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THE JOINT DENSITY OF THE MAXIMUM AND ITS LOCATION FOR A WIENER PROCESS WITH DRIFT

L. A. SHEPP,* Bell Laboratories, Murray Hill, New Jersey

Abstract

We give a simple expression for the joint probability density of: (a) the maximum value \( Y = \max [X(t), 0 \leq t \leq T] \); (b) its location \( \hat{\theta} = \hat{\theta}(X) \), \( 0 \leq \hat{\theta} \leq T \); (c) the endpoint \( X(T) \), where \( X(t) = X_c(t) \) is a Wiener process with drift, \( X_c(t) = W(t) + ct \), \( 0 \leq t \leq T \). That is, we find the density \( p(\theta, y, x) = p(\theta, y, x; c, T) \) of \( \theta, Y, X_c(T) \). The distribution of the joint distribution can be obtained in a simple closed form from the joint distribution of the maximum, its location, and the endpoint which we give.

1. Introduction

It is of interest to calculate the distribution of the value of the difference between the maximum and present values of a stock or other security along with the time of occurrence of the maximum. At least for the usual model using a Wiener process with drift for the log of the value of the stock, this distribution can be obtained in a simple closed form from the joint distribution of the maximum, its location, and the endpoint which we give.

Let \( W(t), 0 \leq t \leq T \) be a standard Wiener process, \(-\infty < c < \infty\), and set

\[
X_c(t) = W(t) + ct, \hspace{1cm} 0 \leq t \leq T.
\]

Let \( \hat{\theta} = \hat{\theta}_{T,c} \) be the time of first occurrence of the maximum of \( X_c(t) \), \( 0 \leq t \leq T \). (Actually the maximum is uniquely attained w.p.l since \( P(M(I) = M(J)) = 0 \) for any disjoint intervals \( I, J \) with rational endpoints where \( M(I) \) is the maximum of \( X_c \) over \( I \).) Let \( Y = \max_{0 \leq t \leq T} X_c(t) = X_c(\hat{\theta}) \). We denote by \( p(\theta, y, x; c, T) \) the joint density of \( \hat{\theta}, Y, X_c(T) \), i.e.

\[
p(\theta, y, x; c, T)d\theta dy dx = P(\hat{\theta} \in d\theta, Y \in dy, X_c(T) \in dx).
\]

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* The problem was posed by the economists C. A. Futia, M. B. Goldman, and H. B. Sosin.
Note that $d\theta, dy, dx$ denote small intervals about the points $\theta, y, x$ throughout the paper. We begin by reducing the problem to the case with $c = 0$, i.e., no drift, by using a very simple argument based on the Radon–Nikodym theorem. Denote Wiener measure on function space by $\mu_w$ and let $\mu_c$ be the measure induced by $X_c$. Then as is well known [1], [6] the Radon–Nikodym derivative is given by

\begin{equation}
\frac{d\mu_c}{d\mu_w}(X) = \exp(cX(T) - \frac{1}{2}c^2T).
\end{equation}

But then with $A$ as the set of functions $X$ in $C[0, T]$ with $\theta(X) \in d\theta$, $Y(X) \in dy$, $X(T) \in dx$, we have

\begin{equation}
p(\theta, y, x; c, T)d\theta dy dx = \mu_c(A) = \int_A \left( \frac{d\mu_c}{d\mu_w} \right) d\mu_w
\end{equation}

\begin{align*}
&= \exp(cx - \frac{1}{2}c^2T) \int_A d\mu_w = \exp(cx - \frac{1}{2}c^2T)\mu_w(A) \\
&= \exp(cx - \frac{1}{2}c^2T)p(\theta, y, x; 0, T)d\theta dy dx
\end{align*}

which reduces the problem to finding $p(\theta, y, x; c, T)$ for $c = 0$.

We give a simple argument to determine $p(\theta, y, x; 0, T)$ in Section 2 and in the remainder of this section give some remarks and corollaries concerning the marginal distributions.

If the variance parameter of the Wiener process is $\sigma^2$ instead of 1, then by a simple scaling argument the density is

\begin{equation}
p(\theta, y, x; c, T, \sigma^2) = p(\theta, y/\sigma, x/\sigma; c/\sigma, T)/\sigma^2
\end{equation}

\begin{align*}
&= \frac{1}{\pi \sigma^4} \frac{y(y-x)}{\theta^{3/2}(T-\theta)^{3/2}} \exp \left\{ \left( -\frac{y^2}{2\theta} - \frac{(y-x)^2}{2(T-\theta)} + \frac{c^2T}{2} \right) \sigma^{-2} \right\}.
\end{align*}

We set $\sigma = 1$ for simplicity in the remainder of the paper.

The marginal distribution of $\theta$ and $Y$ is obtained by integrating (1.5) on $x < y$; which for $c = 0$ at least is possible to do in closed form,

\begin{equation}
p(\theta, y; 0, T) = \frac{1}{\pi} \frac{y \exp(-y^2/2\theta)}{\theta^{3/2}(T-\theta)^{1/2}}, \quad y > 0, 0 < \theta < T.
\end{equation}

Integrating on $y > 0$, or $0 \leq \theta \leq T$, gives the one-dimensional marginal density of $\theta$ and $Y$,

\begin{equation}
p(\theta; 0, T) = \frac{1}{\pi} \frac{1}{\sqrt{\theta(T-\theta)}}, \quad 0 < \theta < T
\end{equation}
in accord with the arcsine density ([3], p. 79), and

\[(1.8) \quad p(y; 0, T) = \sqrt{\frac{2}{\pi T}} \exp\left(-\frac{y^2}{2T}\right), \quad y > 0\]

the half-normal density.

The marginal distribution of \(Y\) and \(X_e(T)\) is obtained by integrating (1.5) over \(0 < \theta < T\),

\[(1.9) \quad p(y, x; c, T) = \frac{2(2y - x)\exp\left(-\frac{1}{2T}(2y - x)^2\right)}{\sqrt{2\pi} T^{3/2}} \exp(cx - \frac{1}{2}c^2T); \quad y > 0, \quad y > x\]

by using the identity, \(a > 0, \ b > 0,\)

\[
\int_0^T \frac{ab \exp\left(-\frac{1}{2\theta} a^2 - \frac{1}{2(T - \theta)} b^2\right)}{\pi\theta^{3/2}(T - \theta)^{3/2}} \, d\theta = \frac{2(a + b)\exp\left(-\frac{1}{2T}(a + b)^2\right)}{\sqrt{2\pi} T^{3/2}}
\]

This identity is easy to check by taking Laplace transforms of both sides ([5], p. 41, 5.30). Alternatively ([4], p. 195) the latter density may be obtained more directly from

\[(1.10) \quad \int_y^\infty p(y, x; 0, T)dy = P(Y \geq y, X(T) \in dx)/dx = P(X(T) \in 2y - dx)/dx\]

by the reflection principle for the Wiener process [2], and the argument in (1.4).

It appears remarkable that \(W(\hat{\theta})/\sqrt{\hat{\theta}}\) and \(\hat{\theta}\) are independent. Indeed, the density of these variables may be obtained from (1.6) and is easily seen to factor. Closely related is the somewhat surprising observation that the conditional density of \(Y\) given \(\hat{\theta}\), from (1.6),

\[(1.11) \quad p(y \mid \theta) = \frac{y}{\theta} \exp(-y^2/2\theta)\]

does not depend on \(T\). Of course the conditional density of \(Y\) given \(\hat{\theta}\) and \(X(T)\) does depend on \(T\). It is easy to check (and is well known) that the conditional distribution of \(\hat{\theta}\) given \(X_e(T) = 0\) is uniform on \([0, T]\).

2. The case of zero drift

We will use the reflection principle ([2], p. 393) and a limiting procedure to obtain the density \(p(\theta, y, x; 0, T)\) directly. First note that for \(0 < t_1 < \theta < t_2 < T\), the following probabilities coincide, up to terms of higher order in \(dy\),
\[ P(\tilde{\theta} \in (t_1, t_2), W(\tilde{\theta}) \in dy, W(T) \in dx) = \]
\[ P \left( \max_{0 < t < t_1} W(t) < y, \max_{t_1 < t < t_2} W(t) < y, \max_{t_1 < t < t_2} W(t) \in dy, W(T) \in dx \right). \]

Thus by independence of the increments, conditioning on \( x_i = W(t_i), i = 1, 2, \)
\[ \int_{\theta = t_1}^{t_2} p(\theta, y, x) d\theta dy dx = \int_{x_1 = -\infty}^{y} \int_{x_2 = -\infty}^{y} P \left( \max_{0 < t < t_1} W(t) < y, W(t_1) \in dx_1 \right) \]
\[ \times P \left( \max_{0 < t < t_2-t_1} W'(t) \in dy - x_i, W(t_2-t_1) \in dx_2 - x_1 \right) \]
\[ \times P \left( \max_{0 < t < T-t_2} W''(t) < y - x_2, W(T-t_2) \in dx - x_2 \right) \]

where \( W'(t) = W(t + t_i) - x_i, W''(t) = W(t + t_2) - x_2 \) are new (standard) Wiener processes independent of \( W. \) Using the reflection principle to explicitly express each of the three factors on the right we obtain
\[ \int_{t_1}^{t_2} p(\theta, y, x) d\theta dy dx = \int_{-\infty}^{y} \int_{-\infty}^{y} \left[ \frac{\exp \left( -\frac{x_1^2}{2t_1} \right) - \exp \left( -\frac{1}{2} \frac{(2y-x_1)\theta}{t_1} \right)}{\sqrt{2\pi t_1}} \right] dx_1 \]
\[ \times \left[ \frac{2}{(t_2-t_1)} \frac{(2y-x_1-x_2)}{\sqrt{2\pi(t_2-t_1)}} \exp \left( -\frac{1}{2} \frac{(2y-x_1-x_2)^2}{t_2-t_1} \right) dx_2 dy \right] \]
\[ \times \left[ \frac{\exp \left( -\frac{1}{2} \frac{(x-x_2)^2}{T-t_2} \right) - \exp \left( -\frac{1}{2} \frac{(2y-x_2-x)^2}{T-t_2} \right)}{\sqrt{2\pi(T-t_2)}} \right] dx. \]

Dividing by \( h = t_2 - t_1 \) and letting \( h \to 0 \) we obtain
\[ p(\theta, y, x; 0, T) = \lim_{h \to 0} \frac{1}{h^{3/2}} \frac{1}{\sqrt{\theta(T-\theta)}} \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{y} dx_1 \int_{-\infty}^{y} dx_2 \]
\[ \times \left[ \exp \left( -\frac{x_1^2}{2\theta} \right) - \exp \left( -\frac{1}{2} \frac{(2y-x_1)^2}{\theta} \right) \right] \]
\[ \times \left[ (2y-x_1-x_2)exp \left( -\frac{1}{2} \frac{(2y-x_1-x_2)^2}{h} \right) \right] \]
\[ \times \left[ \exp \left( -\frac{1}{2} \frac{(x-x_2)^2}{T-\theta} \right) - \exp \left( -\frac{1}{2} \frac{(2y-x_2-x)^2}{T-\theta} \right) \right]. \]

Let \( x_i = y - u_i \sqrt{h}, i = 1, 2 \) to obtain
\[ p(\theta, y, x; 0, T) = \lim_{h \to 0} \frac{1}{h^{5/2}} \frac{1}{\sqrt{\theta(T-\theta)}} \frac{2}{(2\pi)^{3/2}} \int_{0}^{\infty} du_1 \sqrt{h} \int_{0}^{\infty} du_2 \sqrt{h} \]
\[ \times \left[ \exp \left( -\frac{1}{2} \frac{y^2}{\theta} \right) \frac{2u_1 \sqrt{h}y}{\theta} \right] \]
\[ \times \left[ (u_1 + u_2) \sqrt{h} \exp \left( -\frac{1}{2} (u_1 + u_1)^2 \right) \right] \]
\[ \times \left[ \exp \left( -\frac{1}{2} \frac{(y-x)^2}{T-\theta} \right) \frac{2(y-x)u_2 \sqrt{h}}{T-\theta} \right] \]
\[ = \frac{1}{\pi} \frac{y(y-x)}{\theta^{3/2}(T-\theta)^{3/2}} \exp \left( -\frac{y^2}{2\theta} - \frac{(y-x)^2}{2(T-\theta)} \right). \]

This proves (1.5) in the case \( c = 0 \). We have assumed that the density \( p(\theta, y, x; 0, T) \) actually exists but that follows easily by the above reasoning.

A more indirect proof which avoids the use of the approximate formula (2.1) but involves much more calculation can be based on the use of the identity

\[ P(\bar{\theta} > t, W(\bar{\theta}) \in dy, W(T) \in dx) = \int_{t}^{T} P(\theta, y, x)d\theta dy dx, \]

\[ = \int_{y}^{\infty} P \left( \max_{0 \leq s \leq T} W(s) < y, W(t) \in dz, \max_{t \leq s \leq T} W(s) \in dy, W(T) \in dx \right). \]

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References