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Strong Decomposition of Random Variables

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Abstract A random variable X is called strongly decomposable into (strong) components Y, Z, if $X = Y + Z$ where $Y = \phi(X)$, $Z = X - \phi(X)$ are independent nondegenerate random variables and ϕ is a Borel function. Examples of decomposable and indecomposable random variables are given. It is proved that at least one of the strong components Y and Z of any random variable X is singular. A necessary and sufficient condition is given for a discrete random variable X to be strongly decomposable. Phenomena arising when ϕ is not Borel are discussed. The Fisher information (on a location parameter) in a strongly decomposable X is necessarily infinite.

Keywords Absolute continuity · Component · Fisher information · Singularity

1 Introduction

The classical theory of decomposition of random variables known also as the arithmetic of probability distributions deals with the representation of a random variable

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X as the sum of two independent random variables Y and Z,

$$X = Y + Z. \tag{1}$$

If $f(t)$, $f_1(t)$, $f_2(t)$ are the characteristic functions of X, Y, Z, respectively, then (1) is equivalent to

$$f(t) = f_1(t) f_2(t), \quad t \in \mathbb{R}. \tag{2}$$

A seminal result due to Cramér [3] that states that the components Y, Z of a Gaussian random variable X are necessarily Gaussian laid a foundation of a research area connected to the theory of functions, probability and statistics. The monographs [9, 10] are standard references. Very well written expository papers [2, 11, 12] review relatively recent results in the arithmetic of probability distributions.

In this paper we introduce and study a stronger than (1) form of decomposition when Y and Z are not only independent but also functions of X. In Sect. 2 examples of strongly decomposable and indecomposable random variables are given. In Sect. 3 it is proved that for any random variable X at least one of its strong components Y, Z is singular and a necessary and sufficient condition is obtained for a discrete random variable to be decomposable. An interesting statistical property of any strongly decomposable X is proved in Sect. 4; namely, the Fisher information on a parameter θ contained in an observation of $\theta + X$ is necessarily infinite.

2 Definition and Examples

Let U be a random variable or a random vector. Then $P_U(B) = P(U \in B)$ denotes *the distribution* of U. Recall that a Borel set B is a *support* of U if $P(U \in B) = 1$. Let S_U denote the *closed support* of U, that is, the set of all x satisfying $P(U \in G) > 0$ for all open sets G containing x, and let $D_U := \{x \mid P(U = x) > 0\}$ denote *the discrete support* of U. Note that $D_U \subseteq S_U$ and recall that S_U is the smallest closed set supporting U and that U is discrete if and only if D_U is a support of U.

Definition A random variable X is *strongly decomposable* with (strong) components Y, Z and *decomposition function* ϕ if

(i) $X = Y + Z$
(ii) Y, Z are independent nondegenerate random variables
(iii) ϕ is a Borel function and $Y = \phi(X),\ Z = X - \phi(X)$

The conditions (i)–(iii) are very restrictive and make strong decomposition a rare phenomenon compared to decomposition in sense of (1) referred in what follows as *weak*. Observe that the map $x \curvearrowright (\phi(x), x - \phi(x))$ is a bijection of $\mathbf{R}$ onto $\Phi = \{(\phi(x), x - \phi(x)) \mid x \in \mathbf{R}\}$ with inverse $(u, v) \curvearrowright u + v$ for $(u, v) \in \Phi$. Hence, we see that Φ is an injective curve in the plane and by (ii), we have $D_{(Y,Z)} = D_Y \times D_Z$ and $S_{(Y,Z)} = S_Y \times S_Z$. So by (i), we see that the injective curve Φ contains the "rectangle" $D_Y \times D_Z$ and that Φ contains the closed "rectangle" $S_Y \times S_Z$ a.s. (i.e. $P((Y, Z) \in (S_Y \times S_Z) \setminus \Phi) = 0$). This observation shows that Φ is Peano-like curve. Observe that if $\psi(y) := y - \sqrt{2}\phi(\sqrt{2}y)$, then the curve Φ is the graph of ψ rotated 45°

which imposes severe restrictions on the components Y, Z and on the decomposition function ϕ. In Sect. 3, we shall see that the independence condition (ii) can be relaxed to weak independence (see Theorem 1), and that it is the Borel measurability of the decomposition function ϕ which is the real restriction (see Theorem 2).

2.1 Gaussian, Poisson and Binomial Random Variables are Indecomposable

Suppose that a Gaussian X has strong components Y, Z. Due to the Cramér theorem, Y, Z are also Gaussian. Then Y and $X = Y + Z$ as linear functions of a bivariate Gaussian vector (Y, Z), have a bivariate Gaussian distribution and, by virtue of a well known property of the latter,

$$E(Y|X) = a_0 + a_1 X \quad \text{for some } a_0, a_1.$$

Since $Y = Y(X)$, we have $Y = a_0 + a_1 X$ and $Z = X - Y = b_0 + b_1 X$ for some b_0, b_1. However, $a_0 + a_1 X$ and $b_0 + b_1 X$ are independent if and only if $a_1 b_1 = 0$ implying that one of Y, Z is degenerate.

Using a result by Raikov [13] (see also a monograph [9, Chap. 5]) claiming that weak components Y, Z of a Poisson random variable X are necessarily Poisson random variables (possibly, shifted) and arguing as in case of a Gaussian X, it is easy to show that a Poisson X is indecomposable.

Similar arguments prove that a binomial random variable X is indecomposable.

2.2 Uniform and Exponential Random Variables are Decomposable

Let X be a random variable uniformly distributed on (0, 1). It is well known (see, for example, [5]) that in the dyadic expansion of X,

$$X = \sum_{1}^{\infty} (X_n / 2^n),$$

$X_1, X_2, \ldots$ are independent identically distributed random variables with

$$P(X_i = 0) = P(X_i = 1) = 1/2.$$

Let $A \bigcup A' = \mathbb{N}$ be a partition. On setting

$$Y = \sum_{n \in A} (X_n / 2^n), \qquad Z = \sum_{n \in A'} (X_n / 2^n)$$

one gets a strong decomposition

$$X = Y + Z.$$

If A is finite, Y is discrete and Z is absolutely continuous. If both A, A' are infinite, Y and Z are continuous singular random variables. Moreover, their distributions are mutually singular.

Lewis [8] showed that in any weak decomposition of a uniformly distributed X at least one of the components is not absolutely continuous.

If $X_1, X_2, \ldots$ are independent random variables taking values 0 and 1, then the series

$$X = \sum_1^\infty (X_n/2^n)$$

(converging with probability 1) is strongly decomposable.

Let now X be an exponential random variable with a parameter λ. Denote by $Y = [X]$ the integer part of X and by $Z = \{X\}$ the fractional part of X. Then $X = Y + Z$. Furthermore, for $0 < z < 1$ and $n = 0, 1, \ldots$

$$P(Z < z|Y = n) = (1 - e^{-\lambda z})/(1 - e^{-z}), \quad 0 \le z < 1$$

does not depend on n so that Y, Z are strong components of X. Note that only one of the components is absolutely continuous.

It is worth noticing that there are many nonexponential (positive) random variables X such that X/t is strongly decomposable for any $t > 0$. Indeed, let Y be a random variable supported by the set $\mathbb{N}$ and Z an independent of Y random variable with distribution concentrated on $[0, 1)$. Then $X = Y + Z$ has the property that X/t is strongly decomposable for any $t > 0$.

3 Singularity of Strong Components

In all the examples above, a random variable X either had no strong components at all (normal X) or one of the components (or both) was either singular (uniform X) or discrete (exponential X, uniform X). It turns out that these examples are manifestations of a general fact: at least one strong component of an arbitrary random variable X is singular, and we may actually replace the independence condition (ii) by *weak independence*.

Recall that a Borel measure μ on $\mathbf{R}^k$ is called *diffuse* if $\mu(\{x\}) = 0$ for all $x \in \mathbf{R}^k$ or equivalently, if $\mu(F) = 0$ for every countable set $F \subseteq \mathbf{R}^k$; a measure μ_1 is *absolutely continuous* with respect to μ_2, $\mu_1 \ll \mu_2$, if $\mu_2(N) = 0$ implies $\mu_1(N) = 0$; μ_1 and μ_2 are *singular*, $\mu_1 \perp \mu_2$, if there exists a Borel set $B \subseteq \mathbf{R}^k$ satisfying $\mu_1(B) = 0$ and $\mu_2(\mathbf{R}^k \setminus B) = 0$.

We say that random variables X and Y are *weakly independent* if $P_X \otimes P_Y \ll P_{(X,Y)}$ (for independent X, Y, $P_{(X,Y)} = P_X \otimes P_Y$).

Note in passing that in a recent paper [6], X and Y were called *quasi-independent* if $P_X(A)P_Y(B) > 0$ implies $P_{(X,Y)}(A \times B) > 0$. Quasi-independence is a weaker property than weak independence.

Theorem 1 *Let $\phi : \mathbf{R} \to \mathbf{R}$ be a Borel function and $X = Y + Z$ the decomposition where $Y = \phi(X)$ and $Z = X - \phi(X)$. Let μ_1 and μ_2 be σ-finite Borel measures on $\mathbf{R}$ and suppose that one of the measures μ_1 and μ_2 is diffuse. Set $\mu(B) = \mu_1 \otimes \mu_2(T^{-1}(B))$, the image measure of $\mu_1 \otimes \mu_2$ under the bijective linear map $T(x, y) = (x - y, y)$. Then*

(i) $P_{(Y+Z,Y)} \perp \mu_1 \otimes \mu_2$ *and* $P_{(Y,Z)} \perp \mu$.

(ii) *$P_{(Y+Z,Z)} \perp \lambda_2$ and $P_{(Y,Z)} \perp \lambda_2$.*
(iii) *If $\phi^{-1}(u)$ is at most countable for P_Y-almost all $u \in \mathbf{R}$, then $P_{(Y,Z)} \perp \mu_1 \otimes \mu_2$.*
(iv) *If Y and Z are weakly independent and $\mu_1 \otimes \mu_2 \ll \mu$, then either $P_Y \perp \mu_1$ or $P_Z \perp \mu_2$.*
(v) *If Y and Z are weakly independent, then either $P_Y \perp \lambda_1$ or $P_Z \perp \lambda_1$.*

Remark (a) The set $C := \{u \in \mathbf{R} \mid \phi^{-1}(u)$is at most countable$\}$ is not necessarily a Borel set. However, since ϕ is Borel measurable, C is co-analytic and consequently, measurable with respect to any Borel measure on $\mathbf{R}$ (σ-finite or not); see [7].

(b) In particular, a Gaussian X has only Gaussian components and, thus, is strongly indecomposable (as shown by different arguments in Sect. 2.1). However, the next theorem shows the a Gaussian X admits a representation of the form $X = Y + Z$ with Y and Z independent and Gaussian and with $Y = \phi(X)$ and $Z = X - \phi(X) = \phi^{-1}(Y) - Y$ for some (non-measurable) bijection ϕ of $\mathbf{R}$ onto $\mathbf{R}$. In particular, we see that it is the Borel measurability of the decomposition function which makes strong components different from weak components.

The authors are unaware of general conditions for X to have only absolutely continuous weak components.

It is of some interest to find out if there are random variables having only strong components. In particular, does there exist a weak decomposition of a uniform random variable which is not strong?

Proof Let $G := \{(x, \phi(x)) \mid x \in \mathbf{R}\}$ denote the graph of ϕ. Since μ_2 is σ-finite, we see that $F := \{x \mid \mu_2(\{\phi(x)\}) > 0\}$ is at most countable. Since ϕ is Borel measurable, we have that G is a Borel set and so by Fubini's theorem we have

$$\mu_1 \otimes \mu_2(G) = \int_{\mathbf{R}} \mu_2(\{\phi(x)\})\mu_1(dx) = \int_F \mu_2(\{\phi(x)\})\mu_1(dx).$$

If μ_2 is diffuse we have $F = \emptyset$ and if μ_1 is diffuse, we have $\mu_1(F) = 0$ by countability of F. Hence, in either case $(\mu_1 \otimes \mu_2)(G) = 0$ and since $P((Y + Z, Y) \in G) = 1$, we have $P_{(Y+Z,Y)} \perp \mu_1 \otimes \mu_2$. Since $T(x, y) = (x - y, y)$ is a bijective linear map with inverse $T^{-1} = (x + y, y)$, we have $(Y, Z) = T(Y + Z, Z)$ and thus $P_{(Y,Z)} \perp \mu$. Hence, part (i) is proved and since T has determinant 1, (ii) follows from (i) with $\mu_1 = \mu_2 = \lambda_1$.

Suppose that $\phi^{-1}(u)$ is at most countable for P_Y-a.a. $u \in \mathbf{R}$. Then there exists a Borel set $S \subseteq \mathbf{R}$ such that $P(Y \in S) = 1$ and $\phi^{-1}(u)$ is at most countable forall $u \in S$. Since μ_2 is σ-finite and the sets $\phi^{-1}(u)$ are for different u mutually disjoint, we see that $F := \{u \mid \mu_2(\phi^{-1}(u)) > 0\}$ is at most countable. Set $\phi := \{(\phi(x), x - \phi(x)) \mid x \in \mathbf{R}\}$. Since ϕ is a Borel function and $\phi = \{(u, v) \mid u = \phi(u + v)\}$, we see that ϕ is a Borel set and since $\phi^u \subseteq \phi^{-1}(u)$ for all $u \in \mathbf{R}$, ϕ^u is at most countable for all $u \in S$. So by Fubini's theorem we have

$$\begin{aligned}(\mu_1 \otimes \mu_2)(\Phi \cap (S \times \mathbf{R})) &= \int_S \mu_2(\Phi^u)\mu_1(du) \le \int_S \mu_2(\phi^{-1}(u))\mu_1(du) \\ &= \int_{F \cap S} \mu_2(\phi^{-1}(u))\mu_1(du).\end{aligned}$$

Since F is at most countable, we see that $\mu_1 \otimes \mu_2(\phi \cap (S \otimes \mathbf{R})) = 0$ if μ_1 is diffuse and since $\phi^{-1}(u)$ is at most countable for all $u \in F$, we see that $\mu_1 \otimes \mu_2(\phi \cap (S \otimes \mathbf{R})) = 0$ if μ_2 is diffuse. Hence, in either case, we have $(\mu_1 \otimes \mu_2)(\phi \cap (S \otimes \mathbf{R})) = 0$ and since $(Y, Z) \in \phi$ and $P(Y \in S) = 1$, $P((Y, Z) \in \phi \cap (S \otimes \mathbf{R})) = 1$. Hence, $P_{(Y,Z)} \perp \mu_1 \otimes \mu_2$ which proves (iii).

Suppose that $P_Y \otimes P_Z \ll (Y, Z)$ and $\mu_1 \otimes \mu_2 \ll \mu$. Since the distribution of (Y, Z) is singular with respect to μ, there exists a Borel set $W \subseteq \mathbf{R}^2$ such that $\mu(W) = P_{(Y,Z)}(\mathbf{R}^2 \setminus W) = 0$. Hence, $(\mu_1 \otimes \mu_2)(W) = (P_Y \otimes P_Z)(\mathbf{R}^2 \setminus W) = 0$ and hence by Fubini's theorem, $\mu_1(\mathbf{R} \setminus A) = P_Y(\mathbf{R} \setminus B) = 0$ where $A = \{u \mid \mu_2(W_u) = 0\}$ and $B = \{u \mid P_Z(W_u) = 1\}$. If $A \cap B \neq \emptyset$, there exists $u \in \mathbf{R}$ such that $\mu_2(W_u) = 0$ and $P_Z(W_u) = 1$. If $A \cap B = \emptyset$, we have $B \subseteq \mathbf{R} \setminus A$ and thus $\mu_1(B) = 0$ and $P_Y(B) = 1$. Hence, we see that either $P_Z \perp \mu_2$ or $P_Y \perp \mu_1$ which proves (iv), and (v) follows from (iv) when $\mu_1 = \mu_2 = \lambda_1$. □

Theorem 2 *Let P_1 and P_2 be Borel probability measures on $\mathbf{R}$ and $Q(B) := (P_1 \otimes P_2)(S^{-1}(B))$ denote the image probability measure of $P_1 \otimes P_2$ under the linear bijection $S(x, y) = (x + y, x)$. Suppose that Q is absolutely continuous with respect to the product of two diffuse σ-finite Borel measures on $\mathbf{R}$. Then there exist a probability space $(\Omega, \mathcal{F}, P)$, random variables Y, Z and X and a bijection $\phi : \mathbf{R} \to \mathbf{R}$ such that*

(i) *Y and Z are independent with distributions P_1 and P_2.*
(ii) *$X(\omega) = Y(\omega) + Z(\omega)$ and $Y(\omega) = \phi(X(\omega))$ $\forall \omega \in \Omega$.*
(iii) *$Z(\omega) = \phi^{-1}(Y(\omega)) - Y(\omega)$ $\forall \omega \in \Omega$.*

Remark (a) Suppose that (P_1, P_2) satisfies the conditions of Theorem 2. Then it follows easily that P_1 and P_2 are diffuse and by (i), Y and Z are independent, continuous random variables with distributions P_1 and P_2. By (iii), Z is a function of Y which at the first glance seems to contradict the independence and continuity of Y and Z. In fact, it does not and just means that ϕ is non-measurable (as a matter of fact, the function ϕ is extremely non-measurable and owes its existence to the axiom of choice).

(b) Suppose that P_1 and P_2 are absolutely continuous with respect to λ_2. Then so is Q and the condition of Theorem 2 is satisfied. Hence, we see that every weak decomposition of X into absolutely continuous components can be realized as a strong decomposition on some probability space, provided that we drop the assumption of Borel measurability of the decomposition function.

Proof Let μ_1 and μ_2 be diffuse σ-finite Borel measures on $\mathbf{R}$ such that Q is absolute continuous wrt. the product measure $\mu = \mu_1 \otimes \mu_2$.

If $B \subseteq \mathbf{R}^2$ and $u \in \mathbf{R}$, we set $B^u := \{y \in \mathbf{R} \mid (u, y) \in B\}$, the u-section of B, and denote by B^* the set of all $u \in \mathbf{R}$ such that B^u is uncountable. Denote by $\mathcal{S}$ the collection of all Borel sets $B \in \mathcal{B}(\mathbf{R}^2)$ for which B^* is uncountable. If $B \in \mathcal{B}(\mathbf{R}^2)$, we have (see [7, page 496])

(a): $B^u \in \mathcal{B}(\mathbf{R})$ and B^* is analytic and $\operatorname{card} B^u = \mathbf{c}$ for all $u \in B^*$
(b): $\operatorname{card} \mathcal{S} = \mathbf{c}$ and $\operatorname{card} S^* = \mathbf{c}$ for all $S \in \mathcal{S}$

where $\mathbf{c} := 2^{\aleph_0}$ denotes the cardinality of the continuum. Let $B \in \mathcal{B}(\mathbf{R}^2) \setminus \mathcal{S}$ be given. Then B^* is at most countable and B^u is at most countable for all $u \in \mathbf{R} \setminus B^*$. Since μ_1 and μ_2 are continuous, we have $\mu_1(B^*) = 0$ and $\mu_2(B^u) = 0$ for all $u \in \mathbf{R} \setminus B^*$. Hence, due to Fubini's theorem, $\mu(B) = 0$ and since Q is absolutely continuous with respect to μ,

(c): $Q(B) = 0 \ \forall B \in \mathcal{B}(\mathbf{R}^2) \setminus \mathcal{S}$

By the axiom of choice there exists a well-ordering $\preceq$ on $\mathbf{R}$ such that $\operatorname{card}\{x \in \mathbf{R} \mid x \preceq a\} < \mathbf{c}$ for all $a \in \mathbf{R}$. If $A \subseteq \mathbf{R}$ is a non-empty set, let $\operatorname{Min} A$ denote a minimal element in A with respect to the well-ordering $\preceq$.

Let $\mathcal{A}$ denote the set of all uncountable Borel sets and let Lim denote the set of all limit ordinals $<\mathbf{c}$. Since $\operatorname{card}\mathcal{A} = \operatorname{card}(\mathrm{Lim})$, one may "enumerate" $\mathcal{A}$ by ordinals in Lim, say $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathrm{Lim}\}$. Let $A \in \mathcal{A}$ be given. Since $(A \times \mathbf{R})^u = \mathbf{R}$ if $u \in A$ and $(A \times \mathbf{R})^u = \emptyset$ if $u \notin A$, we have $(A \times \mathbf{R})^* = A$ and $A \times \mathbf{R} \in \mathcal{S}$ for all $A \in \mathcal{A}$. Hence, for $\mathcal{S}_0 := \{A \times \mathbf{R} \mid A \in \mathcal{A}\}$, we have $\mathcal{S}_0 \subseteq \mathcal{S}$ and since $\operatorname{card}(\mathcal{S} \setminus \mathcal{S}_0) = \mathbf{c}$, one may "enumerate" $\mathcal{S}$ by ordinals $<\mathbf{c}$, say $\mathcal{S} = \{S_\alpha \mid \alpha < \mathbf{c}\}$ such that $S_\alpha = A_\alpha \times \mathbf{R}$ for all $\alpha \in \mathrm{Lim}$.

Let $\alpha < \mathbf{c}$ be a given ordinal and let $x \in S_\alpha^*$ be a given number. By (a) and (b), we have $\operatorname{card} S_\alpha^* = \mathbf{c} = \operatorname{card} S_\alpha^x$ and since $\operatorname{card}\{\beta \mid \beta < \alpha\} < \mathbf{c}$, we see the sets $S_\alpha^* \setminus \{u_\beta \mid \beta < \alpha\}$ and $S_\alpha^x \setminus \{v_\beta \mid \beta < \alpha\}$ are non-empty for all families $(u_\beta)_{\beta<\alpha} \subseteq \mathbf{R}$ and $(v_\beta)_{\beta<\alpha} \subseteq \mathbf{R}$. Hence, we may define the vectors $\{(x_\alpha, y_\alpha) \mid \alpha < \mathbf{c}\}$ uniquely, by transfinite induction, as follows

(d): $x_0 = \operatorname{Min} S_0^*$ and $y_0 = \operatorname{Min} S_0^{X_0}$
(e): $x_\alpha = \operatorname{Min} S_\alpha^* \setminus \{x_\beta \mid \beta < \alpha\}$ and $y_\alpha = \operatorname{Min} S_\alpha^{x_\alpha} \setminus \{y_\beta \mid \beta < \alpha\} \ \forall 0 < \alpha < \mathbf{c}$

Since $x_0 \in S_0^*$ and $y_0 \in S_0^{x_0}$, we have $(x_0, y_0) \in S_0$. Let $0 < \alpha < \mathbf{c}$ be given. Since $x_\alpha \in S_\alpha^* \setminus \{x_\beta \mid \beta < \alpha\}$ and $y_\alpha \in S_\alpha^{x_\alpha} \setminus \{y_\beta \mid \beta < \alpha\}$, we have $(x_\alpha, y_\alpha) \in S_\alpha$, $x_\alpha \neq x_\beta$ for all $\beta < \alpha$ and $y_\alpha \neq y_\beta$ for all $\beta < \alpha$. Hence, we have

(f): $(x_\alpha, y_\alpha) \in S_\alpha \ \forall \alpha < \mathbf{c}$
(g): $x_\alpha \neq x_\beta$ and $y_\alpha \neq y_\beta \ \forall \alpha \neq \beta < \mathbf{c}$

Moreover,

(h): $\mathbf{R} = \{x_\alpha \mid \alpha < \mathbf{c}\} = \{y_\alpha \mid \alpha < \mathbf{c}\}$

Proof of the first part of (h) Suppose that $\mathbf{R} \neq \{x_\alpha \mid \alpha < \mathbf{c}\}$. Then there exists a number $u \in \mathbf{R}$ satisfying $u \neq x_\beta$ for all $\beta < \mathbf{c}$. Let us define $\Upsilon := \{\alpha < \mathbf{c} \mid u \in S_\alpha^*\}$. Let $\gamma \in \mathrm{Lim}$ be given and define $B_\gamma := \mathbf{R} \times A_\gamma$. Since $B^x = A_\gamma$ for all $x \in \mathbf{R}$, $B_\gamma \in \mathcal{S}$ and all $u \in \mathbf{R} = B_\gamma^*$. Since $\operatorname{card}\mathrm{Lim} = \mathbf{c}$, $\operatorname{card}\Upsilon = \mathbf{c}$. For a given $\alpha \in \Upsilon$, due to $u \in S_\alpha^* \setminus \{x_\beta \mid \beta < \mathbf{c}\}$, we have $x_\alpha \preceq u$, that is, $\Upsilon \subseteq \{\alpha \mid x_\alpha \preceq u\}$. By (g), $\alpha \curvearrowright x_\alpha$ is an injective and since $\operatorname{card}\{x \in \mathbf{R} \mid x \preceq u\} < \mathbf{c}$, we see that $\mathbf{c} = \operatorname{card}\Upsilon < \mathbf{c}$ which is impossible. Thus, $\mathbf{R} = \{x_\alpha \mid \alpha < \alpha\}$.

Proof of the second part of (h) Suppose that $\mathbf{R} \neq \{y_\alpha \mid \alpha < \mathbf{c}\}$. Then there exists a number $v \in \mathbf{R}$ such that $v \neq y_\beta$ for all $\beta < \mathbf{c}$. Define $\Theta := \{\alpha < \mathbf{c} \mid v \in S_\alpha^{x_\alpha}\}$ and let $\alpha \in \Theta$ be given. Since $v \in S_\alpha^{x_\alpha} \setminus \{y_\beta \mid \beta < \mathbf{c}\}$, $y_\alpha \preceq v$ and thus $\{y_\alpha \mid \alpha \in \Theta\} \subseteq \{x \in \mathbf{R} \mid y_\alpha \preceq v\}$. By (g), $\alpha \curvearrowright x_\alpha$ is injective and since $\operatorname{card}\{x \in \mathbf{R} \mid x \preceq v\} < \mathbf{c}$, we see that card $\Theta < \mathbf{c}$. Let now $\gamma \in \mathrm{Lim}$ be given. Then $S_\gamma = A_\gamma \times \mathbf{R}$ and since $S_\gamma^x = \mathbf{R}$ for all $x \in A_\gamma$ and $S_\gamma^x = \emptyset$ for all $x \in \mathbf{R} \setminus A_\gamma$, $S_\gamma^* = A_\gamma$. Since $x_\gamma \in S_\gamma^* = A_\gamma$, we have $v \in \mathbf{R} = S_\gamma^{x_\gamma}$ so that $\gamma \in \Theta$, that is, $\mathrm{Lim} \subseteq \Theta$. Since $\mathrm{Lim} = \mathbf{c}$, we have $\mathbf{c} = \mathrm{card}\,\Theta < \mathbf{c}$ which is impossible. Thus, we see that $\mathbf{R} = \{y_\alpha \mid \alpha < \mathbf{c}\}$. By (g) and (h), we see that $\alpha \curvearrowright x_\alpha$ and $\alpha \curvearrowright y_\alpha$ are bijections of $\{\alpha \mid \alpha < \mathbf{c}\}$ onto $\mathbf{R}$. Hence, $\phi(x_\alpha) := y_\alpha$ is a well-defined bijection of $\mathbf{R}$ onto $\mathbf{R}$. Let $\Phi = \{(x, \phi(x)) \mid x \in \mathbf{R}\}$ denote the graph of ϕ and set

$$\Omega = \{(\phi(x).x - \phi(x)) \mid x \in \mathbf{R}\} \subseteq \mathbf{R}^2 \quad \text{and} \quad \mathcal{F} = \mathcal{B}(\Omega).$$

Then $(\Omega, \mathcal{F})$ is a measurable space. Let $B \in \mathcal{B}(\mathbf{R}^2)$ be a given set with $B \supseteq \omega$. Since $\omega = T(\phi) \subseteq B$, $\phi \subseteq T^{-1}(B)$ and $\phi \cap T^{-1}(B^c) = \emptyset$ where $B^c := \mathbf{R}^2 \setminus B$ denotes the complement of B. By (f) we have $(x_\alpha, \phi(x_\alpha)) = (x_\alpha, y_\alpha) \in S_\alpha$. Hence, $\phi \cap S \neq \emptyset$ for all $S \in \mathcal{S}$ and since $T^{-1}(B^c) \cap \phi = \emptyset$, we see that $T^{-1}(B^c) \in \mathcal{B}(\mathbf{R}^2) \setminus \mathcal{S}$. Thus, due to (c)

$$0 = Q(T^{-1}(B^c)) = (P_1 \otimes P_2)(S^{-1}(T^{-1}(B^c))) = (P_1 \otimes P_2)(B^c)$$

so that $(P_1 \otimes P_2)(B) = 1$ for all $B \in \mathcal{B}(\mathbf{R}^2)$ with $B \supseteq \omega$. But then $(P_1 \otimes P_2)^*(\omega) = 1$ where $(P_1 \otimes P_2)^*$ denotes the outer $(P_1 \otimes P_2)$-measure and since $\mathcal{F} = \mathcal{B}(\omega) = \{B \cap \omega \mid B \in \mathcal{B}(\mathbf{R}^2)\}$, we see that

$$P(F) := (P_1 \otimes P_2)^*(F) \quad \forall F \in \mathcal{F}$$

defines a probability measure on $(\Omega, \mathcal{F})$ satisfying $(P_1 \otimes P_2)(B) = P(B \cap \omega)$ for all $B \in \mathcal{B}(\mathbf{R}^2)$. Hence, if to define

$$Y(\omega) := \omega_1, \qquad Z(\omega) := \omega_2 X(\omega) := \omega_1 + \omega_2 \quad \forall \omega = (\omega_1, \omega_2) \in \Omega$$

then Y, Z and X will be random variables on the probability space $(\Omega, \mathcal{F}, P)$ with

$$P(Y \in A, Z \in B) = P(\Omega \cap (A \times B)) = P_1 \otimes P_2(A \times B) = P_1(A)P_2(B)$$

for all $A, B \in \mathcal{B}(\mathbf{R})$. In particular, we see that claim (i) in Theorem 2 and the first equality in (ii) hold.

Let now $\omega = (\omega_1, \omega_2) \in \Omega$ be given. Then there exists $x \in \mathbf{R}$ such that $\omega = (\phi(x), x - \phi(x))$. Hence, $X(\omega) = x$ and $Y(\omega) = \phi(x) = \phi(X(\omega))$ and since ϕ is a bijection of $\mathbf{R}$ onto itself, we have $x = \phi^{-1}(Y(\omega))$ and $Z(\omega) = x - \phi(x) = \phi^{-1}(Y(\omega)) - Y(\omega)$ which proves claim (iii) and the second equality in (ii). □

Theorem 3 *Let X be a discrete random variable with discrete support D_X and probability mass function $p(u) = P(X = u)$ for $u \in D_X$. Then X is strongly decomposable if and only if there exist sets $A, B \subseteq \mathbf{R}$ with at least two elements in each and functions $q, r : \mathbf{R} \to \mathbf{R}$ such that*

(i) *$P(X = a + b) > 0$ for all $(a, b) \in A \times B$.*
(ii) *For every $u \in D_X$ there exists a unique solution to the equation*

$$x = a + b \quad \textit{and} \quad (a, b) \in A \times B \tag{3}$$

and the unique solution (a, b) satisfies $p(u) = q(a)r(b)$.

Proof Suppose that X is strongly decomposable and let $Y=\phi(X)$ and $Z=X-\phi(X)$ be strong components of X. Since $\gamma(x)=(\phi(x),x-\phi(x))$ is a bijection of $\mathbf{R}$ onto $\Phi=\{(\phi(x),x-\phi(x))\mid x\in\mathbf{R}\}$ with inverse $\gamma^{-1}(x,y)=x+y$, we have $D_{(Y,Z)}=\gamma(D_X)\subseteq\phi$ and $D_X=\gamma^{-1}(D_{(Y,Z)})$ and since X is discrete and Y and Z are independent, Y and Z are discrete random variables satisfying $D_{(Y,Z)}=D_Y\times D_Z\subseteq\phi$ and $D_X=D_Y+D_Z$. Hence, we see that (i) and (ii) holds with $A=D_Y$, $B=D_Z$, $q(x)=P(Y=x)$ and $r(x)=P(Z=x)$.

Suppose that there exist sets $A,B\subseteq\mathbf{R}$ with at least two elements in each and functions $q,r:\mathbf{R}\to\mathbf{R}$ satisfying (i) and (ii). Let $u\in D_X$ and $\gamma(u)=(\phi(u),u-\phi(u))$ be the unique solution of (3). Since D_X is at most countable, ϕ can be extended to a Borel function on $\mathbf{R}$ so that $Y=\phi(X)$ and $Z=X-\phi(X)$ become random variables such that $X=Y+Z$.

Let now $(a,b)\in A\times B$ be given and set $u=a+b$. By (i), $u\in D_X$ and we see that (a,b) is the unique solution of (3). Hence, by (ii) $P((Y,Z)=(a,b))=P(X=a+b)=q(a)r(b)>0$ for all $(a,b)\in A\times B$. In particular, $A\times B\subseteq D_{(Y,Z)}$ and since $D_{(Y,Z)}=\gamma(D_X)\subseteq A\times B$, we have $D_{(Y,Z)}=A\times B$ whence Y and Z are independent random variables with $D_Y=A$ and $D_Z=B$. Since A and B each has at least two elements, we see that Y and Z are non-degenerate so that X is strongly decomposable with components Y and Z. □

Example 1 Let X be a geometric random variable with parameter p, $0<p<1$,

$$P(X=k)=(1-p)^k p,\quad k=0,1,2,\ldots.$$

With $A=\{0,1,2,\ldots\}$ let us set

$$B=\{0,2,4,\ldots\},\qquad C=\{0,1\}.$$

Every $k\in A$ is uniquely represented as

$$k=l+m,\quad l\in B,\ m\in C. \tag{4}$$

Set

$$q(l)=[(1-p)^l+(1-p)^{l+1}]p,\quad l\in B, \tag{5}$$

$$r(0)=1/(2-p),\qquad r(1)=(1-p)/(2-p). \tag{6}$$

Relations (4–6) imply (ii) in Theorem 3 proving that a geometric random variable is strongly decomposable.

4 The Fisher Information in a Strongly Decomposable Random Variable

Remind that the Fisher information on a (location) parameter θ contained in an observation of $\theta+\xi$ (shortly the Fisher information in ξ) where ξ is a random variable with distribution function F is defined as

$$I_\xi=\sup\frac{[\int\psi'(x)dF(x)]^2}{\int[\psi(x)]^2dF(x)},$$

where the supremum is taken over all smooth ψ with a compact support. As proved by Huber (see, e.g., [4, Chap. 4]), $I_\xi < \infty$ if and only if F is absolutely continuous and $F' = f$ is such that $\int (f'/f)^2 f dx < \infty$ in which case $I_\xi = \int (f'/f)^2 f dx$.

In the examples of strongly decomposable X in Sect. 2.2, the density function of (uniform and exponential) X has a discontinuity points implying that $I_X = \infty$. It turns out that the latter relation holds for any strongly decomposable random variable.

Theorem 4 *For any strongly decomposable* X, $I_X = \infty$.

Proof Let $X = Y + Z$ be a strong decomposition. On one side, the Fisher information on θ contained in $(\theta + Y, Z)$ is not less than that in $\theta + Y + Z = \theta + X$ and is strictly less unless $I_Y = \infty$. This is a special case of monotonicity of the Fisher information: the information in any statistic never exceeds the information in the observation. For a location parameter, the Stam inequality [14] or [1, Chap. 5] quantifies this principle:

$$\frac{1}{I_X} \geq \frac{1}{I_Y} + \frac{1}{I_Z}.$$

Furthermore, due to independence of $\theta + Y$ and Z and additivity of the Fisher information, the information in $(\theta + Y, Z)$ is simply I_Y since the distribution of Z does not depend on θ. Thus, $I_X \leq I_Y$ with a strict inequality unless $I_Y = \infty$.

On the other side, Y is a strong component of X and the monotonicity of the Fisher information implies $I_X \geq I_Y$. Hence $I_X = I_Y = \infty$. □

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