Existence of Submatrices with All Possible Columns

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Let $M$ be a matrix with entries from $\{1, 2, \ldots, s\}$ with $n$ rows such that no matrix $M'$ formed by taking $k$ rows of $M$ has $s^k$ distinct columns. Let $f(k; n, s)$ be the largest integer for which there is an $M$ with $f(k; n, s)$ distinct columns. It is proved that $f(k; n, s) = s^k - \sum_{i=0}^{n-k} \binom{s}{i} (s - 1)^{n-i}$. This result is related to a conjecture of Erdős and Szekeres that any set of $2^{k-1} + 1$ points in $R^k$ contains a set of $k$ points which form a convex polygon.

1. Introduction

The theorems provided in this note are motivated by questions like the following:

Suppose an $n$ set $x_1, x_2, \ldots, x_n$ is colored by $s$ colors in $m$ distinct ways. How large need $m$ be to guarantee that there is a $k$ set colored in all possible (i.e., $s^k$) ways?

Suppose that $S$ is a class of subsets of a set $X$ and that $\{x_1, x_2, \ldots, x_n\}$ is an $n$-element subset of $X$ for which $m$ of the sets $A \cap \{x_1, x_2, \ldots, x_n\}$, $A \in S$, are distinct. How large need $m$ be to guarantee that there is a $k$-element set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ for which there are $2^k$ distinct sets $A \cap \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, $A \in S$?

The first of these questions is new, but the second has been considered previously. It has in fact been solved quite precisely by Sauer [4] in response to a query of Erdős. An earlier independent solution was given in [5] in connection with a probabilistic application, but the result of [5] was not the best possible. In Section 2 of this note Theorem 2.1 gives a general result by a new method which implies these earlier results and covers the fresh ground indicated by question (1.1).

The third section gives a geometrical interpretation to a special case of Theorem 2.1, and shows the relationship of the present work to a long-standing conjecture of Erdős and Szekeres (see [1, p. xxii]).
2. Main Results

Let $M$ be a matrix with entries from an $s$-symbol alphabet $\{1, 2, \ldots, s\}$. Now let $f(k; n, s)$ be the largest integer such that there is a matrix $M$ with $n$ rows and $f(k; n, s)$ distinct columns such that no matrix $M'$ formed by taking $k$ of the rows of $M$ has $s^k$ distinct columns.

To note the relationship of $f(k; n, s)$ to question (1.1) one defines a correspondence between matrices and sets of colorings as follows: $M = (a_{ij})$, where $a_{ij} = b$ and $b$ is the color of $x_i$ in the $j$th coloring of $\{x_1, x_2, \ldots, x_n\}$. For any subset of elements $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subseteq \{x_1, x_2, \ldots, x_n\}$ there is a corresponding subset of $k$ rows of $M$ which forms a submatrix $M'$. Further, since any coloring of $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ corresponds to a column of $M$, the number of distinct colorings of $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ equals the number of distinct columns of $M'$. In the notation of (1.1) we therefore have $m = f(k; n, s) + 1$.

The main result can now be stated quite succinctly.

Theorem 2.1.

$$f(k; n, s) = s^n - \sum_{j=k}^{n} \binom{n}{j}(s - 1)^{n-j}. \quad (2.1)$$

**Proof.** First it will be shown by construction that $f(k; n, s) \geq s^n - \sum_{j=k}^{n} \binom{n}{j}(s - 1)^{n-j}$, and then the opposite inequality will be proved afterward by relating the general case to the first construction.

Define $M$ to be the matrix consisting of all columns such that no column contains $k$ or more ones. Since $\sum_{j=k}^{n} \binom{n}{j}(s - 1)^{n-j}$ is precisely the number of columns with $k$ or more ones, we see that $M$ has $s^n - \sum_{j=k}^{n} \binom{n}{j}(s - 1)^{n-j}$ columns. But since no $k$-row submatrix of $M$ contains the column of all ones we have $f(k; n, s) \geq s^n - \sum_{j=k}^{n} \binom{n}{j}(s - 1)^{n-j}$.

To obtain the opposite inequality we suppose that a matrix $M$ has no $k$-row submatrix with $s^k$ columns. To describe the columns which are missing from $M$, let $C_1, C_2, \ldots, C_\tau$ where $\binom{\tau}{k} = \tau$ be a list of the $k$-element subsets of the row indices. For each $i = 1, 2, \ldots, \tau$ there is a submatrix $M_i$ formed by the $C_i$ rows of $M$. Also by the hypothesis there is a $k$-vector $v_i$ which is not a column of $M_i$. Now for each such $v_i$ let $Z_i$ be the set of columns of the $n \times s^n$ matrix which equal $v_i$ when restricted to the index set $C_i$. Finally observe that none of the columns of $Z = \bigcup_{i=1}^{\tau} Z_i$ is a column of $M$.

If $\nu$ denotes the number of columns of $M$ then $\nu \leq s^n - |\bigcup_{i=1}^{\tau} Z_i|$, (where $|\bigcup_{i=1}^{\tau} Z_i|$ denotes the number of the columns in the union $\bigcup_{i=1}^{\tau} Z_i$).

The proof will be completed by obtaining a lower bound on $|\bigcup_{i=1}^{\tau} Z_i|$. To do this we define a function on column vectors $w = (w_1, w_2, \ldots, w_n)$ as follows:

$$\Phi(w) = w', \quad \text{where } w' = (w'_1, w'_2, \ldots, w'_n) \quad (2.2)$$
and
\[ w_i' = 1 \quad \text{if } w \in Z_i \text{ and } j \in C_i \text{ for some } i = 1, 2, \ldots, \tau, \]
\[ = w_j \quad \text{otherwise}. \quad (2.3) \]

The function \( \Phi \) has several elementary but valuable properties which we first note and then prove:

\[ | \Phi(Z) | \leq | Z | \quad \text{for } Z = \bigcup_{i=1}^{\tau} Z_i. \quad (2.4) \]

\( \Phi(Z_i) \) contains all columns of the \( n \times s^n \) matrix which when restricted to \( C_i \) equal the \( k \)-column vector \((1, 1, \ldots, 1)\). \quad (2.5)

\( \Phi(Z) \) contains all \( n \)-columns which contain \( k \) or more ones. \quad (2.6)

\[ | \Phi(Z) | \geq \sum_{j=k}^{n} \binom{n}{j}(s-1)^{n-j}. \quad (2.7) \]

The proof of (2.4) is immediate since \( \Phi \) is a function, and (2.5) is just a consequence of (2.3). To prove (2.6) note that if \( w \) has \( k \) or more ones, then there is a \( C_i \), restricted to \( C_i \) which \( w \) has all ones, and hence \( w \in \Phi(Z_i) \), by (2.3) and the definition of \( Z_i \). Finally (2.7) comes from (2.6) and easy counting.

The last calculation is that

\[ v_1 \leq s^n - | Z | \leq s^n - | \Phi(Z) | \leq s^n - \sum_{j=k}^{n} \binom{n}{j}(s-1)^{n-j}, \quad (2.8) \]

which completes the proof.

The preceding method also permits a precise understanding of those extreme matrices which lack \( k \)-row submatrices with a complete column set. Such matrices are characterized by a "missing" column vector.

**Theorem 2.2.** Suppose \( M \) is an \( n \)-row matrix with \( s^n - \sum_{j=k}^{n} \binom{n}{j}(s-1)^{n-j} \) distinct columns and which has no \( k \)-row submatrix with \( s^k \) distinct columns. Then there is an \( n \) vector \( v \) such that for each column \( w \) of \( M \) one has \( w_i \neq v_i \) for at least \( k \) values of the index \( i \).

**Proof.** In the notation of the previous proof, we note that if there is no \( v \) as required above then there are \( v_i \) and \( v_j \) such that \( C_i \cap C_j \neq \emptyset \) yet \( v_i \) and \( v_j \) are not equal on \( C_i \cap C_j \). By the definition of \( \Phi \) and \( Z_i \) we therefore have \( | \Phi(Z_i \cup Z_j) | < | Z_i \cup Z_j | \). Consequently, we have \( | \Phi(Z) | < | Z | \).

But, since \( M \) has \( s^n - \sum_{j=k}^{n} \binom{n}{j}(s-1)^{n-j} \) distinct columns, we note \( | Z | = \sum_{j=k}^{n} \binom{n}{j}(s-1)^{n-j} \). However, by (2.7) we know \( | \Phi(Z) | \geq \sum_{j=k}^{n} \binom{n}{j}(s-1)^{n-j} \) so the inequality \( | \Phi(Z) | < | Z | \) yields a contradiction.
3. Relevance to a Famous Conjecture

Is it true that out of every \(2^{k-2} + 1\) points in the plane one can always select \(k\) points so that they form a convex \(n\)-sided polygon? This problem, posed in the winter of 1932–1933, published in 1935, promulgated daily, is still unsolved for \(k \geq 6\) [1, pp. xxi, 42; 2; 3].

The results of Section 2 are relevant to this conjecture of Erdös and Szekeres, since they provide a sufficient condition that a set contain a convex polygon.

To see this let \(X\) be the plane and \(S\) the class of convex subsets of \(X\). Next define

\[
\Delta(x_1, x_2, ..., x_n) = |\{(x_1, x_2, ..., x_n) \cap A: A \in S\}|
\]

that is, \(\Delta(x_1, x_2, ..., x_n)\) is the number of subsets \(\{x_{i_1}, x_{i_2}, ..., x_{i_k}\} \subseteq \{x_1, x_2, ..., x_n\}\) such that \(\{x_{i_1}, x_{i_2}, ..., x_{i_k}\} = \{x_1, x_2, ..., x_n\} \cap A\) for some \(A \in S\). Let \(A_j, j = 1, 2, ..., \Delta(x_1, x_2, ..., x_n)\), be elements of \(S\) such that each of the sets \(\{x_1, x_2, ..., x_n\} \cap A_j\) is distinct. These \(A_j\) define a \(n \times \Delta(x_1, x_2, ..., x_n)\) matrix as follows:

\[
a_{ij} = 1 \quad \text{if} \quad x_i \in A_j, \\
= 0 \quad \text{if} \quad x_i \notin A_j.
\]

By the definition of the \(A_j\) we know that \(M = (a_{ij})\) has \(\Delta(x_1, x_2, ..., x_n)\) distinct columns so

\[
\Delta(x_1, x_2, ..., x_n) \leq f(k; n, 2)
\]

unless \(M\) has \(k\) rows which have \(2^k\) distinct columns. But since \(\Delta(x_{i_1}, x_{i_2}, ..., x_{i_k}) = 2^k\) if and only if the set \(\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}\) forms a convex polyhedron, we have proved the following:

**Theorem 3.1.** A sufficient condition that the set \(\{x_1, x_2, ..., x_n\} \subseteq \mathbb{R}^2\) contains \(k\) points which form a convex polygon is that

\[
\Delta(x_1, x_2, ..., x_n) > \sum_{j=0}^{k-1} \binom{n}{j}.
\]

To prove the Erdős–Szekeres conjecture it thus suffices to show that (3.4) holds when \(n = 2^{k-2} + 1\). Of course, condition (3.5) has only been proved sufficient and quite possibly the Erdős–Szekeres conjecture can be true without (3.4) being met. Still, there are several possible uses of \(\Delta(x_1, x_2, ..., x_n)\) in this problem and (3.4) pinpoints the most direct one.

To gain another view of Theorem 3.1 one should note that it is possible to give a more geometrical proof which avoids invoking the full strength of Theorem 2.1. For this proof, suppose \(B \in \{(x_1, x_2, ..., x_n) \cap A: A \in S\}\)
and let \( \partial B \) denote the subset of \( B \) equal to the elements of \( B \) on the boundary of the convex hull of \( B \). We note that \( | \partial B | \leq k - 1 \) if \( \{ x_1, x_2, \ldots, x_n \} \) contains no \( k \)-element convex polygon, since, indeed, \( \partial B \) is convex polygon. Next note that there are precisely \( \sum_{j=0}^{k-1} \binom{n}{j} \) subsets of \( \{ x_1, x_2, \ldots, x_n \} \) with fewer than \( k \) elements. Since \( \partial B \) uniquely determines \( B \) we have

\[
\Delta(x_1, x_2, \ldots, x_n) \leq \sum_{j=0}^{k-1} \binom{n}{j}
\]

unless \( \{ x_1, x_2, \ldots, x_n \} \) contains a \( k \)-element subset which forms a convex polygon. This completes a second proof of Theorem 3.1.

4. A Closely Related Problem

In connection with the results given here and the Erdős–Szekeres conjecture the following question seems quite interesting:

What is the minimum value of \( \Delta(x_1, x_2, \ldots, x_n) \) given that \( \{ x_1, x_2, \ldots, x_n \} \) contains a \( k \)-set which forms a convex polygon? \( \text{(4.1)} \)

(The \( x_i \) are assumed noncolinear.)

If this value is called \( g(n, k) \), it is trivial that \( g(n, k) \geq 2^k \), but a substantial improvement on this seems difficult. Still, by consideration of this problem it may be possible to make progress of the yet unreachable conjecture of Erdős and Szekeres.

REFERENCES