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OPTIMAL RECONSTRUCTION OF A FUNCTION FROM ITS PROJECTIONS

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§1. Introduction.

Let $f(x, y)$ be square integrable and supported on the unit disk C. The projection $P_f(t, \theta)$ of f is the integral of f along the line $L(t, \theta)$: $x \cos \theta + y \sin \theta = t$. We find an explicit formula for the unique function $g(x, y)$ supported on C and of minimum L_2 norm, which has the same projections as f in each of n equally spaced directions or views, i.e., $P_g(t, \theta_j) = P_f(t, \theta_j)$, for all t and n equally spaced $\theta_j = j\pi/n$, $j = 0, 1, \cdots, n - 1$. We also show that the unique polynomial $P(x, y)$ of degree $n - 1$ which best approximates f in $L^2(C)$ is determined from the above n projections of f, and give a relatively simple explicit formula for P. The exact conditions on n functions $P_j(t)$, $j = 0, \cdots, n—1$, to be the n projections $P_f(t, \theta_j)$ of some $f \in L^2(C)$ are found.

These questions arise naturally in attempting to reconstruct the density $f(x, y, z)$ of an object, in each cross-sectional $x - y$ plane with $z = z_0$ fixed, from measurements of $P_f(t, \theta)$ obtained by passing a thin beam of x-rays along lines $L(t, \theta)$ in the $z = z_0$ plane. In the case treated here the x-ray beam is considered to move discretely in θ and then to translate continuously in t.

In a similarly motivated but different situation, considered by R. B. Marr [M], it is supposed that the projections of f are known over the $N(N - 1)/2$ lines which join each pair of N equally spaced points on the circumference of C. Marr found an explicit formula for the *polynomial* $P^{(M)}(x, y)$ of degree $M \leq N - 2$ whose integrals along the given lines best matches the $N(N - 1)/2$ given projections in the sense of minimizing the sum of squares of the differences. He also studied the case where all projections $P(t, \theta)$ are known and, among other results, found the exact conditions for a function $Q(t, \theta)$ to be the projections $P_f(t, \theta)$ for some $f \in L^2(C)$.

Marr's criterion for optimality has the form of finding the *polynomial* g whose projections match certain finitely many given projections with minimum error ϵ. There is of course no reason to restrict g to be a polynomial. In fact if the degree M of the polynomial is $\geq N - 2$, the error ϵ can be made zero. For a general function g, even for the case considered here where *all* line integrals in each of n views are given, there are many functions g with the exact given projections (assuming the given values are actually the projections of some function). One must give further conditions on g to determine it uniquely. Here we use the criterion of minimizing the L_2 norm of g because (a) this allows

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an explicit solution, (b) g is then the function with least oscillations about its mean value, which appears reasonable. It is perhaps worth remarking that the problem of minimizing the weighted L_2 norm, $\iint_C W(x, y)g^2(x, y)\, dx\, dy$, among functions g with the given projections appears very difficult to solve explicitly for all W except the constant weight W considered here. Indeed it would be especially interesting to know the solution for, e.g., the weight $W_k(x, y) = (1 - x^2 - y^2)^k$ which gives less importance to oscillations in g near the circumference of C, $k > 0$, than the constant weight ($W \equiv 1$, $k = 0$) does.

A more realistic optimum reconstruction criterion would suppose the projections of g are given for only *finitely* many lines and seek the minimum L_2 norm *function* with these projections. Unfortunately this is not well posed in the sense that there are *null* functions g having the given finite number of projections, constant along the lines inside C, and zero elsewhere. If we replace lines by strips the problem becomes awkward to solve in an explicit way except by a direct and nonilluminating matrix inversion.

More precisely, the projection of f along the line

$$L(t, \theta) = \{(x, y) : x \cos \theta + y \sin \theta = t\} \tag{1.1}$$

is

$$P_f(t, \theta) = \int_{-\infty}^{\infty} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta)\, ds \tag{1.2}$$

noting that

$$P_f(t, \theta + \pi) = P_f(-t, \theta) \tag{1.3}$$

and the integral in (1.2) only extends between $\pm(1 - t^2)^{\frac{1}{2}}$ since $f(x, y)$ vanishes if $x^2 + y^2 > 1$.

It is easy to see that if f is square integrable $P_f(t, \theta)$ is defined for a.e. t and θ by (1.2) and it is known [R] that P_f determines f up to a null function.

Suppose that $P_f(t, \theta_j) \equiv P_j(t)$ is known only for $\theta_j = j\pi/n, j = 0, 1, \cdots, n - 1$ and all t, i.e., n projections of f are given. Now f is not uniquely determined, and there are many functions $g \in L^2(C)$ with

$$P_g(t, \theta_j) = P_f(t, \theta_j), \quad \text{for} \quad j = 0, 1, \cdots, n - 1 \text{ and all } t. \tag{1.4}$$

We give an explicit formula for the (unique) function g satisfying (1.4) and smoothest in the sense that

$$V(g) = \iint_C (g - \bar{g})^2\, dx\, dy \text{ is minimum} \tag{1.5}$$

where $\bar{g}$ is the average of g over C,

$$\bar{g} = \frac{1}{\pi} \iint_C g(x, y)\, dx\, dy. \tag{1.6}$$

We show g is a sum of ridge functions in directions θ_j, i.e.,

$$g(x, y) = \sum_{j=0}^{n-1} \rho_j(x \cos \theta_j + y \sin \theta_j) \tag{1.7}$$

and in fact g is the best L^2 approximation to f by functions of the form (1.7).

We show in §5 that every polynomial $P(x, y)$ of degree $n - 1$ in (x, y) is of the form (1.7) with ρ_j polynomials in one variable and that the (unique) polynomial of degree $n - 1$ in x and y which best approximates f in $L^2(C)$ is determined from $P_f(t, \theta_j)$, $j = 0, 1, \cdots, n - 1$.

We give in §6 the exact (consistency) conditions on n functions $P_j(t)$, $j = 0, 1, \cdots, n - 1$ to be the n projections of some function $f \in L^2(C)$,

$$P_j(t) = P_f(t, \theta_j), \qquad j = 0, 1, \cdots, n-1, \ -1 \leq t \leq 1. \tag{1.9}$$

§2. Ridge functions.

We note that a $g \in L^2(C)$ satisfying (1.4) and (1.5) satisfies the following problem:

Problem. Find a function $g \in L^2(C)$ satisfying

$$P_g(t, \theta_j) = P_f(t, \theta_j), \qquad j = 0, 1, \cdots, n-1, \ -1 \leq t \leq 1 \tag{2.1a}$$

and

$$\iint_C g^2(x, y)\, dx\, dy \quad \text{is a minimum.} \tag{2.1b}$$

Indeed, the mean $\bar{g}$ of (1.6) satisfies

$$\bar{g} = \frac{1}{\pi} \int_{-1}^{1} P_g(t, \theta_j)\, dt = \frac{1}{\pi} \int_{-1}^{1} P_f(t, \theta_j)\, dt = \bar{f} \tag{2.2}$$

for $j = 0, 1, \cdots, n - 1$ using (1.4) so that $\bar{g}$ is determined by (1.4). (Note that $\int_{-1}^{1} P_f(t, \theta_j)\, dt$ must be independent of j.)

Since (1.5) is given by

$$V(g) = \iint_C g^2(x, y)\, dxdy - \pi(\bar{g})^2 \tag{2.3}$$

we see that minimizing (1.5) subject to (1.4) is the same as (2.1).

Define the inner product and norm in $L^2(C)$ in the usual way

$$(f_1, f_2) = \iint_C f_1 f_2, \qquad |f|^2 = (f, f), \qquad f_i, f \in L^2(C) \tag{2.4}$$

and consider the closed subspace $R^2(C)$, of sums g of ridge functions, ρ_j, with direction θ_j, that is

$$g(x, y) = \sum_{j=0}^{n-1} \rho_j(x, y). \tag{2.5}$$

Here we call a ridge function $\rho(x, y)$ with direction θ a function of the form

$$\rho(x, y) = \rho(x \cos \theta + y \sin \theta). \tag{2.6}$$

Note that if ρ is a ridge function with direction θ and $f = f(x, y) \in L^2(C)$, then from (1.2),

$$(f, \rho) = \int_{-1}^{1} P_f(t, \theta)\rho(t)\, dt. \tag{2.7}$$

We next show that any g satisfying (2.1) is unique and is a sum of ridge functions. Let g_1 be the (Hilbert space) projection of $f \in L^2(C)$ onto $R^2(C)$ so that $g_1 \in R^2(C)$ and satisfies (2.5). The difference, $f_1 = f - g_1$, is then orthogonal to $R^2(C)$,

$$f_1 = f - g_1 \perp R^2(C) \tag{2.8}$$

and so from (2.7), if ρ_j is a ridge function with direction θ_j,

$$0 = (f_1, \rho_j) = \int_{-1}^{1} P_{f_1}(t, \theta_j)\rho_j(t)\, dt. \tag{2.9}$$

Since ρ_j can be an arbitrary function in $L^2(-1, 1)$

$$P_{f_1}(t, \theta_j) = 0 \qquad j = 0, 1, \cdots, n-1, \tag{2.10}$$

or from (2.8)

$$P_f(t, \theta_j) = P_{g_1}(t, \theta_j). \tag{2.11}$$

Any function g with the same projections as f will have its norm reduced by projection onto $R^2(C)$ and so if g satisfies (2.1) we must have $g = g_1$. Since $g = g_1$, g is unique, is a sum of ridge functions, and is the best approximation to f by a function in $R^2(C)$, i.e., by a sum of ridge functions.

Given the projections of f, $-1 \leq t \leq 1$,

$$P_j(t) = P_f(t, \theta_j), \tag{2.12}$$

we know from (2.5) and (2.11) that there exist functions $\rho_j(t)$ such that

$$P_j(t) = P_f(t, \theta_j) = P_g(t, \theta_j) \tag{2.13}$$

where from (2.5) g is the sum of the ridge functions ρ_j,

$$g(x, y) = \sum_{j=0}^{n-1} \rho_j(x \cos \theta_j + y \sin \theta_j). \tag{2.14}$$

However, not every set of n functions $P_j(t)$ satisfies (2.12) for some $f \in L^2(C)$, e.g., their integrals (2.2) must be independent of j among other restrictions (see Theorem, below). Moreover, the functions $\rho_j(t)$ are not unique, although their sum g is unique as was proved. For the special case of *equally-spaced* angles $\theta_0, \cdots, \theta_{n-1}$ we can determine the explicit formula for the solution, g, to problem (2.1), as follows next.

§3. Equally-spaced angles θ_j.

First we consider the projection in the direction θ of a single ridge function restricted to the unit circle. For convenience, we assume the ridge function has direction $\theta_0 = 0$. We have

$$P_\rho(x, \theta) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \rho(x \cos \theta + y \sin \theta)\, dy. \tag{3.1}$$

Now we set $x = \cos \tau$ to obtain

$$P_\rho(\cos \tau, \theta) = \int_{-\sin \tau}^{\sin \tau} \rho(\cos \tau \cos \theta + y \sin \theta)\, dy. \tag{3.2}$$

Then setting $u = \cos \tau \cos \theta + y \sin \theta$, we have

$$P_\rho(\cos \tau, \theta) = \frac{1}{\sin \theta} \int_{\cos (\tau+\theta)}^{\cos (\tau-\theta)} \rho(u)\, du. \tag{3.3}$$

Replacing u by $\cos t$ we obtain the simple relation

$$P_\rho(\cos \tau, \theta) = \frac{1}{\sin \theta} \int_{\tau-\theta}^{\tau+\theta} h(t)\, dt \tag{3.4}$$

where $h(t) = \rho(\cos t) \sin t$.

Then for the projection of the sum of ridge functions given by (2.14) we have

$$P_g(\cos \tau, \theta) = \sum_{i=0}^{n-1} \frac{1}{\sin (\theta - \theta_i)} \int_{\tau-(\theta-\theta_i)}^{\tau+(\theta-\theta_i)} h_i(t)\, dt \tag{3.5}$$

where

$$h_i(t) = \rho_i(\cos t) \sin t. \tag{3.6}$$

The functions $\{\sin w\tau\}$, $w = 1, 2, \cdots$, are orthogonal and complete over $(0, \pi)$. So if we define

$$\Pi_g(\tau, \theta) = P_g(\cos \tau, \theta) = \sum_{w=1}^{\infty} \hat{\Pi}_g(w, \theta) \sin w\tau \tag{3.7}$$

and write

$$h_i(\tau) = \sum_{w=1}^{\infty} \hat{h}_i(w) \sin w\tau \tag{3.8}$$

we find from (3.5) that the Fourier coefficients are related by

$$\hat{\Pi}_g(w, \theta) = 2 \sum_{i=0}^{n-1} \frac{\sin w(\theta - \theta_i)}{w \sin (\theta - \theta_i)} \hat{h}_i(w). \tag{3.9}$$

For the case of equi-spaced projection angles, setting $\theta = \theta_k = k\pi/n$ and

$$\hat{\Pi}_g(w, \theta_k) = \hat{\Pi}_k(w), \tag{3.10}$$

we wish to find for each w, $w = 1, 2, 3, \cdots$, a solution $\{\hat{h}_k(w)\}$ of the set of equations

$$\hat{\Pi}_k(w) = \frac{2}{w} \sum_{j=0}^{n-1} \frac{\sin w(k-j)\frac{\pi}{n}}{\sin (k-j)\frac{\pi}{n}} \hat{h}_j(w), \qquad k = 0, 1, 2, \cdots, n-1 \tag{3.11}$$

where we take, according to (3.9)

$$\frac{\sin w(k-j)\frac{\pi}{n}}{\sin (k-j)\frac{\pi}{n}} = w \quad \text{for} \quad j = k. \tag{3.12}$$

According to (3.9), $\hat{\Pi}(w, \theta)$ as a function of θ is a trigonometric polynomial of degree $w - 1$, so the solution of (3.11) is clearly not unique for $w < n$.
If w is a multiple of n, the matrix is diagonal and

$$\hat{h}_k(mn) = \tfrac{1}{2}\hat{\Pi}_k(mn), \qquad m = 1, 2, 3, \cdots. \tag{3.13}$$

For $w \geq n$, (3.11) has a unique solution which we obtain as follows: We consider the sum

$$S_m(w) = \frac{1}{n} \sum_{k=0}^{n-1} \hat{\Pi}_k(w) \frac{\sin w(m-k)\frac{\pi}{n}}{\sin (m-k)\frac{\pi}{n}}. \tag{3.14}$$

We have from (3.11)

$$S_m(w) = \frac{2}{w} \sum_{j=0}^{n} S_{m,j}(w)\hat{h}_j(w), \tag{3.15}$$

where

$$S_{m,j}(w) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\sin w(m-k)\frac{\pi}{n}}{\sin (m-k)\frac{\pi}{n}} \frac{\sin w(k-j)\frac{\pi}{n}}{\sin (k-j)\frac{\pi}{n}}. \tag{3.16}$$

It may be verified that

$$S_{m,j}(w) = (2\mu + 1) \frac{\sin w(m-j)\frac{\pi}{n}}{\sin (m-j)\frac{\pi}{n}} - \mu(\mu + 1)n\delta_{mj} \tag{3.17}$$

where

$$\delta_{mj} = \begin{cases} 1, & m = j \\ 0, & m \neq j \end{cases}$$

and

$$\mu = \mu(w, n) = [w/n], \tag{3.18}$$

the largest integer contained in w/n, i.e.,

$$w = \mu n + v, \qquad 0 \leq v \leq n - 1. \tag{3.19}$$

Thus we have from (3.17), (3.14), and (3.11) the result

$$\mu(\mu + 1)n\hat{h}_m(w) = (2\mu + 1)\frac{w}{2}\,\hat{\Pi}_m(w) - \frac{w}{2n}\sum_{k=0}^{n-1}\hat{\Pi}_k(w)\,\frac{\sin w(m - k)\frac{\pi}{n}}{\sin (m - k)\frac{\pi}{n}}, \tag{3.20}$$

$$w = 1, 2, 3, \cdots$$
$$m = 0, 1, 2, \cdots, n - 1.$$

So for $w \geq n$ we have the unique solution to (3.11).

For $1 \leq w \leq n - 1$, we have $\mu = 0$, and hence we must have according to (3.20)

$$\hat{\Pi}_m(w) = \frac{1}{n}\sum_{k=0}^{n-1}\hat{\Pi}_k(w)\,\frac{\sin w(m - k)\frac{\pi}{n}}{\sin (m - k)\frac{\pi}{n}}, \qquad 1 \leq w \leq n - 1, \tag{3.21}$$

$$m = 0, 1, 2, \cdots, n - 1.$$

These are the consistency requirements on the Fourier coefficients of equispaced projections of any function integrable over the unit circle and vanishing outside the unit circle.

Comparing (3.21) and (3.9) we see that we may take

$$\hat{h}_m(w) = \frac{w}{2n}\,\hat{\Pi}_m(w), \qquad 1 \leq w \leq n - 1 \qquad m = 0, 1, 2, \cdots n - 1. \tag{3.22}$$

In fact (3.22) is valid for $w = n$, agreeing with (3.20) and (3.13). We note that the particular solution (3.22) minimizes $\sum_m |\hat{h}_m(w)|^2$ over all solutions, for it follows from (3.21) and (3.9) that

$$\sum_{k=0}^{n-1}|\hat{\Pi}_k(w)|^2 = \frac{2n}{w}\sum_{k=0}^{n-1}\hat{h}_k(w)\hat{\Pi}_k{}^*(w), \qquad 1 \leq w \leq n \tag{3.23}$$

where the asterisk denotes the complex conjugate and we have included the case $w = n$, using (3.13). Then Schwarz's inequality applied to the sum on the right gives the desired result. Furthermore, since $\hat{h}_m(w)$ is uniquely determined for $w \geq n$ we can assert that the solution we have given minimizes

$$\begin{aligned}\sum_m \int_0^\pi |h_m(\tau)|^2\,d\tau &= \sum_m \int_0^\pi |\rho_m(\cos\tau)|^2 \sin^2\tau\,d\tau \\ &= \sum_m \iint_{x^2+y^2<1} |\rho_m(x)|^2\,dx\,dy\end{aligned} \tag{3.24}$$

as well as

$$(3.25)\qquad \iint_{x^2+y^2<1} |g(x, y)|^2\, dx\, dy \quad \text{subject to (2.1a) with } \theta_j = \frac{j\pi}{n}.$$

§4. An expression for g.

Now we would like to obtain a closed-form expression for $h_j(\tau)$ and hence a closed-form solution to the problem (2.1)

First we summarize the relations.

$$(4.1)\qquad \Pi_j(\tau) \equiv P_g\left(\cos\tau, j\frac{\pi}{n}\right) = \sum_{w=1}^{\infty} \hat{\Pi}_j(w) \sin w\tau, \qquad 0 \le \tau \le \pi$$

$$(4.2)\qquad h_j(\tau) \equiv \rho_j(\cos\tau)\sin\tau = \sum_{w=1}^{\infty} \hat{h}_j(w) \sin w\tau, \qquad -\pi \le \tau \le \pi$$

$$(4.3)\qquad \hat{h}_j(w) = \frac{w}{2n}\hat{\Pi}_j(w) \qquad 1 \le w \le n$$

$$(4.4)\qquad \begin{aligned} \hat{h}_j(w) &= \frac{(2\mu+1)w}{2\mu(\mu+1)n}\hat{\Pi}_j(w) \\ &\quad - \frac{w}{2\mu(\mu+1)n^2}\sum_{k=0}^{n-1}\hat{\Pi}_k(w)\frac{\sin w(j-k)\frac{\pi}{n}}{\sin(j-k)\frac{\pi}{n}}, \qquad w \ge n \end{aligned}$$

where

$$\mu = \left[\frac{w}{n}\right], \qquad (w = \mu n + v, 0 \le v \le n-1).$$

We have $h_j(\tau) = \rho_j(\cos\tau)\sin\tau$ which is an odd function so that the Fourier series in (4.2) is valid for negative τ. Then if we agree to extend $\Pi_j(\tau)$ as an odd function i.e.,

$$(4.5)\qquad \Pi_j(-\tau) = -\Pi_j(\tau)$$

which is natural in accordance with the Fourier series in (4.1), we can express $h_j(\tau)$ in terms of convolution operators on the projections $\Pi_k(\tau)$. Let us write

$$(4.6)\qquad h_j(\tau) = \sum_{k=0}^{n-1} h_{jk}(\tau)$$

where $y_{jk}(\tau)$ is the component of $y_j(\tau)$ that depends only on $\Pi_k(\tau)$. We have from (4.3) and (4.4)

$$(4.7)\qquad \begin{aligned} \hat{h}_{jj}(w) &= \frac{w}{2n}\hat{\Pi}_j(w), \qquad 0 \le w < n \\ &= \frac{1}{2}\left(1 + \frac{v(n-v)}{n^2\mu(\mu+1)}\right)\hat{\Pi}_j(w), \qquad n \le w < \infty \end{aligned}$$

and for $k \neq j$

$$\begin{aligned}(4.8)\qquad &\hat{h}_{jk}(w) = 0, \qquad 0 \leq w < n \\ &\hat{h}_{jk}(w) = \frac{w}{2n^2\mu(\mu+1)} \frac{\sin(w(j-k)\pi/n)}{\sin((j-k)\pi/n)} \hat{\Pi}_k(w), \qquad n \leq w < \infty.\end{aligned}$$

From (4.7) we see that the coefficient of $\Pi_j(w)$ tends to $\frac{1}{2}$ as $w \rightarrow \infty$ so that

$$(4.9)\qquad h_{jj}(\tau) = \tfrac{1}{2}\Pi_j(\tau) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}\Pi_j(t)K_0(\tau - t)\,dt$$

where

$$(4.10)\qquad K_0(\tau) = -\sum_{w=-(n-1)}^{n-1}\left(1 - \frac{|w|}{n}\right)e^{iw\tau} + 2\sum_{w=n}^{\infty}\frac{v(n-v)}{n^2\mu(\mu+1)}\cos w\tau.$$

Performing the indicated sums, we find

$$\begin{aligned}(4.11)\qquad K_0(\tau) = {}& -\frac{1}{n}\left(\frac{\sin(n\tau/2)}{\sin \tau/2}\right)^2 + [\cos(n\tau/2) - |\sin(n\tau/2)|\,T(n\tau/2)] \\ & \cdot[\cos(\tau/4)\sin(n\tau/2) - 2n\sin(\tau/4)\cos(n\tau/2)]/[4n^2(\sin(\tau/4))^3],\end{aligned}$$

where $T(x)$ is the periodic triangular function defined by

$$\begin{aligned}(4.12)\qquad T(x) &= \pi - 2\,|x|, \qquad -\pi \leq x < \pi \\ T(x + 2\pi) &= T(x).\end{aligned}$$

For $k \neq j$, we see from (4.8) that only the high-frequency components ($|w| > n$) of $\Pi_k(\tau)$ affect $h_j(\tau)$. We have

$$\begin{aligned}(4.13)\qquad h_{jk}(\tau) = {}& [(1/2\pi)\sin((j-k)\pi/n)]\int_{-\pi}^{\pi}[\Pi_k(t + (j-k)\pi/n) \\ & - \Pi_k(t - (j-k)\pi/n)]K_1(\tau - t)\,dt, \qquad k \neq j.\end{aligned}$$

where

$$\begin{aligned}(4.14)\qquad K_1(\tau) = {}& \frac{|\sin(n\tau/2)|}{4n\sin(\tau/2)}T(n\tau/2)\left[\frac{\sin(n\tau/2)}{n\sin(\tau/2)} - 2\cos\{(n-1)\tau/2\}\right] \\ & + [\cos(\tau/2) - (\sin n\tau)/(2n\sin(\tau/2))]/4n\sin(\tau/2).\end{aligned}$$

Remark. For the special case of a centrally symmetric function f, we have

$$(4.15)\qquad P_f(t, \theta_j) = P(t) = P(-t)$$

or

$$(4.16)\qquad \Pi_j(\tau) = \Pi(\tau).$$

We have

$$(4.17)\qquad h_j(\tau) = h(\tau) = \tfrac{1}{2}\Pi(\tau) + \int_0^{\pi}\Pi(t)K_2(\tau - t)\,dt$$

where

$$K_2(\tau) = [\cos n\tau - \cos\tau \sin n\tau/(n \sin\tau)]\mathrm{sgn}(\sin(n\tau)/(4 \sin\tau). \tag{4.18}$$

We have explicitly determined a closed-form expression for the solution g to problem (2.1), namely g is given by (2.14), where ρ_i are given by (4.2) in terms of h_i, and h_i in turn are given by (4.6), (4.9), and (4.13). The solution is complicated, but involves only finite sums of convolutions with the explicit kernels K_0 and K_1 of (4.11) and (4.12) which involve only trigonometric functions.

It should be mentioned that since $g(x, y)$ for fixed (x, y) depends linearly on f, there exists a kernel-measure $\mu_g(dx', dy')$ for which

$$g(x, y) = \iint_C f(x', y')\mu_g(dx', dy'). \tag{4.19}$$

Because of the direct component $\frac{1}{2}\Pi_i(\tau)$ in $h_i(\tau)$ and the requirement

$$\Pi_i(\tau) = P_f(\cos\tau, \theta_i) \tag{4.20}$$

and since the right side of (4.20) is a line integral of f, it follows that the kernel-measure μ_g in (4.19) is not absolutely continuous with respect to Lebesgue measure on C, i.e., has a nonvanishing singular component. A reconstruction scheme based on approximating the explicit formula for g when the integrals of f are known for only finitely many strips would have the difficulty of determining the appropriate contribution of the singular line integral, i.e., the first term in (4.9), to the ridge functions, when the individual line integrals are not known for all lines.

§5. The optimum polynomial approximation.

Next we explicitly determine the polynomial $p(x, y)$ of degree $n - 1$ which best approximates f in $L^2(C)$, being given n views of f. We show that p is obtained from g by simply truncating the series (4.2) to $w \leq n$.

Let q be a polynomial of degree N in x and y, i.e.,

$$q(x, y) = \sum_{j+k} \sum_{\leq N} a_{jk}x^j y^k. \tag{5.1}$$

A ridge polynomial of degree N with direction θ is of the form

$$\rho(x, y) = \rho(x\cos\theta + y \sin\theta) \tag{5.2}$$

where $\rho(x)$ is a polynomial of degree N in x.

LEMMA 1. *Every polynomial of degree $n - 1$ in x and y is a sum of n ridge polynomials (with directions $\theta_0, \theta_1, \cdots, \theta_{n-1}$ which are distinct* mod π) *of degree $n - 1$.*

Proof. If the statement of Lemma 1 were false there would exist a polynomial $q(x, y)$ of degree $n - 1$ in x and y which is orthogonal in $L^2(C)$ to the subspace generated by the ridge polynomials of degree $n - 1$ with direction $\theta = \theta_j$, $j = 0, \cdots, n-1$. But then from (2.7) for all $j = 0, \cdots, n-1$

$$0 = (q, \rho) = \int_{-1}^{1} P_q(t, \theta_j)\rho(t)\,dt \tag{5.3}$$

and every polynomial $\rho(t)$ of degree $n - 1$ in t. From (1.2) we see that $P_q(t, \theta_j)$ is the product of a polynomial in t of degree $n - 1$ and the function $(1 - t^2)^{\frac{1}{2}}$. Thus from (5.3), we obtain for each $j = 0, 1, \cdots, n-1$

$$P_q(t, \theta_j) \equiv 0, \qquad -\infty < t < \infty. \tag{5.4}$$

Fixing t, we see from (1.2) that $P_q(t, \theta)$ is a trigonometrical polynomial in θ of degree $n - 1$. But since (1.2) also shows that

$$P_q(t, \theta + \pi) = P_q(-t, \theta) \tag{5.5}$$

we see from (5.4) that $P_q(t, \theta)$ has $2n$ zeros, $\theta = \theta_j$ and $\theta = \theta_j + \pi$, $j = 0, 1, \cdots, n-1$ and so $P_q(t, \theta) \equiv 0$ in θ and hence also in t. But then $Q \equiv 0$ as is well-known [R] and Lemma 1 is proved.

Let $f \in L^2(C)$ and let $p = p(x, y)$ denote the best approximation to f by a polynomial in x and y of degree $n - 1$ in $L^2(C)$, so that p is the Hilbert space projection onto the subspace $S^2(C)$ of $L^2(C)$ generated by polynomials in x and y of degree $n - 1$. Since $S^2(C) \subset R^2(C)$, the space of sums of ridge functions, by Lemma 1, p is the projection of g onto $S^2(C)$.

From (3.7), (3.8) and the defining relations

$$U_{w-1}(t) = \frac{\sin w\tau}{\sin \tau}, \qquad w = 1, 2, \cdots \tag{5.6}$$

$$t = \cos \tau \tag{5.7}$$

of the Chebyshev polynomials of the second kind, U_w, we can write g, using Lemma 1, in the form

$$g(x, y) = \sum_{j=0}^{n-1} \sum_{w=1}^{\infty} \hat{h}_j(w) U_{w-1}(x \cos \theta_j + y \sin \theta_j). \tag{5.8}$$

We next show that $p(x, y)$ is the polynomial obtained by truncating the series in (4.2) to $w \leq n$, i.e., $p(x, y) = p_0(x, y)$ where

$$p_0(x, y) = \sum_{j=0}^{n-1} \sum_{w=1}^{n} \hat{h}_j(w) U_{w-1}(x \cos \theta_j + y \sin \theta_j). \tag{5.9}$$

Indeed we verify directly that for $1 \leq k \leq n < w$, we have

$$\iint_C U_{k-1}(x \cos \theta_j + y \sin \theta_j) U_{w-1}(x \cos \theta_l + y \sin \theta_l)\,dx\,dy = 0 \tag{5.10}$$

for $j = l$ as well as for $j \neq l$ (Note: For $j = l$, (5.10) becomes

$$\int_{-1}^{1} U_{k-1}(t) U_{w-1}(t) \sqrt{1 - t^2}\,dt = 0 \tag{5.11}$$

which is the orthogonality property of the Chebyshev polynomials.) It follows from (5.10) that

$$(g - p_0) \perp S^2(C) \tag{5.12}$$

and so p_0 is the projection of g onto $S^2(C)$. Since the projection of g onto $S^2(C)$ is unique we must have $p_0 = p$.

We have proved that the polynomial $p(x, y)$ of degree $n - 1$ which best approximates f in L^2 over C is the sum of the ridge polynomials

$$p(x, y) = \sum_{i=0}^{n-1} \rho_i^{(n)}(x \cos \theta_i + y \sin \theta_i) \tag{5.13}$$

where from (4.3), and (5.9),

$$\rho_i^{(n)}(\cos \tau) \sin \tau = \frac{1}{n\pi} \int_0^\pi \Pi_i(\sigma) \sum_{w=1}^{n} w \sin w\sigma \sin w\tau \, d\sigma. \tag{5.14}$$

Extending Π_i to be odd as before we may express $\rho_i^{(n)}$ as a convolution in the manner of (4.9),

$$\rho_i^{(n)}(\cos \tau) \sin \tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Pi_i(\sigma) K(\tau - \sigma) \, d\sigma \tag{5.15}$$

where as an easy calculation shows

$$K(u) = \frac{1}{2}\left(\frac{\sin (n + \frac{1}{2})u}{\sin \frac{1}{2}u} - \frac{1}{n}\left[\frac{\sin (nu/2)}{\sin (u/2)}\right]^2\right). \tag{5.16}$$

We note that in contrast to the complicated nature of the explicit formula for g, the explicit formula (5.13) for $p(x, y)$, where $\rho_i^{(n)}$ are given by (5.15) is reasonably simple. It is clear from the fact that the troublesome first term in (4.9) is not present in the expression (5.15) for the ridge function summands of p that the kernel-measure $\mu_P(dx', dy')$ is absolutely continuous with respect to Lebesgue measure in contrast to μ_g in (4.19).

§6. Consistency conditions.

We next give the exact conditions on a sequence of functions $P_i(t)$ to be the projections of some function $f \in L^2(C)$. The results are stated at the end of the section.

We have given explicit formulas for the $h_i(\tau)$ and hence the components $\rho_i(t)$ of a sum of ridge functions $g(x, y)$ having the same projections as $f(x, y)$ in the directions $\theta_k = k\pi/n$, $k = 0, 1, \cdots, n - 1$. We have from (2.7), (2.14), and (3.6)

$$\begin{aligned} \iint_C |g(x, y)|^2 \, dx \, dy &= \sum_{i=0}^{n-1} \int_{-1}^{1} P_i(t)\rho_i(t) \, dt \\ &= \sum_{i=0}^{n-1} \int_0^\pi \Pi_i(\tau) h_i(\tau) \, d\tau. \end{aligned} \tag{6.1}$$

(We assume that P_j and ρ_j are real-valued). Now

$$\int_0^\pi \Pi_j(\tau)h_j(\tau)\,d\tau = \frac{\pi}{2}\sum_{w=1}^{\infty}\hat{\Pi}_j(w)\hat{h}_j(w). \tag{6.2}$$

We have from (4.3) and (4.4)

$$\sum_{j=0}^{n-1}\hat{\Pi}_j(w)\hat{h}_j(w) = \frac{w}{2n}\sum_{j=0}^{n-1}[\hat{\Pi}_j(w)]^2, \qquad 0 < w < n \tag{6.3}$$

and, for $w \geq n$,

$$\begin{aligned}\sum_{j=0}^{n-1}\hat{\Pi}_j(w)\hat{h}_j(w) &= \frac{w}{2n}\,\frac{2\mu+1}{\mu(\mu+1)}\sum_{j=0}^{n-1}[\hat{\Pi}_j(w)]^2 \\ &\quad - \frac{w}{2n^2\mu(\mu+1)}\sum_{j=0}^{n-1}\hat{\Pi}_j(w)\sum_{k=0}^{n-1}\hat{\Pi}_k(w)\,\frac{\sin(w(j-k)\pi/n)}{\sin((j-k)\pi/n)}.\end{aligned} \tag{6.4}$$

It can be shown that for $j = 0, 1, \cdots, n-1$

$$\sum_{k=0}^{n-1} x_k\,\frac{\sin(w(j-k)\pi/n)}{n\sin((j-k)\pi/n)} = \lambda x_j\,, \qquad w \geq n \tag{6.5}$$

has nontrivial solutions for two eigenvalues $\lambda = \mu$ and $\lambda = \mu + 1$ when v in (3.19) is nonzero; When $v = 0$, w is a multiple of n and there is only one eigenvalue $\lambda = \mu$ since the sum on the left reduces to μx_j. Since a quadratic form is bounded by its maximum and minimum eigenvalues, we must have

$$\begin{aligned} n\mu\sum_{j=0}^{n-1}[\hat{\Pi}_j(w)]^2 &\leq \sum_{j=0}^{n-1}\hat{\Pi}_j(w)\sum_{k=0}^{n-1}\hat{\Pi}_k(w)\,\frac{\sin(w(j-k)\pi/n)}{\sin((j-k)\pi/n)} \\ &\leq n(\mu+1)\sum_{j=0}^{n-1}[\hat{\Pi}_j(w)]^2, \qquad w \geq n.\end{aligned} \tag{6.6}$$

From (6.4)

$$\frac{w}{2n(\mu+1)}\sum_{j=0}^{n-1}[\hat{\Pi}_j(w)]^2 \leq \sum_{j=0}^{n-1}\hat{\Pi}_j(w)\hat{h}_j(w) \leq \frac{w}{2n\mu}\sum_{j=0}^{n-1}[\hat{\Pi}_j(w)]^2, \qquad w \geq n. \tag{6.7}$$

For the upper bound in (6.7) we see easily using (3.19) that $w/2n\mu$ achieves its maximum for $w \geq n$ at $w = 2n - 1$. Since this upper bound dominates (6.3) we have

$$\sum_{j=0}^{n-1}\hat{\Pi}_j(w)\hat{h}_j(w) \leq \left(1 - \frac{1}{2n}\right)\sum_{j=0}^{n-1}[\hat{\Pi}_j(w)]^2, \qquad w > 0. \tag{6.8}$$

Equality holds in (6.8) only for $w = 2n - 1$ and only if (since $\mu = 1$ from (3.19)) the left-hand equality holds in (6.6), requiring

$$\sum_{k=0}^{n-1}\hat{\Pi}_k(w)\,\frac{\sin(w(j-k)\pi/n)}{n\sin((j-k)\pi/n)} = \mu\hat{\Pi}_j(w) = \hat{\Pi}_j(w) \tag{6.9}$$

for $j = 0, 1, \cdots, n-1$ and $w = 2n-1$. It may be shown that there is only one (normalized) eigenvector of (6.9) with eigenvalue 1, namely

$$\hat{\Pi}_k(2n-1) = c, \qquad k = 0, 1, 2, \cdots, n-1 \tag{6.10}$$

i.e., equality can hold in (6.8) if and only if

$$\hat{\Pi}_j(\tau) = c \sin(2n-1)\tau, \qquad j = 0, 1, \cdots, n-1. \tag{6.11}$$

Finally from (6.1), (6.2), and (6.8)

$$\iint_C g^2 \leq \left(1 - \frac{1}{2n}\right) \sum_{j=0}^{n-1} \int_0^\pi \Pi_j(\tau)^2 \, d\tau = \left(1 - \frac{1}{2n}\right) \sum_{j=0}^{n-1} \int_{-1}^1 \frac{P_j(t)^2}{\sqrt{1-t^2}} \, dt. \tag{6.12}$$

We note that

$$\int_{-1}^1 \frac{P_f(t, \theta)^2}{\sqrt{1-t^2}} \, dt \leq 2 \iint_C f^2 \, dx \, dy. \tag{6.13}$$

To see this note that from (1.2) and Schwarz's inequality,

$$P_f(t, \theta)^2 \leq 2\sqrt{1-t^2}\, P_{f^2}(t, \theta) \tag{6.14}$$

and (6.13) follows by dividing by $\sqrt{1-t^2}$ and integrating over t. Equality holds in (6.13) if and only if f is a ridge function with direction θ.

We supposed at the outset that the functions $\{P_j(t)\}$ were projections of some function $f \in L^2(C)$, which implies that $P_j(t)$ vanish for $|t| > 1$, satisfy

$$\int_{-1}^1 \frac{[P_j(t)]^2}{\sqrt{1-t^2}} \, dt < \infty \tag{6.15}$$

from (6.13) and obey the consistency conditions (3.21) for the Fourier coefficients of Π_j for $w < n$. These are in fact the only requirements on a set of functions $\{P_j(t)\}$ to be the projections at equi-spaced angles of some $f \in L^2(C)$ since we have constructed a sum of ridge functions restricted to C which have as projections the given $P_j(t)$. We summarize the results of this section in the following theorem.

THEOREM. *Let n functions $P_j(t)$, $j = 0, 1, \cdots, n-1$ be given satisfying*

(i) $$P_j(t) = 0, \qquad |t| > 1,$$

(ii) $$\int_{-1}^1 [P_j(t)]^2 (1-t^2)^{-\frac{1}{2}} \, dt < \infty,$$

(iii) $$\hat{\Pi}_j(w) = \sum_{k=0}^{n-1} \hat{\Pi}_k(w) \frac{\sin(w(j-k)\pi/n)}{n \sin((j-k)\pi/n)}, \qquad w = 1, 2, \cdots, n$$

where $\sin(w(j-k)\pi/n)/\sin((j-k)\pi/n) \equiv w$ *for* $k = j$, *and*

$$\hat{\Pi}_j(w) = \int_0^\pi P_j(\cos \tau) \sin w\tau \, d\tau, \qquad w = 1, 2, \cdots.$$

Then there exists a function $g(x, y)$ *vanishing for* $x^2 + y^2 > 1$ *for which, setting* $\theta_j = j\pi/n$,

$$\text{(iv)} \qquad \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} g(t\cos\theta_j - s\sin\theta_j\,,\ t\sin\theta_j + s\cos\theta_j)\,ds = P_j(t), \qquad j = 0, 1, \cdots, n-1$$

$$\text{(v)} \qquad \iint_{x^2+y^2<1} g^2(x, y)\,dx\,dy \le \left(1 - \frac{1}{2n}\right) \sum_{j=0}^{n-1} \int_{-1}^{1} \frac{[P_j(t)]^2}{\sqrt{1-t^2}}\,dt$$

where equality holds in (v) *if and only if*

$$\text{(vi)} \qquad P_j(\cos\tau) = c\sin(2n-1)\tau, \qquad j = 0, 1, \cdots, n-1.$$

Remark. The function $g(x, y)$ corresponding to the extremal P_j in (vi) may be shown to be a multiple of the Legendre polynomial, [AS, 22.10.11].

$$g(x, y) = c(n - \tfrac{1}{2})\mathcal{P}_{n-1}(2r^2 - 1), \qquad r^2 = x^2 + y^2. \tag{6.16}$$

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