The Annals of Probability
1985, Vol. 13, No. 3, 744–757

THE STATIONARY DISTRIBUTION OF REFLECTED BROWNIAN MOTION IN A PLANAR REGION

By J. M. Harrison,[1] H. J. Landau, and L. A. Shepp

Stanford University and AT&T Bell Laboratories

Suppose given a smooth, compact planar region S and a smooth inward pointing vector field on ∂S. It is known that there is a diffusion process Z which behaves like standard Brownian motion inside S and reflects instantaneously at the boundary in the direction specified by the vector field. It is also known Z has a stationary distribution p. We find a simple, general explicit formula for p in terms of the conformal map of S onto the upper half plane. We also show that this formula remains valid when S is a bounded polygon and the vector field is constant on each side. This polygonal case arises as the heavy traffic diffusion approximation for certain two-dimensional queueing and storage processes.

1. Introduction and summary. In this paper we calculate the stationary distribution for a particular type of two-dimensional diffusion process. The process is denoted by $Z = \{Z(t), t \geq 0\}$. Its state space is a compact planar region S, and it behaves in the interior of S like standard Brownian motion (uncorrelated components with zero drift and unit variance). At the boundary Z reflects instantaneously, and the direction of reflection may vary with location. This boundary behavior is the distinguishing feature of the process under study, and it will be explained further shortly.

We first treat the case of a smooth state space and smoothly varying direction of reflection, pictured in Figure 1. In this case one can build directly on the work of Stroock-Varadhan (1971). We take as given a bounded and simply connected region G of the form

$$G = \{z \in R^2 \colon \phi(z) \geq 0\} \tag{1.1}$$

where

(1.2) $\phi\colon R^2 \to R$ is twice continuously differentiable, is bounded with bounded first and second-order partials, and satisfies $|\nabla\phi(z)| \geq 1$ on $\{z\colon \phi(z) = 0\}$.

Hereafter let us summarize (1.1) and (1.2) with the statement that G is a C^2 domain. Our state space S is the closure of G, so

$$\partial S = \partial G = \{z \in R^2 \colon \phi(z) = 0\}. \tag{1.3}$$

To specify the reflection field on the boundary, let θ be a continuously differen-

Received January 1984; revised July 1984.

[1] Stanford University. This work was done partially while J. M. Harrison was a consultant at Bell Laboratories, and was partially supported by NSF grant ECS-8017867.

AMS 1980 *subject classifications.* Primary 60J65; secondary 60K30.

Key words and phrases. Diffusion process, reflecting barrier, invariant measures, conformal mapping, boundary value problem.

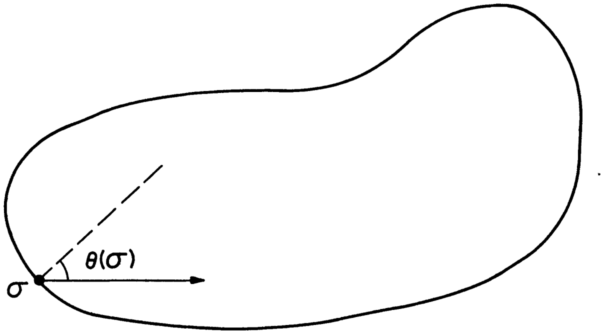

FIG. 1. *Angle of reflection at a boundary point*

tiable function on ∂G, with $|\theta(\sigma)| < \pi/2$ for all $\sigma \in \partial G$. Interpret $\theta(\sigma)$ as an *angle of reflection* at boundary point σ, measured clockwise from the inward pointing normal as shown in Figure 1. To understand the boundary behavior of Z, consider a process that behaves like standard Brownian motion in G and jumps a distance ε whenever ∂G is struck, this jump being in the direction specified by $\theta(\cdot)$. One may think of Z as the limit of this process as $\varepsilon \downarrow 0$; a precise mathematical description will be given in Section 2. There we also show that the Markov process Z has a unique stationary distribution, that this stationary distribution concentrates all its mass on the interior G of S, that it also has a density function p, and that p is given by the formula

$$p(z) = c \operatorname{Re}\{\exp L(z)\}, \quad z \in G. \tag{1.4}$$

Here c is a normalization constant chosen so that p integrates to one, and L is a certain complex-valued function of the complex variable z. To be specific, let F be any conformal mapping of S to the upper half-plane, and let σ_0 be the (unique) point of ∂G such that $F(\sigma_0) = \infty$. Then L is the *analytic* function

$$L(z) = \frac{1}{\pi} \int_{\partial G} \left[\frac{\theta(\sigma) - \theta(\sigma_0)}{F(z) - F(\sigma)}\right] dF(\sigma) - i\theta(\sigma_0), \quad z \in G. \tag{1.5}$$

Note that F is real-valued on ∂G (it maps ∂G onto the boundary of the upper half-plane), so the real and imaginary parts of the integral in (1.5) can each be defined in the ordinary Riemann-Stieltjes sense; each integral is to be computed moving counterclockwise around the boundary.

In Section 3 we turn to the type of case pictured in Figure 2. Here S is a polygon with vertices $\sigma_1, \cdots, \sigma_K$ (in counterclockwise order). For $k = 1, \cdots, K - 1$ we define *side* k as the open line segment between σ_k and σ_{k+1}. Similarly, side K is the line between σ_K and σ_1, excluding the end points. Let ξ_k denote the interior angle made by the two sides meeting at vertex k, as shown in Figure 2. Also given are angles $\theta_1, \cdots, \theta_K$ satisfying $|\theta_k| < \pi/2$. For future purposes, set $\theta_0 = \theta_K$. The object of our study is a strong Markov process Z with the following four properties. (a) It behaves like standard Brownian motion in G. (b) It reflects instantaneously at angle θ_k on side k, as shown in Figure 2. (c) It spends no time at the vertices of S. (d) It has continuous sample paths. (Note in particular that

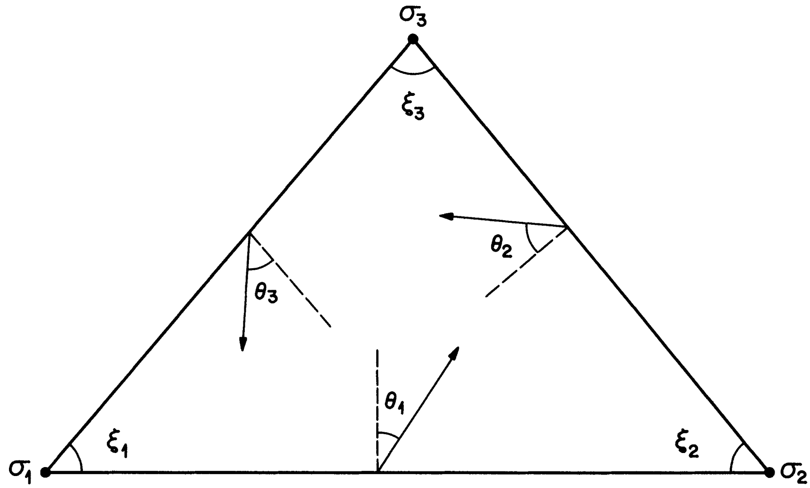

FIG. 2. *An example of the polygonal case*

no directions of reflection have been associated with the vertices.) It turns out that these four properties, translated into precise mathematical language, uniquely determine the distribution of Z if

$$\theta_{k-1} < \theta_k + 2\xi_k \quad \text{for all} \quad k = 1, \cdots, K. \tag{1.6}$$

On the other hand, there is no diffusion process with the stated properties if (1.6) fails. This follows from the work of Varadhan and Williams (1983), as we shall explain in Section 3.

As stated earlier, formula (1.4)–(1.5) gives the stationary distribution of Z when S is a smooth region and $\theta(\cdot)$ varies smoothly over the boundary. But this formula continues to make sense, at least formally, in many situations where S and $\theta(\cdot)$ are not smooth; one naturally suspects that the formula remains valid in such situations. We prove in Section 3 that this suspicion is correct for the polygonal case, and that the general formula also simplifies considerably in that case. As the conformal mapping F, we take the inverse of the standard Schwarz-Christoffel map from the upper half-plane to S, chosen so that σ_0 is on side K and hence $\theta(\sigma_0) = \theta_K$. (See pages 189–196 of Nehari (1952) for a discussion of Schwarz-Christoffel mappings.) On each side of S, the integrand in (1.5) is a constant times the exact differential of $\log[F(z) - F(\sigma)]$. Thus the integral can be evaluated explicitly, and (1.4)–(1.5) eventually reduces to

$$p(z) = c \operatorname{Re}\{\exp(-i\theta_K) \textstyle\prod_{k=1}^{K} [F(z) - F(\sigma_k)]^{\alpha_k}\}, \quad z \in G, \tag{1.7}$$

where

$$\alpha_k = (1/\pi)(\theta_k - \theta_{k-1}) \quad \text{for} \quad k = 1, \cdots, K. \tag{1.8}$$

Note that (1.7) can be re-expressed, using real variables only, as

$$p(z) = c \cos[\textstyle\sum_{k=1}^{K} \alpha_k \gamma_k(z) - \theta_K] \prod_{k=1}^{K} |F(z) - F(\sigma_k)|^{\alpha_k}. \tag{1.9}$$

Here $\gamma_1(z), \cdots, \gamma_K(z)$ are the phase angles pictured in Figure 3, and

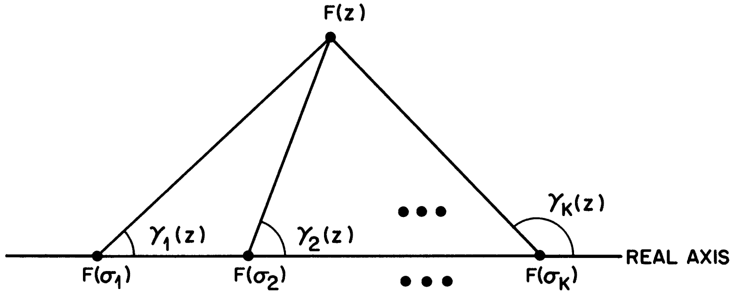

FIG. 3. *Phase angles figuring in final solution for the polygonal case*

$|F(z) - F(\sigma_k)|$ is the length of the line connecting $F(z)$ and $F(\sigma_k)$. The density function p is harmonic, with singularities at vertices of S, and from (1.9) we see that its value on ∂G is given by

$$p(\sigma) = c \cos \theta_j \prod_{k=1}^{K} |F(\sigma) - F(\sigma_k)|^{\alpha_k} \quad \text{on side } j \tag{1.10}$$

$(j = 1, \cdots, K)$. A notable feature of (1.10), and more generally of (1.4)–(1.5), is that boundary values of p depend only on the restriction of F to ∂G, which is real-valued.

We conjecture that (1.4)–(1.5) remains valid for a much broader class of processes than considered here. For example, to unify the two cases discussed above, one might consider a bounded and piecewise smooth state space, with piecewise continuous angle of reflection. Also, extensions to unbounded regions are doubtless possible. These potential directions for future research will be discussed briefly in Section 4.

The original motivation for our study of reflecting Brownian motion comes from queueing and storage theory. Harrison (1978) showed that reflecting Brownian motion on the quadrant, with normal reflection at one axis and oblique reflection at the other, is the natural diffusion approximation for a tandem queue. More precisely, the two-dimensional queue length process, properly normalized, converges weakly to the indicated diffusion under conditions of heavy traffic (approximate equality of average arrival rate and average service rates). This result was greatly generalized by Reiman (1983), who considered the multidimensional queue length process associated with a general open network of n service facilities. Reiman showed that the normalized queue length process converges weakly, under heavy traffic conditions, to reflecting Brownian motion on the n-dimensional positive orthant, the direction of reflection being constant over each boundary hyperplane. A similar sort of limit theorem was proved by Wenocur (1982) for the multidimensional inventory process associated with a production network. By considering systems with finite storage capacity at each production facility, Wenocur obtained a diffusion limit whose state space is a *bounded* polygonal region, again with constant direction of reflection over each boundary hyperplane. This earlier work shows what diffusion process should be used to approximate the heavy traffic behavior of a multidimensional queueing or storage process, but it does not show how to calculate interesting quantities associated with the approximating diffusion. Using our formula (1.9) and the program described by Trefethen (1980), one can numerically evaluate the stationary distribution of the approximating diffusion in a variety of interesting two-dimensional cases. There are also interesting special cases where the conformal mapping F can be written out explicitly, which allows direct numerical evaluation of (1.9); one of these was studied earlier by Harrison-Shepp (1983). For more on the theory of reflecting Brownian motion, and the associated analytical problems, see Harrison-Reiman (1981a, 1981b), Foschini (1982), Harrison-Shepp (1983), Foddy (1983), Williams (1983), and Varadhan-Williams (1983).

Independently of the work described in the preceding paragraph, Newell (1979) studied diffusion approximations for tandem queueing systems, focusing on the stationary queue length distribution and certain closely related quantities. Newell did not discuss approximation of the queue length process by a diffusion process. Instead, using a general approach familiar in mathematical physics, he directly formulated a partial differential equation whose solution approximates the time-dependent distribution of the queue length process under heavy traffic conditions. This is a diffusion equation (heat equation), with auxiliary conditions of an unusual and difficult type. Newell was able to solve the steady-state version of his equation for certain special cases, thus obtaining an approximate equilibrium distribution for the original queueing system. We began our study by verifying that, except for what seems to be a typographical error, Newell's steady-state solution is in fact a stationary distribution for the reflecting Brownian motion that approximates his queue length process. Generalizing this result by stages, we eventually arrived at formula (1.4)–(1.5). Thus Newell's analysis provided the essential energy for our investigation, although most readers would see no connection between the two works in final form.

2. Bounded smooth region with smoothly varying angle. Let G be a bounded C^2 domain, S its closure, and θ a function on ∂G meeting the conditions spelled out in Section 1. Throughout this paper, we denote by $C^2(S)$ the set of all real-valued functions that are twice continuously differentiable on a domain containing S. For each $f \in C^2(S)$ and $\sigma \in \partial G$, let

$$Df(\sigma) = (\partial/\partial n)f(\sigma) + \tan\theta(\sigma)(\partial/\partial\sigma)f(\sigma), \tag{2.1}$$

where $\partial/\partial n$ signifies the *inward-pointing* normal derivative, and $\partial/\partial\sigma$ is the tangential derivative along the boundary in the counterclockwise direction. Thus, except for a constant depending on σ, $Df(\sigma)$ is the directional derivative of f in the direction of reflection pictured in Figure 1.

To define Z precisely, we adopt the language and mathematical machinery of Stroock-Varadhan (1971). Let Ω consist of all continuous functions mapping $[0, \infty) \to S$. Endowing Ω with the topology of uniform convergence on finite time intervals, let $\mathscr{F}$ be the Borel σ-algebra on Ω. Finally, let Z be the identity map

$$Z(t, \omega) = \omega(t) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \omega \in \Omega. \tag{2.2}$$

Let $\mathscr{C}$ be the class of all functions $f \in C^2(S)$ such that

$$Df(\sigma) \geq 0 \quad \text{for all} \quad \sigma \in \partial G. \tag{2.3}$$

We wish to associate with each starting state $z \in S$ a probability measure P_z on $(\Omega, \mathscr{F})$ such that

$$P_z\{Z(0) = z\} = 1, \quad z \in S, \quad \text{and} \tag{2.4}$$

$$\begin{gathered} f(Z(t)) - \frac{1}{2}\int_0^t \Delta f(Z(s))\, ds, \quad t \geq 0, \\ \text{is a submartingale on } (\Omega, \mathscr{F}, P_z) \quad \text{for all} \quad z \in S \quad \text{and} \quad f \in \mathscr{C}, \end{gathered} \tag{2.5}$$

where Δ is the Laplacian. Any such family $\{P_z, z \in S\}$ will be called a *solution of the submartingale problem* (2.4)–(2.5).

THEOREM 2.6. *The submartingale problem* (2.4)–(2.5) *has a unique solution. Moreover, the family of probability measures* $\{P_z, z \in S\}$ *has the strong Markov property.*

PROOF. This is a variation of the central result proved by Stroock and Varadhan (1971). Because our diffusion process is time homogeneous, we state the submartingale problem using functions of state only, whereas Stroock and Varadhan studied processes with time-dependent drift and diffusion coefficients, so they were obliged to consider test functions depending on both state and time. Their argument can be modified to prove the current proposition exactly as in Varadhan-Williams (1983).

A probability measure π on S is said to be a *stationary distribution* (or invariant probability measure) for $\{P_z, z \in S\}$ if

$$\int_S \pi(dz) E_z[f(Z(t))] = \int_S \pi(dz) f(z) \tag{2.7}$$

for all $t > 0$ and all bounded, measurable $f\colon S \to R$. Weiss (1981) developed the following analytical characterization of stationary distributions. We shall comment briefly on the proof after his theorem has been stated precisely.

THEOREM 2.8. *There exists a unique stationary distribution. Moreover, a probability measure* π *is the stationary distribution if and only if* $\pi(\partial G) = 0$ *and*

$$\int_G \pi(dz) \Delta f(z) \leq 0 \quad \textit{for all} \quad f \in \mathscr{C}. \tag{2.9}$$

The proof of existence depends critically on the assumed boundedness of S.

Starting with any probability measure μ on S, let

$$\pi_t(A) = \frac{1}{t}\int_0^t \int_S \mu(dz)P_z\{Z(t) \in A\}$$

for $t \geq 0$. One may interpret $\pi_t(A)$ as the expected fraction of time spent in set A up to time t when the initial state is randomized according to μ. The set of all probability measures on a compact set is itself weakly compact, so $\{\pi_n,\ n = 1, 2, \cdots\}$ converges weakly along a subsequence to a probability measure π. A standard argument then shows that π satisfies (2.7). To prove uniqueness of the stationary distribution, one uses the uniform ellipticity of Laplace's operator plus standard results from ergodic theory. From the results of Stroock and Varadhan (1971), it follows that $P_z\{Z(t) \in \partial G\} = 0$ for all $z \in S$ and $t \geq 0$, and hence $\pi(\partial G) = 0$. To see that the stationary distribution satisfies (2.9), let $f \in \mathscr{C}$ be arbitrary, and note that (2.5) implies

$$E_z\left[f(Z(t)) - \frac{1}{2}\int_0^t \Delta f(Z(s))\, ds\right] \geq f(z) \tag{2.10}$$

for all $z \in S$. Now integrate both sides of (2.10) with respect to $\pi(dz)$, and use (2.7) and Fubini's theorem to obtain

$$0 \geq \int_G \pi(dz)E_z\left[\int_0^t \Delta f(Z)(s))\, ds\right] = \int_0^t \int_G \pi(dz)E_z[\Delta f(Z(s))\, ds]. \tag{2.11}$$

The inner integral on the last line equals $\int_G \pi(dz)\Delta f(z)$ by (2.7), so we conclude that (2.9) is necessary for π to be the stationary distribution; the proof of sufficiency is much more difficult, and we will not discuss it.

It remains only to verify that the probability measure identified in Section 1 satisfies Weiss' condition (2.9). As the proof will suggest, we originally deduced from (2.9) that a well-behaved stationary density p would have to satisfy

$$\Delta p(z) = 0 \quad \text{in} \quad G, \tag{2.12}$$

$$(\partial/\partial n)p(\sigma) = (\partial/\partial\sigma)[p(\sigma)\tan\theta(\sigma)] \quad \text{on} \quad \partial G. \tag{2.13}$$

Formula (1.4)–(1.5) was then obtained by solving this boundary value problem.

LEMMA 2.14. *The function L defined by* (1.5) *is bounded and analytic in G with* $\operatorname{Im} L = -\theta$ *on ∂G.*

PROOF. Suppose first that G is the unit disk and F is the standard conformal mapping

$$F(z) = i\left(\frac{1+z}{1-z}\right), \quad |z| \leq 1. \tag{2.15}$$

Thus ∂G is the unit circle C, and $\sigma_0 = 1$. Specializing (1.5), we have

$$(2.16)\qquad \begin{aligned} L(z) &= \frac{1}{\pi} \int_{\partial G} \left[\frac{\theta(\sigma) - \theta(\sigma_0)}{F(z) - F(\sigma)} \right] dF(\sigma) - i\theta(\sigma_0) \\ &= \frac{1}{\pi} \int_C [\theta(\sigma) - \theta(1)] \frac{(1 - z)\, d\sigma}{(z - \sigma)(1 - \sigma)} - i\theta(1). \end{aligned}$$

Since θ is assumed Lipschitz-continuous on ∂G, we see from the last line of (2.16) that the integral defining $L(z)$ is in fact convergent for $|z| < 1$. Hereafter set $g(\sigma) = \theta(\sigma) - \theta(1)$ and define

$$(2.17)\qquad T(z) = \frac{1}{2\pi i} \int_C \left(\frac{\sigma + z}{\sigma - z} \right) g(\sigma) \frac{d\sigma}{\sigma}, \quad |z| \le 1,$$

with the understanding that the integral is singular when $|z| = 1$. Equation (2.17) is the familiar Poisson integral formula, so T is a bounded analytic function with Re $T = g$ on C, cf. Chapter 4 of Ahlfors (1979). Now the kernel appearing on the last line of (2.16) can be written as

$$\frac{(1 - z)d\sigma}{(z - \sigma)(1 - \sigma)} = \left(\frac{\sigma + 1}{\sigma - 1} \right) \frac{d\sigma}{2\sigma} - \left(\frac{\sigma + z}{\sigma - z} \right) \frac{d\sigma}{2\sigma},$$

so (2.16) can be rewritten $L(z) = iT(1) - iT(z) - i\theta(1)$. Of course $T(1) = g(1) = 0$, so this reduces to $L(z) = -iT(z) - i\theta(1)$. From our previous characterization of T it follows that L is a bounded analytic function with Im $L(\sigma) = -g(\sigma) - \theta(1) = -\theta(\sigma)$ on C, as desired.

In the general case, we can always represent $F(z)$ as $i[1 + \Phi(z)]/[1 - \Phi(z)]$, where Φ is a conformal mapping from S to the unit disk such that $\Phi(\sigma_0) = 1$. That is, F can always be represented as the composition of the conformal mapping (2.15) with such a Φ. Let us make the change of variable $w = \Phi(z)$, let $s = \Phi(\sigma)$ denote a generic point on the unit circle, and set $L^*(w) = L(z)$ and $\theta^*(s) = \theta(\sigma)$. From the Lipschitz-continuity of θ and the description (1.3) of ∂G it follows that θ^* is Lipschitz-continuous on C, so the argument above shows that L^* is bounded and analytic with Im $L^* = -\theta^*$ on C. The desired conclusion for L then follows directly.

THEOREM 2.18. *The unique stationary distribution is* $\pi(dz) = p(z)dz$, *where* p *is defined by* (1.4)–(1.5).

PROOF. Viewing c as an arbitrary positive constant for the moment, we take $p(z) = c$ Re$\{\exp L(z)\}$ and $q(z) = c$ Im$\{\exp L(z)\}$, meaning that

$$(2.19)\qquad p(z) = c \exp\{\operatorname{Re} L(z)\}\cos\{\operatorname{Im} L(z)\},$$

$$(2.20)\qquad q(z) = c \exp\{\operatorname{Re} L(z)\}\sin\{\operatorname{Im} L(z)\}.$$

From Lemma (2.14) we know that $\exp L(z)$ is itself bounded and analytic, so p

and q are both harmonic, and that $\operatorname{Im} L(\sigma) = -\theta(\sigma)$ on ∂G. Now $|\theta(\sigma)| < \pi/2$ by assumption, so $|\operatorname{Im} L(z)| < \pi/2$ for all $z \in G$ by the maximum principle. Thus p is strictly positive and bounded, insuring that c can be set to make p a probability density. From (2.19), (2.20) and (2.14),

$$q(\sigma)/p(\sigma) = \tan\{\operatorname{Im} L(\sigma)\} = -\tan\theta(\sigma) \quad \text{on} \quad \partial G. \tag{2.21}$$

Finally, since p and q are conjugate functions, we have (remember that $\partial/\partial n$ denotes the *inward-pointing* normal derivative)

$$(\partial/\partial n)p(\sigma) = -(\partial/\partial n)q(\sigma). \tag{2.22}$$

Now multiply both sides of (2.21) by $p(\sigma)$, take the tangential derivative of both sides, and then substitute (2.22) to obtain (2.13). Summarizing the development up to here, we have seen that (1.4)–(1.5) does in fact define a probability density p, and that p satisfies (2.12)–(2.13).

Let $\pi(dz) = p(z)dz$, and let $f \in \mathscr{C}$ be arbitrary. Using (2.12) and Green's second identity, and again remembering that $\partial/\partial n$ is an *inward-pointing* normal derivative, we have

$$\begin{aligned} \int_G \pi(dz)\Delta f(z) &= \int_G p(z)\Delta f(z)\,dz \\ &= \int_{\partial G}\left[f(\sigma)\frac{\partial}{\partial n}p(\sigma) - p(\sigma)\frac{\partial}{\partial n}f(\sigma)\right]d\sigma. \end{aligned} \tag{2.23}$$

But $(\partial/\partial n)f(\sigma) \geq -\tan\theta(\sigma)(\partial/\partial\sigma)f(\sigma)$ by the definition of $\mathscr{C}$ Integrating by parts around ∂G, we thus have

$$\begin{aligned} \int_{\partial G} p(\sigma)\frac{\partial}{\partial n}f(\sigma)\,d\sigma &\geq -\int_{\partial G} p(\sigma)\tan\theta(\sigma)\frac{\partial}{\partial\sigma}f(\sigma)\,d\sigma \\ &= \int_{\partial G} f(\sigma)\frac{\partial}{\partial\sigma}[p(\sigma)\tan\theta(\sigma)]\,d\sigma. \end{aligned} \tag{2.24}$$

Combining (2.23) and (2.24) gives

$$\int_G \pi(dz)\Delta f(z) \leq \int_{\partial G}\left\{\frac{\partial}{\partial n}p(\sigma) - \frac{\partial}{\partial\sigma}[p(\sigma)\tan\theta(\sigma)]\right\}f(\sigma)\,d\sigma.$$

By (2.13), the integrand on the right is identically zero, so π satisfies (2.9), and the proof is complete.

3. The polygonal case. All notation and assumptions are as described earlier in Section 1. In particular, recall that G is the interior of a polygon S with vertices $\sigma_1, \cdots, \sigma_K$. In the current context, we define

$$Df(\sigma) = (\partial/\partial n)f(\sigma) + \tan\theta_k(\partial/\partial\sigma)f(\sigma) \quad \text{on side} \quad k, \tag{3.1}$$

with Df undefined at vertices. Also, let $\mathscr{C}$ be the set of all $f \in C^2(S)$ such that

(3.2) f is a constant over a neighborhood of each vertex

(3.3) $Df \geq 0$ on each side of S.

To define the diffusion process Z, we now seek a family of measures $\{P_z, z \in S\}$ on $(\Omega, \mathscr{F})$ such that, for each $z \in S$,

$$P_z\{Z(0) = z\} = 1, \tag{3.4}$$

$$f(Z(t)) - \frac{1}{2}\int_0^t \Delta f(Z(s))\, ds, \quad t \geq 0, \tag{3.5}$$

is a submartingale on $(\Omega, \mathscr{F}, P_z)$ for each $f \in \mathscr{C}$, and

$$E_z\left[\int_0^\infty 1_{\{Z(s)=\sigma_k\}}\, ds\right] = 0 \quad \text{for} \quad k = 1, \cdots, K. \tag{3.6}$$

Condition (3.6) requires that the residence time of Z at each vertex σ_k be zero almost surely, regardless of starting state. This condition is an essential complement to (3.5), because the latter involves only test functions that are flat around vertices, and thus (3.5) does not specify the behavior of Z at the vertices. The following is proved by Varadhan and Williams (1983) for the case where S is a wedge; the extension to a polyhedron is accomplished by an easy localization argument, which we omit.

THEOREM 3.7. *The submartingale problem* (3.4)–(3.6) *has a solution if and only if the problem data satisfy* (1.6). *In this case, the solution is unique, and the family* $\{P_z, z \in S\}$ *has the strong Markov property.*

Hereafter we assume (1.6) holds. It follows from the results of Varadhan and Williams (1983) that for any starting state $z \in S$, vertex σ_k is hit with probability 1 if $\theta_k < \theta_{k-1}$, and σ_k is hit with probability zero otherwise. We shall have no need for this result in our study.

Stationary distributions are defined exactly as in Section 2. The following analytical characterization is identical to (2.8), except that now $\mathscr{C}$ is defined differently. This theorem is a composite of results by Weiss (1981) and Williams (1983), as we shall explain.

THEOREM 3.8. *There exists a unique stationary distribution. Moreover, a probability measure* π *is the stationary distribution if and only if* $\pi(\partial G) = 0$ *and*

$$\int_G \pi(dz)\Delta f(z) \leq 0 \quad \textit{for all} \quad f \in \mathscr{C}. \tag{3.9}$$

Existence of a stationary distribution is proved exactly as in the case of smooth data, using the boundedness of S. By imitating the analysis in Chapter 7 of Williams (1983), one can show that Z is recurrent in the fine topology, and thus its invariant measure is unique. Because Z spends no time on ∂G, regardless of starting state, the stationary distribution must have $\pi(\partial G) = 0$. The necessity of (3.9) is proved exactly as in the case of smooth data, and Theorem 3 of Weiss (1981) establishes sufficiency. (Weiss treats a more general problem with piecewise smooth data, assuming that the corresponding submartingale problem is well posed.)

Consider now the density function p defined by (1.7). Let us first check that p is integrable over G, so c can be set to make p a probability density. This comes down to the question of integrability around an arbitrary vertex σ_k. Recall from Figure 2 that ξ_k is the interior angle made by the two sides meeting at σ_k. A fundamental property of the Schwarz-Christoffel mapping is that

$$|F(z) - F(\sigma_k)| = O(|z - \sigma_k|^{\pi/\xi_k}) \quad \text{as} \quad |z - \sigma_k| \to 0.$$

Combining this with (1.9) gives

$$p(z) = O(|z - \sigma_k|^{\beta_k}) \quad \text{as} \quad |z - \sigma_k| \to 0,$$

where $\beta_k = (\theta_k - \theta_{k-1})/\xi_k$. Our key restriction (1.6) says that $\beta_k > -2$, so p is integrable over any neighborhood of σ_k, as required.

In proving that Weiss' condition (3.9) is satisfied by $\pi(dz) = p(z)dz$, we shall show that

$$\Delta p(z) = 0 \quad \text{in} \quad G, \tag{3.10}$$

$$(\partial/\partial n)p(\sigma) = \tan\theta_k(\partial/\partial\sigma)p(\sigma) \quad \text{on side } k, \quad \text{and} \tag{3.11}$$

$$\int_\Gamma \frac{\partial}{\partial n} p(\sigma)\, d\sigma = p(b)\tan\theta(b) - p(a)\tan\theta(a), \tag{3.12}$$

where a and b are boundary points located on side i and side j, respectively, $\theta(a) = \theta_i$ and $\theta(b) = \theta_j$, and Γ is any differentiable curve that leads from a to b through G. One might think that (3.10)–(3.11), being analogous to (2.12)–(2.13) for the case of smooth data, would uniquely determine p, but this is not the case. (Note that these conditions are always satisfied by the uniform density.) The extra requirement (3.12) is essential here.

THEOREM 3.13. *The unique stationary distribution is $\pi(z) = p(z)dz$, where p is defined by* (1.7).

PROOF. Let us define the complex-valued function

$$\begin{aligned} L(z) &= \log\{\exp(-i\theta_K) \textstyle\prod_{k=1}^{K} [F(z) - F(\sigma_k)]^{\alpha_k}\} \\ &= \textstyle\sum_{k=1}^{K} \alpha_k \log[F(z) - F(\sigma_k)] - i\theta_K \end{aligned} \tag{3.14}$$

for $z \in G$. Let H be the upper half-plane and R the real line, and let $L^*(w) = L(z)$, where $w = F(z)$. Also set $\theta^*(u) = \theta(\sigma)$ on R, where $u = F(\sigma)$, with the obvious convention that $\theta(\sigma) = \theta_k$ on side k. From (3.14) we have

$$L^*(w) = \textstyle\sum_{k=1}^{K} (1/\pi)(\theta_k - \theta_{k-1})\log(w - u_k) - i\theta_K, \tag{3.15}$$

where $u_k = F(\sigma_k)$. Recall that $\log(w)$ is analytic in the upper half-plane with $\operatorname{Im}\log(u) = 0$ for $u < 0$, and $\operatorname{Im}\log(u) = \pi$ for $u < 0$. Combining this with (3.15), we see that L^* is analytic in the upper half-plane with $\operatorname{Im} L^*(u) = -\theta^*(u)$ on R. Thus L is analytic in G with $\operatorname{Im} L(\sigma) = -\theta_k$ on side k.

Noting that $p(z) = c\operatorname{Re}\{\exp L(z)\}$ by (3.14) and (1.7), let us define $q(z) = c\operatorname{Im}\{\exp L(z)\}$. Proceeding exactly as in the proof of Theorem (2.18), we find that

p and q are conjugate harmonic functions satisfying

$$-q(\sigma) = \tan\theta_k p(\sigma) \quad \text{on side } k. \tag{3.16}$$

Now take the tangential derivative of each side of (3.16), and substitute

$$-(\partial/\partial\sigma)q(\sigma) = (\partial/\partial n)p(\sigma) \tag{3.17}$$

as in the proof of (2.18). This gives (3.11). Next, let Γ be any differentiable curve that leads from a to b through G, where $a, b, \in \partial G$ are not vertices. From (3.16) we have

$$p(b)\tan\theta(b) - p(a)\tan\theta(a) = -[q(b) - q(a)]. \tag{3.18}$$

But $[q(b) - q(a)]$ can be written as the integral of the tangential derivative of q over Γ. Using (3.17) to substitute for the tangential derivative, we finally arrive at (3.12). Thus p has been shown to satisfy (3.10)–(3.12).

Finally, to prove that (3.9) is satisfied by $\pi(dz) = p(z)dz$, let $f \in \mathscr{C}$ be arbitrary. Because f is flat around each vertex, we can cut the corners of G, as pictured in Figure 4, so that f has constant value c_k over a neighborhood containing the small triangle eliminated near σ_k. Let $a_1, \cdots, a_K$ and $b_1, \cdots, b_K$ be as shown in Figure 4, and let Γ_k be the line segment leading from a_k to b_k. Finally, let W be the open polygonal region bounded by $\Gamma_1, \cdots, \Gamma_K$ and the sides of S, as shown in Figure 4. Note that $\Delta f = 0$ in each of the small triangles that make up $G - W$, since f is constant over each such triangle. Thus, using (3.10) and Green's second identity exactly as in the proof of (2.8), we have

$$\begin{aligned} \int_G \pi(dz)\Delta f(z) &= \int_G p(z)\Delta f(z)\,dz = \int_W p(z)\Delta f(z)\,dz \\ &= \int_{\partial W}\left[f(\sigma)\frac{\partial}{\partial n}p(\sigma) - p(\sigma)\frac{\partial}{\partial n}f(\sigma)\right]d\sigma. \end{aligned} \tag{3.19}$$

For the last stage of the argument, let $\theta(\cdot)$ be defined on ∂W so that it is continuously differentiable, $|\theta(\cdot)| < \pi/2$, and $\theta(\sigma) = \theta_k$ at each point $\sigma \in \partial W$ that lies on side k. Recall that

$$(\partial/\partial n)f(\sigma) \geq -\tan\theta(\sigma)(\partial/\partial\sigma)f(\sigma) \tag{3.20}$$

on each side of S by definition of $\mathscr{C}$. This weak inequality extends to all $\sigma \in \partial W$,

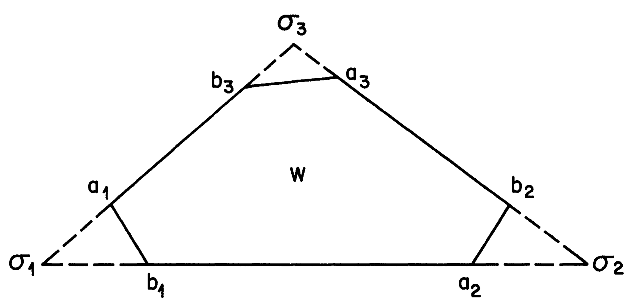

FIG. 4. *Polygonal region W formed by cutting the corners of G*

because both the normal and tangential derivatives of f are zero on $\Gamma_1, \cdots, \Gamma_K$. Just as in the proof of (2.8), we substitute (3.20) into (3.19) and integrate by parts around ∂W to arrive at

$$\int_G \pi(dz)\Delta f(z) \le \int_W \left\{\frac{\partial}{\partial n} p(\sigma) - \frac{\partial}{\partial \sigma}[p(\sigma)\tan\theta(\sigma)]\right\} f(\sigma)\, d\sigma. \tag{3.21}$$

By (3.11), the quantity in braces vanishes where ∂W coincides with ∂G, so (3.21) reduces to

$$\int_G \pi(dz)\Delta f(z) \le \sum_{k=1}^K c_k \int_{\Gamma_k} \left\{\frac{\partial}{\partial n} p(\sigma) - \frac{\partial}{\partial \sigma}[p(\sigma)\tan\theta(\sigma)]\right\} d\sigma \tag{3.22}$$

But each of the integrals on the right side of (3.22) is zero by (3.12), so (3.22) reduces to (3.8), and the proof is complete.

4. Concluding remarks. To unify the two cases treated in this paper, one might consider a piecewise smooth region of the type studied by Weiss (1981). Let G be the union and/or intersection of finitely many bounded C^2 domains, with the restriction that there are no cusps on ∂G, and let $\theta(\cdot)$ be Lipschitz continuous on each section of ∂G, with $|\theta(\cdot)| < \pi/2$. Let S be the closure of G, and let F be a conformal mapping from S to the upper half-plane such that $F(\sigma_0) = \infty$ for some $\sigma_0 \in \partial G$ that is not a singularity. The submartingale problem for this case is stated very much as in Section 3, using test functions f that are flat around singularities of ∂G. We conjecture that if the submartingale problem is well posed, then there is a unique stationary distribution, and its density function is given by formula (1.4)–(1.5). This submartingale problem (existence and uniqueness of a diffusion with the specified data) has not been studied as yet. If our conjecture is correct, then it can only be well posed when the function p defined by (1.4)–(1.5) is integrable over G; it may well be that integrability of this function is *necessary and sufficient* for the submartingale problem to have a (unique) solution.

In Sections 2 and 3 we have only used the boundedness of S to prove that a stationary distribution exists; then direct calculations show that the stationary density is given by a particular function p. For unbounded regions, we conjecture that this same function p is the unique stationary distribution if it is integrable, and that no stationary distribution exists if it is not integrable. If the angles of reflection provide sufficient restorative force, it is certainly possible to have a stationary distribution with an unbounded region: see Harrison-Shepp (1983) for a concrete example of this (S is the semi-infinite strip and the angle of reflection is constant on each side).

Readers interested in the behavior of reflected Brownian motion around singular boundary points may consult Dynkin (1964, 1967) and Varadhan-Williams (1983) for more on this fascinating and intricate subject. We also refer to Gakhov (1966) and Smirnov (1964) for extensive discussion of oblique derivative boundary value problems of the type encountered in Sections 2 and 3.

Acknowledgement. We are indebted to S. R. S. Varadhan and R. J. Williams for many helpful conversations.

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J. M. HARRISON
GRADUATE SCHOOL OF BUSINESS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

H. J. LANDAU
L. A. SHEPP
AT&T BELL LABORATORIES
600 MOUNTAIN AVENUE
MURRAY HILL, NEW JERSEY 07974