# Probability Matching Tolerance Intervals for Distributions in Exponential Families

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**Abstract**

Tolerance intervals are widely used in industrial applications. So far attention has been mainly focused on the construction of tolerance intervals for continuous distributions. In this paper we introduce a unified analytical approach to the construction of tolerance intervals for distributions, both discrete and continuous, in exponential families with quadratic variance functions. These tolerance intervals are shown to have desirable probability matching properties and outperform existing tolerance intervals in the literature.

**Keywords:** Coverage probability, Edgeworth expansion, exponential family, probability matching, tolerance interval.

## 1 Introduction

Statistical tolerance intervals are important in many industrial applications ranging from engineering to pharmaceutical industry. See, for example, Hahn and Chandra (1981), Hahn and Meeker (1991), Katori et al. (1995) and Hauck et al. (2005). The goal of a tolerance interval is to contain at least a specified proportion of the population, $\beta$, with a specified degree of confidence, $1-\alpha$. More specifically, let $X$ be a random variable with cumulative distribution function $F$. An interval $(L(X), U(X))$ is said to be a $\beta$-content, $(1-\alpha)$-confidence tolerance interval for $F$ (called a $(\beta, 1-\alpha)$ tolerance interval for short) if

$$P\{[F(U(X)) - F(L(X))] \geq \beta\} = 1-\alpha. \quad (1)$$

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One-sided tolerance bounds can be defined analogously. A bound $L(X)$ is said to be a $(\beta, 1-\alpha)$ lower tolerance bound if $P\{1 - F(L(X)) \geq \beta\} = 1 - \alpha$ and a bound $U(X)$ is said to be a $(\beta, 1-\alpha)$ upper tolerance bound if $P\{F(U(X)) \geq \beta\} = 1 - \alpha$.

Ever since the pioneering work of Wilks (1941 and 1942), construction of tolerance intervals for continuous distributions has been extensively studied. See, for example, Wald and Wolfowitz (1946), Easterling and Weeks (1970), Odeh and Owen (1980), Kocherlakota and Balakrishnan (1986), Vangel (1992), Mukerjee and Reid (2001) and Krishnamoorthy and Mathew (2004). Comparing to the continuous distributions, literature on tolerance intervals for discrete distributions is sparse. This is mainly due to the difficulty in deriving explicit expression for the tolerance intervals in the discrete case. Zacks (1970) proposed a criterion to select tolerance limits for monotone likelihood ratio families of discrete distributions. The most widely used tolerance intervals to date for Poisson and Binomial distributions were proposed by Hahn and Chandra (1981). The intervals are constructed by a two-step procedure. See Hahn and Meeker (1991) for a survey of these intervals.

Although the tolerance intervals are useful and important, the properties of the tolerance intervals such as their coverage probability have not been studied as in-depth as those of confidence intervals. As we shall see in Section 2 the tolerance intervals given in Hahn and Chandra (1981) tend to be very conservative in terms of their coverage probability. Techniques for the construction of tolerance intervals in the literature often vary from distribution to distribution.

In the present paper, we introduce a unified analytical approach using the Edgeworth expansions for the construction of tolerance intervals for both the continuous and discrete distributions in exponential families with quadratic variance functions. The Edgeworth expansions provide an accurate and useful tool in analyzing the coverage properties, and reveal uniform structure across all the distributions, both discrete and continuous, under consideration. We show that these tolerance intervals enjoy desirable probability matching properties and outperform existing tolerance intervals in the literature. The most satisfactory aspects of our results are the constancy of the phenomena and uniformity in the final resolutions of these problems.

Edgeworth expansions are a powerful tool for studying complicated probabilistic quantities. For example it is a major technical tool for understanding the properties of resampling methods such as the bootstrap and jackknife. See, for example, Barndorff-Nielsen and Cox (1989) and Hall (1992). Edgeworth expansions have also been used very successfully for the construction of confidence intervals in discrete distributions. See Hall (1982), Brown, Cai and DasGupta (2002 and 2003), and Cai (2005).

We begin in Section 2 by briefly reviewing the existing tolerance intervals for Binomial and Poisson distributions and showing that these tolerance intervals have serious deficiency in terms of the coverage probability. The serious deficiency of these intervals calls for better alternatives. We shall consider in this paper tolerance intervals of a particular form whose motivation is given in Section 2.1. After Section 3.1 in which basic notations and definitions of natural exponential family are summarized, we introduce the first-order and second-order probability matching tolerance intervals in Section 3. As in the case of confidence intervals, the coverage probability of the tolerance intervals for the lattice distributions such as Binomial and Poisson distributions contains two components: oscillation and systematic bias. The oscillation in the coverage probability, which is due to the lattice structure of the distributions, is unavoidable for any non-randomized procedures. The systematic bias, which is large for many existing tolerance intervals, can be nearly eliminated. We show that our new tolerance intervals have better coverage properties in the sense that they have nearly vanishing systematic bias in all the distributions under consideration.

In Section 4, two-sided tolerance intervals are constructed by using one-sided upper and lower probability matching tolerance bounds. In addition to the coverage properties, parsimony in expected length of the two-sided intervals is also discussed. Section 5 is an appendix containing detailed technical derivations of the tolerance intervals. The derivations are based on the two term Edgeworth expansions and Cornish-Fisher expansions.

# 2 Tolerance Intervals: Existing Methods

As mentioned in the introduction, we shall construct tolerance intervals for both continuous and discrete distributions in the exponential families. However, the emphasis of our discussion is on the discrete distributions. In this section we review the existing tolerance intervals for two important discrete distributions, the Binomial distribution and Poisson distribution. These tolerance intervals will be used for comparison with the new intervals constructed in the present paper.

The most widely used method for constructing tolerance intervals for the Binomial and Poisson distributions was proposed by Hahn and Chandra (1981). Suppose $x$ is the observed value of a random variable $X$ having a Binomial distribution $B(n, \theta)$ or a Poisson distribution $Poi(\theta)$ and one wishes to construct a tolerance interval based on $x$. The method introduced by Hahn and Chandra (1981) for constructing a $(\beta, 1-\alpha)$ tolerance interval $(L(x), U(x))$ has two steps.

(i). Construct a two-sided $(1-\alpha)$-level confidence interval $(l, u)$ for $\theta$, where $l$ and $u$ depends on $x$.

(ii). Find the minimum number $U(x)$ and the maximum number $L(x)$ such that

$$p_u(X \leq U(x)) \geq (1+\beta)/2 \quad \text{and} \quad p_l(X > L(x)) \geq (1+\beta)/2.$$

Similarly, a lower $(\beta, 1-\alpha)$ tolerance bound $L(x)$ can be constructed by finding a lower $(1-\alpha)$ confidence bound of $\theta$, say $l$, and then deriving the maximum value $L(x)$ such that $p_l(X > L(x)) \geq \beta$.

For this two-step procedure, it is clear that the choice of the confidence interval used in Step 1 is important to the performance of the resulting tolerance interval. For any $0 < \gamma < 1$, let $z_\gamma = \Phi^{-1}(1-\gamma)$ be the upper $\gamma$ quantile of a standard normal distribution. Hahn and Meeker (1991) suggested the following $(1-\alpha)$ level confidence intervals for the Binomial case,

$$(l, u) = \hat{\theta} \pm z_{\alpha/2}(\frac{\hat{\theta}(1-\hat{\theta})}{n})^{1/2} \tag{2}$$

and

$$(l, u) = \left((1 + \frac{(n-x+1)F_{(\alpha/2;2n-2x+2,2x)}}{x})^{-1}, (1 + \frac{n-x}{(x+1)F_{(\alpha/2;2x+2,2n-2x)}})^{-1}\right), \tag{3}$$

where $F_{(a;r_1,r_2)}$ denotes the upper $a$ quantile of the $F$ distribution with $r_1$ and $r_2$ degrees of freedom. For the Poisson distribution, the suggested $(1-\alpha)$ confidence intervals in Hahn and Meeker (1991) are

$$(l, u) = \hat{\theta} \pm z_{\alpha/2}(\frac{\hat{\theta}}{n})^{1/2} \tag{4}$$

and

$$(l, u) = \left(0.5\chi^2_{(\alpha/2;2x)}/n,\ 0.5\chi^2_{(1-\alpha/2;2x+2)}/n\right), \tag{5}$$

where $\chi^2_{(a;r_1)}$ is $a$ quantile of the chi-square distribution with $r_1$ degrees of freedom. The confidence bounds for one-sided tolerance intervals in both Binomial and Poisson distributions are given analogously.

Figures 1 presents the coverage probabilities of both the two-sided and one-sided tolerance intervals for the Binomial and Poisson distributions. It can be seen easily from the plots that these tolerance intervals are too conservative with higher coverage probability than the nominal level for both distributions.

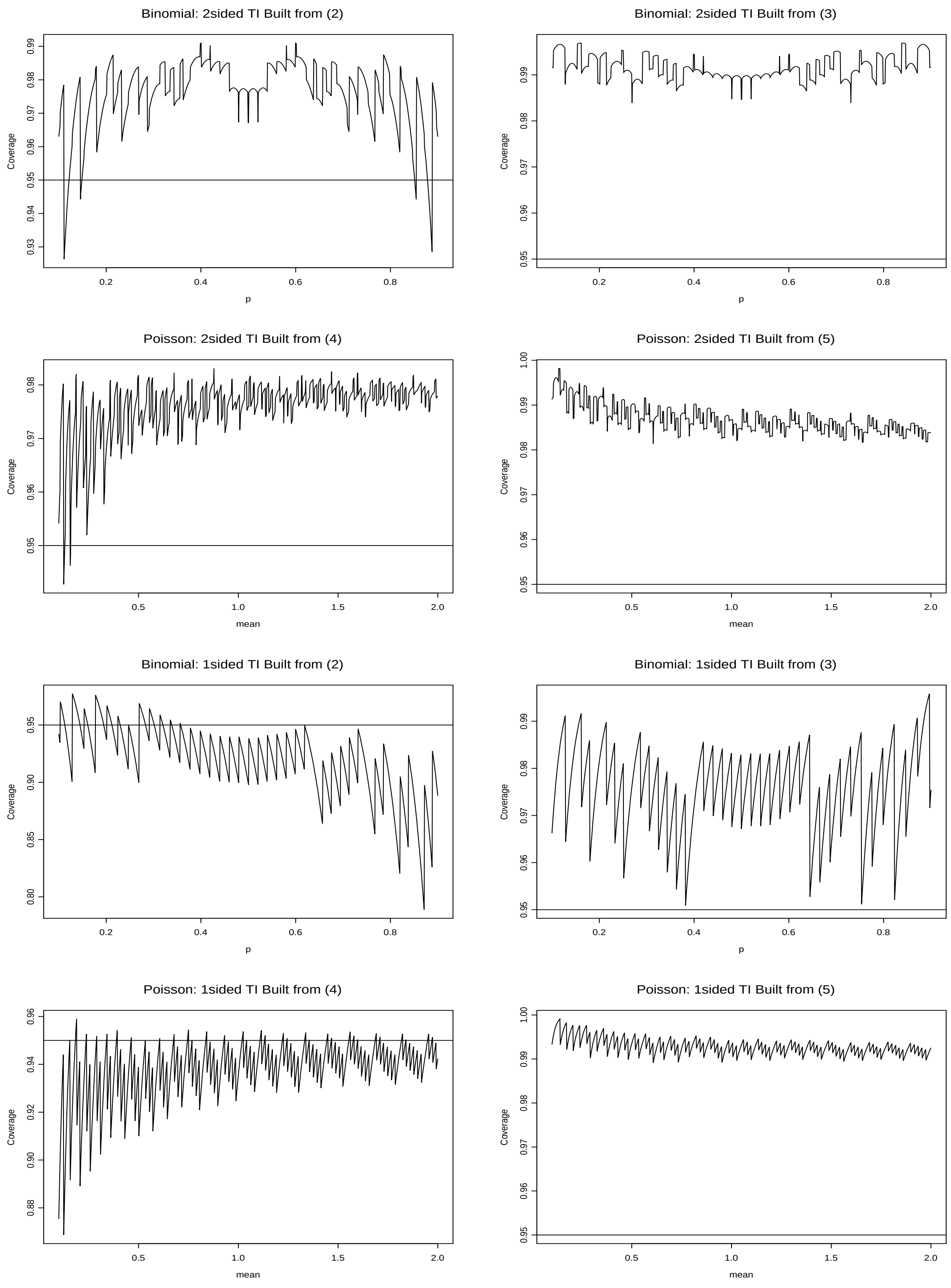


Figure 1: Coverage probabilities of the 90%-content, 95% level two-sided (top two rows) and one-sided (bottom two rows) tolerance intervals for the Binomial and Poisson distributions with $n = 50$.

As in the case of confidence intervals, the coverage probability of the tolerance intervals contains two components: oscillation and systematic bias. The oscillation in the coverage probability, which is due to the lattice structure of the Binomial and Poisson distributions, is unavoidable for any non-randomized procedures. However, the systematic bias for the existing tolerance intervals are significantly larger than we expected. In Section 3 we shall introduce new tolerance intervals using the Edgeworth expansion. These new tolerance intervals have better coverage properties in the sense that they have nearly vanishing systematic bias. Figure 4 presents the coverage probabilities of the proposed two-sided tolerance intervals for the Binomial and Poisson distributions. Compared with Figure 1, the new tolerance intervals certainly have much better performance than the existing intervals in the sense that the actual coverage probability is much closer to the nominal level. The detailed derivation of our tolerance intervals is given in the next two sections.

## 2.1 Motivation from Normal Tolerance Interval

We shall consider in Section 3 tolerance intervals of a particular form. The motivation of considering tolerance intervals of this form comes from normal tolerance intervals. Let $X_1, ..., X_n$ be a sample from a normal distribution $N(\mu, \sigma^2)$. Wald and Wolfowitz (1946) introduced the $\beta$-content, $(1-\alpha)$-confidence tolerance interval

$$[\bar{X} - \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}ts, \bar{X} + \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}ts], \tag{6}$$

where $\bar{X}$ and $s$ are sample mean and sample standard deviation, respectively, $\chi^2_{n-1,\alpha}$ is the $\alpha-$quantile of the chi-squared distribution with $n-1$ degrees of freedom and $t$ is the solution of the equation

$$\int_{\frac{1}{\sqrt{n}}-t}^{\frac{1}{\sqrt{n}}+t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \beta.$$

To make a better analogy between the NEF-QVF families and normal cases, we first attempt to rewrite the tolerance interval in (6) in terms of $X = \sum_{i=1}^n X_i$ under $N(n\mu, n\sigma^2)$. Note that (6) implies

$$1 - \alpha \approx P(\Phi_{\mu,\sigma}(\bar{X} + \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}rs) - \Phi_{\mu,\sigma}(\bar{X} - \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}rs) \geq \beta)$$

where $\Phi_{\mu,\sigma}$ denotes the cdf of the $N(\mu, \sigma^2)$ distribution. We would like to re-express the above quantities in terms of $\Phi_{n\mu,\sqrt{n}\sigma}$, the cdf of $N(n\mu, n\sigma^2)$. Let Z denote a standard normal random variable. Then

$$\begin{aligned}\Phi_{\mu,\sigma}(\bar{X} \pm \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}rs) &= P(\sigma Z < \bar{X} - \mu \pm \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}rs|X) \\ &= P(n\mu + \sqrt{n}\sigma Z < n\mu + \sqrt{n}(\bar{X} - \mu) \pm \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r\sqrt{n}s|X) \\ &= \Phi_{n\mu,\sqrt{n}\sigma}(n\mu + \sqrt{n}(\bar{X} - \mu) \pm \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r\sqrt{n}s). \qquad (7)\end{aligned}$$

If $\mu$ were known, then we could replace the tolerance interval in (6) by

$$[n\mu + \sqrt{n}(\bar{X} - \mu) - \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r\sqrt{n}s, n\mu + \sqrt{n}(\bar{X} - \mu) + \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r\sqrt{n}s], \qquad (8)$$

under the $N(n\mu, n\sigma^2)$ distribution. Of course $\mu$ is unknown in practice and we need to replace $\mu$ in (8) by a function of $X$. A simple approach is to replace $\mu$ in the lower and upper limits by, respectively, the lower and upper limits of a $\beta$-confidence confidence interval for $\mu$,

$$[\bar{X} - z_{(1-\beta)/2}\frac{s}{\sqrt{n}}, \bar{X} - z_{(1-\beta)/2}\frac{s}{\sqrt{n}}].$$

This leads the following tolerance interval

$$[X - (\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r + (1 - \frac{1}{\sqrt{n}})z_{(1-\beta)/2})\sqrt{n}s, X + (\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r + (1 - \frac{1}{\sqrt{n}})z_{(1-\beta)/2})\sqrt{n}s]$$

under the $N(n\mu, n\sigma^2)$ distribution.

For the NEF-QVF families, by the Central Limit Theorem, and adopting the method in the normal case by identifying $d_0 + d_1X/n + d_2X^2/(n^2)$ as $s^2$, an approximate $\beta$-content, $(1-\alpha)$-confidence tolerance lower bound and upper bound are

$$X - (\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r + (\sqrt{n} - 1)z_{(1-\beta)/2})\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2)}$$

and

$$X + (\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}r + (\sqrt{n} - 1)z_{(1-\beta)/2})\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2)},$$

respectively. More generally, we shall consider tolerance bounds of the form

$$\begin{aligned}L(X) &= X + a - b\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2) + c} \\ U(X) &= X + a + b\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2) + c}\end{aligned}$$

with suitably chosen constants $a$, $b$ and $c$.

# 3 Probability-Matching Tolerance Intervals

In this section we construct one-sided probability-matching tolerance intervals in the natural exponential family (NEF) with quadratic variance functions (QVF) by using the Edgeworth expansion. After Section 3.1 in which basic notations and definitions of natural exponential family are given, we introduce the first-oder and second-order probability matching tolerance intervals in Section 3.2.

## 3.1 Natural Exponential Family

NEF-QVF families consist of six important distributions, three continuous: normal, gamma, and NEF-GHS distributions; three discrete: Binomial, Negative Binomial, and Poisson (see, e.g., Morris (1982) and Brown (1986)). We will consider tolerance intervals for both the continuous and the discrete NEF-QVF distributions, although our primary focus is on the discrete case.

We first state some basic facts about the NEF-QVF families for use in the rest of this article. The distributions in a natural exponential family have the form

$$f(x|\xi) = e^{\xi x - \psi(\xi)} h(x)$$

where $\xi$ is called the natural parameter. The mean $\mu$, variance $\sigma^2$ and cumulant generating function $\phi_\xi$ are respectively

$$\mu = \psi'(\xi), \quad \sigma^2 = \psi''(\xi), \;\text{ and }\; \phi_\xi(t) = \psi(t+\xi) - \psi(\xi).$$

The cumulants are given as $K_r = \psi^{(r)}(\xi)$. Let $\beta_3$ and $\beta_4$ denote the skewness and kurtosis. In the subclass with a quadratic variance function (QVF), the variance $\psi''(\xi)$ depends on $\xi$ only through the mean $\mu$, and indeed,

$$\sigma^2 \equiv V(\mu) = d_0 + d_1\mu + d_2\mu^2, \tag{9}$$

for suitable constants $d_0$, $d_1$, and $d_2$. We denote the discriminate by

$$\Delta = d_1^2 - 4d_0d_2. \tag{10}$$

The notation $\Delta$ will be later used in the statements of theorems for both the discrete and the continuous cases, although for all the discrete cases $\Delta$ happens to be equal to 1. Note that $\frac{d\mu}{d\xi} = \psi''(\xi) = \sigma^2$, so

$$K_3 = \psi^{(3)}(\xi) = \frac{dV}{d\mu}\cdot\frac{d\mu}{d\xi} = (d_1 + 2d_2\mu)\sigma^2 \text{ and } K_4 = \psi^{(4)}(\xi) = \frac{dK_3}{d\mu}\cdot\frac{d\mu}{d\xi} = \Delta\sigma^2 + 6d_2\sigma^4.$$

Hence,

$$\beta_3 = K_3/\sigma^3 = (d_1 + 2d_2\mu)\sigma^{-1} \quad \text{and} \quad \beta_4 = K_4/\sigma^4 = \Delta\sigma^{-2} + 6d_2. \tag{11}$$

Discrete NEF-QVF families consist of the Binomial, Negative Binomial, and Poisson distributions. Let us list the important facts for the three distributions separately.

- Binomial, $\mathrm{B}(1,p)$: $\xi = \log(p/q)$, $\psi(\xi) = \log(1+e^{\xi})$, and $h(x) = 1$. Also $\mu = p$, $V(\mu) = pq = \mu - \mu^2$. Thus $d_0 = 0$, $d_1 = 1$, $d_2 = -1$,

$$\beta_3 = \frac{1-2\mu}{(\mu(1-\mu))^{1/2}}, \quad \text{and} \quad \beta_4 = \frac{1-6\mu+6\mu^2}{\mu(1-\mu)}.$$

- Negative Binomial, $\mathrm{NB}(1,p)$, the number of successes before the first failure: $\xi = \log p$, $\psi(\xi) = -\log(1-e^{\xi})$, and $h(x) = 1$. And $\mu = p/q$, $V(\mu) = p/q^2 = \mu + \mu^2$. Thus $d_0 = 0$, $d_1 = 1$, $d_2 = 1$,

$$\beta_3 = \frac{1+2\mu}{(\mu(1+\mu))^{1/2}}, \quad \text{and} \quad \beta_4 = \frac{1+6\mu+6\mu^2}{\mu(1+\mu)}.$$

- Poisson, $\mathrm{Poi}(\lambda)$: $\xi = \log\lambda$, $\psi(\xi) = e^{\xi}$, and $h(x) = 1/x!$. And $\mu = \lambda$, $V(\mu) = \mu$. Thus here $d_0 = 0$, $d_1 = 1$, $d_2 = 0$,

$$\beta_3 = \frac{1}{\mu^{1/2}}, \quad \text{and} \quad \beta_4 = \frac{1}{\mu}.$$

Continuous NEF-QVF families consist of the normal, gamma , and NEF-GHS distributions.

- Normal, $\mathrm{N}(\theta,1)$: $\xi = \theta$, $\psi(\xi) = \xi^2/2 + \log\sqrt{2\pi}$, and $h(x) = e^{x^2/2}$. Also $\mu = \theta$, $V(\mu) = 1$. Thus here $d_0 = 1$, $d_1 = 0$, $d_2 = 0$, $\beta_3 = 0$, and $\beta_4 = 0$.

- Gamma, $\mathrm{Gam}(r,\lambda)$ (with $r$ known): $\xi = -1/\lambda$, $\psi(\xi) = \log\Gamma(r)(-1/\xi)^r$, and $h(x) = e^{(r-1)\log(x)}$. And $\mu = r\lambda$, $V(\mu) = \mu^2/r$. Thus in this case, $d_0 = 0$, $d_1 = 0$, $d_2 = 1/r$, $\beta_3 = 2r^{-\frac{1}{2}}$, and $\beta_4 = 6/r$.

- $NEF{-}GHS(r,\lambda)$ (with $r$ known): The pmf is $f(x) = (1+\lambda^2)^{-r/2}\exp\{x\tan^{-1}(\lambda)\}f_{r,0}(x)$, where $f_{r,0}(x)$ is the density of a generalized hyperbolic secant distribution with parameter $r$ which is defined as

$$f_{r,0}(x) = \frac{2^{r-2}}{\Gamma(r)}\prod_{j=0}^{\infty}\{1 + \frac{x^2}{(r+2j)^2}\}^{-1}.$$

In this case, $\xi = \tan^{-1}(\lambda)$, $\psi(\xi) = -r/2\log(1+\tan^2\xi)$, and $h(x) = f_{r,0}(x)$. And $\mu = r\lambda$, $V(\mu) = r + \mu^2/r$. Thus here $d_0 = r$, $d_1 = 0$, $d_2 = 1/r$,

$$\beta_3 = 2\mu/(r(r^2+\mu^2))^{\frac{1}{2}}, \quad \text{and} \quad \beta_4 = (2r^2+6\mu^2)/(r^3+r\mu^2).$$

## 3.2 One-sided Tolerance Interval

We now introduce the first-order and second-order probability matching one-sided tolerance intervals. Let $X = \sum_{i=1}^{n} X_i$, where $X_i$ are iid observations from one of the six distributions discussed in Section 3.1. We shall denote the distribution of $X$ by $F_{n,\mu}$ and focus our discussion on the lower tolerance intervals. The upper tolerance intervals can be constructed analogously. Two-sided tolerance intervals will be discussed in Section 4.

Similar to the confidence intervals, the coverage probability of a lower $(\beta, 1-\alpha)$ tolerance interval admits a two-term Edgeworth expansion of the general form of

$$P(1-F_{n,\mu}(L(X)) \geq \beta) = 1-\alpha+S_1\cdot n^{-\frac{1}{2}}+Osc_1\cdot n^{-\frac{1}{2}}+S_2\cdot n^{-1}+Osc_2\cdot n^{-1}+O(n^{-\frac{3}{2}}) \quad (12)$$

where the first $O(n^{-\frac{1}{2}})$ term, $S_1 n^{-\frac{1}{2}}$, and the first $O(n^{-1})$ term, $S_2 n^{-1}$, are respectively the first and second order smooth terms, and $Osc_1 \cdot n^{-\frac{1}{2}}$ and $Osc_2 \cdot n^{-1}$ are the oscillatory terms. (The oscillatory terms vanish in the case of continuous distributions.) The smooth terms capture the systematic bias in the coverage probability. See Bhattacharya and Rao (1976) and Hall (1992) for details on Edgeworth expansions.

We call a tolerance interval *first-order probability matching* if the first order smooth term $S_1 n^{-\frac{1}{2}}$ is vanishing and call the interval *second-order probability matching* if both smooth terms $S_1 n^{-\frac{1}{2}}$ and $S_2 n^{-1}$ vanish. Note that the oscillatory terms are unavoidable for any nonrandomized procedures in the case of lattice distributions. See Ghosh (1994) and Ghosh (2001) for general discussions on probability matching confidence sets.

Motivated by the discussion given in Section 2.1, we shall consider an approximate $\beta$-content, $(1-\alpha)$-confidence lower tolerance bound of the form

$$L(X) = X + a - b\sqrt{n(d_0 + d_1 X/n + d_2 X^2/n^2) + c}, \quad (13)$$

where $d_0, d_1$ and $d_2$ are the same constants as in (9) and $a$, $b$ and $c$ are constants depending on $\alpha$ and $\beta$ such that

$$L(X) < L(Y) \text{ if } X < Y. \quad (14)$$

**Remark:** The quantity $a$ in (13) "re-centers" the tolerance interval. As we will see later that $a$ is important to the performance of the tolerance interval. The quantity $c$ in (13) plays the role of "boundary correction". The effect of $c$ can be significant when $\mu$ is near the boundaries.

We shall use the Edgeworth expansion to choose the constants $a$, $b$ and $c$ so that the resulting tolerance intervals are first-order and second-order probability matching. The first step in the derivation is to invert the constraint $1 - F(L(X)) \geq \beta$ to a constraint on $X$ of the form $X \leq u(\mu, \beta)$. Then the coverage probability of the tolerance interval can be expanded using the Edgeworth expansion. The optimal choice of the values $a$, $b$ and $c$ can then be solved by setting the smooth terms in the expansion to zero. The algebra involved here is more tedious than for deriving the probability matching confidence interval. The detailed proof is given in the Appendix.

**Theorem 1** *The tolerance interval given in (13) is first-order probability matching for the six distributions in the NEF-QVF if*

$$a = \frac{1}{6}[(z_{1-\beta}^2 - 1)(1 + 2d_2\hat{\mu}) + (1 + 3z_\alpha z_{1-\beta} + 2z_\alpha^2)(d_1 + 2d_2\hat{\mu})], \tag{15}$$

$$b = z_\alpha + z_{1-\beta}, \tag{16}$$

*and $c = 0$, where $\hat{\mu} = X/n$ and $\hat{\sigma} = \sqrt{d_0 + d_1\hat{\mu} + d_2\hat{\mu}^2}$. The tolerance interval (13) is second-order probability matching with $a$ and $b$ given as in (15) and (16), and $c$ given by*

$$\begin{aligned} c = {} & \frac{1}{36(z_\alpha + z_{1-\beta})} \{(-1 + 18d_0d_2 + 2(-8 + 9d_1)d_2\hat{\mu} + 2d_2^2\hat{\mu}^2)z_{1-\beta}^3 \\ & +24d_2(d_0 + \hat{\mu}(d_1 + d_2\hat{\mu}))z_{1-\beta}^2 z_\alpha + z_{1-\beta}[1 + 2d_2\hat{\mu}(20 + 5d_2\hat{\mu} + 24d_2\hat{\mu}z_\alpha^2) + 3d_1^2(2 + z_\alpha^2) \\ & +18d_0d_2(-3 + 2z_\alpha^2) + 6d_1d_2\hat{\mu}(-5 + 8z_\alpha^2)] + z_\alpha[d_1^2(7 + 2z_\alpha^2) \\ & +2d_1d_2\hat{\mu}(5 + 13z_\alpha^2) + 2d_2(9d_0(-1 + z_\alpha^2) + d_2\hat{\mu}^2(5 + 13z_\alpha^2))]\}. \end{aligned} \tag{17}$$

**Remark:** We have focused above on the construction of lower tolerance intervals. The first order and second order $\beta$-content, $(1-\alpha)$-confidence upper tolerance intervals can be constructed analogously, which are

$$X + a + b\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2)} \tag{18}$$

and

$$X + a + b\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2) + c}, \tag{19}$$

respectively, with the same $a, b$ and $c$ as the lower tolerance intervals.

For all six distributions in the NEF-QVF, $b = z_\alpha + z_{1-\beta}$. It is useful to give the expressions of the constants $a$ and $c$ individually for each of the six distributions.

1. Binomial: $a = \frac{1}{6}(1 - 2\hat{\mu})(z_\alpha + z_{1-\beta})(2z_\alpha + z_{1-\beta})$, and

$$c = -\frac{1}{18}(13z_\alpha^2 + 11z_\alpha z_{1-\beta} + z_{1-\beta}^2 + 5)(\hat{\mu} - \hat{\mu}^2) + \frac{1}{36}(2z_\alpha^2 + z_\alpha z_{1-\beta} - z_{1-\beta}^2 + 7).$$

2. Poisson: $a = \frac{1}{6}(z_\alpha + z_{1-\beta})(2z_\alpha + z_{1-\beta})$ and $c = \frac{1}{36}(7 - z_{1-\beta}^2 + z_\alpha z_{1-\beta} + 2z_\alpha^2)$.

3. Negative Binomial: $a = \frac{1}{6}(1+2\hat{\mu})(z_\alpha+z_{1-\beta})(2z_\alpha+z_{1-\beta})$ and $c = \frac{1}{18}(13z_\alpha^2+11z_\alpha z_{1-\beta}+z_{1-\beta}^2 + 5)(\hat{\mu} + \hat{\mu}^2) + \frac{1}{36}(2z_\alpha^2 + z_\alpha z_{1-\beta} - z_{1-\beta}^2 + 7)$.

4. Normal, $N(\mu, \sigma^2)$ (with $\sigma$ known): $a = \frac{1}{6}(z_{1-\beta}^2 - 1)$ and $c = \frac{z_{1-\beta}-z_{1-\beta}^3}{36(z_\alpha+z_{1-\beta})}$.

5. Gamma, $Gam(r, \lambda)$ (with $r$ known): $a = \frac{\hat{\mu}}{3r}(z_\alpha + z_{1-\beta})(2z_\alpha + z_{1-\beta}) + \frac{1}{6}(z_{1-\beta}^2 - 1)$ and

$$\begin{aligned} c &= \frac{1}{36r^2(z_\alpha + z_{1-\beta})}\left\{2\hat{\mu}^2(z_{1-\beta}^3 - 19z_\alpha + 12z_{1-\beta}^2 z_\alpha + 37z_\alpha^3 + z_{1-\beta}(36z_\alpha^2 - 7))\right. \\ &\quad \left. +r^2(z_{1-\beta} - z_{1-\beta}^3) - 8r\hat{\mu}(2z_{1-\beta}^3 + 6\hat{\mu}z_\alpha(z_\alpha^2 - 1) + z_{1-\beta}(3\hat{\mu}(z_\alpha^2 - 1) - 5))\right\}. \end{aligned}$$

6. NEF-GHS$(r, \lambda)$ (with $r$ known): $a = \frac{\hat{\mu}}{3r}(2z_\alpha + z_{1-\beta})(z_\alpha + z_{1-\beta}) + \frac{1}{6}(z_{1-\beta}^2 - 1)$ and

$$\begin{aligned} c &= \frac{1}{36r^2(z_\alpha + z_{1-\beta})}\left\{-8r\hat{\mu}z_{1-\beta}(2z_{1-\beta}^2 - 5)\right. \\ &\quad +2\hat{\mu}^2(5z_{1-\beta} + z_{1-\beta}^3 + 5z_\alpha + 12z_{1-\beta}^2 z_\alpha + 24z_{1-\beta}z_\alpha^2 + 13z_\alpha^3) \\ &\quad \left. +r^2(17z_{1-\beta}^3 + 24z_{1-\beta}^2 z_\alpha + 18z_\alpha(z_\alpha^2 - 1) + z_{1-\beta}(53 - 36z_\alpha^2))\right\}. \end{aligned}$$

Figures 2 and 3 plot the coverage probabilities of the first-order and second-order probability matching $(0.9, 0.95)$ lower tolerance intervals for $n = 50$. It is clear from Figure 2 that for the three discrete distributions, the first and second order probability matching tolerance intervals have nearly vanishing systematic bias, and the second order probability matching interval has noticeably smaller systematic bias than the first order probability matching interval for the Negative Binomial distribution. By comparing Figure 2 with Figure 1 we can see that the coverage probability performance of these one-sided intervals is better than the existing one-sided tolerance intervals.

For the continuous distributions, the systematic biases of the new tolerance intervals are small. Note that for the normal distribution, the coverage probability of the new tolerance interval is a constant for a fixed sample size when the mean varies. Thus, we plot the coverage probability for different sample sizes instead of different means for the normal distribution in Figure 3. The performance of the coverage probability of the first and second order probability matching intervals for the normal distribution is similar and both have systematic biases less than 0.02. For the gamma distribution, the second order probability matching tolerance interval has smaller systematic bias than the first order probability matching interval.

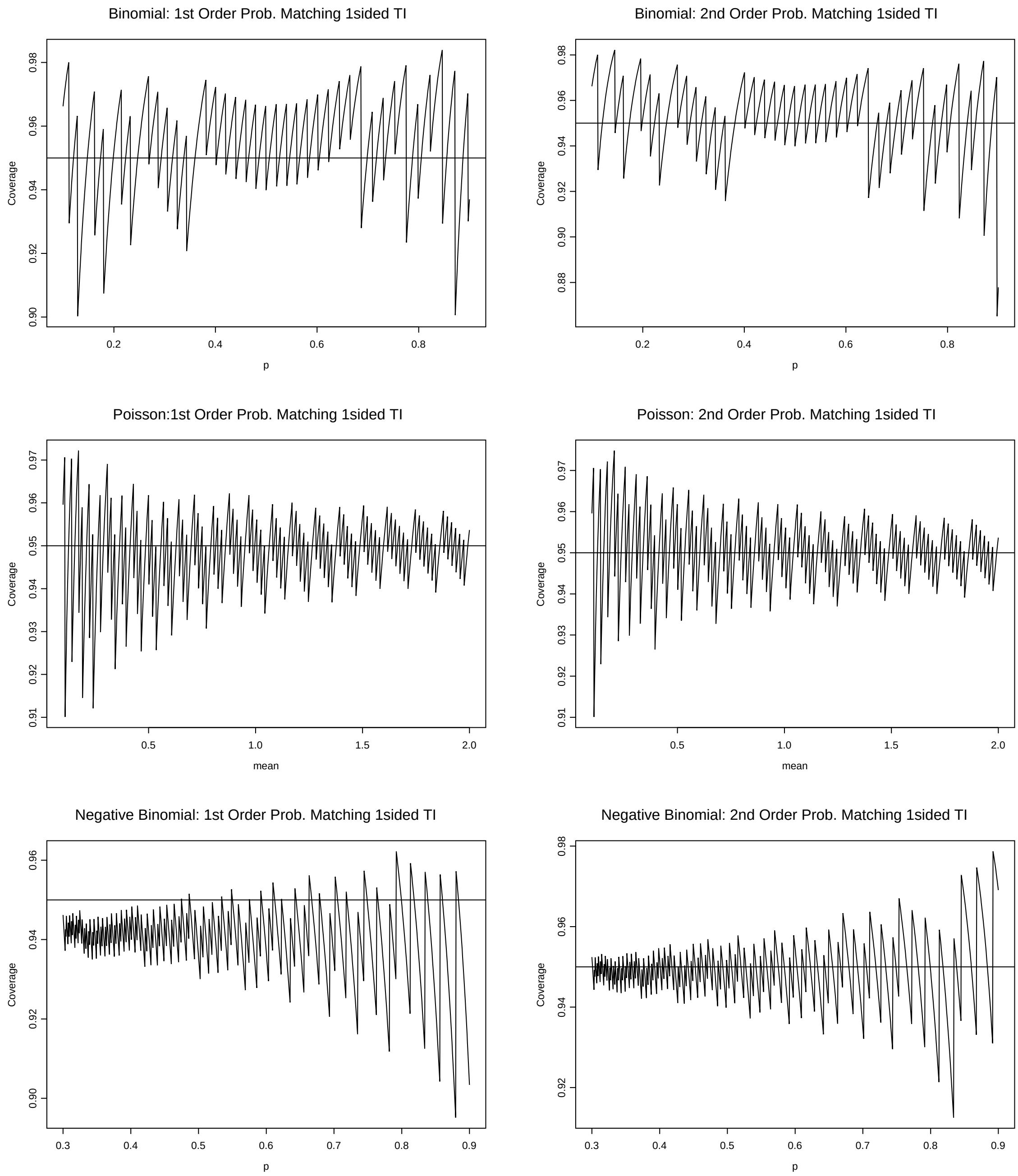


Figure 2: Coverage probabilities of the 90%-content, 95% level of the first order and second order probability matching lower tolerance bounds for the Binomial, Poisson and Negative Binomial distributions with $n = 50$.

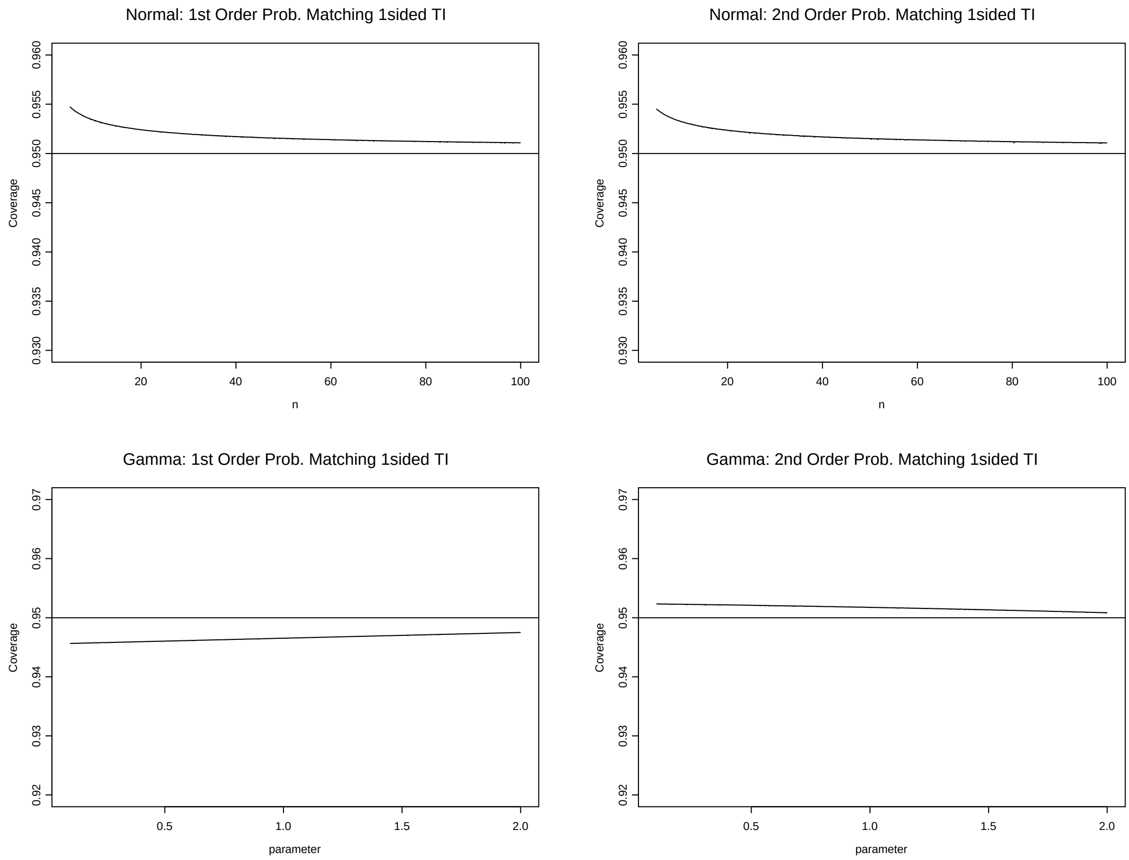


Figure 3: Coverage probabilities of the 90%-content, 95% level of the first order and second order probability matching lower tolerance bounds for a normal distribution (top row) and a Gamma distribution (bottom row) with $r = 2$ and $n = 50$.

# 4 Two-sided Tolerance Interval

We have derived the optimal one-sided first-order and second-order probability matching tolerance intervals in Section 3. It is also natural to consider two-sided tolerance intervals. However, it is difficult to obtain the optimal choices for the values of $a, b$ and $c$ for a two-sided tolerance interval using the same approach. A key step in the derivation of the one-sided intervals given in Section 3 is the inversion of the constraint $1 - F(L(X)) \geq \beta$ to $X \leq u(\mu, \beta)$. Similarly, for a two-sided tolerance interval, it is necessary to invert the constraint

$$F_{n,\mu}(U(X)) - F_{n,\mu}(L(X)) \geq \beta \tag{20}$$

in terms of $X$. However, it is difficult to solve the inequality theoretically.

We thus take an alternative approach to construct two-sided tolerance intervals by using one-sided upper and lower tolerance bounds. Let $U_{(1+\beta)/2}(X)$ and $L_{(1+\beta)/2}(X)$ be the upper and lower probability matching $((1+\beta)/2, 1-\alpha)$ tolerance bounds, respectively. We propose to use the interval

$$(L_{(1+\beta)/2}(X),\ U_{(1+\beta)/2}(X)) \tag{21}$$

as a $\beta$-content, $(1-\alpha)$-confidence two-sided tolerance interval.

Figures 4 and 5 plot the coverage probabilities of two-sided $(0.9, 0.95)$ tolerance intervals built from the first-order and second-order probability matching tolerance bounds. The coverage probabilities for the two-sided tolerance intervals are calculated exactly for the three discrete distributions and are approximated by numerical calculations for the normal and gamma distributions. By comparing Figure 4 with Figure 1 it is clear that the performance of these two-sided intervals is better than the existing two-sided tolerance intervals in the case of Binomial and Poisson distributions. The coverage probability of the proposed two-sided tolerance intervals for the three discrete distributions oscillating in the center from 0.95 to 0.96 with the systematic bias less than 0.01. In contrast, the coverage probability of the two-sided tolerance intervals in Figure 1 oscillates in the center from 0.975 to 0.99 with the systematic bias greater than 0.025. The proposed two-sided tolerance intervals for continuous distributions also have good performance with the systematic bias less than 0.005.

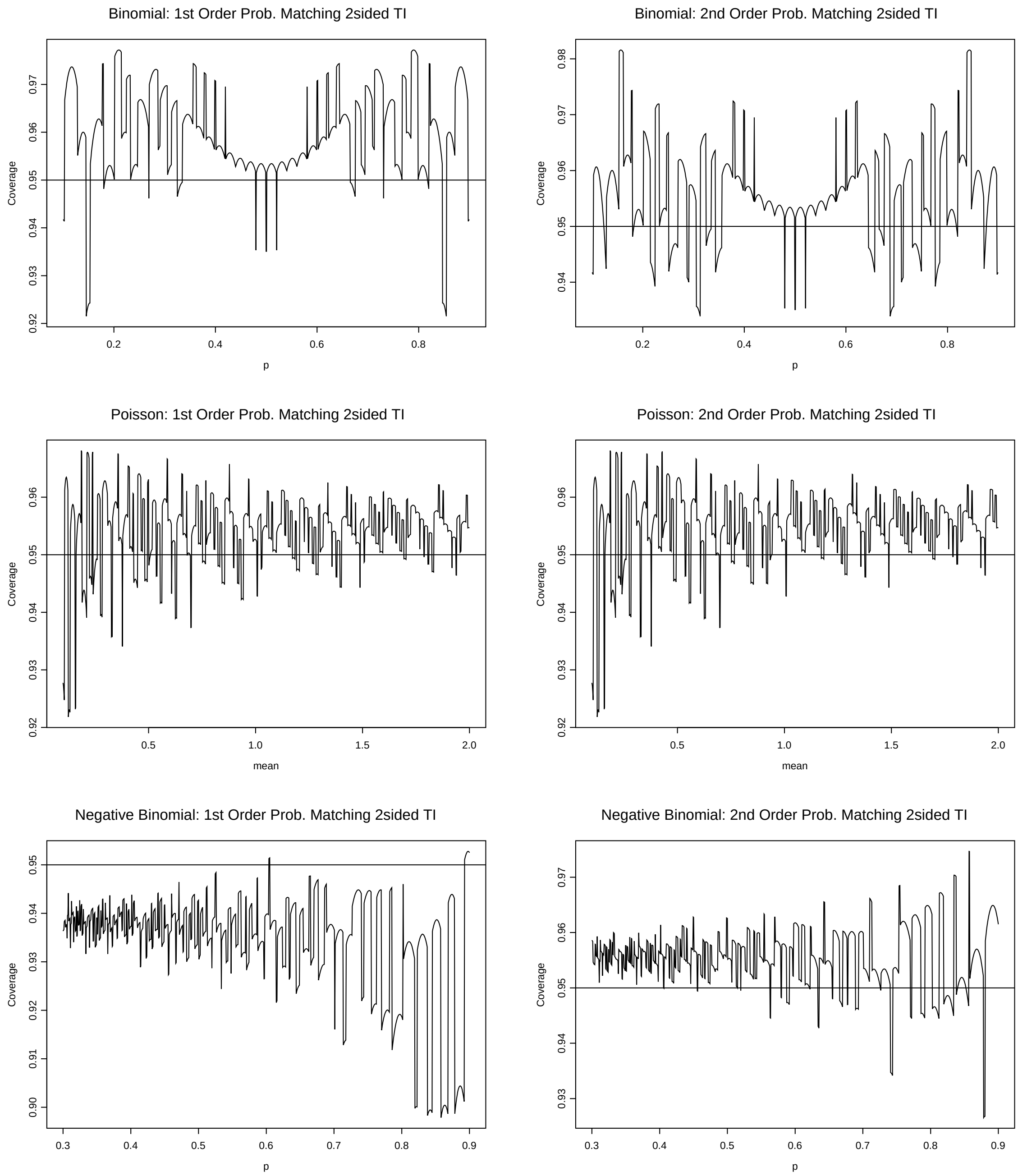


Figure 4: Coverage probabilities of the 90%-content, 95% level of the first order and second order two-sided probability matching tolerance intervals for the Binomial, Poisson and Negative Binomial distributions with $n = 50$.

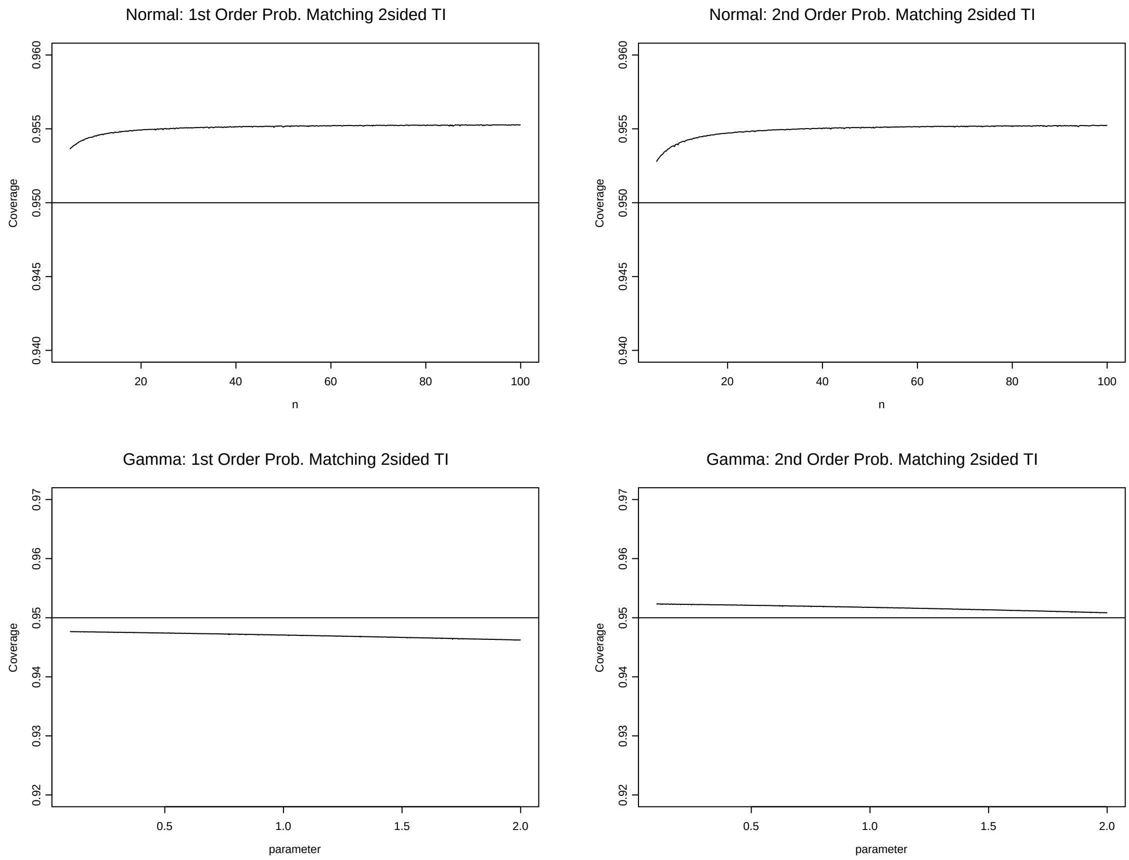


Figure 5: Coverage probabilities of the two-sided 90%-content, 95%-confidence tolerance intervals built from first order (left column) and second order (right column) probability matching tolerance bounds for the normal (top row) and gamma (bottom row) distributions.

In addition to coverage probability, parsimony in length is also an important issue for two-sided tolerance intervals. Figure 6 compares the expected length of the two new tolerance intervals with that of the two intervals discussed in Section 2. It is clear that the expected length of the proposed tolerance intervals is less than that of the existing tolerance intervals. Thus, based on both coverage probability and expected length, the tolerance intervals derived from our analytical approach outperform the existing tolerance intervals in the literature.

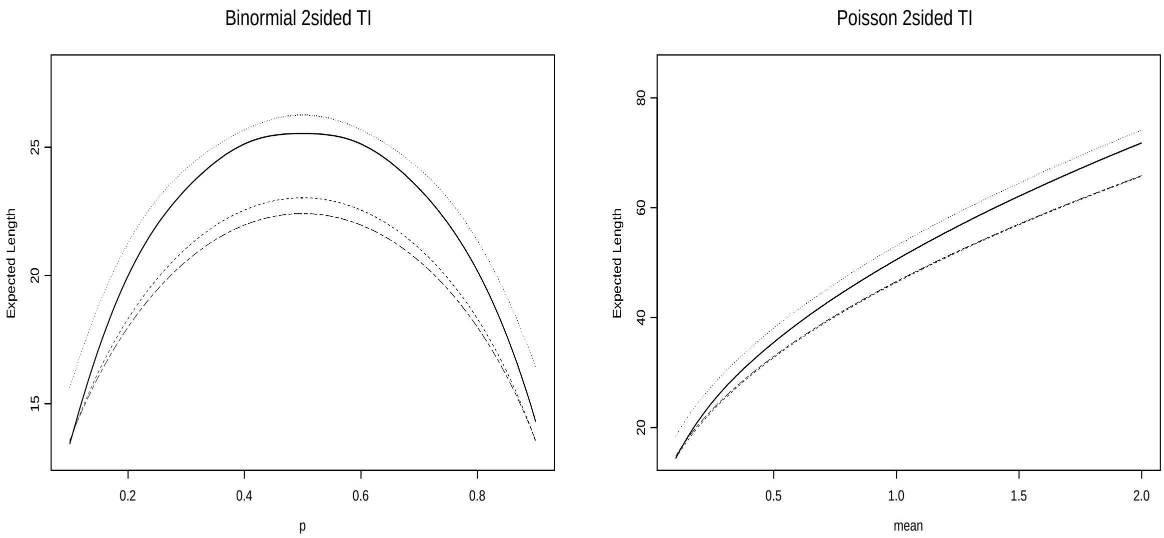


Figure 6: Expected lengths of the 90%-content, 95% level of the two-sided tolerance interval based on (2)(solid, binomial) and (3)(dotted, binomial), the tolerance interval based on (4)(solid, Poisson) and (5)(dotted, Poisson), the first order probability matching two-sided tolerance interval (dashed) and the second order probability matching two-sided tolerance interval (long-dashed) for Binomial (left panel) and Poisson (right panel) distributions with $n = 50$. For the Poisson distribution, the dashed and long-dashed lines are almost overlapped.

# 5 Appendix: Proof of Theorem 1

We begin by introducing notation and a technical lemma. All three discrete distributions under consideration are lattice distributions with the maximal span of one. Lemma 1 below gives the Edgeworth expansion and Cornish-Fisher expansion for these distributions. The first part is from Brown et al. (2003). The expansions for the three continuous distributions are same, except that there are no oscillation terms. For details on the Edgeworth expansion and Cornish-Fisher expansion, see Esseen (1945), Petrov (1975), Bhattacharya and Rao (1976), and Hall (1992).

Let $X_1, ..., X_n$ be iid observations from one of the three discrete distributions in the NEF-QVF families. Denote the mean of $X_1$ by $\mu$ and standard deviation by $\sigma$. Let $\beta_3 = K_3/\sigma^3$ and $\beta_4 = K_4/\sigma^4$ be the skewness and kurtosis of $X_1$, respectively. Set $X = \sum_1^n X_i$ and $Z_n = n^{\frac{1}{2}}(\bar{X} - \mu)/\sigma$ where $\bar{X} = X/n$. Let $F_n(z) = P(Z_n \leq z)$ be the cdf of $Z_n$ and Let $f_{n,\mu,\beta} = \inf\{x : P(X \leq x) \geq 1 - \beta\}$ be the upper $\beta$ quantile of the distribution of $X$.

**Lemma 1** *Suppose* $z = z_0 + c_1 n^{-\frac{1}{2}} + c_2 n^{-1} + O(n^{-\frac{3}{2}})$ *where* $z_0$, $c_1$ *and* $c_2$ *are constants. Then the two term Edgeworth expansion for* $F_n(z)$ *is given by*

$$F_n(z) = \Phi(z_0) + p_1(z)\phi(z_0)n^{-\frac{1}{2}} + p_2(z)\phi(z_0)n^{-1} + Osc_1 \cdot n^{-\frac{1}{2}} + Osc_2 \cdot n^{-1} + O(n^{-\frac{3}{2}}) \quad (22)$$

*where* $Osc_1$ *and* $Osc_2$ *are bounded oscillatory functions of* $\mu$ *and* $z$ *and*

$$\begin{aligned}
p_1(z) &= c_1 + \frac{1}{6}\beta_3(1 - z_0^2) && (23)\\
p_2(z) &= c_2 - \frac{1}{2}z_0 c_1^2 + \frac{1}{6}(z_0^3 - 3z_0)\beta_3 c_1 - \frac{1}{24}\beta_4(z_0^3 - 3z_0) - \frac{1}{72}\beta_3^2(z_0^5 - 10z_0^3 + 15z_0) && (24)\\
p_3(z) &= -c_1 + \frac{1}{6}\beta_3(z_0^2 - 3). && (25)
\end{aligned}$$

*The two-term Cornish-Fisher expansion for* $f_{n,\mu,\beta}$ *is given by*

$$\begin{aligned}
f_{n,\mu,\beta} &= n\mu - z_{1-\beta}(n\sigma^2)^{\frac{1}{2}} + \frac{1}{6}(1 + 2d_2\mu)(z_{1-\beta}^2 - 1)\\
&\quad + [\frac{1}{72}(z_{1-\beta}^3 - z_{1-\beta}) + \frac{1}{9}(d_2\mu + d_2^2\mu^2)(2z_{1-\beta}^3 - 5z_{1-\beta}) - \frac{\sigma^2}{4}d_2(z_{1-\beta}^3 - 3z_{1-\beta})](n\sigma^2)^{-\frac{1}{2}}\\
&\quad + Osc_3 + Osc_4 \cdot n^{-\frac{1}{2}} + O(n^{-1}) \qquad (26)
\end{aligned}$$

*where* $Osc_3$ *and* $Osc_4$ *are bounded oscillatory functions of* $\mu$ *and* $\beta$.

We shall focus on the smooth terms and ignore the oscillatory terms in (22) and (26) in the following calculations.

**Proof of Theorem 1:** It follows from (14) that $1 - F(L(X)) \geq \beta$ is equivalent to $L(X) \leq f_{n,\mu,\beta}$ which is also equivalent to

$$X \leq u(\mu, \beta), \quad (27)$$

where

$$u(\mu, \beta) = \frac{1}{(1 - b^2 d_2 n^{-1})}\left\{-a + \frac{1}{2}b^2 d_1 + f_{n,\mu,\beta} + bD_n\right\} \quad (28)$$

with

$$\begin{aligned}
D_n &= \Big\{d_0 n + n^{-1} f_{n,\mu,\beta}(nd_1 + d_2 f_{n,\mu,\beta}) - ad_1 + \frac{1}{4}b^2 d_1^2 + c - 2ad_2 f_{n,\mu,\beta} n^{-1}\\
&\quad + (a^2 - b^2 c)d_2 n^{-1} - b^2 d_0 d_2\Big\}^{\frac{1}{2}}. \qquad (29)
\end{aligned}$$

The coverage of the tolerance interval is then

$$P(1 - F_{n,\mu}(L(X)) \geq \beta) = P(X \leq u(\mu, \beta)) = P(Z_n \leq z_n) \quad (30)$$

where $Z_n = (X - n\mu)/\sqrt{n\sigma^2}$ and $z_n = (u(\mu, \beta) - n\mu)/\sqrt{n\sigma^2}$.

To derive the optimal choices for $a$, $b$ and $c$, we need the Edgeworth expansion of $P(Z_n \le z_n)$ as well as the expansion of the quantile $f_{n,\mu,\beta}$ given in Lemma 1. By (26), the term $d_0 n + n^{-1} f_{n,\mu,\beta}(nd_1 + d_2 f_{n,\mu,\beta})$ in (29) is equal to

$$n\sigma^2 - (n\sigma^2)^{\frac{1}{2}}(d_1 + 2d_2\mu)z_{1-\beta} + \frac{1}{6}(d_1 + 2d_2\mu)(1 + 2d_2\mu)(z_{1-\beta}^2 - 1) + \sigma^2 d_2 z_{1-\beta}^2 + O(n^{-\frac{1}{2}}). \tag{31}$$

It then follows from (26), (29) and (31), and the Taylor expansion

$$(x + \epsilon)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\epsilon - \frac{1}{8}x^{-\frac{3}{2}}\epsilon^2 + O(x^{-\frac{5}{2}}\epsilon^3)$$

for large $x$ and small $\epsilon$ that

$$\begin{aligned} D_n \;=\;& (n\sigma^2)^{\frac{1}{2}} - \frac{1}{2}(d_1 + 2d_2\mu)z_{1-\beta} + \left\{ -\frac{1}{2}(d_1 + 2d_2\mu)a + \frac{1}{8}b^2 d_1^2 + \frac{1}{2}c \right. \\ & \left. + \frac{1}{12}(1 + 2d_2\mu)(d_1 + 2d_2\mu)(z_{1-\beta}^2 - 1) - \frac{1}{2}b^2 d_0 d_2 - \frac{1}{8}(d_1^2 - 4d_0 d_2)z_{1-\beta}^2 \right\} (n\sigma^2)^{-\frac{1}{2}} + O(n^{-1}). \end{aligned}$$

Note that $(1 - b^2 d_2 n^{-1})^{-1} = 1 + b^2 d_2 n^{-1} + O(n^{-2})$. Using this and the above expansion for $D_n$, we have the following expansion for $z_n$,

$$\begin{aligned} z_n \;=\;& (b - z_{1-\beta}) \\ & + \left\{ \frac{1}{6}(1 + 2d_2\mu)(z_{1-\beta}^2 - 1) - \frac{1}{2}(d_1 + 2d_2\mu)z_{1-\beta}b + (\frac{1}{2}d_1 + d_2\mu)b^2 - a \right\} \sigma^{-1} n^{-\frac{1}{2}} \\ & + \left\{ \frac{1}{4}d_2(3z_{1-\beta} - z_{1-\beta}^3)\sigma^2 + \frac{1}{9}(d_2\mu + d_2^2\mu^2)(2z_{1-\beta}^3 - 5z_{1-\beta}) \right. \\ & + \frac{1}{72}(z_{1-\beta}^3 - z_{1-\beta}) + (b - z_{1-\beta})b^2 d_2\sigma^2 + [-\frac{1}{2}a(d_1 + 2d_2\mu) + \frac{1}{8}b^2 d_1^2 - \frac{1}{2}b^2 d_0 d_2 + \frac{1}{2}c \\ & \left. + \frac{1}{12}(1 + 2d_2\mu)(d_1 + 2d_2\mu)(z_{1-\beta}^2 - 1) - \frac{1}{8}(d_1^2 - 4d_0 d_2)z_{1-\beta}^2]b \right\} \sigma^{-2} n^{-1} \\ & + O(n^{-\frac{3}{2}}) \\ \equiv\;& (b - z_{1-\beta}) + c_1 n^{-\frac{1}{2}} + c_2 n^{-1} + O(n^{-\frac{3}{2}}). \end{aligned} \tag{32}$$

It then follows from the Edgeworth expansion (22) for $P(Z_n \le z_n)$ given in Lemma 1 that $b$ needs to be chosen as $b = z_\alpha + z_{1-\beta}$ in order for the coverage probability of the tolerance interval to be close to the nominal level $1 - \alpha$. With this choice of $b$ and using the notation in (12) for the Edgeworth expansion of $P(Z_n \le z_n)$, the coefficients for the smooth terms are:

$$S_1 = [c_1 + \frac{1}{6}\beta_3(1 - z_\alpha^2)]\phi(z_\alpha) \tag{33}$$

$$S_2 = \left\{ c_2 - \frac{1}{2}z_\alpha c_1^2 + \frac{1}{6}(z_\alpha^3 - 3z_\alpha)\beta_3 c_1 - \frac{1}{24}\beta_4(z_\alpha^3 - 3z_\alpha) - \frac{1}{72}\beta_3^2(z_\alpha^5 - 10z_\alpha^3 + 15z_\alpha) \right\} \phi(z_\alpha). \tag{34}$$

**First-order probability matching interval:** To make the tolerance interval first-order probability matching, we need $S_1 \equiv 0$, or equivalently $c_1 = \frac{1}{6}\beta_3(z_\alpha^2 - 1)$. This leads to the choice of

$$\begin{aligned} a &= \frac{1}{6}[(z_{1-\beta}^2 - 1)(1 + 2d_2\mu) + 3z_\alpha(z_\alpha + z_{1-\beta})(d_1 + 2d_2\mu) + \sigma\beta_3(1 - z_\alpha^2)] \\ &= \frac{1}{6}[(z_{1-\beta}^2 - 1)(1 + 2d_2\mu) + (1 + 3z_\alpha z_{1-\beta} + 2z_\alpha^2)(d_1 + 2d_2\mu)] \end{aligned} \tag{35}$$

However, $\mu$ is unknown. We shall replace $\mu$ by $\hat{\mu}$ in $a$ and set $c = 0$. It is straightforward to verify that there is no first-order effect by replacing $\mu$ with $\hat{\mu}$ in (35) and that

$$X + a - b\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2)}. \tag{36}$$

with $a$ and $b$ given in (15) and (16) is first-order probability matching lower bound.

**Second-order probability matching interval:** To make the interval second-order probability matching, we need both $S_1 \equiv 0$ and $S_2 \equiv 0$. We can find the value of $c$ from equations (32 ), (33) and (34). However, $a$ was assumed to be a constant not depending on $X$ in the original derivation of $z_n$. But in (36), $a$ is a function of $X$ and this will have a second-order effect. We thus need to consider tolerance bound of the form (13) with $a$ given in (15) and $b$ given in (16), and redo the analysis to find the optimal $c$. Set $h_1 = \frac{1}{6}(d_2((z_{1-\beta}^2 - 1) + 3(z_\alpha + z_{1-\beta})z_\alpha)) + 2d_2/2(1 - z_\alpha^2)$ and $h_2 = \frac{1}{6}[(z_{1-\beta}^2 - 1) + 3d_1z_\alpha(z_\alpha + z_{1-\beta}) + d_1(1 - z_\alpha^2)]$. Then (13) can be rewritten as

$$L(X) = X[1 + 2h_1n^{-1}] + h_2 - (z_\alpha + z_{1-\beta})\sqrt{n(d_0 + d_1X/n + d_2X^2/n^2) + c}.$$

It follows from (14) and some algebra that $1 - F(L(X)) \geq \beta$ if and only if $X \leq u^*(\mu, \beta)$ where

$$u^*(\mu, \beta) = \frac{f_{n,\mu,\beta} + \frac{1}{2}d_1(z_\alpha + z_{1-\beta})^2 - h_2 + 2h_1n^{-1}f_{n,\mu,\beta} - 2h_1h_2n^{-1} + (z_\alpha + z_{1-\beta})D_n^*}{(1 + 2h_1n^{-1})^2 - d_2(z_\alpha + z_{1-\beta})^2n^{-1}} \tag{37}$$

with

$$\begin{aligned} D_n^* &= \Big\{d_0n + n^{-1}f_{n,\mu,\beta}(nd_1 + d_2f_{n,\mu,\beta}) + \frac{d_1^2}{4}(z_\alpha + z_{1-\beta})^2 - h_2d_1 + c - d_0d_2(z_\alpha + z_{1-\beta})^2 \\ &\quad + 4h_1d_0 + [4d_0h_1^2 + 2f_{n,\mu,\beta}(h_1d_1 - h_2d_2) + (h_2^2d_2 - 2h_1d_1h_2) \\ &\quad + (4h_1 - (z_\alpha + z_{1-\beta})^2d_2)c_n]n^{-1} + 4h_1^2cn^{-2}\Big\}^{\frac{1}{2}}. \end{aligned} \tag{38}$$

It then follows from the expansion (26) for $f_{n,\mu,\beta}$ that

$$\begin{aligned} D_n^* &= (n\sigma^2)^{\frac{1}{2}} + \frac{1}{2}(d_1 + 2d_2\mu)z_{1-\beta} + \left\{ \frac{1}{12}(1 + 2d_2\mu)(d_1 + 2d_2\mu)(z_{1-\beta}^2 - 1) \right. \\ &\quad \left. + z_\alpha(z_\alpha - 2z_{1-\beta})(\frac{1}{8}d_1^2 - \frac{1}{2}d_0d_2) + \frac{1}{2}c - \frac{1}{2}d_1h_2 + 2d_0h_1 - d_2\mu h_2 + d_1h_1\mu \right\} (n\sigma^2)^{-\frac{1}{2}} \\ &\quad + O(n^{-1}). \end{aligned}$$

Note that $\frac{1}{(1+2h_1n^{-1})^2 - d_2(z_\alpha + z_{1-\beta})^2 n^{-1}} = 1 - [4h_1 - (z_\alpha + z_{1-\beta})^2 d_2]n^{-1} + O(n^{-2})$. Set $z_n^* = (u^*(\mu,\beta) - n\mu)/\sqrt{n\sigma^2}$. It then follows from (37), after some algebra,

$$\begin{aligned} z_n^* &= z_\alpha + \frac{1}{6}(d_1 + 2d_2\mu)(z_\alpha^2 - 1)\sigma^{-1}n^{-\frac{1}{2}} + \left\{ -\frac{\sigma^2 d_2}{12}[z_{1-\beta}^2(3z_{1-\beta} + 4z_\alpha) + 2z_\alpha^3 + z_{1-\beta}(-9 + 6z_\alpha^2)] \right. \\ &\quad + \frac{1}{72}[(1 + 16d_2\mu(1 + d_2\mu))z_{1-\beta}^3 - z_{1-\beta}(1 - 36c + 4d_2\mu(10 + 16d_2\mu + 3d_2\mu z_\alpha^2) \\ &\quad \left. + 3d_1(2 + z_\alpha^2)(d_1 + 4d_2\mu)) + 3z_\alpha(12c + 12d_1d_2\mu z_\alpha^2 + d_1^2(2 - 5z_\alpha^2) - 4d_2^2\mu^2(2 + z_\alpha^2))] \right\} (n\sigma^2)^{-1} \\ &\quad + O(n^{-\frac{3}{2}}). \end{aligned}$$

The Edgeworth expansion in Lemma 1 then leads to the choice of $c$ given as

$$\begin{aligned} c &= \frac{1}{36(z_\alpha + z_{1-\beta})} \left\{ (-1 + 18d_0d_2 + 2(-8 + 9d_1)d_2\mu + 2d_2^2\mu^2)z_{1-\beta}^3 \right. \\ &\quad + 24d_2(d_0 + \mu(d_1 + d_2\mu))z_{1-\beta}^2 z_\alpha + z_{1-\beta}[1 + 2d_2\mu(20 + 5d_2\mu + 24d_2\mu z_\alpha^2) + 3d_1^2(2 + z_\alpha^2) \\ &\quad + 18d_0d_2(-3 + 2z_\alpha^2) + 6d_1d_2\mu(-5 + 8z_\alpha^2)] + z_\alpha[d_1^2(7 + 2z_\alpha^2) \\ &\quad \left. + 2d_1d_2\mu(5 + 13z_\alpha^2) + 2d_2(9d_0(-1 + z_\alpha^2) + d_2\mu^2(5 + 13z_\alpha^2))] \right\}. \end{aligned} \tag{39}$$

We shall replace $\mu$ in (39) by $\hat{\mu}$ since $\mu$ is unknown. It can be verified directly that resulting tolerance interval is second-order probability matching. ∎

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