Some Inequalities Concerning Random Subsets of a Set

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Abstract—A class of plausible inequalities between conditional probabilities is considered involving random subsets of a set which arise, for example, in studying the capacity of a multiple access channel. Some of the inequalities are true and can be proved by the Fortuin-Kasteleyn-Ginibre (FKG) inequality. Some others are true but seem to need other methods. Still others are false. The results also apply to studying the conditional probability of the failure of a device designed to achieve reliability by using redundant parts, given side information about the failure of some of its components.

I. INTRODUCTION

To improve reliability devices are often designed with redundant parts, some of which may be defective without causing breakdown. Suppose the condition for device breakdown is that there must be at least \( a_i \) defective items in component \( A_i \), \( i=1, \ldots, m \), of the device; otherwise the device works. Suppose also that we know that there are at least \( b_j \) defective items in component \( B_j \), \( j=1, \ldots, k \), where the components \( A_i, B_j \) are arbitrary subsets of the parts. It would seem that as \( b_j \) increases the probability of failure of the device would also increase. We show in fact that this is true provided the probabilities that each item is defective are independent, and also that \( k=1 \). It is surprising, however, that this is false for \( k=2 \) in general.

Let \( N \) denote a set of items \( \{1, \ldots, n\} \) and let \( D \) denote the random subset of defective items in \( N \) where item \( i \) has probability \( p_i \) of being defective, probability \( q_i=1-p_i \) of being good, and the states of the items are independent. Let \( A_i, i=1, \ldots, m \), and \( B \) be fixed subsets of \( N \). Define the events

\[
E = \bigcap_{i=1}^{m} \{ |A_i \cap D| \geq a_i \}, \quad F_B = |B \cap D| \geq b, \quad G_b = |B \cap D| = b.
\]

We prove the following inequalities.

**Theorem 1:**

1. \( P(E_F_B) \geq P(E) \).
2. \( P(E_{G_b}) \geq P(E_{G_{b-1}}) \).
3. \( P(F_B|E) \geq P(F_{b-1}) \).

Is there something more fundamental underlying these inequalities? For example, could it be true that

\[
P \left( F \left( \bigcap_{j=1}^{k} \{ |B_j \cap D| \geq y_j \} \right) \right) \geq P \left( \bigcup_{j=1}^{k} \{ |B_j \cap D| \geq y_j \} \right),
\]

for \( y_j \geq y_j', j=1, \ldots, k \), which says that the inequality holds whenever one set of conditions contains the other set? The following example shows that this is not always the case. Let \( N=\{1,2\}, A=\{1\}, B_1=\{1,2\}, B_2=\{2\} \). Then

\[
P( |A \cap D| \geq 1 | |B_1 \cap D| \geq 1, |B_2 \cap D| \geq 1 ) = p < p/(1-q^2) = P( |A \cap D| \geq 1 | |B_1 \cap D| \geq 1 )
\]

Define \( F_b = \bigcap_{j=1}^{k} \{ |B_j \cap D| \geq b_j \} \) where \( b \) is the vector \((b_1, \ldots, b_k)\). Let \( b-1 \) denote the vector \((b_1-1, \ldots, b_k)\). The next question is whether we can extend Theorem 1 when \( x \) replaces \( x \) for the subscripts of \( F \) and \( G \). It turns out that even though those inequalities in Theorem 1 look similar, only a) can be extended.

**Theorem 2:** \( P(E_{F_b}) \geq P(E) \).

We now give counterexamples to the possible extensions of b)-e). We assume that \( p_i=p, A=\{1\} \) and \( a=1 \) throughout. For b), c), and e), let

\[
B_1 = \{1,2,3\}, \quad B_2 = \{2\}, \quad B_3 = \{3\}, \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = 1.
\]

b) \( P(E_{G_b}) = 0 < p = P(E_{G_{b-1}}) \).

c) \( P(E_{F_b}) = p < 1 - q^2 = P(E_{F_{b-1}}) \).

d) \( P(F_{b-1}|E) < p^2q^{-1} = P(F_{b-1}) \).

e) \( P(F_{b-1}|E) = p/3 < p^2q^{-1} = P(F_{b-1}) \).
For d) let \( B_1 = (1, 2), B_2 = (2), b_1 = 1, b_2 = 0. \) Then

\[
P(E|F_b) = \frac{p}{1 - q} < 1 = P(E|G_b).
\]

We prove Theorem 2 by using the Fortuin–Kasteleyn–Ginibre (FKG) inequality and Theorem 1 by using a coupling argument. It seems that the two theorems, in spite of their similarity, are tailored for the two different methods. In particular we cannot find an FKG proof for the last four inequalities in Theorem 1.

We extend our results to a Poisson model and generalize an inequality of Mikhailov and Tsybakov [2] which is instrumental in studying the capacity of a multiple access channel.

Finally we show that Theorem 1 fails but Theorem 2 holds when \( N \) is a set of independent nonnegative integer-valued (rather than \( \{0, 1\} \)-valued) random variables.

II. THE PROOF OF THEOREM 1

We first prove b). Define \( X = \{ S \subseteq B: |S| = b\} \) and \( Y = \{ S \subseteq B: |S| = b - 1\} \). We show that we can generate a probability space, and a couple \( (D, D') \) of random subsets of \( N \) such that \( \bar{D} \subseteq \bar{D}' \) and \( \bar{D}' \) individually have the conditional distributions of \( D \) given \( G_{b-1} \) and \( G_b \), respectively (the present construction is due to C. L. Mallows). As soon as we have constructed such a couple, b) will follow at once, since

\[
P(E|D_{b-1}) = \sum_{i=1}^{m} \{ |A_i \cap \bar{D}| \geq a_i \} \leq P\left( \bigcap_{i=1}^{m} \{ |A_i \cap \bar{D}'| \geq a_i \} \right) = P(E|G_b).
\]

Define

\[
p'_i = \frac{p_i / q_i}{\sum_{j \in B} p_j / q_j}, \quad \text{if } i \in B.
\]

Then a random \( x \in X \) has probability

\[
f(x) = \prod_{i \in x} p'_i
\]

and a random \( y \in Y \) has probability

\[
f(y) = \frac{\prod_{i \in y} p'_i}{\sum_{y} \prod_{i \in y} p'_i}.
\]

For a given \( y \) define \( P(x, y) = 0 \) if \( x \not\supseteq y \) and

\[
P(x, y) = \sum_{x' \subseteq Z} \frac{f(x')/|x' \setminus y|}{f(x')} \quad \text{if } x \supseteq y
\]

where \( Z = \{ x' \in X: x' \cap y \supseteq x \} \). Denote the \( x \) so generated by \( y \).

Then \( P(x, y) = \sum_{x' \subseteq Z} f(x')/|x' \setminus y| \) and \( f(y) \) is increasing in \( y \) and \( f(y) \) is increasing in \( y' \) such that \( y \supseteq y' \) and \( |y' \setminus y| = |x' \setminus y| \). Therefore \( P(x, y) = f(x) \sum_{y' \subseteq Z} f(y) = f(x) \); i.e., \( x \) is randomly generated.

A random set \( \bar{D} \subseteq N \) satisfying \( |B \cap \bar{D}| = b - 1 \) can be generated by adding a random set \( W \) from \( N \setminus B \) to a random set \( y \in Y \). Clearly, \( x + y \) is a random set \( \bar{D}' \subseteq N \) satisfying \( |B \cap \bar{D}'| = b \). By our construction, as already observed, if \( E \) occurs under \( D \), then \( E \) occurs under \( D' \) and b) is proved.

Next we prove c) and d):

\[
P(E|F_b) = \frac{\sum_{i=0}^{\infty} P(E, G_{b+i})}{P(F_b)} = \frac{\sum_{i=0}^{\infty} P(E, G_{b+i})}{P(F_b)} = \frac{\sum_{i=0}^{\infty} P(E, G_{b+i})}{P(F_{b-i})}
\]

Theorem 2 is proved.
IV. The Poisson Model

Consider the number of defectives in a given interval \( N \). Suppose that we partition \( N \) into \( n \) equally spaced subintervals such that the probability of each subinterval containing a defective is \( p \) and containing more than one defective is zero. Then Theorems 1 and 2 also apply to this case. If we let \( n \to \infty \) and \( np \to \lambda \), then we have Theorems 1 and 2 when the number of defectives is a Poisson variable.

The Poisson model has been widely used in the analysis of the capacity of a multiple access channel. In particular, Mikhailov and Tsybakov [2] proved a special case of Theorem 1 c), i.e., for \( b = 1 \) or 2 and \( a_i = 2 \), and used it to obtain an upper bound (the best so far) for the capacity. In a recent paper of Tsybakov, Mikhailov, and Likhanov [3], Theorem 1 b) was proved for the special case \( a_i = k \) for a fixed \( k \). This inequality was then crucially used to obtain upper bounds for the capabilities of channels which can simultaneously transmit up to \( k-1 \) messages.

V. The Case of Independent Random Variables

In this section \( N = (X_1, \ldots, X_n) \) is a set of independent non-negative integer-valued random variables with probability density function (pdf) \( P_1, \ldots, P_n \). We first show that Theorem 1 b)-e) are no longer true even if \( P = P \) for all \( i \).

Let \( X \) and \( Y \) be independent identically distributed (i.i.d.) and

\[
\begin{align*}
P(X=1) &= P(X=9) = 0.30 \\
P(X=2) &= P(X=8) = 0.15 \\
P(X=3) &= P(X=7) = 0.04 \\
P(X=4) &= P(X=6) = 0.01.
\end{align*}
\]

Then

\[
\begin{align*}
P(X \geq 2 | Y = 7) &= \frac{19}{34} < 1 = P(X \geq 2 | X + Y = 6) \\
P(X \geq 2 | Y \geq 7) &= \frac{2003}{2503} < \frac{1211}{1511} = P(X \geq 2 | X + Y \geq 6) \\
P(X \geq 2 | X \geq 6) &= \frac{1211}{1511} < 1 = P(X \geq 2 | X + Y = 6) \\
P(X + Y \geq 7 | X \geq 2) &= \frac{7509}{7555} < \frac{6009}{6055} = P(X + Y \geq 7) \\
P(X + Y \geq 6 | X \geq 2) &= \frac{7555}{7555} = P(X + Y \geq 6).
\end{align*}
\]

All inequalities in Theorem 1 b)-e) are reversed.

Next we show that Theorem 2 remains true under the current probability model. Let \( x = (x_1, \ldots, x_n) \) where \( x_i \) is the random value of \( X_i \). Let

\[
\begin{align*}
x \lor y &= (\min \{ x_1, y_1 \}, \ldots, \min \{ x_n, y_n \}) \\
x \land y &= (\max \{ x_1, y_1 \}, \ldots, \max \{ x_n, y_n \}).
\end{align*}
\]

Let \( P(x) = (P_1(x_1), \ldots, P_n(x_n)) \). Then it is easily seen that

\[
P(x) P(y) = P(x \lor y) P(x \land y)
\]

where the multiplication stands for inner product. Furthermore, \( E \) and \( F \) are both increasing in \( x \). Hence the FKG inequality still applies.

This proof can be extended to the continuous variable case by a standard "limit" argument.

VI. Concluding Remarks

Theorems 1 and 2 provide an interesting contrast in appreciating the power and the limitations of the FKG inequality. While the FKG inequality proves Theorem 2 effortlessly, saving a lengthy and painstaking argument of the type used in proving Theorem 1 a), there does not seem to be an easy way of applying it to those similar inequalities in Theorem 1 b) to e).

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REFERENCES


Measures of Mutual and Causal Dependence Between Two Time Series

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Abstract—New measures are proposed for mutual and causal dependence between two time series, based on information theoretical ideas. The measure of mutual dependence is shown to be the sum of the measure of unidirectional causal dependence from the first time series to the second, the measure of unidirectional causal dependence from the second to the first, and the measure of instantaneous causal dependence. The measures are applicable to any kind of time series: continuous, discrete, or categorical.

I. INTRODUCTION

In areas such as econometrics, engineering, medicine, ecology, biology, education, and psychology it is frequently of interest to measure dependence between two time series. For example, we may wish to measure, in some way, the amount by which the prime interest rate influences unemployment, or, as another example, the amount by which the existence of a death penalty influences the number of severe crimes.

For the special case that the variables of the two time series are continuous, Geweke [5] proposed certain measures that are based on autoregressive (AR) and autoregressive with external inputs (ARX) modeling of the two time series. The unemployment rate, for example, is modeled first as an AR process and then, using the values of the prime interest rate, with an ARK model. The two associated innovation processes. This approach is based on the framework developed by Granger [6], who built on an earlier work of Wiener [14]. Related work on the problem of testing whether some sort of causal dependence exists between two time series is presented in Sims [12], Caines and Chan [1], and Caines et al. [2].

In this correspondence we propose new measures for mutual and causal dependence which are applicable to any kind of time series, be it continuous, discrete or categorical. Our measures are based on a class of probabilistic models which includes the AR and ARX classes of models used by Geweke as a particular case.

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