ADVANCES IN MATHEMATICS 26, 206–222 (1977)

A Variational Problem for Random Young Tableaux

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Stanley posed the problem of minimizing the functional

$$H(f) = \int_0^\infty dx \int_0^{f(x)} dy \log(f(x) - y + f^{-1}(y) - x) \tag{1}$$

over nonincreasing nonnegative f on $(0, \infty)$ of integral unity. We show that the minimum is unique and has the value $-\frac{1}{2}$, as was conjectured by Stanley. The minimizing function f_0, $H(f_0) = -\frac{1}{2}$, is given parametrically by

$$f_0(x) = (2/\pi)(\sin\theta - \theta\cos\theta), \qquad x = f_0(x) + 2\cos\theta, \quad 0 \leqslant \theta \leqslant \pi \tag{2}$$

for $0 \leqslant x \leqslant 2$; and $f_0(x) = 0$ for $x \geqslant 2$. Closely related unpublished results have been obtained by Hammersley. We also find the minimum of $H(f)$ subject to the constraints $f(0) \leqslant a$ and $f^{-1}(0) = \inf(x\colon f(x) = 0) \leqslant b$ where a and b are given. Proofs of the results for the case of constraints are complicated and will be given elsewhere. Let λ_n be the shape of the random Young tableau with n unit squares obtained from sampling from the Schensted distribution where

$$P(\lambda_n) = n!/\pi^2(\lambda_n), \tag{3}$$

where $\pi(\lambda_n)$ is the product of the n hook lengths of λ_n. Consider the stochastic processes

$$\bar{\lambda}_n(t) = (1/n^{1/2})\,\lambda_n(t\,n^{1/2}), \qquad n \geqslant 1, \quad t \geqslant 0, \tag{4}$$

where $\lambda_n(t)$ is the height of the tableau λ_n at a horizontal distance t from the corner. We show

$$\bar{\lambda}_n \to f_0 \tag{5}$$

in the sense of weak convergence in a certain metric, where f_0 is the deterministic function in (2).

Let $l(\sigma_n)$ denote the length of the longest increasing subsequence of a random permutation σ_n of $1, 2,\ldots, n$. Hammersley showed that

$$l(\sigma_n)/n^{1/2} \to c \qquad \text{in probability, } n \to \infty. \tag{6}$$

Schensted showed that $l(\sigma_n)$ has the same distribution as $\lambda_n(0)$ under the distribution (3) on λ_n. It has long been conjectured (apparently first by Baer and Brock) that $c = 2$. We show here that $c \geqslant 2$ as a by-product of (5).

* This paper was presented as a Special Invited Paper at the Annual Meeting of the Institute of Mathematical Statistics August 1975.

ISSN 0001-8708

1. Introduction

Baer and Brock [1] conjectured $c = 2$ on the basis of extensive computer calculations (see also [3]). Using a clever imbedding of the problem into the Poisson process, and results of Kingman [5], Hammersley [3] showed the existence of the limit (5). Hammersley [3] made several attacks on determining the actual value of c, including an unpublished one similar to the present attack (but apparently not obtaining uniqueness).

The present attack relies on a combinatorial identity of Schensted [7], expressing the probability distribution of $l(\sigma_n)$ in terms of Young tableaux. A Young shape [7] of size n is an array of n unit squares, left and bottom justified (note that the array is usually top justified, but we prefer to think of the shape as a nonnegative function) whose column lengths are nonincreasing from left to right, as in Fig. 1. With each square $S \in \lambda = \lambda_n$, let $L(S; \lambda)$ denote the hook length of S in λ, i.e., the number of squares in λ directly to the right and directly above S, counting S itself exactly once. The hook lengths are written in each square of Fig. 1 as an illustration. Let $\pi = \pi(\lambda)$ denote the hook product, i.e., the product of all the integers $L(S; \lambda)$, $S \in \lambda$. Then, Schensted [7]

$$P(l(\sigma_n) = k) = \sum_{\lambda(0)=k} \frac{n!}{\pi(\lambda)^2}, \qquad k = 1, 2, \ldots, n, \tag{1.1}$$

where the sum is taken over all shapes $\lambda = \lambda_n$ with n squares where the first (leftmost) column has length $\lambda(0) = k$. From (1.1) we see that if we randomly choose a shape λ_n of size n according to the distribution

$$P(\lambda_n) = n!/(\pi(\lambda_n))^2 \tag{1.2}$$

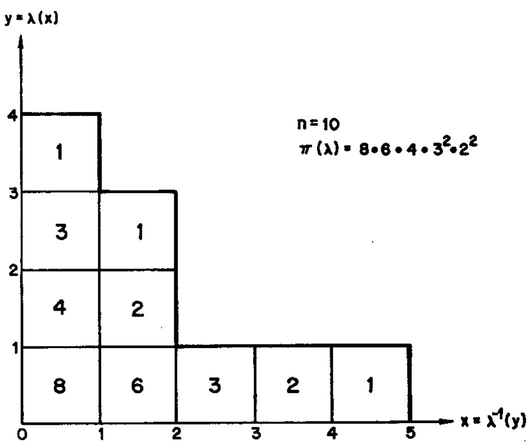

Fig. 1. A Young tableau for $n = 10$ with hook lengths $P(\lambda) = 10!/\pi^2(\lambda)$.

(note [7] that these probabilities actually sum to unity) then $\lambda_n(0)$, the length of the first column of λ_n, is a random variable with the same distribution as $l(\sigma_n)$. A direct mapping of $\sigma_n \to \lambda_n$ is given by Schensted [7], but we make no use of this. Since $\pi(\lambda_n)$ varies greatly with λ_n it is expected and we shall prove that $P(\lambda_n)$ is concentrated very sharply around one specific shape λ_{0n} foreach n, where

$$\pi(\lambda_{0n}) = \min_{\lambda_n} \pi(\lambda_n). \tag{1.3}$$

We would then further expect that $l(\sigma_n)$ would be concentrated around $\lambda_{0n}(0)$.

The problem (1.3) of minimizing $\pi(\lambda_n)$ thus arises in finding the most likely Young tableaux under the distribution (1.2). The problem (1.3) also arises in finding the maximal degree of an irreducible representation of the symmetric group S_n. Frobenius [6] gave a one–one correspondence between a shape λ (whose column lengths correspond to a partition of n) and an irreducible representation of the symmetric group as a group of matrices (with complex entries) of common size $f^\lambda \times f^\lambda$. The size f^λ is called the degree of the representation corresponding to λ in the Frobenius correspondence, and using a formula of Frobenius, it was shown [9], that the degree f^λ is given by $f^\lambda = n!/\pi(\lambda)$. The question of finding the shape of maximum degree f^λ thus again reduces to the problem (1.3) of minimizing $\pi(\lambda)$, $\lambda = \lambda_n$, and was apparently first mentioned in [10, Footnote 9], as was brought to our attention by Stein. In [11], the computer calculations of λ_{0n} in [1] are extended from $n = 36$ to $n = 75$. For $n = 75$, [11] shows that λ_{0n} has column lengths 14, 11, 10, 8, 7, 6, 5, 4, 3, 2, 2, 1, 1, 1 which does begin to have the general shape predicted by our asymptotic result (2), (4), and (5) and shown in Fig. 3.

Thus it is of interest to find shapes $\lambda = \lambda_n$ minimizing $\pi(\lambda_n)$ in (1.3). We express this problem as a calculus of variations minimization of a functional. Note

$$\begin{aligned}\log \pi(\lambda) &= \sum_{S\in\lambda} \log L(S;\lambda) \\ &= \sum_{S\in\lambda} \log(\lambda(x_S) - y_S + \lambda^{-1}(y_S) - x_S),\end{aligned} \tag{1.4}$$

where $\lambda(x)$ is the height of the shape λ at x, when λ is placed at the origin in the first quadrant of the x–y plane, (x_S, y_S) is the center of the square $S \in \lambda$, and λ^{-1} is the inverse function to λ. Since log is a convex function, and S has unit area,

$$\log(\lambda(x_S) - y_S + \lambda^{-1}(y_S) - x_S) \geqslant \iint_S \log(\lambda(x) - y + \lambda^{-1}(y) - x)\, dx\, dy. \tag{1.5}$$

Normalizing λ to have unit area by setting

$$f_\lambda(x) = (1/n^{1/2})\,\lambda(n^{1/2}x), \qquad x \geqslant 0 \tag{1.6}$$

where n is the number of squares in $\lambda = \lambda_n$, and defining for any $f \in \mathscr{F}$, where $\mathscr{F}$

is the class of nonnegative nonincreasing functions on $(0, \infty)$ of integral unity,

$$H(f) = \int_0^\infty \int_0^{f(x)} \log(f(x) - y + f^{-1}(y) - x)\, dy\, dx \tag{1.7}$$

we have from (1.2), (1.4)–(1.7),

$$\begin{aligned} P(\lambda) &\leqslant n! \exp\left[-2 \sum_{S\in\lambda} \iint_S \log(\lambda(x) - y + \lambda^{-1}(y) - x)\, dy\, dx\right] \\ &= n! \exp\left[-2 \int_0^\infty \int_0^{\lambda(x)} \log(\lambda(x) - y + \lambda^{-1}(y) - x)\, dy\, dx\right] \\ &= n! \exp\left[-2n \int_0 \int_0^{f_\lambda(x)} \log n^{1/2}(f_\lambda(x) - y + f_\lambda^{-1}(y) - x)\, dy\, dx\right] \\ &= (n!/n^n) \exp[-2nH(f_\lambda)] \sim (2\pi n)^{1/2}\, e^{-2n(H(f_\lambda)+\frac{1}{2})}. \end{aligned} \tag{1.8}$$

Since the inequality in (1.8) is reasonably tight it is natural to expect (since $P(\lambda) \leqslant 1$) that $\min H(f) = -\frac{1}{2}$. Further if Λ_n is any set of λ_n for which $\lambda = \lambda_n \in \Lambda_n$ implies that

$$H(f_\lambda) \geqslant -\tfrac{1}{2} + \delta, \qquad \delta > 0, \tag{1.9}$$

where δ is independent of n, then since the number $p(n)$ of shapes λ of order n is the number of partitions of n, [4],

$$p(n) = \exp O(n^{1/2}), \tag{1.10}$$

we see from (1.8)–(1.10) that

$$P(\Lambda_n) \leqslant p(n)(2\pi n)^{1/2} e^{-2n\delta} \to 0, \qquad \text{exponentially.} \tag{1.11}$$

In particular if Λ_n is the set of $\lambda = \lambda_n$ for which $\lambda_n(0) \leqslant (2 - \epsilon)\, n^{1/2}$, or $f_\lambda(0) \leqslant 2 - \epsilon$ by (1.6) we will show that (1.9) holds. It then follows using (1.11) that $P(l(\sigma_n)/n^{1/2} \geqslant 2 - \epsilon) = P(\lambda_n(0)/n^{1/2} \geqslant 2 - \epsilon) \to 1$ for every $\epsilon > 0$, i.e., that $c \geqslant 2$ in (6). The reason that the present method fails to yield $c = 2$ is that there are $f \in \mathscr{F}$ with $f(0)$ arbitrarily large but with $H(f)$ arbitrarily close to $-\frac{1}{2}$. Of course such $f(x)$ fall quickly to 2 in the neighborhood of $x = 0$. Thus the continuous function problem of minimizing $H(f)$ no longer is a good approximation to the discrete problem of minimizing $\pi(\lambda_n)$.[1]

We show that in a somewhat complicated metric d on $\mathscr{F}$, the stochastic processes,

$$\bar{\lambda}_n(x) = (1/n^{1/2})\, \lambda_n(xn^{1/2}), \qquad x \geqslant 0 \tag{1.12}$$

[1] Vershik and Kerov (cf. *Dokl. Acad. Nauk* 233 (1977), 1024–1028) show $c = 2$ by a separate combinatorial argument based on the notion of Young *diagrams*.

converge as $n \to \infty$ weakly in the sense of [2] to the deterministic (i.e., constant) process $f_0(x)$ in (2). It is probably also true as a consequence of weak convergence in the metric d defined below, but it is not proved, that

$$P(|\bar{\lambda}_n(x) - f_0(x)| \geqslant \epsilon) \to 0, \qquad n \to \infty \tag{1.13}$$

for every $x \geqslant 0$. Of course if (1.13) could be proved for $x = 0$ then $c = 2$ would follow immediately since $f_0(0) = 2$ by (2).

To define the metric d, let for $f \in \mathscr{F}$, Tf denote the function on $(-\infty, \infty)$ defined by

$$Tf(\xi) = \min(x, f(x)) \qquad \text{if} \quad \xi = x - f(x), \qquad x \geqslant 0 \tag{1.14}$$

setting $Tf(\xi) = 0$ if $-\infty < \xi < -f(0)$. It will be seen in Section 2 that $g = Tf$ is nonnegative and integrates to unity. We define

$$d(f_1, f_2) = Q(Tf_1 - Tf_2)^{1/2} \tag{1.15}$$

where

$$Q(g) = \tfrac{1}{4} \int_{-\infty}^{\infty} |\hat{g}(\omega)|^2 \, |\omega| \, d\omega \tag{1.16}$$

and $\hat{g}$ is the Fourier transform of g. Since $Q^{1/2}$ is a norm, it is easy to check that d is a metric on $\mathscr{F}$.

We will obtain in Section 2 the lower bound for $H(f)$, $f \in \mathscr{F}$

$$\begin{aligned} H(f) &\geqslant H(f_0) + Q(Tf - Tf_0) \\ &= -\tfrac{1}{2} + d^2(f, f_0) \end{aligned} \tag{1.17}$$

by (1.15). Let Λ_n be the set of $\lambda = \lambda_n$ for which $d(\bar{\lambda}_n, f_0) > \delta$. By (1.17) we see that (1.9) holds and (1.11) follows, $f_\lambda = \bar{\lambda}_n \to f_0$ weakly in $(\mathscr{F}, d)$, i.e.,

$$P(d(\bar{\lambda}_n, f_0) \geqslant \delta) \to 0 \qquad \text{exponentially.} \tag{1.18}$$

We will show (2.35)–(2.41) that if $f(0) \leqslant 2 - \epsilon$ then $Q(Tf - Tf_0) \geqslant \delta > 0$ which, using (1.17) gives (1.9) and shows $c \geqslant 2$.

We state in Section 3 the more precise result determining explicitly the value of the infimum $H_0(a, b)$ of $H(f)$ over $f \in \mathscr{F}$ subject to the constraints $f(0) \leqslant a$ and $f^{-1}(0) = \inf\{x : f(x) = 0\} \leqslant b$. Note that by Schensted's correspondence $H_0(a, b)$ gives an asymptotic index via (1.8) of the probability that the longest increasing subsequence and the longest decreasing subsequence have lengths less than $an^{1/2}$ and $bn^{1/2}$, respectively. The unique minimizing functions $f_0(x; a, b)$ are given explicitly. Graphs of some representative $f_0(x; a, b)$ are seen in Fig. 3. The graph labeled 1 in Fig. 3 is that of the unconstrained minimizing f_0 of (2). Proofs of the results stated in Section 3 will be given elsewhere.

It is of course natural to expect that for appropriate normalizing constants $c_n \to \infty$ (perhaps $c_n = n^{1/4}$ would do) the stochastic processes

$$\tilde{\lambda}_n(t) = c_n(\bar{\lambda}_n(t) - f_0(t)), \qquad t \geqslant 0 \tag{1.19}$$

would tend weakly to a nonzero limiting process $W(t)$, $t \geqslant 0$, as $n \to \infty$. It would be of interest to know what the process W is. It is clear only that W integrates pathwise to zero and that $W(t) \geqslant 0$ for $t \geqslant 2$. Perhaps $W(t) = 0$ for $t \geqslant 2$ and is the Wiener process in $0 \leqslant t \leqslant 2$ conditioned to integrate to zero over [0, 2] and to vanish at 0 and 2, but this is just a guess.

We mention that Greene [12, Theorem 3.1] has given an interpretation of the length of the kth column, for $k > 1$, of the Young tableau, λ, in terms of increasing subsequences of the corresponding permutation, σ, under the Schensted correspondence. Namely, he shows that the number of entries in the first k columns of λ is the maximum length of the union of k increasing subsequences of σ.

2. The Variational Problem

Let $\mathscr{F}$ be the class of nonincreasing nonnegative functions $f(x)$ on $(0, \infty)$ with

$$\int_0^\infty f(x)\, dx = 1. \tag{2.1}$$

Consider the region R in the x–y plane defined by $0 \leqslant y \leqslant f(x)$, $0 \leqslant x \leqslant \infty$ illustrated in Fig. 2.

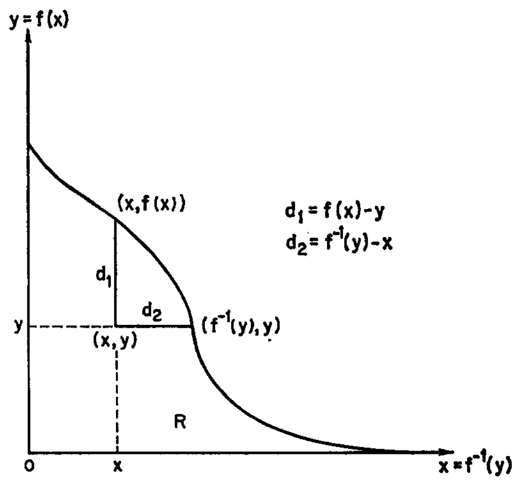

Fig. 2. The hook length for a continuous Young tableau f.

The hook distance of a point (x, y) in R is defined as $d_1 + d_2$ where

$$d_1 = f(x) - y, \qquad d_2 = f^{-1}(y) - x.$$

We are interested in minimizing the functional

$$\begin{aligned} H(f) &= \iint_R \log\{d_1(x, y) + d_2(x, y)\}\, dx\, dy \\ &= \iint_R \log\{f(x) - y + f^{-1}(y) - x\}\, dx\, dy. \end{aligned} \tag{2.2}$$

It is important to make the right change of variables in (2.2). It is instructive to do this in stages. Assume for the moment that $f(0) = a < \infty$, $f^{-1}(0) = b < \infty$, and that f is absolutely continuous. Setting $y = f(t)$ in (2.2) and using the change of variables

$$x - f(x) = \xi, \qquad x = h(\xi); \qquad t - f(t) = \eta, \qquad t = h(\eta) \tag{2.3}$$

we obtain after a short calculation

$$\begin{aligned} H(f) &= -\int_0^b f'(t)\, dt \int_0^t \log(f(x) - f(t) + t - x)\, dx \\ &= \int_{-a}^b (1 - h'(\eta))\, d\eta \int_{-a}^\eta \log(\eta - \xi)\, h'(\xi)\, d\xi \\ &= \int_{-a}^b d\eta \int_{-a}^\eta \log(\eta - \xi)\, h'(\xi)\, d\xi \\ &\quad - \tfrac{1}{2} \int_{-a}^b h'(\eta)\, d\eta \int_{-a}^b h'(\xi)\, d\xi \log|\eta - \xi|. \end{aligned} \tag{2.4}$$

The further transformation

$$g(\xi) = \begin{cases} h(\xi) & \xi < 0 \\ h(\xi) - \xi & \xi \geqslant 0 \end{cases} \tag{2.5}$$

gives a great simplification. It is easy to verify that there is a one–one correspondence between $f \in \mathscr{F}$ and g given by

$$\begin{aligned} x &= g(\xi) + \tfrac{1}{2}\xi + \tfrac{1}{2}|\xi|, \\ f(x) &= g(\xi) - \tfrac{1}{2}\xi + \tfrac{1}{2}|\xi|, \end{aligned} \tag{2.6}$$

where the inverse $g = Tf$ is given by (1.14). The function g has the properties that

$$\begin{aligned} 0 \leqslant g'(\xi) \leqslant 1 \quad &\text{for} \quad \xi < 0, \\ -1 \leqslant g'(\xi) \leqslant 0 \quad &\text{for} \quad \xi > 0, \end{aligned} \tag{2.7}$$

$$\int_{-\infty}^{\infty} g(\xi)\, d\xi = \int_{0}^{\infty} f(x)\, dx = 1. \tag{2.8}$$

Even functions g correspond to functions f with $f = f^{-1}$ as follows from (2.6) by replacing ξ by $-\xi$. By (2.7), g is absolutely continuous whether or not f is, and from (2.4), (2.5), after a calculation

$$H(f) = I(g) \triangleq Q(g) + L(g) \tag{2.9}$$

where (since g' vanishes outside of $(-a, b)$)

$$Q(g) = -\tfrac{1}{2} \int_{-\infty}^{\infty} g'(x)\, dx \int_{-\infty}^{\infty} g'(t)\, dt \log | x - t | \tag{2.10}$$

is a quadratic functional of g' and

$$L(g) = -\int_{-\infty}^{\infty} g'(x)(x \log | x | - x)\, dx \tag{2.11}$$

is a linear functional of g'. Note that since there is a one–one correspondence between $f \in \mathscr{F}$ and g we may regard the problem (1) of minimizing $H(f)$ as that of minimizing the much simpler functional $I(g) = Q(g) + L(g)$ in (2.10)–(2.11) over $g \in \mathscr{G}$ where $\mathscr{G}$ is the class of functions satisfying (2.7) and (2.8), i.e., $g \in \mathscr{G}$ has $| g' | \leqslant 1$, and g is unimodal with mode at zero, nonnegative, and of integral unity. Showing uniqueness of the solution to the problem of minimizing $H(f)$ is of course the same as that for minimizing $I(g)$.

We next express the functional $I(g)$ of the Fourier $\hat{g}$ and Hilbert $\tilde{g}$ transforms of g,

$$\hat{g}(x) = \int g(t)\, e^{-ixt}\, dt \tag{2.12}$$

$$\tilde{g}(x) = \frac{1}{\pi} \int \frac{g(t)}{x - t}\, dt. \tag{2.13}$$

Formally integrating by parts, which can be justified easily using the fact that g is unimodal and integrable,

$$\tilde{g}(x) = \frac{1}{\pi} \int g'(t) \log | x - t |\, dt \tag{2.14}$$

so that from (2.10)

$$Q(g) = -\frac{\pi}{2} \int g'(x)\, \tilde{g}(x)\, dx. \tag{2.15}$$

Since the Fourier transform of $\tilde{g}$ is $i(\operatorname{sgn}\omega)\,\hat{g}(\omega)$ and that of g' is $i\omega\hat{g}(\omega)$ we obtain from Parseval's identity,

$$Q(g) = \tfrac{1}{4}\int |\hat{g}(\omega)|^2\,|\omega|\,d\omega \tag{2.16}$$

so that Q is a positive definite quadratic form. Thus integrating by parts in (2.11) gives from (2.9)

$$I(g) = \int g(x)\log|x|\,dx + \tfrac{1}{4}\int |\hat{g}(\omega)|^2\,|\omega|\,d\omega. \tag{2.17}$$

We can obtain a crude lower bound for $I(g)$ using (2.7),

$$\begin{aligned} I(g) > L(g) &= -1 - \int g'(x)\operatorname{sgn} x\,|x|\log|x|\,dx \\ &\geqslant -1 - \int_{-1}^{1} |x|\log|x|\,dx = -\tfrac{3}{2} \end{aligned} \tag{2.18}$$

since $Q(g) > 0$. To get the best lower bound $(-\frac{1}{2})$ introduce a comparison function $g_0 \in \mathscr{G}$ and write for an arbitrary $g \in \mathscr{G}$, from (2.15)

$$\begin{aligned} Q(g - g_0) &= -\frac{\pi}{2}\int (g'(x) - g_0'(x))(\tilde{g}(x) - \tilde{g}_0(x))\,dx \\ &= Q(g) + Q(g_0) + \pi\int g'(x)\,\tilde{g}_0(x)\,dx. \end{aligned} \tag{2.19}$$

Thus,

$$I(g) = Q(g) + L(g) = Q(g - g_0) - Q(g_0) - R(g, g_0) \tag{2.20}$$

where

$$R(g, g_0) = \int g'(x)(x\log|x| - x + \pi\tilde{g}_0(x))\,dx. \tag{2.21}$$

We will exhibit a $g_0 \in \mathscr{G}$ for which

$$\begin{aligned} p(x) \triangleq x\log|x| - x + \pi\tilde{g}_0(x) &= 0 \quad \text{for} \quad |x| \leqslant 2 \\ &> 0 \quad \text{for} \quad x > 2 \\ &< 0 \quad \text{for} \quad x < -2 \end{aligned} \tag{2.22}$$

$$g_0(x) = 0 \qquad \text{for} \quad |x| > 2. \tag{2.23}$$

Assuming (2.22) and (2.23) hold for some $g_0 \in \mathscr{G}$, note that the integrand in (2.21) is identically zero in x when $g = g_0$, i.e., $R(g_0, g_0) = 0$ so that from (2.20)

$$I(g_0) = Q(0) - Q(g_0) - R(g_0, g_0) = -Q(g_0). \tag{2.24}$$

Further, since $g \in \mathcal{G}$ satisfies (2.7) the integrand of (2.21) for any $g \in \mathcal{G}$ is non-positive because of (2.22) and so $R(g, g_0) \leqslant 0$. Thus from (2.24), for any $g \in \mathcal{G}$

$$I(g) \geqslant I(g_0) + Q(g - g_0) \tag{2.25}$$

which gives (1.17) where $g_0 = Tf_0$ and f_0 is the function in (2.6) corresponding to g_0. Since $Q(g - g_0) \geqslant 0$ and $Q(g - g_0) = 0$ only for $g - g_0 = 0$ from (2.16), the uniqueness of the minimizing g_0 (or f_0) follows.

It remains to produce a $g_0 \in \mathcal{G}$ satisfying (2.22) and (2.23). The function g_0 was found by one of us (BFL) and although we considered giving a motivation for the choice of g_0 it seems best to simply give g_0 and verify its properties.

Consider the function

$$\begin{aligned} G_0(z) = \frac{2}{\pi}\left(\left(1 - \frac{z^2}{4}\right)^{1/2} + \frac{iz}{2}\right) - \frac{iz}{\pi}\log\left(\frac{z}{i}\right) \\ - \frac{iz}{\pi}\log\left(\left(1 - \frac{z^2}{4}\right)^{1/2} + \frac{iz}{2}\right), \end{aligned} \tag{2.26}$$

which is analytic in the upper half-plane $\operatorname{Im} z > 0$. On the real axis $z = x$,

$$\left(1 - \frac{z^2}{4}\right)^{1/2} = \begin{cases} \left(1 - \dfrac{x^2}{4}\right)^{1/2}, & |x| \leqslant 2 \\ -i \operatorname{sgn} x \left(\dfrac{x^2}{4} - 1\right)^{1/2}, & |x| > 2 \end{cases} \tag{2.27}$$

as can be seen by indenting around $z = +2$. Thus on the real axis $z = x$

$$\log\left(\left(1 - \frac{z^2}{4}\right)^{1/2} + i\frac{z}{2}\right) = \begin{cases} i \sin^{-1}\dfrac{x}{2}, & |x| \leqslant 2 \\ \dfrac{i\pi}{2}\operatorname{sgn} x + \log\left(\dfrac{|x|}{2} - \left(\dfrac{x^2}{4} - 1\right)^{1/2}\right), & \\ & |x| > 2. \end{cases} \tag{2.28}$$

We then verify that on the real axis $z = x$,

$$G_0(x) = \begin{cases} \dfrac{2}{\pi}\left(\left(1 - \dfrac{x^2}{4}\right)^{1/2} + \dfrac{ix}{2}\right) - \dfrac{|x|}{2} - \dfrac{ix}{\pi}\log|x| & \\ \qquad + \dfrac{x}{\pi}\sin^{-1}\dfrac{x}{2}, \quad |x| < 2 & \\ \dfrac{2}{\pi}\left(\dfrac{ix}{2} - i(\operatorname{sgn} x)\left(\dfrac{x^2}{4} - 1\right)^{1/2}\right) - \dfrac{ix}{\pi}\log|x| & \\ \qquad - \dfrac{ix}{\pi}\log\left(\dfrac{|x|}{2} - \left(\dfrac{x^2}{4} - 1\right)^{1/2}\right), \quad |x| > 2. & \end{cases} \tag{2.29}$$

It is easy to check from (2.29) that $G_0(z) = O(1/|z|)$ as $|z| \to \infty$ in $\operatorname{Im} z \geqslant 0$ and so [8, p. 125], the real and imaginary parts of $G_0(x)$ are conjugate Hilbert transforms, i.e.,

$$g_0(x) = \operatorname{Re} G_0(x) = \begin{cases} \dfrac{2}{\pi}\left(1 - \dfrac{x^2}{4}\right)^{1/2} + \dfrac{x}{\pi} \sin^{-1} \dfrac{x}{2} - \dfrac{|x|}{2}, & |x| \leqslant 2 \\ 0, & |x| > 2 \end{cases} \tag{2.30}$$

has Hilbert transform

$$\tilde{g}_0(x) = \operatorname{Im} G_0(x) = \begin{cases} \dfrac{x}{\pi} - \dfrac{x}{\pi} \log |x|, & |x| \leqslant 2 \\ \dfrac{2}{\pi}\left(\dfrac{x}{2} - (\operatorname{sgn} x)\left(\dfrac{x^2}{4} - 1\right)^{1/2}\right) - \dfrac{x}{\pi} \log |x| & \\ \qquad - \dfrac{x}{\pi} \log\left(\dfrac{|x|}{2} - \left(\dfrac{x^2}{4} - 1\right)^{1/2}\right), & |x| > 2. \end{cases} \tag{2.31}$$

It is easy to check from (2.30) and (2.31) that g_0 satisfies (2.23) and that in (2.22),

$$p(x) = x\left[\log\left(\frac{|x|}{2} + \left(\frac{x^2}{4} - 1\right)^{1/2}\right) - \frac{2}{|x|}\left(\frac{x^2}{4} - 1\right)^{1/2}\right], \qquad |x| > 2. \tag{2.32}$$

It is easy to check that $p(x)/x$ has a positive derivative for $|x| \geqslant 2$ and since it vanishes for $x = 2$, (2.22) follows easily. We must check also that $g_0 \in \mathscr{G}$, i.e., (2.7) and (2.8) hold. Since

$$g_0'(x) = \frac{1}{2}\left(\frac{2}{\pi} \sin^{-1} \frac{x}{2} - \operatorname{sgn} x\right), \qquad |x| \leqslant 2, \tag{2.33}$$

we verify directly that (2.7) holds and

$$-\int x g_0'(x) = \int g_0(x)\, dx = 1, \tag{2.34}$$

i.e., $g_0 \in \mathscr{G}$. Since we have exhibited $g_0 \in \mathscr{G}$ satisfying (2.22) and (2.23), the argument below (2.23) shows that (2.25) holds for every $g \in \mathscr{G}$. By the one-one correspondence between $g = Tf$ and f we have proved (1.17). Note that (2) follows from (2.30) by setting $\xi = 2 \cos \theta$, $0 \leqslant \theta \leqslant \pi$.

We next prove that if $f(0) \leqslant 2 - \epsilon$ then $Q(Tf - Tf_0) \geqslant \delta > 0$ which by (1.17) gives $c \geqslant 2$ as pointed out at the end of Section 1. Note first that $g = Tf$ satisfies

$$g(\xi) = 0, \qquad \xi \leqslant -(2 - \epsilon) \tag{2.35}$$

because from (1.14), $g(\xi) = Tf(\xi) = 0$ for $\xi < -f(0)$. Since $g_0'(\xi) < 0$ for $-2 < \xi < -(2 - \epsilon)$ by (2.33) we see that $K(t)$ defined by

$$K(t) = \int_0^\infty (g'(t-u) - g_0'(t-u))\, e^{-u}\, du \tag{2.36}$$

satisfies, for $-2 < t < -(2 - \epsilon)$,

$$K(t) = -\int_0^{2+t} g_0'(t-u)\, e^{-u}\, du > e^{-(2+t)} g_0(t), \tag{2.37}$$

whence for any $g = Tf$ with $f(0) \leqslant 2 - \epsilon$, there is a fixed $\delta_1 > 0$ with

$$\int K^2(t)\, dt \geqslant \delta. \tag{2.38}$$

From (2.16) and the inequality $2 \mid \omega \mid \leqslant 1 + \omega^2$,

$$\begin{aligned} Q(Tf - Tf_0) &= Q(g - g_0) \\ &= \tfrac{1}{4} \int \mid \hat{g}(\omega) - \hat{g}_0(\omega)|^2 \mid \omega \mid d\omega \\ &\geqslant \tfrac{1}{4} \int \mid \hat{g}(\omega) - \hat{g}_0(\omega)|^2 \frac{2 \mid \omega \mid}{1 + \omega^2} \mid \omega \mid d\omega \qquad (2.39) \\ &= \tfrac{1}{2} \int \mid i\omega\hat{g}(\omega) - i\omega\hat{g}_0(\omega)|^2 \left| \frac{1}{1 + i\omega} \right|^2 d\omega \\ &= \pi \int K^2(t)\, dt > \delta, \end{aligned}$$

where we have used (2.36) and Parseval's identity in the last line. This shows that $f(0) \leqslant 2 - \epsilon$ implies that

$$H(f) \leqslant -\tfrac{1}{2} + \delta \tag{2.40}$$

so that (1.9) holds, which yields $c \geqslant 2$.

We remark that the linear part in (2.20) gives an inequality similar to (2.25) which may be useful in estimating $H(f) = I(g)$,

$$I(g) \geqslant I(g_0) - R(g, g_0). \tag{2.41}$$

This immediately follows from (2.20) and (2.24) since $Q(g - g_0) \geqslant 0$.

3. Minimization Under Constraints

Let $\mathscr{F}_{a,b}$ denote the set of $f \in \mathscr{F}$ for which

$$f(0) \leqslant a, \qquad f^{-1}(0) = \inf[x: f(x) = 0] \leqslant b. \tag{3.1}$$

If $0 < a \leqslant \infty$, $0 < b \leqslant \infty$, and $ab \geqslant 1$, $\mathscr{F}_{a,b}$ is not empty and we denote

$$H_0(a, b) = \inf_{\mathscr{F}_{a,b}} H(f), \tag{3.2}$$

$$H_0(a) = \inf_b H_0(a, b), \tag{3.3}$$

$$H_0 = \inf_a H_0(a). \tag{3.4}$$

It follows from the results of Section 2 that

$$H_0 = -\tfrac{1}{2} = H_0(a, b), \qquad a \geqslant 2, \quad b \geqslant 2, \tag{3.5}$$

since $f_0 \in \mathscr{F}_{a,b}$ for $a \geqslant 2$, $b \geqslant 2$ and achieves the minimum of $H(f)$ with no constraints.

To state the explicit formula for $H_0(a, b)$ and $H_0(a)$ it is enough to consider the case $a \leqslant b$ by symmetry and by (3.5) we may also assume that $a \leqslant 2$. Let

$$b_0 = b_0(a) = \frac{1}{a} - \frac{a}{4} + \left(2 + \frac{a^2}{2}\right)^{1/2}, \qquad a \leqslant 2. \tag{3.6}$$

Then, for $a \leqslant 2$,

$$\begin{aligned} H_0(a) &= H_0(a, b_0) \\ &= H_0(a, b), \qquad b \geqslant b_0 . \end{aligned} \tag{3.7}$$

For the case $a \leqslant 2$, $a \leqslant b \leqslant b_0(a)$ the values of $H_0(a, b)$ and the unique minimizing functions $f_0(x; a, b) \in \mathscr{F}_{a,b}$ are given as follows:

Starting with new fundamental parameters α and β, $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$, define

$$c = \frac{1 + \alpha^2}{2\alpha}, \qquad d = \frac{1 + \beta^2}{2\beta}, \qquad \gamma = \frac{1 - \beta^2}{2\beta} - \frac{1 - \alpha^2}{2\alpha}, \tag{3.8a}$$

$$A = \frac{3}{4} + \frac{(1 - \alpha^2)(1 - \beta^2)}{4\alpha\beta} - \frac{\alpha^2 + \beta^2}{4}, \tag{3.8b}$$

$$a = \frac{1 - \beta^2 + 2\alpha\beta}{2\beta A^{1/2}}, \qquad b = \frac{1 - \alpha^2 + 2\alpha\beta}{2\alpha A^{1/2}}. \tag{3.9}$$

For a and b satisfying

$$0 < a \leqslant 2, \qquad a \leqslant b \leqslant b_0(a) \tag{3.10}$$

there is a unique pair $0 < \alpha \leqslant \beta \leqslant 1$ satisfying (3.8) and (3.9). For (a, b) satisfying (3.10), the unique minimizing function $f_0(x; a, b) \in \mathscr{F}_{a,b}$ satisfies the constraints (3.1) with equality, i.e.,

$$f_0(0; a, b) = a, \qquad f_0^{-1}(0; a, b) = b, \tag{3.11}$$

whereas if, e.g., $a \leqslant 2$, $b > b_0(a)$ then $f_0^{-1}(0; a, b) = b_0$. Thus for (a, b) satisfying (3.10) we have

$$\begin{aligned} H_0(a, b) = {} & \frac{1}{4A} - \frac{3}{2} - \frac{1}{2}\log(4\alpha\beta A) - \frac{1}{4}(b^2 - a^2)\log\frac{\beta}{\alpha} \\ & + \frac{(a+b)^2}{4}\log\frac{(\alpha/\beta) + 2 + (\beta/\alpha)}{(1+\alpha\beta)^2} + \frac{1}{2}(ab - 1)\log\alpha\beta \end{aligned} \tag{3.12}$$

and

$$f_0(x; a, b) = a, \qquad 0 \leqslant x \leqslant \frac{(1-\alpha)^2}{2\alpha A^{1/2}}, \tag{3.13}$$

$$f_0(b^-; a, b) = \frac{(1-\beta)^2}{2\beta A^{1/2}}, \tag{3.14}$$

$$f_0(x; a, b) = 0, \qquad x \geqslant b, \tag{3.15}$$

$$\begin{aligned} f_0(x; a, b) = {} & \frac{d - \cos\theta}{A^{1/2}} + \frac{1}{\pi A^{1/2}}(\sin\theta + \theta(\alpha - \beta + \cos\theta)) \\ & + 2(\cos\theta - d)\tan^{-1}\frac{\beta\sin\theta}{1 - \beta\cos\theta} \\ & - 2(\cos\theta + c)\tan^{-1}\frac{\alpha\sin\theta}{1 + \alpha\cos\theta}, \end{aligned} \tag{3.16}$$

where $0 \leqslant \theta \leqslant \pi$ and

$$x = f_0(x; a, b) + \frac{\cos\theta - \gamma}{A^{1/2}}. \tag{3.17}$$

The formulas simplify somewhat in the extreme cases $b = a$ and $b = b_0(a)$. For $b = a$ we have $\alpha = \beta$ and

$$\begin{aligned} H_0(a, a) = {} & \frac{\alpha^2}{1 + \alpha^2 - \alpha^4} - \frac{3}{2} - \frac{1}{2}\log(1 + \alpha^2 - \alpha^4) \\ & + \frac{a^2 - 1}{2}\log\alpha^2 + 2a^2\log\frac{2}{1 + \alpha^2}, \end{aligned} \tag{3.18}$$

where α is determined by setting $\alpha = \beta$ in (3.9) as

$$a^2 = \frac{(1+\alpha^2)^2}{1 + \alpha^2 - \alpha^4}, \qquad 1 \leqslant a \leqslant 2. \tag{3.19}$$

In the limiting case $a = 1$, we have $\alpha = 0$ and

$$H_0(1, 1) = -\tfrac{3}{2} + 2 \log 2 \tag{3.20}$$

which corresponds to $f_0(x; 1, 1) = 1$, $0 < x < 1$, which is of course the sole member of $\mathscr{F}_{1,1}$. In general, $f_0(x; a, a)$ does not greatly simplify.

For the case $a \leqslant 2$, $b = b_0(a)$, we have $\beta = 1$ and

$$H_0(a) = H_0(a, b_0) = -1 + \frac{a^2}{8} + \log \frac{a}{2} - \left(1 + \frac{a^2}{4}\right) \log \frac{2a^2}{4 + a^2} \tag{3.21}$$

$$\begin{aligned} f_0(x; a, b_0) &= \frac{1}{\pi A^{1/2}} \left\{\alpha\theta + \sin\theta - 2(c + \cos\theta) \tan^{-1} \frac{\alpha \sin\theta}{1 + \alpha\sin\theta}\right\}, \\ x &= f_0(x; a, b_0) + \frac{\cos\theta - \gamma}{A^{1/2}}, \qquad \frac{(1-\alpha)^2}{2\alpha A^{1/2}} \leqslant x \leqslant b_0 \end{aligned} \tag{3.22}$$

where $0 \leqslant \theta \leqslant \pi$, and

$$f_0(x; a, b_0) = \begin{cases} a, & 0 \leqslant x \leqslant \dfrac{1}{a} + \dfrac{3a}{4} - \left(2 + \dfrac{a^2}{2}\right)^{1/2} \\ 0, & x \geqslant b_0 \end{cases} \tag{3.23}$$

and from (3.8) and (3.9),

$$\alpha^2 = \frac{2a^2}{4 + a^2}, \qquad b_0 = \frac{1}{a} - \frac{a}{4} + \left(2 + \frac{a^2}{2}\right)^{1/2} \tag{3.24}$$

Thus when $b \geqslant b_0(a)$, $a \leqslant 2$ the optimal $f_0(x; a, b)$ is continuous at $x = b$, where it vanishes. However, under the constraint $f(0) < 2$ the optimal function is constant over an interval including the origin. This interval is relatively small even for $a = 2^{1/2}$ having a length $(5(2^{1/2})/4)\, 3^{1/2} = 0.0357$. For comparison, the triangular function $f(x) = 2^{1/2} - x$, $0 \leqslant x \leqslant 2^{1/2}$ is not optimal for the case $a = b = 2^{1/2}$. We have by direct calculation

$$H(f) = \tfrac{3}{2}(\log 2 - 1) = -0.4602792, \tag{3.25}$$

whereas from (3.18) and (3.21)

$$\begin{aligned} H_0(2^{1/2}, 2^{1/2}) &= \tfrac{9}{4} \log 3 + \tfrac{9}{2} \log 2 - 5 \log(1 + 3^{1/2}) + 2(3^{1/2}) \\ &\quad - \tfrac{9}{2} = -0.4701211, \end{aligned} \tag{3.26}$$

$$H_0(2^{1/2}) = -\tfrac{3}{4} - 2 \log 2 + \tfrac{3}{2} \log 3 = -0.4883759, \tag{3.27}$$

$$b_0(2^{1/2}) = \tfrac{1}{4} 2^{1/2} + 3^{1/2} = 2.085604. \tag{3.28}$$

It is perhaps worth remarking that the overall optimal function, $f_0 \in \mathscr{F}$, given in (2) satisfies

$$f_0(x; 2, 2) = 2 - (3\pi x/2)^{2/3} + O(x), \qquad x \to 0 \tag{3.29}$$

and f_0 is symmetric about the line $y = x$. Some graphs of $f_0(x; a, b)$ are given in Fig. 3.

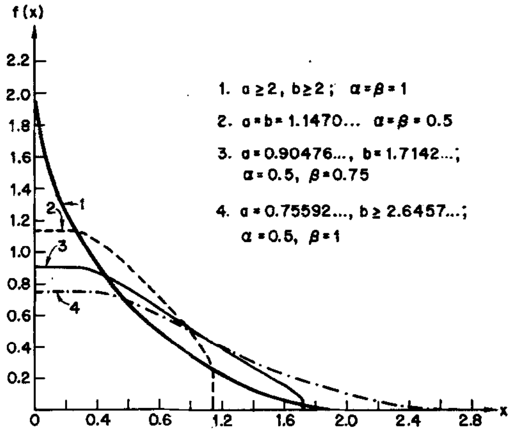

FIG. 3. $f_0(x; a, b)$ for various values of a, b.

The proofs of the results in this section are omitted because they are cumbersome. The basic method is the same as the proof for the unconstrained case, i.e., to find a function $g_0(x; a, b)$ with conditions analogous to those in (2.22) and (2.23). We hope to give full proofs elsewhere.

Note added in proof. The result $c = 2$ has now been proved by A. M. Vershik and C. V. Kerov.

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