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FIRST-PASSAGE TIME FOR A PARTICULAR STATIONARY PERIODIC GAUSSIAN PROCESS

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Abstract

We find the first-passage probability that $X(t)$ remains above a level a throughout a time interval of length T given $X(0) = x_0$ for the particular stationary Gaussian process X with mean zero and (sawtooth) covariance $\rho(\tau) = 1 - \alpha\,|\,\tau\,|, |\,\tau\,| \leqq 1$, with $\rho(\tau + 2) = \rho(\tau)$, $-\infty < \tau < \infty$. The desired probability is explicitly found as an infinite series of integrals of a two-dimensional Gaussian density over sectors. Simpler expressions are found for the case $a = 0$ and also for the unconditioned probability that $X(t)$ be non-negative throughout $[0, T]$. Results of some numerical calculations are given.

FIRST PASSAGE; LEVEL-CROSSING PROBABILITY; GAUSSIAN PERIODIC PROCESS

1. Introduction

Let $X(t)$ be a stationary Gaussian process with $EX(t) = 0$ and $E[X(t)X(t')] = \rho(t - t')$. Denote by $Q_X(a, T\,|\,x_0)$ the conditional probability that $X(t) \geqq a$ for all t satisfying $0 \leqq t \leqq T$, given that $X(0) = x_0$. The first-passage time probability $\hat{Q}_X(a, T\,|\,x_0)dT$, namely the probability that $X(t)$ first cross the level a in the interval $T \leqq t \leqq T + dT$ given that $X(0) = x_0$, can be obtained simply from $Q_X(a, T\,|\,x_0)$ by differentiation:

$$\hat{Q}_X(a, T\,|\,x_0) = -(d/dT)Q_X(a, T\,|\,x_0).$$

Other quantities describing the excursions of $X(t)$ about the level a, such as the distribution of the interval between a-crossings, can also be derived from knowledge of the a-level exceedence $Q_X(a, T\,|\,x_0)$. See [1] for a general review of the level-crossing problem.

The determination of Q_X is not an easy matter in general. To the best of our knowledge it is known in only three non-trivial cases: (i) when $\rho(\tau) = e^{-|\tau|}$; (ii) when the covariance is the triangular function $\rho(\tau) = 1 - |\,\tau\,|$ for $|\,\tau\,| \leqq 1$, $\rho(\tau) = 0$, $|\,\tau\,| \geqq 1$; (iii) $\rho(\tau) = \frac{3}{2}\exp[-|\,\tau\,|/3]\,[1 - \frac{1}{3}\exp[-2\,|\,\tau\,|/\sqrt{3}]]$. References for these cases can be found in [1]. A recent extension of the work cited there for Case (iii) is given in [3].

This note points out the solution (16) and (20) for Q_X for a new case — the periodic 'sawtooth' covariance

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$$\rho(\tau) = \rho(-\tau) = \rho(\tau + 2)$$

$$\rho(\tau) = \begin{cases} 1 - \alpha\tau, & 0 \leqq \tau \leqq 1 \\ \alpha\tau + 1 - 2\alpha, & 1 \leqq \tau \leqq 2 \end{cases} \tag{1}$$

$$0 \leqq \alpha \leqq 2.$$

Simpler expressions for $Q_X(0, T \mid x_0)$ and the unconditioned probability that $X(t)$ be non-negative in $[0, T]$ are given in (23)–(25) and (28) respectively. These results may be useful in connection with inequalities given in [5] to find bounds for the first-passage probability of more general Gaussian processes.

2. The deterministic nature of half of $X(t)$

The Gaussian process with covariance (1) is periodic with period 2 and satisfies the following curious relation:

$$X(t+1) = X(0) + X(1) - X(t), \quad -\infty < t < \infty. \tag{2}$$

Thus in any period of the process, $X(t)$ is completely determined in one half of a period by its values on the other half of the period.

Actually (2) holds for a wider class of periodic processes than the one just described. Let ξ_0, ξ_n, η_n, $n = 1, 2, \cdots$, be independent random variables with mean zero and variance one. Let the periodic process $Y(t)$ be defined by

$$Y(t) \doteq a_0\xi_0 + \sum_1^\infty a_n [\xi_n \cos \pi n t + \eta_n \sin \pi n t] \tag{3}$$

where we assume that $\Sigma \mid a_j \mid^2 < \infty$. For this process,

$$\rho(\tau) = a_0^2 + \sum_1^\infty a_n^2 \cos \pi n \tau, \tag{4}$$

and from (3), it follows directly that

$$Y(t) + Y(t+1) = 2a_0\xi_0 + \sum_1^\infty [1 + (-1)^n] a_n [\xi_n \cos \pi n t + \eta_n \sin \pi n t]. \tag{5}$$

The odd terms in the summation vanish identically so that if

$$a_{2n} = 0, \quad n = 1, 2, \cdots \tag{6}$$

then

$$Y(t) + Y(t+1) = 2a_0\xi_0 = Y(0) + Y(1)$$

which is of the form (2). The condition (6) means that $\rho(\tau) - \rho(\frac{1}{2})$ has odd symmetry about $\tau = \frac{1}{2}$.

For the covariance (1), one finds the Fourier series expansion

$$\rho(\tau) = 1 - \frac{\alpha}{2} + \sum_{0}^{\infty} \frac{4\alpha}{\pi^2(2n+1)^2} \cos(2n+1)\pi\tau,$$

so that $X(t)$ has a representation of the form (3) with $a_0 = \sqrt{(1-\frac{1}{2}\alpha)}$, $a_{2n+1} = \sqrt{(4\alpha)}/\pi(2n+1)$, $a_{2n} = 0$, $n = 1, 2, \cdots$, and the ξ's and η's Gaussian, so that indeed (2) holds for this process. A direct proof of (2) can also bè obtained by merely verifying that $E[X(t+1) + X(t) - X(1) - X(0)]^2 = 0$. Since $X(t)$ can be taken sample continuous, it then follows that (2) holds almost surely.

3. The Markov-like nature of $X(t)$, $0 \leqq t \leqq 1$

Let $0 \leqq t_0 < t_1 < \cdots < t_k < \cdots < t_n \leqq 1$ be $n + 1$ points in the unit interval. The joint density of $X(t_0), X(t_1), \cdots, X(t_n)$ is given by

$$\begin{aligned} p_X(x_0, x_1, \cdots, x_n) &= \frac{2\exp\{-\frac{1}{2}(x_0 + x_n)^n/2[2 - \alpha(t_n - t_0)]\}}{\sqrt{(2\pi 2[2 - \alpha(t_n - t_0)])}} \\ &\quad \times \prod_{1}^{n} \frac{\exp\{-\frac{1}{2}(x_j - x_{j-1})^2/2\alpha(t_j - t_{j-1})\}}{\sqrt{(2\pi 2\alpha(t_j - t_{j-1}))}}. \end{aligned} \tag{7}$$

This can be established readily by considering the Gaussian random variables $Z_0 \equiv X(t_0) + X(t_n)$, $Z_j \equiv X(t_j) - X(t_{j-1})$, $j = 1, 2, \cdots, n$. Direct calculation using (1) shows that the Z's are independent and that

$$EZ_0^2 = 2[2 - \alpha(t_n - t_0)], \quad EZ_j^2 = 2\alpha(t_j - t_{j-1}), \quad j = 1, 2, \cdots, n.$$

The Jacobian $\partial(Z_0, \cdots, Z_n)/\partial(X(t_0), \cdots, X(t_n)) = 2$, whence (7) follows.

From (7) one computes directly that

$$\begin{aligned} &p_X(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n-1} \mid x_0, x_k, x_n) \\ &\quad = p_X(x_1, x_2, \cdots, x_{k-1} \mid x_0, x_k) p_X(x_{k+1}, x_{k+2}, \cdots, x_{n-1} \mid x_k, x_n). \end{aligned} \tag{8}$$

Here $p_X(x_1, x_2, \cdots, x_{k-1} \mid x_0, x_k)$ is the joint density of $X(t_1), X(t_2), \cdots, X(t_{k-1})$ given that $X(t_0) = x_0$ and that $X(t_k) = x_k$, with a similar meaning for the other factors in (8). From (8) it follows that if $0 < T < 1$ and A is an event defined on $(0, T)$ and B is an event defined on $(T, 1)$, then

$$\Pr[A \cap B \mid X(0), X(T), X(1)] = \Pr[A \mid X(0), X(T)] \Pr[B \mid X(T), X(1)]. \tag{9}$$

This Markov-like property will be of use to us in the next section.

4. The a-level exceedence probability, $Q_X(a, T \mid x_0)$

It is convenient to introduce the notation

$$Q_X(a, b, T \mid \xi, \eta) \equiv \Pr[a \leqq X(t) \leqq b, 0 \leqq t \leqq T \mid X(0) = \xi, X(T) = \eta] \tag{10}$$

$$q_X(a,b,\eta,T\mid\xi)\,d\eta \tag{11}$$
$$\equiv \Pr[a \leqq X(t) \leqq b, 0 \leqq t \leqq T, \eta \leqq X(T) \leqq \eta + d\eta \mid X(0) = \xi].$$

Now let $W(t)$ denote a standard Wiener process, i.e., a Gaussian process with $EW(t)W(t') = \min(t,t')$. Then provided $0 \leqq T \leqq 1$, it is true that

$$Q_X(a,b,T\mid\xi,\eta) = \sqrt{(2\pi 2\alpha T)}\exp[(\xi-\eta)^2/4\alpha T]q_W(a,b,\eta,2\alpha T\mid\xi). \tag{12}$$

This can be seen as follows. Divide the interval $(0, T)$ into n equal parts by the points $t_j \equiv jT/n$, $j = 0,1,\cdots,n$. Then

$$\begin{aligned} Q_X(a,b,T\mid\xi,\eta) &= \lim_{n\to\infty}\int_a^b dx_1\cdots\int_a^b dx_{n-1}p_X(x_1,\cdots,x_{n-1}\mid X(0)=\xi, X(T)=\eta) \\ &= \sqrt{(2\pi 2\alpha T)}e^{\frac{1}{2}(\xi-\eta)^2/2\alpha T}\lim_{n\to\infty}\int_a^b dx_1\cdots\int_a^b dx_{n-1}\prod_{j=1}^{n}\frac{\exp[-\frac{1}{2}[x_j-x_{j-1}]^2/(2\alpha T/n)]}{\sqrt{(2\pi 2\alpha T/n)}} \end{aligned} \tag{13}$$

as is seen from (7). Here we write $x_0 = \xi$, $x_n = \eta$. But the integrand now is seen to be the conditional density $p_W(x_1,x_2,\cdots,x_n \mid W(0)=\xi)$ for the sampled Wiener process $W(j(2\alpha T/n))$, $j=1,2,\cdots,n$, given that $W(0)=\xi$. The indicated limit in the last member of (13) is thus $q_W(a,b,\eta,2\alpha T\mid\xi)$ and (12) follows.

The quantity $q_W(a,b,\eta,T\mid\xi)$ is known [2]:

$$\begin{aligned} q_W(a,b,\eta,T\mid\xi) = \frac{1}{\sqrt{(2\pi T)}}\sum_{-\infty}^{\infty}[&\exp(-[\eta-\xi+2k(b-a)]^2/T) \\ &-\exp(-[\eta+\xi-2a+2k(b-a)]^2/2T]), \end{aligned} \tag{14}$$
$$a \leqq \xi, \eta \leqq b.$$

We proceed to express $Q_X(a,T\mid x_0)$ in terms of it.

We have for all $T \geqq 0$,

$$Q_X(a,T\mid x_0) = \int_a^\infty d\eta\, p_X(\eta\mid x_0)\,Q_X(a,\infty,T\mid x_0,\eta). \tag{15}$$

When $0 \leqq T \leqq 1$, (7), (12) and (14) permit explicit expression of the integrand and elementary manipulation yields

$$Q_X(a,T\mid x_0) = \Phi_c\left[\frac{a+x_0(\alpha T-1)}{\sqrt{(\alpha T(2-\alpha T))}}\right] - e^{\frac{1}{2}(x_0^2-a^2)}\Phi_c\left[\frac{x_0+a(\alpha T-1)}{\sqrt{(\alpha T(2-\alpha T))}}\right], \tag{16}$$
$$0 \leqq T \leqq 1, \ a \leqq x_0,$$

where the complementary error function is given by

$$\Phi_c(x) = \frac{1}{\sqrt{(2\pi)}} \int_x^{\infty} e^{-\frac{1}{2}t^2}\, dt. \tag{17}$$

When $\alpha = 1$, (16) agrees with previous results.

To compute the a-level exceedence probability for intervals of duration greater than one, we use the Markov-like property (9) and the relationship (2). Let $\mathscr{C}$ denote the condition $X(0) = x_0$, $X(T) = x_T$, $X(1) = x_1$, and as before assume that $0 \leqq T \leqq 1$. Then

$$\begin{aligned}
&\Pr[X(t) \geqq a, 0 \leqq t \leqq 1 + T \mid \mathscr{C}] \\
&\quad = \Pr[\{X(t) \geqq a, 0 \leqq t \leqq T\} \cap \{X(t) \geqq a, T \leqq t \leqq 1\} \\
&\quad \cap \{X(t) \geqq a, 1 \leqq t \leqq 1 + T\} \mid \mathscr{C}] \\
&\quad = \Pr[\{X(0) + X(1) - a \geqq X(t) \geqq a, 0 \leqq t \leqq T\} \cap \{X(t) \geqq a, T \leqq t \leqq 1\} \mid \mathscr{C}] \\
&\quad = \Pr[x_0 + x_1 - a \geqq X(t) \geqq a, 0 \leqq t \leqq T \mid X(0) = x_0, x(T) = x_T] \\
&\qquad \times \Pr[X(t) \geqq a, T \leqq t \leqq 1 \mid X(T) = x_T, X(1) = x_1] \\
&\quad = Q_X[a, x_0 + x_1 - a, T \mid x_0, x_T]\, Q_X[a, \infty, 1 - T \mid x_T, x_1].
\end{aligned} \tag{18}$$

Here we have used (2) to obtain the third member of (18), (9) to obtain the fourth and (10) and the stationarity of $X(t)$ to obtain the final member. Now

$$\begin{aligned}
Q_X(a, 1 + T \mid x_0) &= \int_a^{\infty} dx_T \int_a^{\infty} dx_1 \Pr[X(t) \geqq a, 0 \leqq t \leqq 1 + T \mid \mathscr{C}]\, p_X(x_T, x_1 \mid x_0) \\
&= \frac{2e^{\frac{1}{2}x_0^2}}{\sqrt{(2(2-\alpha))}} \int_a^{\infty} dx_1 \int_a^{x_1 + x_0 - a} dx_T e^{-(\frac{1}{2}(x_0 + x_1)^2)/(2(2-\alpha))} \\
&\qquad \times q_W(a, x_0 + x_1 - a, x_T, 2\alpha T \mid x_0) q_W(a, \infty, x_1, 2\alpha(1 - T) \mid x_T) \\
&\qquad\qquad x_0 \geqq a, \quad 0 < T < 1,
\end{aligned} \tag{19}$$

on using (7), (18) and (12). Finally, (14) can be used to express the integrand of (19) in terms of elementary functions. The resultant form for the exceedence probability is

$$\begin{aligned}
QX(a, T \mid x_0) &= \sum_{i=1}^{4} \sum_{k=-\infty}^{\infty} e^{D_{ik}} \int_{a_{ik}}^{\infty} dx \int_{b_{ik}}^{x - c_{ik}} dy\, e^{-\frac{1}{2}[A_{ik}x^2 + B_{ik}y^2 + 2C_{ik}xy]}, \\
&\qquad 0 \leqq \alpha < 2, \quad 1 \leqq T < 2, \quad x_0 \geqq a_0.
\end{aligned} \tag{20}$$

Here the various constants are polynomials in x_0. While not very enlightening for theoretical purposes, the form (20) is useful for computing since only a few terms of the k sum give an appreciable contribution to the results.

When $T=2$ and $\alpha<2$, (19) must be replaced by

$$Q_X(a,2\,|\,x_0)=\frac{2e^{\frac{1}{4}x_0^2}}{\sqrt{(2(2-\alpha))}}\int_a^\infty dx_1 e^{-\frac{1}{2}(x_0+x_1)^2/2(2-\alpha)} \times q_W(a,x_0+x_1-a,x_1,2\alpha\,|\,x_0). \tag{21}$$

By using (14) this result can be expressed as an infinite sum of terms involving elementary functions and complementary error functions. Again, the answer is complicated in form, but suitable for use in numerical calculations.

4. The probability that $X(t)$ be non-negative

From the preceding formulas one can find more manageable expressions for the unconditioned zero exceedence

$$\begin{aligned} Q_X(T;\alpha) &\equiv \Pr[X(t)\geqq 0,\ 0\leqq t\leqq T] \\ &= \int_0^\infty dx_0 Q_X(0,T\,|\,x_0)\frac{e^{-\frac{1}{2}x_0^2}}{\sqrt{(2\pi)}}. \end{aligned} \tag{22}$$

From (16), a straightforward calculation gives

$$Q_X(T;\alpha)=(1/2\pi)[\arccos(\alpha T-1)-\sqrt{(\alpha T(2-\alpha T))}],\quad 0\leqq T\leqq 1, \tag{23}$$

while from (14) and (20), after a calculation indicated in the appendix, we find that

$$Q_X(T;\alpha)=[1+e^{\pi\sqrt{(\alpha/(2-\alpha))}}]^{-1},\quad T\geqq 2. \tag{24}$$

For the remaining T values we have the more complicated expression

$$\begin{aligned} &Q_X(1+T;\alpha)= \\ &\frac{\sqrt{c}}{2\pi}\int_{-1}^{1}dx\sum{}'\Bigg[\frac{(x+k+1)(1-T)+(1-x)T}{[c+(2x+k)^2][cT(1-T)+(x+k+1)^2(1-T)+(1-x)^2T]^{\frac{1}{2}}} \\ &\qquad -\frac{(x+k+1)(1-T)+(x+1)T}{(c+k^2)[cT(1-T)+(x+k+1)^2(1-T)+(1+x)^2T]^{\frac{1}{2}}}\Bigg], \\ &\qquad\qquad 0\leqq T\leqq 1,\quad c\equiv\alpha/(1-\alpha), \end{aligned} \tag{25}$$

where the sum is over all odd values of k.

While the integral indicated here can be carried out in elementary terms, the result is quite complicated and for numerical purposes the form (25) is satisfactory.

A graph of $Q_X(T;\alpha)$ for $\alpha = 1$ is given in Figure 1 computed from (23), (24) and (25). In using the latter, 20 terms of the sum were taken and the integration

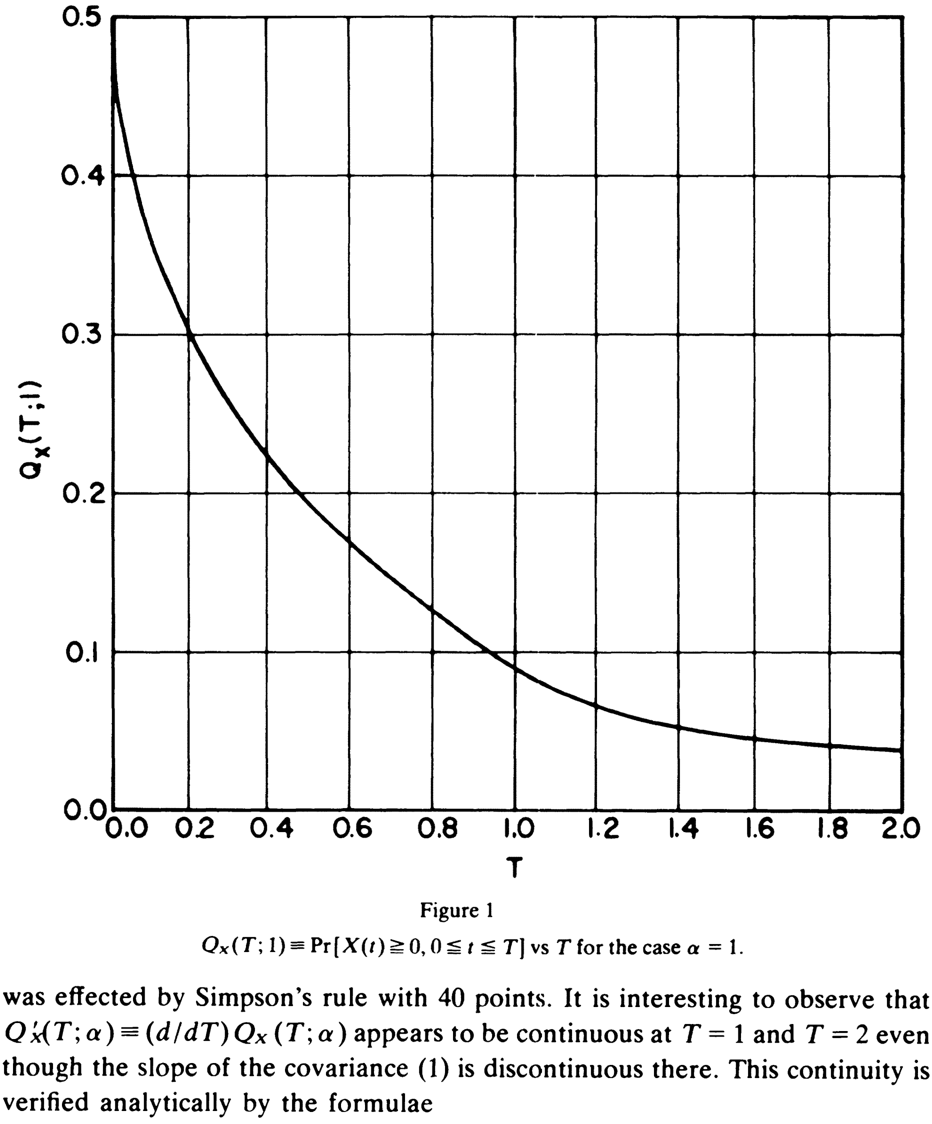

Figure 1

$Q_X(T;1) \equiv \Pr[X(t) \geqq 0, 0 \leqq t \leqq T]$ vs T for the case $\alpha = 1$.

was effected by Simpson's rule with 40 points. It is interesting to observe that $Q'_X(T;\alpha) \equiv (d/dT)Q_X(T;\alpha)$ appears to be continuous at $T = 1$ and $T = 2$ even though the slope of the covariance (1) is discontinuous there. This continuity is verified analytically by the formulae

$$Q'_X(1+;\alpha) = Q'_X(1-;\alpha) = -(1/2\pi)\sqrt{(\alpha(2-\alpha))} \tag{26}$$

$$Q'_X(2+;\alpha) = Q'_X(2-;\alpha) = 0. \tag{27}$$

We indicate the derivation in the appendix.

A graph of $Q'_X(T;\alpha)$ for $\alpha = 1$ is given in Figure 2. It was computed by taking finite difference quotients of $Q_X(T;\alpha)$ at 200 points in the interval $(0.9, 1.1)$. We

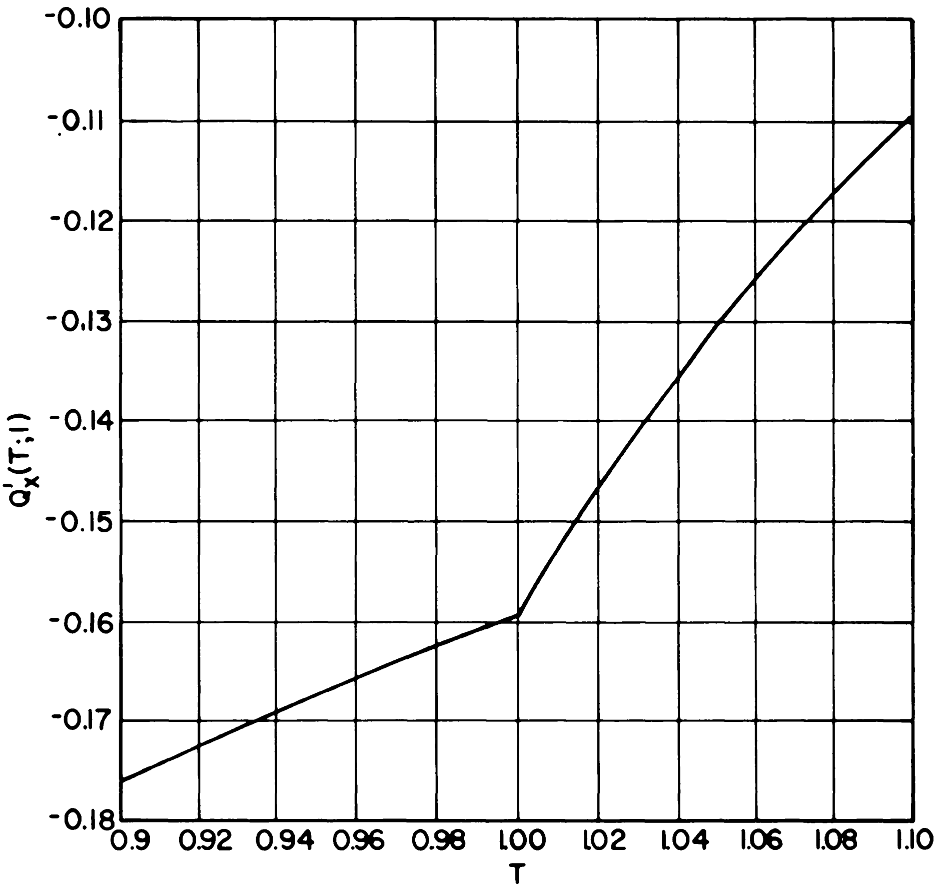

Figure 2

$Q_X'(T;1) \equiv \frac{d}{dT} \Pr[X(t) \geqq 0, 0 \leqq t \leqq T]$ vs T for the case $\alpha = 1$.

used Simpson's rule with 200 points in the integral in (25) to compute $Q_X(T;\alpha)$ for T in $(1.0, 1.02)$ to retain sufficient accuracy for the difference quotient; in $(1.02, 1.1)$ only 20 points were needed to effect the integration. Figure 2 convincingly indicates that the second derivative of $Q_X(T, \alpha)$ is discontinuous at $T = 1$.

Another well-studied stationary Gaussian process having a covariance with a discontinuity in slope at $T = 1$ is the zero mean process $Y(t)$ with triangular covariance $\rho_Y(\tau) = EY(0)\,Y(\tau) = \max(1 - |\tau|, 0)$. We use the continuity of $Q_X'(T;1)$ at $T = 1$ just established to show that the zero exceedence probability for the Y process, $Q_Y(T) \equiv \Pr[Y(t) \geqq 0,\ 0 \leqq t \leqq T]$, also has a continuous derivative with respect to T at $T = 1$. To prove this, we make use of an inequality ([5], Theorem 1) which states that for zero mean stationary Gaussian processes $Y_1(t)$ and $Y_2(t)$ with $EY_1(t)^2 = EY_2(t)^2$, $Q_{Y_1}(T) \geqq Q_{Y_2}(T)$ if $\rho_{Y_1}(t) \equiv EY_1(0)Y_1(t) \geqq \rho_{Y_2}(t) \equiv EY_2(0)Y_2(t)$ for $0 \leqq t \leqq T$. We apply this inequality to three zero mean Gaussian processes: (i) $X(t)$ with covariance (1) with $\alpha = 1$; (ii) $Y(t)$ as described above; (iii) $Z(t)$ with covariance $\rho_Z(\tau) = 1 - |\tau|$ for $|\tau| < \frac{3}{2}$, $\rho_Z(\tau + 3) = \rho_Z(\tau)$. We note that $\rho_X(t) \geqq \rho_Y(t) \geqq \rho_Z(t)$, $0 \leqq t \leqq \frac{3}{2}$, with strict equality for $0 \leqq t \leqq 1$. Thus

$$Q_X(T; 1) \geqq Q_Y(T) \geqq Q_Z(T), \quad 0 \leqq T \leqq \tfrac{3}{2},$$

with strict equality for $0 \leqq T \leqq 1$. Since both $Q_X(T; 1)$ and $Q_Z(T)$ have continuous derivatives at $T = 1$, it readily follows that $Q_Y(T)$ must also have a continuous derivative there. That $Q_Z(T)$ has a continuous derivative at $T = 1$

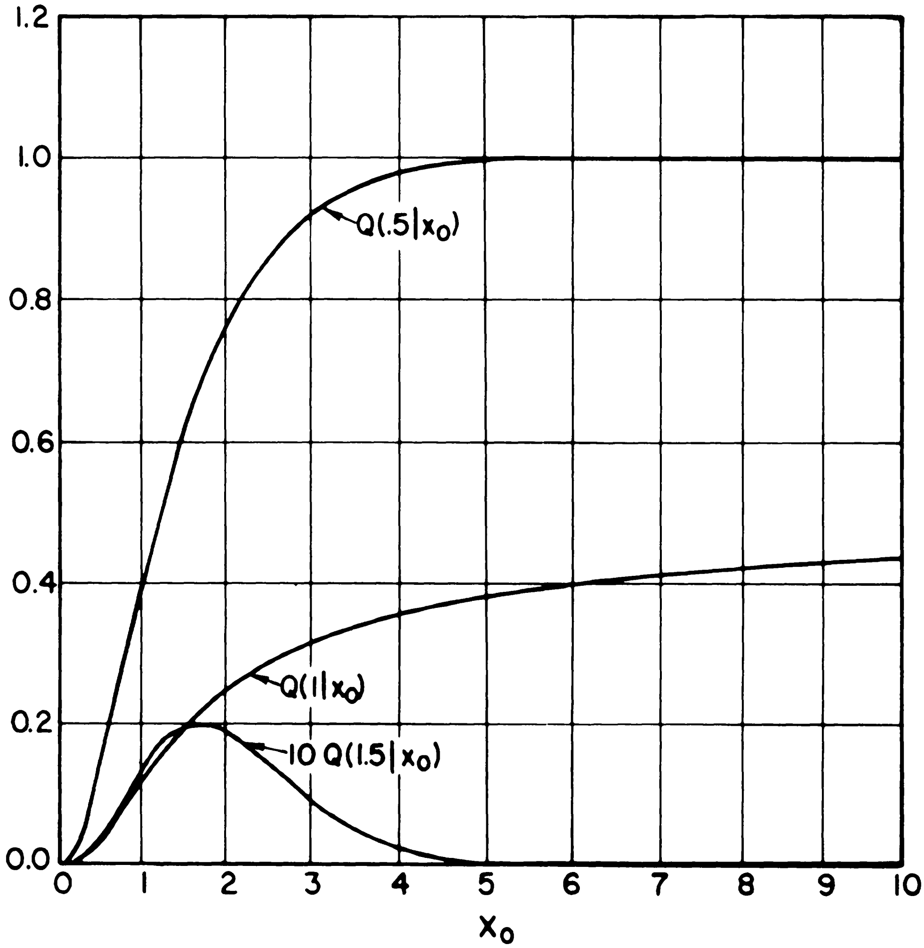

Figure 3

$Q(\alpha \mid x_0) \equiv \Pr[X(t) \geqq 0, 0 \leqq t < \infty \mid X(0) = x_0]$ vs x_0 for $\alpha = 0.5, 1, 1.5$.

can be seen from (23), since it is clear that for $T < \frac{3}{2}$, $Q_Z(T) = Q_X(\frac{2}{3}T, \frac{3}{2}) = (1/2\pi)$ $[\arccos(T-1) - \sqrt{(T(2-T))}]$. An exact expression for $Q_Y(T)$ was found in [4] but it is awkward to use in computations and the continuity of $Q_Y'(T)$ at $T = 1$ had been conjectured but not established.

From Equation (21) we can determine the conditional excursion probability $Q(\alpha \mid x_0) = \Pr(X(t) \geqq 0, 0 \leqq t < \infty \mid X(0) = x_0)$ as the infinite series

$$
\begin{aligned}
Q(\alpha \mid x_0) &= \frac{1}{\sqrt{(\alpha(2-\alpha))}} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{e^{\frac{1}{2}x_0^2}}{\sqrt{A_n}} \left[e^{x_0^2[-1/\alpha + B_n^2/2A_n]} \Phi(x_0(B_n - A_n)/\sqrt{A_n}) - \Phi(-x_0\sqrt{A_n}) \right]
\end{aligned} \tag{28}
$$

where $A_n = (2(2-\alpha))^{-1} + (2n+1)^2/(2\alpha)$, $B_n = (2n+1)/\alpha$, and Φ is the standard normal distribution function. Using (28) we have graphed in Figure 3 $Q(\alpha \mid x_0)$ for $\alpha = 0.5$, 1, 1.5 as a function of x_0. Note in Figure 3 for $\alpha = 1.5$ we have plotted 10 times $Q(1.5 \mid x_0)$ instead of $Q(1.5 \mid x_0)$ itself. For $\alpha = 1.5$, we see that for large increasing x_0, the probability that $X(t)$ remains positive decreases. This apparent paradox is easily explained by noting that for $\alpha > 1$, $X(0)$ and $X(1)$ are anticorrelated, $E(X(1) \mid X(0) = x_0) = (1-\alpha)x_0 \to -\infty$ as $x_0 \to \infty$, while the conditional variance of $X(1)$ remains bounded. Thus as $x_0 \to \infty$, $X(0)$ is positive but $X(1)$ is more likely to be negative.

Acknowledgement

This paper has benefited from comments supplied by a referee. He also points out that the Gaussian process $X(t)$ with covariance (1) has the following simple representation. Let $W(t)$ be the Wiener process of Section 4 with $W(0) = 0$, and let Y be a zero-mean Gaussian random variable, independent of $W(t)$, with variance $1 - \frac{1}{2}\alpha$. Then the process $X(t)$ defined by

$$
\begin{aligned}
X(t) &= Y + W(2\alpha t) - \tfrac{1}{2}W(2\alpha), \quad 0 \leqq t \leqq 1 \\
X(t+1) &= Y - W(2\alpha t) + \tfrac{1}{2}W(2\alpha), \quad 0 \leqq t \leqq 1 \\
X(t+2) &= X(t)
\end{aligned}
$$

is a zero-mean Gaussian process with covariance (1). The basic property (9) can be easily derived from this representation using the known Markov nature of $W(t)$.

Appendix: Outline of computational detail

A. *Derivation of* (24). Use (22) with $T = 2$, (21) with $a = 0$ and (14) to express $Q(2; \alpha)$ as a double integral of an infinite sum. A change of integration variables from x_0, x_1 to u, t defined by $u = x_0 + x_1$, $x_0 = tu$ yields

$$Q_X(2;\alpha) = \frac{1}{2\pi\sqrt{(\alpha(2-\alpha))}} \int_0^1 dt \int_0^\infty du\, u \exp(-\tfrac{1}{4}u^2/(2-\alpha))$$
$$\times \sum_{k=-\infty}^{\infty} \{\exp[-(u^2/\alpha)(k+\tfrac{1}{2}-t)^2] - \exp[-(u^2/\alpha)(k+\tfrac{1}{2})^2]\}.$$

Carry out the u integration and use the identity

$$\frac{s}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{s^2+(k+t)^2} = -1 + \frac{2(1-q\cos 2\pi t)}{(1-q\cos 2\pi t)^2 + (q \sin 2\pi t)^2},$$

$q = e^{-2\pi s}$. The integration on t can then be carried out to yield $Q_X(2;\alpha) = [1 + e^{\pi\sqrt{(\alpha/(2-\alpha))}}]^{-1}$. Since $X(t)$ has period 2, $Q_X(T;\alpha) = Q_X(2;\alpha)$ for $T \geqq 2$ and (24) follows.

B. *Derivation of* (25). Use (22) and (19) to write

$$Q_X(1+T;\alpha) = \frac{1}{\sqrt{(\pi(2-\alpha))}} \int_0^\infty dx_0 \int_0^\infty dx_1 \int_0^{x_0+x_1} dx_T e^{-\frac{1}{4}(x_0+x_1)^2/2(2-\alpha)}$$
$$\times q_W(0, x_0+x_1, x_T, 2\alpha T \mid x_0) q_W(0, \infty, x_1, 2\alpha(1-T) \mid x_T).$$

Use (14) noting that for $b = \infty$ only the $k = 0$ term is to be taken there, so that $q_W(0, \infty, x_1, 2\alpha(1-T) \mid x_T) =$

$$\frac{1}{\sqrt{(4\pi\alpha(1-T))}} \{\exp[-(x_1-x_T)^2/(4\alpha(1-T))] - \exp[-(x_1+x_T)^2/(4\alpha(1-T))]\}.$$

Change variables of integration to x, y, u through the transformation $x_0 + x_1 = u$, $x_0 = xu$, $x_T = yu$. In the terms having $\exp[u^2(1-x+u)^2/4\alpha(1-T)]$ as a factor change y to $-y'$ and k to $-k'$. One then has

$$Q_X(1+T;\alpha) = \frac{1}{4\pi\alpha\sqrt{(\pi(2-\alpha)T(1-T))}}$$
$$\times \int_0^1 dx \int_{-1}^1 dy \int_0^\infty du u^2 \sum_{-\infty}^{\infty} [\exp(-\tfrac{1}{2}u^2 A_k) - \exp(-\tfrac{1}{2}u^2 B_k)]$$

with A_k and B_k quadratic expressions in x and y. Carry out the u integration. Replace x by $1-x$ and in the negative terms of the integrand replace k by $-k$ and y by $-y$. Next carry out the y integration. Equation (25) results.

C. *Derivation of* (26). To obtain $Q'_X(1+;\alpha)$ we differentiate (25) with respect to T and evaluate at $T = 0+$. The derivate with respect to T of the expression in brackets in (25) vanishes at $T = 0+$ if $x + k + 1 \neq 0$. Since $|x| \leqq 1$ in (25) and the sum is over odd k, only the term $k = -1$ need be retained.

Working with only this term it is convenient to calculate $f'(0+)$ as $\lim_{T\to 0+}[f(T)-f(0+)]/T$ and so obtain

$$Q_X'(1+;\alpha)$$

$$= \lim_{T\to 0+}\frac{1}{T}\frac{\sqrt{c}}{2\pi}\int_{-1}^{1} dx\left[\frac{x(1-T)-(x-1)T}{[c+(2x-1)^2][cT(1-T)+x^2(1-T)+(x-1)^2T]^{\frac{1}{2}}}\right.$$

$$\left.-\frac{x(1-T)+(x+1)T}{(c+1)[cT(1-T)+x^2(1-T)+(x+1)^2T]^{\frac{1}{2}}}-\frac{\operatorname{sgn}x}{c+(2x-1)^2}+\frac{\operatorname{sgn}x}{c+1}\right].$$

Now substitute $x = u\sqrt{T}$ and let $T\to 0+$. Straightforward but tedious calculation yields

$$Q_X'(1+;\alpha)=\frac{1}{2\pi}\frac{\sqrt{c}}{c+1}2\int_0^\infty\left(\frac{-4u}{u+\sqrt{(u^2+c+1)}}+\frac{2u^2}{u^2+c+1}\right)\frac{du}{\sqrt{(u^2+c+1)}}$$

$$= -\frac{1}{2\pi}\sqrt{(\alpha(1-\alpha))}.$$

A similar calculation gives (27).

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