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FIRST PASSAGE TIME FOR A PARTICULAR GAUSSIAN PROCESS

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We find an explicit formula for the first passage probability, $Q_a(T \mid x) = P_r(S(t) < a, 0 \leqq t \leqq T \mid S(0) = x)$, for all $T > 0$, where S is the Gaussian process with mean zero and covariance $ES(\tau)S(t) = \max(1-|t-\tau|, 0)$. Previously, $Q_a(T \mid x)$ was known only for $T \leqq 1$.

In particular for $T = n$ an integer and $-\infty < x < a < \infty$,

$$Q_a(T \mid x) = \frac{1}{\varphi(x)} \int_D \cdots \int \det \varphi(y_i - y_{j+1} + a)\, dy_2 \cdots dy_{n+1},$$

where the integral is an n-fold integral on $y_2, \cdots, y_{n+1}$ over the region D given by

$$D = \{a - x < y_2 < y_1 < \cdots < y_{n+1}\}$$

and the determinant is of size $(n+1)\times(n+1)$, $0 < i, j \leqq n$, with $y_0 \equiv 0$, $y_1 \equiv a-x$.

1. Introduction. Let $S = S(t)$, $0 \leqq t \leqq T$ be the Gaussian process with mean zero and covariance

$$ES(\tau)S(t) = 1 - |t-\tau|, \qquad |t-\tau| \leqq 1$$
$$= 0, \qquad |t-\tau| > 1. \tag{1.1}$$

As observed in [5], S can be represented in terms of the standard Wiener process W by

$$S(t) = W(t) - W(t+1), \qquad t \geqq 0. \tag{1.2}$$

The first passage probability

$$Q_a(T \mid x) = P_r(S(t) < a, 0 \leqq t \leqq T \mid S(0) = x) \tag{1.3}$$

was studied by Slepian (1961), Mehr and McFadden (1965), and Shepp (1966). Application to a signal shape problem in radar was found by Zakai and Ziv (1969). We give an explicit formula for $Q_a(T \mid x)$ as an integral ((2.15) below) in T-dimensional space when T is an integer, and an integral ((2.25) below) in $2[T]+2$ dimensional space when T is not an integer.

Slepian found $Q_a(T \mid x)$ for $T \leqq 1$ by deriving a recurrence equation from a certain Markov-like property of S which was later called the reciprocal property by Jamison (1970). Shepp found an equivalent form of Slepian's result by using the Radon–Nikodym derivative of S with respect to the Wiener process and integrating in function space. Both of the above methods break down for $T > 1$: (a) The reciprocal property is not valid for $T > 1$; (b) S is not absolutely continuous with respect to the Wiener process for $T > 1$. The present method relies instead on an identity of Karlin and McGregor (1959).

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In Section 2 we derive the formula for $Q_a(T \mid x)$. When T is an integer the formula is seen to be very similar to the Fredholm formula for the resolvent kernel. We study this similarity in Section 3, showing that the generating function of $EQ_a(n \mid S(0))$, $n = 0, 1, 2, \cdots$ can be given in terms of a resolvent kernel. Unfortunately the resolvent kernel does not seem to be easily obtainable and all attempts to find the generating function in simple form have so far been unsuccessful.

2. Derivation of the formulas. Let $X(t)$, $0 \leqq t \leqq J$, be a real-valued Markov process with continuous sample paths and let $X_0, \cdots, X_n$ be independent copies of X. Suppose $a_0 < \cdots < a_n$ and $b_0 < \cdots < b_n$ and let $db_0, \cdots, db_n$ be infinitesimal intervals about $b_0, \cdots, b_n$ respectively. The result of Karlin and McGregor ([2] page 1149) becomes

$$(2.1)\quad P_r(X_0(t) < \cdots < X_n(t),\ 0 \leqq t \leqq \tau, \quad \text{and} \quad X_i(\tau) \in db_i,\ i = 0, \cdots, n \mid X_i(0) = a_i,\ i = 0, \cdots, n) = \det p_\tau(a_i, b_j) \cdot db_0, \cdots, db_n$$

where $p_\tau(a, b)\, db = P_r(X(\tau) \in db \mid X(0) = a)$ and det stands for the determinant of the $(n+1) \times (n+1)$ transition probability matrix. Specializing (2.1) by taking $X =$ the Wiener process, dividing both sides of (2.1) by $P_r(X_i(\tau) \in db_i,\ i = 0, \cdots, n \mid X_i(0) = a_i,\ i = 0, \cdots, n)$ we obtain

$$(2.2)\quad P_r(W_0(t) < \cdots < W_n(t),\ 0 \leqq t \leqq \tau \mid W_i(0) = a_i,\ W_i(\tau) = b_i,\ i = 0, \cdots, n) = (\det p_\tau(a_i, b_j)) / \textstyle\prod_{i=0}^{n} p_\tau(a_i, b_i),$$

where $W_0, \cdots, W_n$ are independent Wiener processes. The transition probabilities $p_\tau(a, b)$ are given by the well-known formula

$$(2.3)\qquad p_\tau(a, b) = \frac{1}{(2\pi\tau)^{-\frac{1}{2}}} \exp\left[-\tfrac{1}{2} \frac{(a-b)^2}{\tau} \right] \equiv \varphi_\tau(a-b).$$

For simplicity we first consider the case when T is an integer, $T = n$, and argue as follows. From (1.2) and (1.3),

$$\begin{aligned}(2.4)\qquad Q_a(T \mid x) &= P_r(W(t) - W(t+1) < a,\ 0 \leqq t \leqq n \mid W(0) \\ &= 0,\ W(0) - W(1) = x) \\ &= P_r(W(t) < W(t+1) + a < W(t+2) + 2a < \cdots < W(t+n) \\ &\qquad + \mathrm{na},\ 0 \leqq t \leqq 1 \mid W(0) = 0,\ W(0) - W(1) = x).\end{aligned}$$

Integrating out over the values x_i of W at times $i = 0, \cdots, n+1$, and letting Ω denote the event of the last term in (2.4) we have,

$$(2.5)\qquad Q_a(T \mid x) = \int \cdots \int P_r(\Omega,\ W(0) \in dx_0, \cdots, W(n+1) \in dx_{n+1} \mid W(0) = 0,\ W(0) - W(1) = x).$$

Restating (2.5) in terms of conditional probabilities, and noting that in (2.5) we must have $x_0 = 0, x_1 = -x$ because of the conditioning, we get

$$(2.6)\qquad Q_a(T \mid x) = \int \cdots \int P_r(\Omega \mid W(0) = x_0, \cdots, W(n+1) = x_{n+1}, W(0) = 0, W(0)-W(1) = x)P_r(W(0) \in dx_0, \cdots, W(n+1) \in dx_{n+1} \mid W(0) = 0, W(0)-W(1) = x).$$

We introduce the processes W_i, $i = 0, 1, \cdots, n$

$$(2.7)\qquad W_i(t) = W(t+i)+ia, \qquad 0 \leqq t \leqq 1.$$

We have

$$(2.8)\qquad \Omega = \{W_0(t) < W_1(t) < \cdots < W_n(t), 0 \leqq t \leqq 1\}$$

and under the conditioning involved in the first probability on the right side of (2.6),

$$(2.9)\qquad W_i(0) = W(i)+ia = x_i+ia, \qquad W_i(1) = W(i+1)+ia = x_{i+1}+ia.$$

Thus

$$(2.10)\qquad Q_a(T \mid x) = \int \cdots \int P_r(\Omega \mid W_i(0) = x_i+ia, W_i(1) = x_{i+1}+ia, i = 0, 1, \cdots, n, W_0(0) = 0, W_0(0)-W_0(1) = x) \times P_r(W(0) \in dx_0, \cdots, W(n+1) \in dx_{n+1} \mid W(0) = 0, W(0)-W(1) = x).$$

The range of integration is the set where the first probability under the integral is nonzero, that is where the inequalities in (2.8) hold for $t = 0, t = 1$ and $W_i(0) = x_i+ia$, $W_i(1) = x_{i+1}+ia$. The range is therefore the set where $x_i+ia < x_{i+1}+(i+1)a$, $i = 0, \cdots, n$. Since $W(0) = 0$ and $W(1) = W(0)-(W(0)-W(1)) = 0-(x) = -x$ we must have

$$(2.11)\qquad x_0 = 0, \qquad x_1 = -x.$$

The first probability under the integral in (2.10) is given by (2.2) since $x_0 = 0$, $x_1 = -x$, and the conditioned Wiener processes W_i are independent. Thus

$$(2.12)\qquad P_r(\Omega \mid W_i(0) = x_i+ia, W_i(1) = x_{i+1}+ia, i = 0, \cdots, n) = (\det \varphi(x_i+ia-x_{j+1}-ja))/\textstyle\prod_{i=0}^{n} \varphi(x_i+ia-x_{i+1}-ia),$$

where from (2.3)

$$(2.13)\qquad \varphi(u) = \frac{1}{(2\pi)^{-\frac{1}{2}}} \exp\,[-\tfrac{1}{2}u^2].$$

The second probability under the integral in (2.10) is simply

$$(2.14)\qquad \textstyle\prod_{i=1}^{n} \varphi(x_i-x_{i+1})/\varphi(x_0-x_1), \qquad x_0 = 0, x_1 = -x.$$

Putting (2.12) and (2.14) into (2.10) we obtain after the change of variables $y_i = x_i + ia$, $i = 0, \cdots, n+1$, the following formula for $Q_a(T \mid x)$, $-\infty < x < a < \infty$, $T = n$ an integer.

$$Q_a(T \mid x) = \frac{1}{\varphi(x)} \int_D \cdots \int \det \varphi(y_i - y_{j+1} + a)\, dy_2 \cdots dy_{n+1}, \tag{2.15}$$

where the integral is an n-fold integral on $y_2, \cdots, y_{n+1}$ over the region D given by

$$D = \{a - x < y_2 < y_3 < \cdots < y_{n+1}\} \tag{2.16}$$

and the determinant is of size $(n+1) \times (n+1)$, $0 \leqq i, j \leqq n$, with $y_0 \equiv 0$, $y_1 \equiv a - x$.

Of course, $Q_a(t \mid x) = 0$ for $a < x$.

It is easily verified that for $T = 1$ we have

$$Q_a(1 \mid x) = \Phi(a) - \frac{\varphi(a)}{\varphi(x)} \Phi(x) \tag{2.17}$$

agreeing with ([4] page 349). For $T \geqq 2$, the integral does not seem to be simply expressible.

Next we derive the formula for $Q_a(T \mid x)$ in case T is not an integer say $T = n + \theta$, $0 < \theta < 1$, and integer $n \geqq 0$. We have

$$\begin{aligned} Q_a(T \mid x) &= P_r(W(t) - W(t+1) < a, 0 \leqq t \leqq n + \theta \mid W(0) = 0, W(0) \\ &\quad - W(1) = x) = P_r(W(t) < W(t+1) \\ &\quad + a < \cdots < W(t+n+1) + (n+1)a, 0 \leqq t \leqq \theta, \quad \text{and} \\ &\quad W(\tau + \theta) < W(\tau + \theta + 1) + a < \cdots < W(\tau + \theta + n) \\ &\quad + na, 0 \leqq \tau \leqq 1 - \theta \mid W(0) = 0, W(0) - W(1) = x). \end{aligned} \tag{2.18}$$

Integrating out over the values u_i and v_i of W at times i and $i + \theta$, $i = 0, 1, 2, \cdots, n+1$, we have, letting Ω' denote the event of the last term of (2.18),

$$\begin{aligned} Q_a(T \mid x) &= \int \cdots \int P_r(\Omega', W(0) \in du_0, \cdots, W(n+1) \in du_{n+1}, W(\theta) \in dv_0, \\ &\quad \cdots, W(n+1+\theta) \in dv_{n+1} \mid W(0) = 0, W(0) - W(1) = x). \end{aligned} \tag{2.19}$$

Restating (2.19) in terms of conditional probabilities, we get

$$\begin{aligned} Q_a(T \mid x) &= \int \cdots \int P_r(\Omega' \mid W(0) = u_0', \cdots, Wn+1) = u_{n+1}, W(\theta) \\ &\quad = v_0, \cdots, W(n+1+\theta) = v_{n+1}, W(0) = 0, W(0) \\ &\quad - W(1) = x) P_r(W(0) \in du_0, \cdots, W(n+1) \in du_{n+1}, \\ &\quad W(\theta) \in dv_0, \cdots, W(n+1+\theta) \in dv_{n+1} \mid W(0) \\ &\quad = 0, W(0) - W(1) = x). \end{aligned} \tag{2.20}$$

We introduce the processes W_i, $i = 0, \cdots, n+1$; W_j', $j = 0, \cdots, n$

$$W_i(t) = W(t+i)+ia, \qquad 0 \leqq t \leqq \theta \tag{2.21}$$
$$W_j'(\tau) = W(\tau+\theta+j)+ja, \qquad 0 \leqq \tau \leqq 1-\theta.$$

We have $\Omega' = \Omega_1 \cap \Omega_2$ where

$$\Omega_1 = \{W_0(t) < \cdots < W_{n+1}(t), 0 \leqq t \leqq \theta\} \tag{2.22}$$
$$\Omega_2 = \{W_0'(\tau) < \cdots < W_n'(\tau), 0 \leqq \tau \leqq 1-\theta\}$$

and under the conditioning involved in the first probability on the right side of (2.20) we have for $0 \leqq i \leqq n+1$, $0 \leqq j \leqq n$,

$$\begin{aligned} W_i(0) &= W(i)+ia = u_i+ia \\ W_i(\theta) &= W(\theta+i)+ia = v_i+ia \\ W_j'(0) &= W(\theta+j)+ja = v_j+ja \\ W_j'(1-\theta) &= W(j+1)+ja = u_{j+1}+ja. \end{aligned} \tag{2.23}$$

The processes $W_i(t)$ and $W_j'(\tau)$ conditioned to satisfy (2.23) are independent and so the conditional probability of Ω' in (2.20) is the product of the conditional probabilities of Ω_1 and Ω_2. Thus with $u_0 = W(0) = 0$, $u_1 = W(0)-(W(1)-W(0)) = -x$, (2.20) becomes

$$\begin{aligned} Q_a(T \mid x) = \int \cdots \int & P_r(\Omega_1 \mid W_i(0) = u_i+ia, W_i(\theta) = v_i+ia, i = 0, \cdots, n+1) \\ & \times P_r(\Omega_2 \mid W_j'(0) = v_j+ja, W_j'(1-\theta) = u_{j+1} \\ & +ja, j = 0, 1, \cdots, n) P_r(W(0) \in du_0, \cdots, \\ & W(n+1) \in du_{n+1}, W(\theta) \in dv_0, \cdots, \\ & W(n+1+\theta) \in dv_{n+1} \mid W(0) = 0, W(0)-W(1) = x). \end{aligned} \tag{2.24}$$

Using (2.2) to express the first two probabilities under the integral in (2.24) and letting $x_i = u_i+ia$, $y_i = v_i+ia$, $i = 0, \cdots, n+1$ we obtain the final result for $T = n+\theta$, $0 < \theta < 1$, n an integer as

$$\begin{aligned} Q_a(T \mid x) = \frac{1}{\varphi(x)} \int_{D'} \cdots \int & (\det \varphi_\theta(x_i - y_j))(\det \varphi_{1-\theta}(y_i - x_{j+1}+a)) \\ & \times dx_2 \cdots dx_{n+1} dy_0 \cdots dy_{n+1} \end{aligned} \tag{2.25}$$

where the integral is a $2n+2$-fold integral over the region D' given by

$$D' = \{a-x < x_2 < \cdots < x_{n+1} \quad \text{and} \quad y_0 < y_1 < \cdots < y_{n+1}\}. \tag{2.26}$$

The first determinant in (2.25) is of size $(n+2)\times(n+2)$, $0 \leqq i, j \leqq n+1$ while the second is of size $(n+1)\times(n+1)$, $0 \leqq i, j \leqq n$. In each, $x_0 = 0$, $x_1 = a-x$.

One may verify that for $T < 1$, $Q_a(T \mid x)$ agrees with the previous results found in [4] and [5].

3. Remarks on the similarity with Fredholm theory. For large T the expressions (2.15) and (2.25) are unwieldy and apparently not suited for either numerical calculation or asymptotic estimation. For simplicity we restrict attention here to integral T and to the unconditional probabilities,

$$F_n(a) = P_r(S(t) < a, 0 \leqq t \leqq n). \tag{3.1}$$

D. Slepian pointed out the strong similarity between (2.15) and the formulas involved in the Fredholm resolvent. Indeed, if we define

$$K(s, t) = \varphi(s-t+a) \tag{3.2}$$

then it can be seen from (2.15) that

$$F^n(a) = \frac{(-1)^n}{n!} \int_0^\infty \int_0^y \cdots \int_0^y K\begin{pmatrix} 0, u_1, \cdots, u_n \\ y, u_1, \cdots, u_n \end{pmatrix} du_1 \cdots du_n \tag{3.3}$$

in the notation of ([6] page 70). Applying Fredholm theory [6] we find that the generating function

$$F(\lambda, a) = \sum_{n=0}^\infty \lambda^n F_n(a) \tag{3.4}$$

is given as

$$F(\lambda, a) = \int_0^\infty \exp\left[-\lambda \int_0^y H(\lambda, u, u, u)\, du\right] H(\lambda, 0, y, y)\, dy \tag{3.5}$$

where $H = H(\lambda, s, t, y)$ is the resolvent kernel of K, determined uniquely by the resolvent equation

$$H(\lambda, s, t, y) = K(s, t) + \lambda \int_0^y H(\lambda, s, u, y) K(u, t)\, du, \qquad 0 \leqq s \leqq y, \tag{3.6}$$

the parameter a being suppressed in both H and K.

We have included this section in the hope that (3.5) could be used to obtain bounds on the radius of convergence of (3.4) or equivalently to find bounds on

$$\lim_{n\to\infty} n^{-1} \cdot \log F_n(a), \tag{3.7}$$

assuming the limit exists. Unfortunately, we were unable to complete this approach because of the difficulty of estimating H sufficiently closely.

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