impact of price upon adoption.

3. OPTIMAL POLICY IMPLICATIONS

In this section we employ optimal control theory to derive normative implications for monopolistic advertising and pricing policies over a finite time horizon. Let r denote the discount rate, T the end of the planning horizon and $C(X)$ the unit cost, which can be a function of experience $X(t)$. We will assume throughout, unless specified otherwise, that $\mathrm{d}C/\mathrm{d}X \leqslant 0$, which means that the monopolist unit cost declines or remains constant as more and more adoption (penetration) experience is gained (e.g. based on the learning curve premise). Then the objective function is the sum of the discounted profit and is given by:

$$\Pi = \int_0^T \mathrm{e}^{-rt}\{[P(t) - C(X(t))]\dot{X}(t) - A(t)\}\ \mathrm{d}t \tag{8}$$

Thus the monopolist problem is:

$$\underset{P,A}{\text{Max}}\ \Pi = \int_0^T \mathrm{e}^{-rt}\{[P(t) - C(X(t))]\dot{X}(t) - A(t)\}\ \mathrm{d}t$$

s.t.

$$\begin{aligned}
\dot{I}(t) &= [f(A(t)) = +\beta(X(t) - X(t-\tau))]\,[N_o - I(t)], \qquad & I(t_0) &= I_0 \\
\dot{X}(t) &= [N(X(t), I(t), t) - X(t)]\ \exp(-kP(t)), & X(t_0) &= X_0 \\
N(X(t), I(t), t) &= I(t)\int_{a(X(t))F+b}^{\infty} g(W)\ \mathrm{d}W
\end{aligned} \tag{9}$$

This is a dynamic optimization problem with two state variables, $I(t)$ and $X(t)$, and two control variables, $P(t)$ and $A(t)$. The current value Hamiltonian is given by

$$\begin{aligned}
H = {} & [P(t) - C(X(t))]\,[N(X, I, t) - X(t)]\exp(-kP(t)) - A(t) \\
& + m_1[f(A(t)) + \beta(X(t) - X(t-\tau))]\,[N_o - I(t)] \\
& + m_2\ \exp(-kP(t))[N(X, I, t) - X(t)]
\end{aligned} \tag{10}$$

where $m_1(t)$ and $m_2(t)$ are the current value multipliers associated with the kinematic equations. When Pontryagin's maximum principle is applied, m_1 and m_2 satisfy the following system of differential equations:

$$\dot{m}_1 = rm_1 - \frac{\partial H}{\partial I}$$

$$= rm_1 - [P(t) - C(X(t))]\exp(-kP(t))\frac{\partial N}{\partial I}$$

$$+ m_1[f(A(t)) + \beta(X(t) - X(t-\tau))] - m_2\exp(-kP(t))\frac{\partial N}{\partial I} \tag{11}$$

$$\dot{m}_2 = rm_2 - \frac{\partial H}{\partial X} - \frac{\partial H}{\partial X(t-\tau)}\bigg|_{t+\tau} \quad (\text{for } 0 \leqslant t < T-\tau)$$

$$= rm_2 - m_2\exp(-kP(t))\left(\frac{\partial N}{\partial X} - 1\right) + \exp(-kP(t))\frac{\mathrm{d}C}{\mathrm{d}X}[N(X, I, t) - X(t)]$$

$$- \exp(-kP(t))[P(t) - C(X(t))]\left(\frac{\partial N}{\partial X} - 1\right) - m_1\beta[N_o - I(t)]$$

$$+ \beta m_1(t+\tau)[N_o - I(t+\tau)] \tag{12a}$$

$$\dot{m}_2 = rm_2 - \frac{\partial H}{\partial X} \quad (\text{for } T-\tau \leqslant t \leqslant T)$$

$$= rm_2 - m_2\exp(-kP(t))\left(\frac{\partial N}{\partial X} - 1\right) + \exp(-kP(t))\frac{\mathrm{d}C}{\mathrm{d}X}[N(X, I, t) - X(t)]$$

$$- \exp(-kP(t))[P(t) - C(X(t))]\left(\frac{\partial N}{\partial X} - 1\right) - m_1\beta[N_o - I(t)] \tag{12b}$$

with the transversality conditions:

$$m_1(T) = 0, \qquad m_2(T) = 0 \tag{13}$$

The derivative of the Hamiltonian with respect to P and A must vanish on the optimal path. That is,

$$\frac{\partial H}{\partial A} = -1 + m_1[N_o - I(t)]f'(A^*(t)) = 0 \tag{14}$$

and

$$\frac{\partial H}{\partial P} = \exp(-kP^*(t))[N(X, I, t) - X(t)]\{1 - k[P^*(t) - C(X(t)) + m_2]\} = 0 \tag{15}$$

Our task now is to characterize the optimal advertising and pricing paths over time. We do that via the following set of propositions. The mathematical proofs are presented in the Appendix.

3.1. *Advertising strategy*

Proposition 1

If $r = 0$ or is not too large, $f'(A) > 0$ and $f''(A) < 0$ (concave advertising response function), then the optimal strategy is to decrease advertising spending monotonically over time.

Proof. See Appendix

This result may seem intuitively appealing and consistent with observed practices of many firms when introducing new products. An intensive advertising strategy at the introductory stage is vital to convey the information about the new product to potential adopters and to begin the diffusion process. Once some consumers adopt the innovation they become transmitters of information about the new product and thus reduce the need for intensive advertising. Actually, there is less need for advertising as the number of adopters or transmitters of information increases. The proposition is consistent with the findings of others (e.g. Horsky and Simon,[10] and Kalish[19]). From equation (21) (see Appendix) it can be shown that the optimal advertising spending rate A^* decreases faster whenever (1) the advertising effectiveness $f'(A)$ is larger, (2) the price sensitivity parameter k is smaller and (3) the effect of $\partial N/\partial I$ is stronger. It is also worth noting that the time-lagged internal spread of information does not affect the optimal advertising policy although it does influence the nature of the optimal pricing strategies as will be shown later.

3.2. *Pricing strategies*

We consider two major scenarios in deriving pricing strategies. In the first scenario we assume full awareness over the entire planning horizon (i.e. $I(t) = N_o$ for all t) whereas in the second scenario we assume that awareness is diffusing over time. Propositions 2, 3 and 4 deal with the former case whereas Propositions 5 and 6 deal with the latter.

Proposition 2

For the case of full awareness ($I(t) = N_o$), if social influence is very negative at the time of the launch of the innovation ($\partial N/\partial X|_{X\to 0} \ll 0$) but monotonically increasing to a positive level ($\partial N/\partial X|_{X\to X(T)} > 1$), then the optimal price should be monotonically decreasing and then increasing over time.

Proof. See Appendix

Proposition 2 deals with the case of innovations which receive social rejection at the time of their launch or test market (e.g. the new RU486 drug, which induces abortion early in pregnancy without recourse to surgery,[21] and AT&T video telephones). In this case the firm needs to focus on such things as consumer education, quality improvement and advertising to change perception, and it also needs to charge lower prices over some period of time to offset this negative social influence. This decrease in price induces higher demand and helps the innovation in entering and becoming established in the consumption habits of the social system. Once this is accomplished, or once consumers start to have a good experience with the innovation, then price can be increased.

Proposition 3

If social influence is nil over the planning horizon ($\partial N/\partial X = 0$) then there is cost decline ($\mathrm{d}C/\mathrm{d}X < 0$), and if there is full awareness about the innovation ($I(t) = N_o$) then the optimal price should be monotonically decreasing over time.

Proof. See Appendix.

This proposition seems to be intuitive and was derived by others under different formulations.[9,19,22] Since early adopters of the innovation have no effect on future sales, there is no reason for subsidizing them by a low introductory price. Therefore the best strategy is to price discriminate over time, i.e. charge a higher price for early adopters and then decrease the price as more and more individuals adopt the new product (skimming strategy).

Proposition 4

For the case of full awareness ($I(t) = N_o$), if $r = 0$ and/or $\mathrm{d}C/\mathrm{d}X = 0$ (absence of cost decline), and if social influence is strongly positive at the introductory stage of the innovation ($\partial N/\partial X|_{X\to 0} \gg 1$) and monotonically decreasing to a level where it loses its effect ($\partial N/\partial X|_{X\to X(T)} < 1$), then the optimal pricing strategy is characterized by an increasing and then decreasing price path.

Proof. See Appendix.

In this case, early adopters of the innovation have a positive effect on increasing future demand by socially influencing others to adopt. Therefore it is worthwhile to subsidize them by a low introductory price. More precisely, early adopters of the innovation have the effect of making other people desire the innovation and, hence, of pushing their reservation price upwards. The firm should respond to this upward shift of reservation prices by charging higher prices. However, as this social influence effect diminishes, the distribution of reservation prices in the population tends to be more and more stable and the only way to get higher revenues is to reduce price and thus discriminate over time.

Corollary 4.1

Under the conditions specified in Proposition 4, but with social influence still strong at the end of the planning horizon ($\partial N/\partial X|_{X\to X(T)} > 1$), the optimal price is monotonically increasing.

Proof. See Appendix.

So far we have focused on pricing strategies under a full awareness scenario. Note that the pricing strategies derived under this case hold also under the complete model if the word-of-mouth effect concerning the innovation does not exist ($\beta = 0$). To check for this one can simply substitute zero for β in equations (24a) and (24b) (see Appendix) and confirm that this leads to equation (27b), the equation on which the proofs of Propositions 2–4 are essentially based. We now consider the complete model where the word-of-mouth communication effect and the social influence are operating together. In particular, we will analyse the effect of the time lag parameter upon the pricing strategies.

Proposition 5

For the complete model, assume that $r = 0$ and/or $\mathrm{d}C/\mathrm{d}X = 0$, advertising spending decreases optimally over time ($\dot{A}^* < 0$) and nil social influence ($\partial N/\partial X = 0$). If word-of-mouth communication is characterized by a large contact rate (β) and a period of active information dissemination equal to the entire length of the planning horizon ($\tau = T$), and if this communication is more effective than advertising, then optimal price is increasing and then decreasing over time.

Proof. See Appendix.

Proposition 5 partials out the effect of word-of-mouth communication. It is in line with Proposition 4 where we looked at the effect of social influence, and it corresponds to the optimal nature of the advertising spending (Proposition 1). The same argument we used there also holds here. The only difference resides in the fact that word-of-mouth communication expands the market potential by increasing the number of aware customers, whereas positive social influence expands it by pushing customers' reservation prices upward.

Corollary 5.1

The result of Proposition 5 remains correct if social influence is positive but decaying over time to a nil level (i.e. $\partial N/\partial X|_{X\to 0} \gg 1$, $\partial^2 N/\partial X^2 < 0$, and $\partial N/\partial X|_{X\to X(T)} \ll 1$).

Proof. See Appendix.

Although the results presented in Proposition 5 and Corollary 5.1 are similar, it is worth noting that the presence of both positive initial social influence and effective word-of-mouth communication leads to a faster increase followed by a slower decrease of the optimal price. Note also that if the social influence is still strong by the end of the planning horizon ($\partial N/\partial X|_{X\to X(T)} > 1$), then the optimal price is monotonically increasing over the entire time period (see Appendix).

The analysis presented in Propositions 2–4 assumes, in essence, that word-of-mouth communication does not exist ($\tau = 0$). The analysis presented in Proposition 5 assumes, on the other hand, an effective and long period of active information dissemination ($\tau = T$). To gain some insight into the effect of the time lag parameter upon the optimal pricing strategies we will assume now that the activity of spreading information is delayed and limited ($0 < \tau < T$). It becomes immediately obvious (see (24a) and (24b) in the Appendix) that the planning horizon is divided into two subperiods: $t \in [0, T-\tau)$ and $t \in [T-\tau, T]$. To maintain continuity in the exposition we will assume conditions similar to those specified in Proposition 5. Proposition 6 characterizes the pricing strategy for $0 < \tau < T$, and compares the price increase segment for two cases specified in terms of the time lag parameter: case (1), $0 < \tau < T$, and case (2), $\tau = T$ (i.e. Proposition 5).

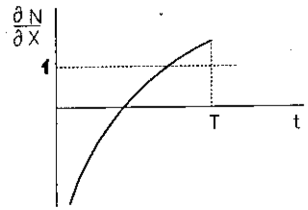
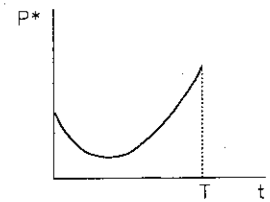
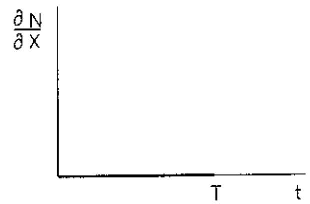
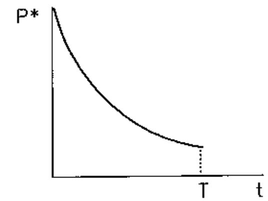
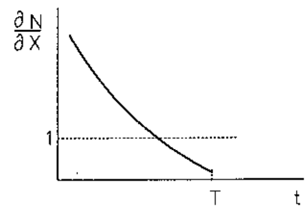
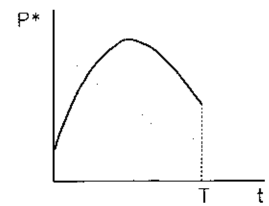
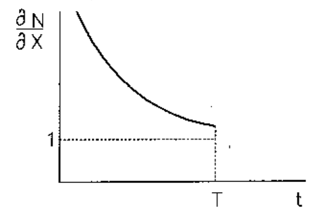
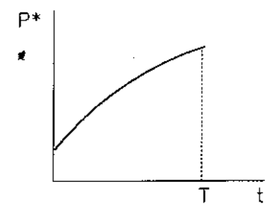
Proposition 6

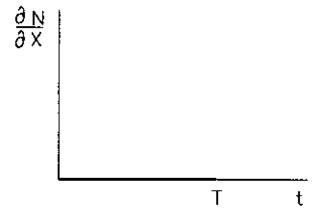
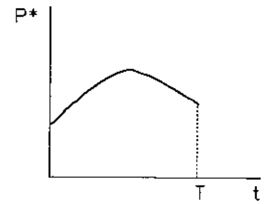
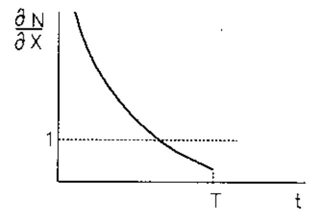
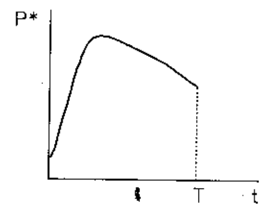
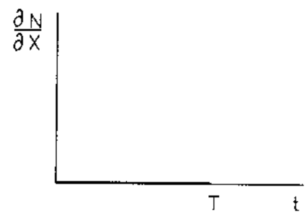
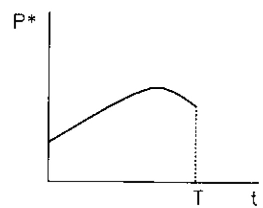
For the complete model assume that $r = 0$ and/or $\mathrm{d}C/\mathrm{d}X = 0$, advertising spending decreases optimally over time ($\dot{A}^* < 0$), and nil social influence ($\partial N/\partial X = 0$). If the word-of-mouth communication is characterized by a *very* large contact rate (β), delayed but for a limited period ($0 < \tau < T$), and if it is much more effective than advertising (particularly relative to Proposition 5), then the optimal price is increasing, at a lower rate than that implied by Proposition 5, and then is decreasing.

Proof. See Appendix.

Proposition 6 indicates that the presence of a delayed and limited word-of-mouth

Table I. Summary of optimal pricing strategies over time

Scenario	Proposition/ Corollary	Assumptions	Social influence function	Optimal price time path
Full awareness or absence of word-of-mouth communication $\tau = 0$	2	$I(t) = N_o \forall t \in [0, T]$ $\partial N/\partial X\|_{X \to 0} \leqslant 0$ $\partial N/\partial X\|_{X \to X(T)} > 1$ $\partial^2 N/\partial X^2 > 0$		
Full awareness or absence of word-of-mouth communication $\tau = 0$	3	$I(t) = N_o \ \forall t \in [0, T]$ $\partial N/\partial X = 0 \ \forall t \in [0, T]$ $dC/dX < 0$		
Full awareness or absence of word-of-mouth communication $\tau = 0$	4	$I(t) = N_o \ \forall t \in [0, T]$ $r = 0$ and/or $dC/dX = 0$ $\partial N/\partial X\|_{X \to 0} \geqslant 1$ $\partial N/\partial X\|_{X \to X(T)} < 1$ $\partial^2 N/\partial X^2 < 0$		
Full awareness or absence of word-of-mouth communication $\tau = 0$	4.1	Same assumptions as in Proposition 4 except $\partial N/\partial X\|_{X \to X(T)} > 1$		

Full model $\tau = T$	5	$r = 0$ and/or $\mathrm{d}C/\mathrm{d}X = 0$ $\dot{A}^* < 0$ large β $\partial N/\partial X = 0 \ \forall t \in [0, T]$		
Full model $\tau = 0$	5.1	Same assumptions as in Proposition 5 except $\partial N/\partial X\|_{X \to 0} \gg 1$ $\partial N/\partial X\|_{X \to X(T)} < 1$ $\partial^2 N/\partial X^2 < 0$		
Full model, $0 < \tau < T$	6	$r = 0$ and/or $\mathrm{d}C/\mathrm{d}X = 0$ $\dot{A}^* < 0$ very large β $\partial N/\partial X = 0 \ \forall t \in [0, T]$		

communication effect essentially hampers the firm's increasing price response to the upward shift of the non-adopters' reservation prices.

Table I provides a summary of the optimal policy implications and their underlying assumptions.

4. DISCUSSION AND CONCLUSIONS

A model of new product diffusion has been proposed. It considers advertising and price effects, consumers' purchasing ability and other phenomena such as dynamic market potential, adopter loss of interest in transmitting information about the innovation and negative/positive social influence. The model is composed of two modules: an awareness module and an adoption module. Awareness is generated by word-of-mouth communication and is also controlled externally by advertising. Adoption, which is conditional on awareness, depends on the growth of the market potential and changes in price. We provide motivations from the diffusion and economics literature leading to the development of the model. We employ optimal control theory to derive advertising and pricing policy implications and find that optimal advertising always decreases over time, whereas the nature of the optimal pricing depends on whether social influence is positive or negative. In the case of nil social influence and no word-of-mouth communication effect, optimal price always decreases over time. If the social influence is positive but its effect is decaying over time, then optimal pricing strategy is generally characterized by increasing and then decreasing paths.

Note that there are multiple factors incorporated in our model that determine the nature of the optimal advertising and pricing strategies. These are the discount factor (r), cost experience ($\mathrm{d}C/\mathrm{d}X$), social influence effect ($\partial N/\partial X$), the time period over which word-of-mouth communication is effective (τ) and the contact rate (β). While Propositions 1–3 are stated quite generally in terms of these variables, Propositions 4–6 require a condition stating that $r=0$ and/or $\mathrm{d}C/\mathrm{d}X=0$. The case where $r>0$ and $\mathrm{d}C/\mathrm{d}x<0$ has not been solved for and, hence, deserves further attention. In addition, other patterns of the social influence function could be considered. Future work on the model will also be concerned with:

(1) extending the model to deal with repeat purchase situations (e.g. Dodson and Muller[16])
(2) extending the model to deal with a competitive situation (e.g. Eliashberg and Jeuland[23])
(3) incorporating individual level behavioural, preferential and perceptual variables in the model (e.g. Chatterjee and Eliashberg[24]).

APPENDIX

The maximization problem and the necessary conditions are reported in Section 3. Here our task is to characterize the advertising and price paths over time by proving the propositions presented and discussed in the text.

A.1. Advertising strategy

From equation (15) we obtain:

$$m_2=\frac{1}{k}-[P-C(X)] \tag{16}$$

Taking the time derivative of (16) we get:

$$\dot{m}_2 = -\dot{P} + \frac{dC}{dX}\dot{X} \tag{17}$$

From equation (14) we obtain:

$$m_1 = \frac{1}{[N_o - I(t)]f'(A)} \tag{18}$$

Taking the time derivative of (18) we get:

$$\dot{m}_1 = \frac{-\dot{A}f''(A)[N_o - I(t)] + f'(A)\dot{I}}{\{[N_o - I(t)]f'(A)\}^2} \tag{19}$$

Now, by substituting (18) and (16) for m_1 and m_2 respectively into (11) we obtain:

$$\dot{m}_1 = \frac{r}{[N_o - I(t)]f'(A)} - [P - C(X)]\exp(-kP(t))\frac{\partial N}{\partial I}$$
$$+ \frac{1}{[N_0 - I(t)]f'(A)}\{f(A) + \beta[X(t) - X(t-\tau)]\}$$
$$- \left(\frac{1}{k} - [P - C(X)]\right)\exp(-kP(t))\frac{\partial N}{\partial I}$$

That is:

$$\dot{m}_1 = \frac{r}{[N_o - I(t)]f'(A)} + \frac{1}{[N_o - I(t)]]f'(A)}\frac{\dot{I}}{N_o - I(t)} - \frac{1}{k}\exp(-kP(t))\frac{\partial N}{\partial I}$$
$$= \frac{1}{[N_o - I(t)]f'(A)}\left(r + \frac{\dot{I}f'(A)}{f'(A)[N_o - I(t)]}\right) - \frac{1}{k}\exp(-kP(t))\frac{\partial N}{\partial I} \tag{20}$$

Now, by equating (19) and (20), we get:

$$\frac{-\dot{A}f''(A)}{f'(A)} = r - \frac{1}{k}[N_o - I(t)]f'(A)\exp(-kP(t))\frac{\partial N}{\partial I}$$

which implies that:

$$\dot{A}^* = \frac{f'(A)}{f''(A)}\left(\frac{1}{k}(N_o - I(t))f'(A)\exp(-kP(t))\frac{\partial N}{\partial I} - r\right) \tag{21}$$

Proof of Proposition 1

$$\left.\begin{array}{l} r = 0 \\ f'(A) > 0 \\ f''(A) < 0 \\ \partial N/\partial I \geqslant 0 \end{array}\right\} \Rightarrow \dot{A}^* < 0$$

i.e. the optimal advertising rate should be decreasing over time.

The specifications $f'(A) > 0$ and $f''(A) < 0$ have been generally assumed on the advertising response function. An internal solution assumption with regard to $A^*(t)$ and the terminal condition on m_1 also imply that $f(A)$ should satisfy $f'(A) \to \infty$ as $A \to 0$ (see Kalish[19]). The requirement that $r = 0$ is stated in Proposition 1.

To complete the proof we have to show that: $\partial N/\partial I \geqslant 0$. Recall that:

$$N(X, I, t) = I(t) \int_{a(X(t))F+b}^{\infty} g(W)\, \mathrm{d}W$$

Hence

$$\frac{\partial N}{\partial I} = \int_{a(X)F+b}^{\infty} g(W)\, \mathrm{d}W \geqslant 0$$

because $g(W)$ represents a non-negative probability density function. Note that if $r > 0$ but is not too large, i.e.

$$r < \frac{1}{k}\,[N_o - I(t)]\, f'(A)\exp(-kP(t))\,\frac{\partial N}{\partial I}$$

then the optimal advertising rate should still be decreasing over time. Q.E.D

A.2. Pricing strategies

We start by replacing m_1 and m_2 in (12a) by (18) and (16), respectively. Suppose first that $t \in [0, T-\tau)$:

$$\begin{aligned}\dot{m}_2 &= r\left(\frac{1}{k} - [P - C(X)]\right) - \left(\frac{1}{k} - [P - C(X)]\right)\exp(-kP(t))\left(\frac{\partial N}{\partial X} - 1\right)\\ &\quad + \exp(-kP(t))\,\frac{\mathrm{d}C}{\mathrm{d}X}\,[N(X, I) - X] - \exp(-kP(t))[P - C(X)]\left(\frac{\partial N}{\partial X} - 1\right)\\ &\quad - \beta\left[\frac{1}{f'(A(t))} - \frac{1}{f'(A(t+\tau))}\right]\\ &= r\left(\frac{1}{k} - [P - C(X)]\right) - \frac{1}{k}\exp(-kP(t))\left(\frac{\partial N}{\partial X} - 1\right) + \frac{\mathrm{d}C}{\mathrm{d}X}\dot{X}\\ &\quad - \beta\left[\frac{1}{f'(A(t))} - \frac{1}{f'(A(t+\tau))}\right]\end{aligned} \tag{22}$$

But from (17)

$$\dot{m}_2 = -\dot{P} + \frac{\mathrm{d}C}{\mathrm{d}X}\dot{X} \tag{23}$$

Hence for $0 \leqslant t < T - \tau$

$$\dot{P} = \frac{1}{k}\exp(-kP(t))\left(\frac{\partial N}{\partial X} - 1\right) - r\left(\frac{1}{k} - [P - C(X)]\right) + \beta\left(\frac{1}{f'(A(t))} - \frac{1}{f'(A(t+\tau))}\right) \tag{24a}$$

The same method applies when $t \in [T-\tau, T]$, and we obtain:

$$\dot{P} = \frac{1}{k}\exp(-kP(t))\left(\frac{\partial N}{\partial X} - 1\right) - r\left(\frac{1}{k} - [P - C(X)]\right) + \frac{\beta}{f'(A(t))} \tag{24b}$$

for $T - \tau \leqslant t \leqslant T$.

Determination of $\partial N/\partial X$. Recall that:

$$N(X, I, t) = I(t) \int_{a(X(t))F+b}^{\infty} g(W)\, \mathrm{d}W \tag{25}$$

where $g(W)$ is the (non-negative) income distribution. Define

$$V(X) = \int_{R(X)}^{\infty} g(W)\, \mathrm{d}W$$

where $R(X) = a(X(t))F + b$. By using the Leibnitz rule, we get:

$$V'(X) = -g(R(X))\frac{\mathrm{d}R}{\mathrm{d}X} = -g(a(X(t))F+b)F\frac{\mathrm{d}a}{\mathrm{d}X}$$

Thus

$$\frac{\partial N}{\partial X} = -I(t)g(a(X)F+b)F\frac{\mathrm{d}a}{\mathrm{d}X} \tag{26}$$

If the innovation is successful in generating positive social influence then $\mathrm{d}a/\mathrm{d}X$ should be negative because it drives the critical income downwards, and thus $\partial N/\partial X \geqslant 0$. If the innovation is unsuccessful then $\mathrm{d}a/\mathrm{d}X$ should be positive, and thus $\partial N/\partial X \leqslant 0$.

Proof of Proposition 2. Here we are considering the case of full awareness ($I(t) = N_o \forall t \in [0, T]$) and the presence of very negative social influence at the introductory stage of the innovation ($\partial N/\partial X|_{X\to 0} \ll 0$). Social influence is assumed, however, to be monotonically increasing to a positive level ($\partial N/\partial X|_{X\to X(T)} > 1$, $\partial^2 N/\partial X^2 > 0 \ \forall t \in [0, T]$).

$I(t) = N_o$ implies the cancellation of all $N_o - I(t)$ terms in equations (12a) and (12b) and hence, the vanishing of all

$$\beta\left(\frac{1}{f'(A(t))} - \frac{1}{f'(A(t+\tau))}\right) \quad \text{and} \quad \frac{\beta}{f'(A(t))}$$

terms from equations (24a) and (24b), respectively. Thus we can rewrite $\dot{P}$ as:

$$\dot{P} = \frac{1}{k}\exp(-kP)\left(\frac{\partial N}{\partial X} - 1\right) - r\left(\frac{1}{k} - [P - C(X)]\right), \quad \text{for } 0 \leqslant t \leqslant T \tag{27a}$$

or equivalently, if we use equation (16), as:

$$\dot{P} = \frac{1}{k}\exp(-kP)\left(\frac{\partial N}{\partial X} - 1\right) - rm_2(t) \tag{27b}$$

From (27b) it is clear that $\dot{P}|_{X\to 0} < 0$ for $\partial N/\partial X|_{X\to 0} \ll 0$. However, since $\partial N/\partial X|_{X\to X(T)} > 1$ (by assumption) and since $m_2(T) = 0$, then $\dot{P}(T) > 0$.

Thus far we have shown that $\dot{P}|_{X\to 0} < 0$ (or $\dot{P}(0) < 0$) and $\dot{P}(T) > 0$. To complete the proof of Proposition 2 we follow arguments similar to Kalish.[9] We need to show, under the price regularity assumption, that if a change in sign of $\dot{P}$ occurs only once in $[0, T]$, it is from negative to positive (i.e. the optimal price path is 'U'-shaped). To accomplish this we need to look at the sign of $\ddot{P}$ at $\dot{P} = 0$ (assuming that $P(t)$ is continuous and twice differentiable). If we can show that $\ddot{P}|_{\dot{P}=0} > 0$, $\forall t \in [0, T]$, then $\dot{P}$ (being negative when $X \to 0$), once it is positive, never becomes negative afterwards and this will establish the proof. To show this, note that:

$$\ddot{P}|_{\dot{P}=0} = \frac{1}{k}\exp(-kP)\left(\frac{\partial^2 N}{\partial X^2}\right)\dot{X} - r\frac{\mathrm{d}C}{\mathrm{d}X}\dot{X} \tag{28}$$

is always positive under the set of assumptions underlying Proposition 2, namely: $\partial^2 N/\partial X^2 > 0$ and $\mathrm{d}c/\mathrm{d}X \leqslant 0$. This establishes the proof of Proposition 2. Q.E.D

Proof of Proposition 3. Here we are considering the case of full awareness and nil social influence (i.e. $I(t) = N_o$ and $\partial N/\partial X = 0$). From (27b) it is clear that $\dot{P}(t) < 0$ for $r = 0$ $\forall t \in [0, T]$ and also that $\dot{P}(T) < 0$ for any $r > 0$ (since $m_2(T) = 0$). To show that $\dot{P}(t) < 0$ also for any $r > 0$, $\forall t \in [0, T]$, we use the same line of argument used to prove Proposition 2. Since $\partial^2 N/\partial X^2 = 0$ (by assumption), (28) reduces to

$$\ddot{P}|_{\dot{P}=0} = -r\frac{\mathrm{d}C}{\mathrm{d}X}\dot{X} > 0 \quad \forall t \in [0, T] \tag{29}$$

Consequently, since $\dot{P}(T) < 0$ then $\dot{P}(t)$ can only be negative over $[0, T]$ under the price regularity assumption. More clearly, suppose that $\dot{P}(t)$ takes some positive value at, say, time $t_1 < T$. Then, since $\dot{P}(T) < 0$, $\ddot{P}|_{\dot{P}=0} < 0$, $\forall t \in [t_1, T]$, which is a contradiction of (29). Therefore, $\dot{P}(t)$ can only be negative over $[0, T]$. This establishes the proof of Proposition 3. Q.E.D.

Proof of Proposition 4 and Corollary 4.1. Here we are considering the case of full awareness and the presence of a strong positive social influence at the introductory stage of the innovation ($\partial N/\partial X|_{X\to 0} \gg 1$). This positive social influence is assumed, however, to be decreasing monotonically over time to a nil or weak level ($\partial^2 N/\partial X^2 < 0$, $\partial N/\partial X|_{X\to X(T)} < 1$).

It is clear from equation (27b), under the set of assumptions specified in Proposition 4, that $\dot{P}|_{X\to 0} > 0$ and $\dot{P}(T) < 0$ (remember that $m_2(T) = 0$). To complete the proof of Proposition 4 we need to show, under the price regularity assumption, that $\dot{P}(t)$ changes its sign from positive to negative once. This is equivalent to showing that $\ddot{P}|_{\dot{P}=0} < 0$, $\forall t \in [0, T]$. This holds in our case because from equation (28) it is clear that $\ddot{P}|_{\dot{P}=0} < 0$ whenever $\partial^2 N/\partial X^2 < 0$ and $r = 0$ and/or $\mathrm{d}C/\mathrm{d}X = 0$.

Also, if we assume that $\partial N/\partial X|_{X\to X(T)} > 1$ (instead of $\partial N/\partial X|_{X\to X(T)} < 1$) then by using the same line of argument we can show that $\dot{P}(t) > 0$ $\forall t \in [0, T]$. This establishes the proof of Corollary 4.1. Q.E.D.

Proof of Proposition 5 and Corollary 5.1. In this case we are considering the complete model and are assuming $r = 0$ and/or $\mathrm{d}C/\mathrm{d}X = 0$, $\dot{A} < 0$, $\partial N/\partial X = 0$, $\partial^2 N/\partial X^2 = 0$ and $\tau = T$. The assumption $\tau = T$ implies that $\dot{P}(t)$ can be represented by equation (24b) for $0 \leqslant t \leqslant T$. From (24b) it is clear that $\dot{P}|_{X\to 0} > 0$ if $\beta/f'(A(0))$ is large enough, i.e.

$$\frac{\beta}{f'(A(0))} > \frac{1}{k}\exp(-kP(0)) + r\left(\frac{1}{k} - P(0) + C(X(0))\right) \tag{30}$$

Also, since $m_1(T) = 0$ and $m_2(T) = 0$, $1/f'(A(T))$ and $[1/k - P(T) + C(X(T))]$ are equal to zero. This implies that $\dot{P}(T) < 0$.

To show that the optimal price follows a bell-curve-type path under the price regularity assumption, we need to show that $\ddot{P}|_{\dot{P}=0} < 0$, $\forall t \in [0, T]$. Since $\tau = T$,

$$\ddot{P}|_{\dot{P}=0} = \frac{1}{k}\exp(-kP(t))\left(\frac{\partial^2 N}{\partial X^2}\right)\dot{X} - r\,\frac{dC}{dX}\,\dot{X} - \frac{\beta \dot{A} f''(A)}{(f'(A))^2} \tag{31}$$

Since $\partial^2 N/\partial X^2 = 0$, $r = 0$ and/or $dC/dX = 0$, $\dot{A} < 0$, and $f''(A) < 0$, then indeed $\ddot{P}|_{\dot{P}=0} < 0$.

Similarly, by repeating the above steps, we can prove the same result when $\partial N/\partial X|_{X \to 0} \gg 1$, $\partial N/\partial X|_{X \to X(T)} < 1$ and $\partial^2 N/\partial X^2 < 0$. This establishes the proof of Corollary 5.1. However, if $\partial N/\partial X|_{X \to X(T)} > 1$ then $\dot{P}(T) > 0$ and the optimal price is always increasing. Q.E.D.

Proof of Proposition 6. The case under consideration is the complete model with $r = 0$ and/or $dC/dX = 0$, $\dot{A} < 0$, $\partial N/\partial X = 0$, $\partial^2 N/\partial X^2 = 0$ and $0 < \tau < T$.

The differential equations now governing the optimal pricing are (24a) and (24b). From (24a) it is clear that $\dot{P}|_{X \to 0} > 0$ if the parameter β is large enough, i.e.

$$\beta\left(\frac{1}{f'(A(0))} - \frac{1}{f'(A(\tau))}\right) > \frac{1}{k}\exp(-kP(0)) + r\left(\frac{1}{k} - P(0) + C(X(0))\right) \tag{32}$$

Note that since $\dot{A}(t) < 0$, $f'(A) > 0$ and $f''(A) < 0$, the multiplier of β in (32) is positive. Hence, for $P(0)$ identical to that under Proposition 5, note that condition (32) is stronger than condition (30).

It can be shown, following arguments similar to the proof of Proposition 5, that $\dot{P}(T) < 0$ and $\ddot{P}|_{\dot{P}=0} < 0$. Hence, under the price regularity assumption, the optimal price follows a bell-curve-type path similar to that of Proposition 5.

To compare the levels of the optimal prices and the rates at which they increase under case (1), $0 < \tau < T$, with those under case (2), $\tau = T$, we need to examine (24a) vis-à-vis (24b) over the same time horizon. The comparison reveals that if the initial prices are identical under the two scenarios for example, then the price increase rate and the price level at any point in time under case (1), $0 < \tau < T$, are less than those under case (2), $\tau = T$. Q.E.D.

A note on sufficiency. If A^* and P^* are optimal they must satisfy the following conditions:

$$(1)\ \frac{\partial^2 H}{\partial A^2} < 0, \quad (2)\ \frac{\partial^2 H}{\partial P^2} < 0 \quad \text{and} \quad (3)\ \left(\frac{\partial^2 H}{\partial A^2}\right)\left(\frac{\partial^2 H}{\partial P^2}\right) - \left(\frac{\partial^2 H}{\partial A \partial P}\right)^2 > 0$$

Note that for our formulation,

$$(1)\quad \frac{\partial^2 H}{\partial A^2} = m_1[N_o - I(t)]f''(A^*) = \frac{f''(A^*)}{f'(A^*)} < 0$$

because of the specifications of the advertising response function:

$$(2)\quad \frac{\partial^2 H}{\partial P^2} = -k\exp(-kP^*(t))[N(X, I, t) - X(t)] < 0, \text{ and}$$

$$(3)\quad \left(\frac{\partial^2 H}{\partial A^2}\right)\left(\frac{\partial^2 H}{\partial P^2}\right) - \left(\frac{\partial^2 H}{\partial A \partial P}\right)^2 > 0$$

because $\partial^2 H/\partial A \partial P = 0$, $\partial^2 H/\partial A^2 < 0$ and $\partial^2 H/\partial P^2 < 0$.

In addition to the conditions above stated we need to show that the maximized Hamiltonian H^0 is concave in the state variables X and I (see Kamien and Schwartz,[25] pp. 204–211). It turns out that this condition does not hold most generally. This may imply some implicit parametric restriction.

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