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A FIRST PASSAGE PROBLEM FOR THE WIENER PROCESS

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We study the stopping time $T = T_{a,c}$, the first time the Wiener process $W(t)$, $0 \leqq t < \infty$ crosses the curve $\pm c(t + a)^{\frac{1}{2}}$, $a > 0$, $c > 0$. In an analogous discrete time problem the moments of T were studied in Blackwell-Freedman (1964), Chow-Robbins-Teicher (1965), and Chow-Teicher (1966). Results obtained showed that for the discrete problem $ET < \infty$ if and only if $c < 1$, and $ET^2 < \infty$ if and only if $c < (3 - 6^{\frac{1}{2}})^{\frac{1}{2}}$. We show that these statements are also valid in our case and find the generalization: $ET^n < \infty$ if and only if c is less than the first positive zero of the Hermite polynomial He_{2n}. We conjecture that the same result holds also in the original discrete case. (Note that $(3 - 6^{\frac{1}{2}})^{\frac{1}{2}}$ is the first zero of He_4.) We also give explicit formulas for the moments of T. Our method is based on the Wald identity for the Wiener process: $E \exp(-\lambda^2 T/2 + \lambda W(T)) = 1$.

We show that *any* stopping time T with $ET < \infty$ satisfies $EW(T) = 0$ and $EW^2(T) = ET$. These identities will be derived as a consequence of the basic properties of the Ito stochastic integral.

1. Stopping times and Ito integrals. Let $W(t, \omega)$, $0 \leqq t < \infty$, $\omega \in \Omega$ be a Wiener process with continuous paths. A finite nonnegative rv $T(\omega)$ is called a stopping time if $\{T \leqq t\} \in \mathfrak{S}\{W(s): s \leqq t\}$, $0 \leqq t < \infty$. For such a T let $\varphi_T(t, \omega) = 1$ for $t \leqq T(\omega)$, $\varphi_T(t, \omega) = 0$ for $t > T(\omega)$.

Lemma 1. *The Ito integral* $I(\varphi_T)(\omega) = \int_0^\infty \varphi_T(t, \omega)\, dW(t)$ *is defined and* $I(\varphi_T) = W(T)$ a.s.

Proof. According to K. Ito (1951), we must show that φ_T is measurable in $t \times \omega$, is nonanticipative, and satisfies $\int_0^\infty \varphi^2(t, \omega) dt < \infty$ a.s. Let $T_n = k/2^n$ on $\{(k-1)/2^n \leqq T < k/2^n\}$. It is easy to check that T_n is both measurable and nonanticipative and that $\varphi_{T_n} \to \varphi_T$ at each (t, ω). Since $T(\omega) = \int_0^\infty \varphi^2(t, \omega)\, dt$ and $T(\omega) < \infty$ a.s. because T is a stopping time, we have checked that $I(\varphi_T)$ is defined. That $I(\varphi_T) = W(T)$ is a consequence of the definition of the Ito integral and the continuity of Wiener paths.

Theorem 1. *Let T be any stopping time with $ET < \infty$. Then $EW(T) = 0$ and $EW^2(T) = ET$.*

Proof. The mean of an Ito integral with finite variance is zero (Ito (1951)). The variance of $I(\varphi_T) = W(T)$ is given by Ito's formula $\int_0^\infty E\varphi_T^2(t, \omega)\, dt = \int_0^\infty P\{T \geqq t\}\, dt = ET$. This proves both assertions.

Caution. It is possible that $ET = \infty$ and $EW^2(T) < \infty$ as the example $T =$ time of crossing level one shows.

The result for discrete time corresponding to Theorem 1 has been proved in great generality by martingale methods [3].

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2. Parabolic level crossing. Let $T = T_{a,c}$ denote the first time $|W(t)| = c(t + a)^{\frac{1}{2}}$. By Theorem 1, $E(c(T + a)^{\frac{1}{2}})^2 = ET$ or $ET = ac^2/(1 - c^2)$ provided that $ET < \infty$. This indicates that $ET = \infty$ for $c \geqq 1$ and $ET < \infty$ for $c < 1$. This much follows easily but in order to obtain higher moments of T it is best to proceed directly to the probability distribution of T.

The analogue for continuous time of the fundamental theorem of sequential analysis is (Dvoretzky-Kiefer-Wolfowitz (1953))

$$Ee^{-\lambda^2 T/2}e^{\lambda W(T)} = 1. \tag{1}$$

Although the conditions under which (1) is valid are not very general, it holds whenever T is a bounded stopping time. In case T is not bounded (1) can sometimes be proved by passage to the limit from bounded times. This we now do. Let $T \wedge n = \min(T, n)$. Using (1) for $T \wedge n$ we have

$$1 = \int_{T \leqq n} e^{-\lambda^2 T/2}e^{\lambda W(T)}\, dP + \int_{T>n} e^{-\lambda^2 n/2}e^{\lambda W(n)}\, dP. \tag{2}$$

The second integral goes to zero because $|W(n)| < c(n + a)^{\frac{1}{2}}$ on $\{T > n\}$, and (1) follows.

We will use (1) to study the distribution of T. Let $p(t) = p_{a,c}(t)$ denote the density of T, for ease of notation. By (1) we have

$$\int_0^\infty p(t)e^{-\lambda^2 t/2}\cosh \lambda c(t + a)^{\frac{1}{2}}\, dt = 1, 0 \leqq \lambda < \infty. \tag{3}$$

Multiply in (3) by $\lambda^\beta \exp(-\lambda^2 a/2)$ and integrate over λ to get

$$\int_0^\infty p(t)(t + a)^\mu\, dt = \int_0^\infty \lambda^\beta e^{-\lambda^2 a/2}\, d\lambda / \int_0^\infty \lambda^\beta e^{-\lambda^2/2}\cosh \lambda c\, d\lambda \tag{4}$$

where $\mu = -(1 + \beta)/2$ and $\beta > -1$.

In order to obtain information about $E(T + a)^\mu$ with $\mu > 0$, we must analytically continue (4) to $\beta < -1$. Expanding in powers of c we get

$$\left(\int_0^\infty p(t)(t + a)^\mu\, dt\right)^{-1} = a^{-\mu} \sum_{m=0}^\infty (-2c^2)^m \mu(\mu - 1) \cdots (\mu - m + 1)/(2m), \tag{5}$$

which we recognize as $a^{-\mu} M(-\mu, \frac{1}{2}, c^2/2)$ where M is the confluent hypergeometric function (Abramowitz-Stegun (1964)). This gives the analytic continuation. For $\mu = n$ a positive integer, $M(-n, \frac{1}{2}, c^2/2) = He_{2n}(c)/He_{2n}(0)$ where He are the Hermite polynomials, $He_n(x) = (-1)^n(\exp(x^2/2))\, d^n/dx^n(\exp(-x^2/2))$.

The moment $ET^\mu = ET^\mu_{a,c}$ is finite or infinite according as $E(T + a)^\mu$ is finite or infinite. By (5) we see that $E(T + a)^\mu < \infty$ if and only if $c < c_0 = c_0(\mu)$ independently of a where c_0 is the first zero of $M(-\mu, \frac{1}{2}, c^2/2)$. Thus the nth moment of T is finite when c is less than the first zero of He_{2n}. The zeros of He and M have been tabulated (Abramowitz-Stegun (1964)), giving $c_0(\mu)$:

μ: $ET^\mu < \infty$	$c_0(\mu)$: $c < c_0$	μ	$c_0(\mu)$
.1	2.0941	1.0	1.0000
.2	1.7545	2.0	.7420
.3	1.5537	3.0	.6167
.4	1.4132	4.0	.5391
.5	1.3069	5.0	.4849
.6	1.2223	6.0	.4444
.7	1.1528	8.0	.3868
.8	1.0942	10.0	.3470
.9	1.0438		

For μ large, $c_0 \sim 1/\mu^{\frac{1}{2}}$ and for μ small, $c_0 \sim (2 \log 1/\mu)^{\frac{1}{2}}$. The moments themselves can be obtained from (5). For example if $c < c_0(2)$ we have $E(T + a)^2 = a^2/(1 - 2c^2 + c^4/3)$, $E(T + a) = a/(1 - c^2)$ and we can solve for ET and ET^2.

(5) gives the Laplace transform of $p(a(e^u - 1))$ and so determines p in principle. Darling-Siegert (1953) give a Laplace transform formula for $P\{|W(t)| < \alpha t^{\frac{1}{2}}, a \leqq t \leqq b\}$ by Markovian methods.

On the basis of the analogy with discrete random walk it seems reasonable to guess that for example, for simple random walk S_n the stopping time N = first $n > n_0$ for which $|S_n| > cn^{\frac{1}{2}}$ will have $EN^k < \infty$ if and only if c is less than the first zero of He_{2k}, generalizing [4]. As in [4], n_0 is taken large enough to avoid stopping with probability one for small c.

We learned from D. L. Burkholder after this note was to be published that Leo Breiman has obtained more general results in his paper, "First exit times from a square root boundary," which will appear in the Proceedings of the 5th Berkeley Symposium. He obtains the conjectured results for the discrete time case as well. Our methods are different.

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