NOTES

CORRECTION TO

"RADON-NIKODYM DERIVATIVES OF
GAUSSIAN MEASURES"

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Introduction. J. R. Klauder kindly pointed out that the first statement of Theorem 11 of my paper [2] is incorrect. It was claimed incorrectly that if \( h = h(t), \ 0 \leq t \leq T \) is a (strictly) increasing absolutely continuous function with \( h(0) = 0 \), then a necessary and sufficient condition that the Gauss–Markov process

\[ X(t) = \frac{1}{(h'(t))^{1/2}} W(h(t)), \quad 0 \leq t \leq T \]

is equivalent to the Wiener process \( W, X \sim W \), is that

\[ \int_0^T \left( \frac{1}{(h'(t))^{1/2}} \right)^2 \, dt < \infty. \]

The case

\[ h(t) = t + t^3, \quad 0 \leq t \leq T = 1 \]

gives an example where (2) fails although \( X \sim W \). We will prove that the condition

\[ \int_0^T h(t) \left( \frac{1}{(h'(t))^{1/2}} \right)^2 \, dt < \infty \]

is necessary and sufficient for \( X \sim W \). Note that (3) satisfies (4) but not (2). Theorem 1 of [2] gives a general condition for a Gaussian process to be equivalent to \( W \) but the condition is difficult to apply in this case. Instead we use the elegant results of M. Hitsuda [1]. Note that [4] gives necessary and sufficient conditions among a restricted class of \( h \) for \( X \sim W \). Of course the exact scale normalization \( 1/(h'(t))^{1/2} \) in (1) is necessary for \( X \sim W \) (e.g., note that \( cW \sim W \) only for \( c = 1 \)).

The error in the argument in [2] that \( X \sim W \) implies (2) occurs in the ninth line from the bottom of page 344 where it is incorrectly claimed that \( v' \in L^2(0, T) \) if \( u'(\min(s, t))v'(\max(s, t)) \in L^2[0, T] \times [0, T] \).

The argument given for the converse assertion, that (2) implies \( X \sim W \), tacitly assumes that \( h \) is bounded and under this assumption is correct since then (2) implies (4) which implies that \( X \sim W \). However for unbounded \( h \), i.e., \( h(T) = \infty \), e.g.,

\[ h(t) = t/(1 - t), \quad 0 \leq t \leq T = 1, \]

\[ v'(s) v'(t) \in L^2(0, T) \]

implies

\[ \int_0^T \frac{1}{h'(t)^{1/2}} \, dt = \infty. \]
if (1) is defined by continuity at \( t = 1 \) so that \( X \) is the pinned Wiener process with \( X(1) = 0 \), then (2) holds but \( X \sim W \) is false since \( W(1) \neq 0 \) w.p.1. Thus the assertion \( "1 \notin sp(K)" \) holds automatically" on page 344 of [2] tacitly assumes bounded \( h \). Of course, Hitsuda’s method avoids the spectral condition altogether and has other advantages [1, page 299].

**Proof that (4) is necessary and sufficient that** \( X \sim W \). If (4) holds then

\[
\begin{align*}
  l(s, u) &= - (h'(u))^{t} (1/(h'(s))^{t})' ; & s > u \\
  &= 0 ; & s \leq u
\end{align*}
\]

is a Volterra kernel in \( L^{2}[0, T] \times [0, T] \) the primes denoting differentiation with respect to \( s \) or \( u \) as indicated in each term by the variable in parentheses. By Theorem 2 of [1], \( Y \sim W \) where \( Y \) is defined in terms of a Wiener process \( W \) by

\[
Y(t) = W(t) - \int_{0}^{t} \int_{0}^{s} l(s, u) dW(u) \, ds
\]

where we have used the argument on the top of page 306 of [1] to interchange the integrals in the second line of (7), and (6) in the third line. Since the last line of (7) is a Gaussian process with the same covariance as \( X \) in (1), it follows that \( X \) and \( Y \) are the same process (induce the same measure). Since \( Y \sim W \) and \( W \) is a Wiener process we have proved that (4) implies \( X \sim W \).

To prove that \( X \sim W \) implies (4), note that the process

\[
X(t) = \frac{1}{(h'(t))^{t}} \int_{0}^{s} (h'(u))^{t} dW(u)
\]

is the same process as \( X \) in (1) as observed above. Since \( X \) is equivalent to a Wiener process, by Theorem 1 of [1] there exists on the same space as \( X \) and \( W \) in (8), another Wiener process \( W \) for which

\[
X(t) = W(t) - \int_{s}^{t} l(s, u) dW(u) \, ds
\]

where \( l \) is a (unique) \( L^{2} \) Volterra kernel. Moreover \( W \) is a Wiener process with respect to the same \( \sigma \)-fields \( \mathcal{F}_{t} \) as \( W \).

Since \( (h'(t))^{t} X(t) = \int_{0}^{s} (h'(u))^{t} dW(u) \) is a martingale with respect to \( \mathcal{F}_{t} \), we have for any \( \tau < t \)

\[
E[X(t)(h'(t))^{t} | \mathcal{F}_{\tau}] = X(\tau)(h'(\tau))^{t}.
\]

From (9) and (10) with \( s \wedge \tau = \min(s, \tau) \), for \( \tau < t \)

\[
(h'(t))^{t} W(\tau) - (h'(t))^{t} \int_{0}^{s \wedge \tau} l(s, u) dW(u) \, ds
\]

\[
= (h'(\tau))^{t} W(\tau) - (h'(\tau))^{t} \int_{0}^{s \wedge \tau} l(s, u) dW(u) \, ds.
\]
Interchanging integrals as before since \( I \in L^2 [0, T] \times [0, T] \) we obtain
\[
(12) \quad W(\tau) ((h'(t))^\sharp - (h'(\tau))^\sharp) = \int_0^\tau ((h'(t))^\sharp \int_0^t I(s, u) \, ds - (h'(\tau))^\sharp \int_0^\tau I(s, u) \, ds) \, dW(u).
\]
Considering \( \tau \) and \( t \) as fixed and noting that \( \int_0^\tau \varphi \, dW = 0 \) for an \( L^2 \) function \( \varphi \) implies \( \varphi \equiv 0 \) a.e., we obtain that for each \( 0 < u < \tau < t \), a.e.
\[
(13) \quad (h'(t))^\sharp - (h'(\tau))^\sharp = (h'(t))^\sharp \int_u^\tau I(s, u) \, ds - (h'(\tau))^\sharp \int_0^\tau I(s, u) \, ds.
\]
Setting \( \tau = u \) we obtain easily that \( h \) is twice differentiable and \( I = I \) in (6). Thus \( l \in L^2 [0, T] \times [0, T] \), and since \( \int_0^T \int_0^t P(s, u) \, ds \, du \) is the left side of (4), we have shown that (4) holds.

We remark that since \( X \sim W \) implies the scale changed processes \( X \) and \( W \) where, for any \( Y \),
\[
(14) \quad Y(t) = \frac{1}{(g'(t))^\sharp} Y(g(t))
\]
are also equivalent, we have \( X \sim W \), for any increasing differentiable function \( g \) with \( g(0) = 0 \). Taking \( g \) to be \( h^{-1} \) and noting that \( X = W \) in this case we see that \( X \sim W \) and only if \( X \sim W \), i.e., the condition (4) must be invariant under the change from \( h \) to \( h^{-1} \). A direct proof of this fact is given in [3].

Other corrections in [2].

1. Israel Bar–David pointed out that (16.2), page 347, should include the additional term:
\[-\frac{1}{2} X^2(0)[R(0, 0)]^{-1}\]
on the right-hand side.

2. In footnote 3, page 332, the name referred to should be I. M. Golosov.

3. (18.19), page 352: change \( X_j \) to \( x_j \).

4. First line of display below (18.19), page 352: change \( T \) to \( T^2 \).

5. Change (18.21), page 352 to read
\[
(18.21) \quad \Delta^2 g_k = \frac{T^3}{n^2} \int_k g_{k+1}.
\]

REFERENCES


