On The Sum Of Symmetric Random Variables

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Let U and V be two symmetric (about zero) random variables with $U + V$ symmetric about C; here C is a constant. It is easy to see that if U and V are mutually independent, or if both U and V satisfy the weak law of large numbers, then $C = 0$. So, intuitively, we would suspect that $C = 0$ in general. However, we show that there exist two random variables U and V symmetric about 0 with $U + V$ symmetric about $C \neq 0$. The example given is closely related to one given by Alejandro D. De Acosta in another context.

Let X and Y be two mutually independent and nondegenerate random variables such that X is symmetric about a and Y is symmetric about b; here a and b are two constants. Kadane and Duncan (1980) observed that if $E(|X|^3) < \infty$ and $E(|Y|^3) < \infty$, then XY is symmetric if and only if $ab = 0$. They conjectured that the statement would still be true even without the moment condition. Chen and Slud (1981) proved that the statement does hold if $E(|X|) < \infty$ or $E(|Y|) < \infty$ and they observed that the *Kadane-Duncan Conjecture* will follow from their main theorem if the following statement holds.

Statement. Let U and V be two symmetric (about zero) random variables with $U + V$ symmetric about C; here C is a constant. Then $C = 0$.

Intuitively, we would suspect that this statement holds since this statement does hold if U and V are mutually independent, or if both U and V satisfy the weak law of large numbers. However, this statement is false in general and the following is a counterexample. This counterexample also reveals the difficulty to establish the *Kadane-Duncan Conjecture*.

Example. Let U and V be two random variables such that the joint characteristic function of U and V is

$$\begin{aligned}\phi(s, t) &= E(\exp\{isU + itV\})\\ &= \exp\Big\{-\int_0^{2\pi} |s\cos\theta + t\sin\theta| d\theta\\ &\quad - i\int_0^{2\pi} (s\cos\theta + t\sin\theta)\\ &\quad \times (\log|s\cos\theta + t\sin\theta|) g(\theta) d\theta\Big\};\end{aligned}$$

here $g(\theta)$ is a real-valued function of θ defined on the interval $[0, 2\pi]$ such that $|g(\theta)| \leq 2/\pi$ for all $0 \leq \theta \leq 2\pi$ (see Feller 1966, p. 542). Then $\phi(s, 0) = E(\exp\{isU\})$ is the characteristic function of U; $\phi(0, t) = E(\exp\{itV\})$ is the characteristic function of V; and $\phi(s, s) = E(\exp\{is(U + V)\})$ is the characteristic function of $U + V$. Now if g is so chosen that g is orthogonal (over the interval $[0, 2\pi]$) to the subspace β spanned by the set $\{\cos\theta, \sin\theta, \cos\theta\log|\cos\theta|, \sin\theta\log|\sin\theta|\}$, but g is not orthogonal to $(\cos\theta + \sin\theta)\log|\cos\theta + \sin\theta|$ (it is possible since the function $(\cos\theta + \sin\theta)\log|\cos\theta + \sin\theta|$ is not in the subspace β). Then

$$\phi(s, 0) = E(\exp\{isU\}) = \exp\{-4|s|\};$$

$$\phi(0, t) = E(\exp\{itV\}) = \exp\{-4|t|\};$$

and

$$\begin{aligned}\phi(s, s) = E(\exp\{is(U + V)\}) &= \exp\Big\{-4\sqrt{2}|s|\\ &\quad - is\int_0^{2\pi} (\cos\theta + \sin\theta)\log|\cos\theta + \sin\theta| g(\theta) d\theta\Big\}.\end{aligned}$$

since g is not orthogonal to the function

$$(\cos\theta + \sin\theta) \log|\cos\theta + \sin\theta|,$$

$$\int_0^{2\pi} (\cos\theta + \sin\theta)\log|\cos\theta + \sin\theta| g(\theta) d\theta = C \neq 0$$

and

$$\phi(s, s) = \exp\{-4\sqrt{2}|s| - isC\}.$$

Therefore, U is symmetric about 0, V is symmetric about 0, but $U + V$ is symmetric about $C \neq 0$. In fact, we have just shown that there exist two identically distributed, symmetric (about 0) Cauchy random variables U and V such that $U + V$ is a Cauchy random variable and symmetric about $C \neq 0$.

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