On Contingent Claims that Insure Ex-Post Optimal Stock Market Timing
Author(s): M. Barry Goldman, Howard B. Sosin, Lawrence A. Shepp
Source: *The Journal of Finance*, Vol. 34, No. 2, Papers and Proceedings, Thirty-Seventh Annual Meeting, American Finance Association, Chicago, Illinois, August 29-31, 1978 (May, 1979), pp. 401-413
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2326979
Accessed: 19/11/2010 16:19

THE JOURNAL OF FINANCE • VOL. XXXIV, NO. 2 • MAY 1979

On Contingent Claims that Insure Ex-post Optimal Stock Market Timing

M. BARRY GOLDMAN, HOWARD B. SOSIN and LAWRENCE A. SHEPP

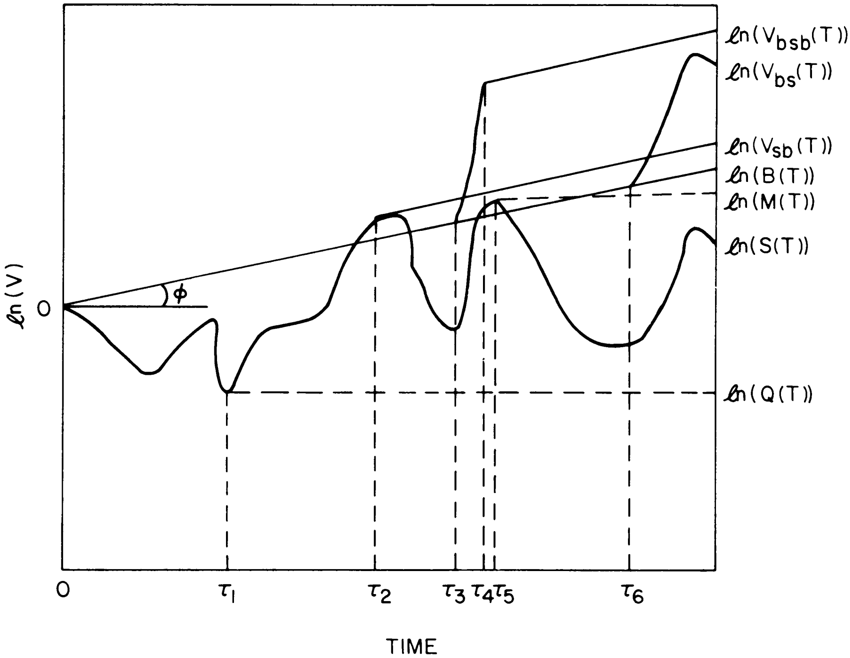
1. Introduction

Up and down the market moves
In and out the people
That's the way the money goes
Pop goes the weasel

WHEN TO INVEST, how much to pay, and when to terminate an investment are important considerations for investors. To date, much academic attention has been focused on the middle part of this problem; there are Capital Asset Pricing Models and Arbitrage Models that predict the relative pricing of risky and riskless assets. In addition to these pricing theories, there is a substantial body of literature concerned with ex-ante timing rules and measurement of timing performance. From the practitioners come charting rules; from the academics come denials as to their efficacy. From mutual fund managers come claims of superior timing; from academics comes evidence that it is lacking. The general failure of published timing rules has so discouraged some investors that they have adopted naive strategies, like *dollar averaging* as guides to moving funds into and out of the market.

Investors' interest in ex-ante timing rules leads naturally to a consideration of the ex-ante cost and properties of contingent claims that, ex-post, guarantee optimal timing. Recent work by Goldman, Sosin and Gatto [4] (henceforth GSG) analyzed European contingent claims that, over a fixed horizon, either endow the purchaser with the right to purchase one share of stock at the security's ex-post realized minimum (i.e., a call on the minimum, named $C_{\min}$), or with the right to sell one share of stock at the security's ex-post realized maximum (i.e., a put on the maximum, named $P_{\max}$). These options offer a partial answer to the ex-post optimal-timing question in that they satisfy the common shareholder dream of "buying at the low" and "selling at the high."

In essence, buying at the ex-post low and selling at the ex-post high provide ways of transfering funds from cash to stock, and from stock to cash. However, since there is usually a positive interest rate, an investor need not start nor end with cash—he can start and end with bonds. With the addition of bonds to the investor's choice set, it is no longer generally optimal ex-post to have purchased a security at its minimum or to have sold it at its maximum. A stock buyer would typically have done better by holding bonds somewhat beyond the time the minimum is achieved, a seller by selling somewhat before the maximum is achieved.



EX-POST INVESTMENT STRATEGIES

FIGURE 1

Notes for Figure 1

1. This figure plots the natural logarithms of a sample path for the pure stock and pure bond portfolios assuming an initial investment of one dollar. It also illustrates the ex-post investment strategies of the three new put options.
2. $Tan(\phi) = r$, the riskless rate of interest. Thus, a dollar invested strictly in riskless bonds will grow to $\exp(rT)$ at time T which in log-space is rT and is represented by $\ln(B(T))$.
3. An investment held strictly in stock will have a terminal value of $S(T)$. Somewhere during the time interval $0 \leq \tau \leq T$, S will achieve a minimum (Q) and a maximum (M). Figure 1 depicts the minimum occurring at τ_1 and the maximum occurring at τ_5 with $\tau_5 > \tau_1$. However, it is entirely consistent with the analysis to have the maximum occur before the minimum.
4. V_{sb} starts in stock at $\tau = 0$ and switches to bonds at τ_2 and then maintains this position until T. The time τ_2 is determined ex-post as the intersection of $S(\tau)$ with a supporting hyperplane of slope r. Alternatively stated, τ_2 maximizes the expression: $S(\tau_2)\exp[r(T - \tau_2)]$.
5. V_{bs} starts in bonds at $\tau = 0$ (and hence grows initially at rate r). The switchover to stock is depicted as occurring at τ_6. Ex-post τ_6 is chosen to maximize the expression:

$$\left[\frac{\exp[r\tau_6]}{S(\tau_6)}\right] S(T)$$

In this paper we consider three new European, ex-post, optimal-timing, contingent claims that are natural extensions of P_{max} and C_{min} and that take explicit account of the time value of money. They are:
(1). A claim that starts with one dollar invested in stock and then optimally (once) switches the proceeds to bonds (with value V_{sb});
(2). A claim that starts with one dollar invested in bonds and then optimally (once) switches the proceeds to stock (with value V_{bs}); and
(3). A claim that starts with one dollar in bonds, switches once to stock and then switches back to bonds (with value V_{bsb}).

Each of these claims will be compared with the naive strategy of simply holding stock throughout the period. Thus, the actual claims we will consider are three new European put options: P_{sb} with exercise price V_{sb}; P_{bs} with exercise price V_{bs}; and P_{bsb} with exercise price V_{bsb}. For a particular sample path of the stock, with an assumed initial stock price of unity, Figure 1 illustrates the ex-post investment strategies of these options.

Using C_{min} and P_{max} as benchmarks, this paper illustrates the mathematical method used to price these new puts and displays some of their equilibrium properties.

Throughout the paper we will adhere to the following notation:

Timing conventions:

0: All options are assumed to be written at time zero,
T: The expiration date of all options,
τ: The current time,
θ: $T - \tau$, the time to expiration.

Remaining notation:

$S(\tau)$: Stock price at time τ, (occasionally abbreviated to S),
r: Risk-free rate of interest,
σ^2: Variance per unit time of log of stock price return,[1]
$M(\tau)$: $\max_{0\leq\delta\leq\tau} S(\delta)$, (occasionally abbreviated to M),
$Q(\tau)$: $\min_{0\leq\delta\leq\tau} S(\delta)$, (occasionally abbreviated to Q),

[1] By "return" we denote $(S_{\theta+\Delta\theta}/S_\theta)$; the proportional gain or loss, $(S_{\theta+\Delta\theta} - S_\theta)/S_\theta$, we call "rate of return."

---

where the first term is the number of shares purchased and $S(T)$ is their terminal value.

6. V_{bsb} starts in bonds at $\tau = 0$ switches to stock at τ_3 and switches back to bonds at $\tau_4 \cdot \tau_3$ and τ_4 are chosen ex-post to maximize the expression

$$\left[\frac{\exp[r\tau_3]}{S(\tau_3)}\right] S(\tau_4)\exp[r(T - \tau_4)]$$

where the first term is the number of shares purchased, the second term is their accumulated sale value. Note, $\tau_3 \leq \tau_4$ by definition.

7. This figure has been drawn with $B(T) > S(T)$. However, it is entirely consistent to have $S(T) \geq B(T)$.

$C_{\min}[S(\tau), Q(\tau), \theta]$: Value of a European option to buy the stock at its realized minimum, when the current stock price is $S(\tau)$, the realized minimum to date is $Q(\tau)$, and the time remaining on the option is θ,

$P_{\max}[S(\tau), M(\tau), \theta]$: The value of a European option to sell the stock at its realized maximum, when the current stock price is $S(\tau)$, the realized maximum to date is $M(\tau)$, and the time remaining on the option is θ,

$N\{\cdot\}$, $N'\{\cdot\}$: Standard normal cumulative distribution and density functions,

E: The expectation operator,

dy: Differential of a Wiener process,

α: Drift term of the rate of return on the stock,

$V_{\max}[S(\tau), M(\tau), \theta]$: The price at time τ of a security whose terminal value is the ex-post realized maximum stock price.

$V_{\min}[S(\tau), Q(\tau), \theta]$: The price at time τ of a security those terminal value is the ex-post realized minimum stock price.

$V_{sb}[S(\tau), m(\tau), \theta]$: The price at time τ of a security that pays out at T the terminal value of a portfolio that starts with stock and optimally switches to bonds, $\exp[m(\tau) + r\theta]$ would be the terminal value of V_{sb} given the stock's sample path to date.

$V_{bs}[S(\tau), n(\tau), \theta]$: The price at time τ of a security that pays out at T the terminal value of a portfolio that starts with bonds and optimally switches to stock. $\exp[-n(\tau)]$ is the number of shares that would have been purchased given the sample path to date.

$V_{bsb}[S(\tau), \hat{m}(\tau), n(\tau), \theta]$: The price at time τ of a security that pays out at T the terminal value of a portfolio that starts with bonds, switches to stock and then switches back to bonds. $\exp[\hat{m}(\tau) + r\theta]$ would be the terminal value of V_{bsb}, and

P_{bs}, P_{sb}, P_{bsb}: European put options with parameters drawn from V_{bs}, V_{sb} and V_{bsb}, whose exercise prices will be $V_{bs}[T]$, $V_{sb}[T]$, and $V_{bsb}[T]$ respectively.

2. Hedging and Valuation

In order to hedge a position, the writer of any option must find a way to invest the proceeds from the sale of the option in an initial portfolio and to then alter the composition of this portfolio as is required to guarantee that in all states of nature (i.e., with probability one) the terminal value of the portfolio is adequate to meet his terminal obligation. This portfolio strategy is termed a *perfect hedge* and has the following properties: (1) the value of the terminal portfolio is *exactly* equal to the terminal obligation, and (2) the hedging policy is *self-financing*—each portfolio revision undertaken is exactly financed by the proceeds from the sale of the previous position.

Recent work by Black and Scholes [1] and Merton [6] has established that in a perfect (i.e., frictionless) market, when the natural logarithm of the underlying stock price follows a Wiener process with drift, (i.e., the underlying assumption of this paper, with $dS/S = \alpha\cdot dt + \sigma\cdot dy$ and equivalently through the use of the Ito model of stochastic differential calculus $d(\ln S) = (\alpha - \sigma^2/2)dt + \sigma\cdot dy)$, the payoff of a European put or call with a fixed exercise price can be identically duplicated by a portfolio consisting of shares of stock and units of riskless bond.

Thus, this portfolio meets the criteria established for a perfect hedge. Cox and Ross [3] have illustrated and Harrison and Kreps [5] have proved that if a contingent claim can be perfectly hedged it should be priced as if it existed in a risk-neutral world. This result implies that the arithmetic mean return (α) of the underlying stock is of no consequence to the pricing of the option and in fact it may be assumed equal to the riskless rate (r). Then the drift of the logarithm of the stock price becomes effectively ($r - \sigma^2/2$) per unit time, and after making this substitution, the option will be priced at its discounted expected terminal value.

In their paper, GSG provide an explicit proof of the hedgeability of $P_{\max}$ and $C_{\min}$. Much the same strategy could be employed here to prove that these new options are also hedgeable. However, recent work by Harrison and Kreps has removed the need to perform this tedious task. In particular, they provide necessary and sufficient conditions for a contingent claim to be redundant (basically it must be a function only of the past history of the underlying stock and must meet certain smoothness conditions). The contingent claims to be considered here are particular cases of the Harrison-Kreps general specification. To save space we suppress a formal proof of hedgeability and refer the interested reader to Harrison-Kreps for details.

Since $P_{\max}$ and $C_{\min}$ are hedgeable, we can make the Cox-Ross transformation (i.e., substituting the risk-free rate of interest for the logarithm of the stock's expected return) in order to price these options. $P_{\max}$ will be priced equal to the probability weighted, conditional (over non-negative payouts) expected value of the realized maximum over the life of the option minus the terminal stock price, discounted back to the present; $C_{\min}$ will be priced equal to the probability weighted, conditional expected value of the terminal stock price minus the realized minimum, discounted back to the present:

$$P_{\max}[S(\tau), M(\tau), \theta] = e^{-r\theta}E_{|M\geqslant S}[M(T) - S(T)]\, Prob(M \geqslant S) \tag{1}$$

$$C_{\min}[S(\tau), Q(\tau), \theta] = e^{-r\theta}E_{|Q\leqslant S}[S(T) - Q(T)]\, Prob(Q \leqslant S) \tag{2}$$

where E is a conditional expectation operator.

A casual examination of (1) and (2) seems to indicate that knowledge of the conditional joint distribution of the maximum (minimum) and the terminal value of a Wiener process with drift (conditioned on the current price, the current maximum (minimum) to date and the length of time remaining to expiration) is required. However, since these options are always exercised, $Prob(M \geqslant S) = 1$, ($Prob(Q \leqslant S) = 1$), can use the distributive property of expectation. Since $e^{-r\theta}E[S(T)] = S(\tau)$ relations (1) and (2) may be rewritten as

$$P_{\max}[S(\tau), M(\tau), \theta] = e^{-r\theta}E[M(T)] - S(\tau). \tag{1$'$}$$

$$C_{\min}[S(\tau), Q(\tau), \theta] = S(\tau) - e^{-r\theta}E[Q(T)]. \tag{2$'$}$$

Thus, knowledge of the joint distribution is unnecessary; all that is required is knowledge of the conditional distribution of the maximum (minimum). The importance of this observation is that in order to value $P_{\max}$ it is sufficient to value a security that pays off the realized maximum ($V_{\max}$) and to then subtract the current stock price; to value $C_{\min}$ it is sufficient to subtract from the current stock price the value of a security that pays off the realized minimum ($V_{\min}$). Hence,

$$P_{\max} = V_{\max} - S \tag{3}$$

$$C_{\min} = S - V_{\min}. \tag{4}$$

Similarly, it can be established that,

$$P_{sb} = V_{sb} - S \tag{5}$$

$$P_{bs} = V_{bs} - S \tag{6}$$

$$P_{bsb} = V_{bsb} - S. \tag{7}$$

Relations 3–7 illustrate that to price these options it is sufficient to compute the expected value of a desired terminal portfolio, discount it back to the present and adjust by the stock price.

V_{sb}: An Absorbing Barrier Problem

To calculate the value V_{sb} we begin by deriving the Cox-Ross transformed distribution of final payouts (conditioned at the initial time) of the stock to bonds (*SB*) option. For any particular sample path, the payout is:

$$\max_{0 \leq \delta \leq T} \left[\frac{S(\delta)}{S(0)} \right] \exp[r(T - \delta)].$$

Equivalently, we can choose δ to maximize

$$x(\delta) \equiv \ln[S(\delta)/S(0)] - r\delta$$

(Notice that the payout is, $\max_\delta \exp(rT)\exp[x(\delta)]$).

The transformed distribution of $\ln[S(\delta)/S(0)]$ at the initial time is normal with transformed mean, $(r - \sigma^2/2)\delta$ and invariant variance, $\sigma^2\delta$.

Accordingly, we want to derive the distribution of $\max_\delta x(\delta)$ or equivalently the distribution of

$$\max_{0 \leq \delta \leq T}[-\delta\sigma^2/2 + \sigma(W(\delta) - W(0)]$$

where W is a standardized Wiener process. Clearly, the density of this distribution is equivalent to the density of absorption of an equivalent Wiener process with drift, for the specified interval T. That is, the density of $z \equiv \max_\delta x(\delta)$ is

$$f_T(z) = \left[N'\left(\frac{z + T\sigma^2/2}{\sigma\sqrt{T}}\right) + \exp(-z)N'\left(\frac{-z + T\sigma^2/2}{\sigma\sqrt{T}}\right) \right] \Big/ \sigma\sqrt{T}$$
$$+ \exp(-z)N\left(\frac{-z + T\sigma^2/2}{\sigma\sqrt{T}}\right) \quad \forall z > 0.$$

The conditional density of z at some time $0 \leqslant \tau \leqslant T$ is:

$$g(Z \mid m(\tau), \tau) = \begin{cases} 0 & \forall z < m(\tau) \\ \lambda(z - m(\tau)) \displaystyle\int_0^{m(\tau)-x(\tau)} f_\theta(z')\,dz' & \text{for } z = m(\tau) \\ f_\theta(z - x(\tau)) & \text{for } z > m(\tau) \end{cases}$$

where $m(\tau) \equiv \max_{0 \leqslant \delta \leqslant \tau} \ln[S(\delta)/S(0)] - r\delta$ and $\lambda(z - m(\tau))$ is a dirac delta function. Finally,

$$V_{sb}[S, m(\tau), \theta] = \exp(-r\theta)E_\theta(pa\breve{y}out)$$

$$= \exp(m(\tau)) \int_0^{m(\tau)-x(\tau)} f_\theta(z')dz' + \int_{m(\tau)}^{\infty} \exp(z) f_\theta(z - x(\tau))\, dz.$$

This last expression may be evaluated as a complicated function of cumulative normal densities.

V_{bs}: *A Reflecting Barrier Problem*

To calculate the value V_{bs} we begin by deriving the Cox-Ross transformed distribution of final payouts (conditioned at the initial time) of the bonds to stock (BS) option. For any particular sample path the payout is

$$\max_{0 \leqslant \delta \leqslant T}[S(T)/S(\delta)]\exp(r\delta) = \max_{0 \leqslant \delta \leqslant T} \exp(rT)\exp(y(\delta))$$

where $y(\delta) \equiv \ln[S(T)/S(\delta)] - r\theta$. Let $\hat{y}(0) = 0$, and let

$$d\hat{y} = \begin{cases} d[\ln(S)] - rdt & \text{if } \hat{y} > 0 \textit{ or } d[\ln(S)] - rdt > 0 \\ 0 & \textit{otherwise.} \end{cases}$$

By construction $\hat{y}(T) = \max_{0 \leqslant \delta \leqslant T} y(\delta)$.

That is, the distribution of $\hat{y}(T)$ is simply derived by transforming the problem into the problem of determining the final position of a particle that follows a Wiener process with drift and is subject to a reflecting barrier from below.[2] Its

[2] Basically what is happening here is that we are trying to determine the switchover point from bonds to stock and the excess (over the riskless rate) return per share of stock purchased at the optimal time and held until the option terminates. For ease of explanation assume that the stock took only k discrete jumps ($J(i)$, $i = 1, k$). Clearly, if the first jump $J(1)$ was smaller than the riskless return from bonds $J(1) < r\Delta$ we would have wanted to be in bonds. But suppose the first jump exceeded the riskless return—would we have wanted to be in stock? Potentially but we must first examine the next increment. If $J(1) + J(2) < 2r\Delta$. Clearly we would have wanted to be in bonds for the first two periods. If $J(1) + J(2) > 2r\Delta$ we consider the third jump, etc. However, whenever we find a period of length $L\Delta$ where $\sum_1^L J(L) < Lr\Delta$ we can simply start checking from interval $L + 1$. That is, we reset our counter to zero ($y = 0$) and start over, i.e., we reflect off a lower barrier of zero.

density (conditioned at the initial time) is well known:

$$h_T(\hat{y}(T)) = \left[N'\left(\frac{\hat{y} + T\sigma^2/2}{\sigma\sqrt{T}}\right) + \exp(-\hat{y})N'\left(\frac{-\hat{y} + T\sigma^2/2}{\sigma\sqrt{T}}\right)\right]\Big/\sigma\sqrt{T} + \exp(-\hat{y})N\left(\frac{-\hat{y} + T\sigma^2/2}{\sigma\sqrt{T}}\right) \qquad \forall\, \hat{y} > 0.$$

Notice that this density is exactly equivalent to $f_T(z)$, the transformed density of outcomes for the option that invests first in stock and then in bonds. The conditional distribution of $\hat{y}$ at some time $0 \leqslant \tau \leqslant T$ is:

$$k(\hat{y}(T)\,|\,\hat{y}(\tau), \tau) = \left[N'\left(\frac{\hat{y}(T) - \hat{y}(\tau) + \theta\sigma^2/2}{\sigma\sqrt{\theta}}\right) + \exp(-\hat{y}(\tau))\, N'\left(\frac{\hat{y}(T) + \hat{y}(\tau) + \theta\sigma^2/2}{\sigma\sqrt{\theta}}\right)\right]\Big/\sigma\sqrt{\theta} + \exp(-\hat{y}(T))\, N\left(\frac{-\hat{y}(T) - \hat{y}(\tau) + \theta\sigma^2/2}{\sigma\sqrt{\theta}}\right).$$

Equivalently, one could compute these distributions by appropriately reversing the direction of time and using the absorbing barrier method.

Finally,

$$V_{bs}[S(\tau), n(\tau), \theta] = \exp(-r\theta)E(pa\check{y}out)$$

where $n(\tau) = \ln S(\tau) - \hat{y}(\tau)$ or

$$V_{bs}[S(\tau), n(\tau), \theta] = \int_0^\infty \exp(\hat{y})k(\hat{y}\,|\ln(S(\tau)) - n(\tau), \theta)\, d\hat{y}$$

which can be evaluated as a complicated function of cumulative normal densities.

V_{bsb}: *An Absorbing and Reflecting Barrier Problem*

To calculate the value of V_{bsb} it is apparent that the distribution of $\hat{z} = \max_{0 \leqslant \delta \leqslant T} \hat{y}(\delta)$ must be computed. This problem is identical to the Page [7] problem of detecting a one-sided shift in a location parameter, and to the Robbins [8] problem of maximum waiting time. For the reader who wishes to inspect this distribution, Sweet and Hardin [10] have solved for it in terms of an infinite series. The Sweet and Hardin solution has an error in it which is corrected in the Robbins paper.

For the two previous put options we had valuation formulae which were functions of $m(\tau)$ and $n(\tau)$ respectively. $m(\tau)$ can be loosely thought of as a provisional profit (or a lower bound on profit). $n(\tau)$ can be thought of as a provisional buying price (or the maximum buying price, relative to the riskless asset that the investor need pay). For the option that invests first in bonds, then stock and finally bonds (BSB), we need to keep track of both the provisional profit and $n(\tau)$. However, provisional profit is now defined as $\hat{m}(\tau) = \max_{0\le\delta\le\tau} \hat{y}(\delta)$. Notice that $n(\tau)$ and $\hat{m}(\tau)$ do not change simultaneously.

Using the method previously described

$$
\begin{aligned}
V_{bsb}[S(\tau), \hat{m}(\tau), n(\tau), \theta] &= E_\theta[\exp(\hat{m}(T)] \\
&= \exp(\hat{m}(\tau))\, Prob[\hat{m}(\tau) \geq \hat{\hat{m}}(\theta)] \\
&\quad + E_{\theta,\hat{\hat{m}}>\hat{m}}[\exp(\hat{\hat{m}}(\theta)]\, Prob[\hat{\hat{m}} > \hat{m}]
\end{aligned}
$$

where $\hat{\hat{m}}(\theta) \equiv \max_{\tau\le\delta\le T} \hat{y}(\delta)$. Obviously $\hat{m}(T) = \max[\hat{m}(\tau), \hat{m}(\theta)]$.

3. Properties of These Options

In this section we investigate the properties of $P_{\max}$, P_{bs}, P_{sb}, and P_{bsb}. Properties of $C_{\min}$ have been relegated to a footnote since it is the only call in a crowd of puts and not really comparable.[3] GSG prove that $P_{\max}$ is convex in S, first decreasing and then increasing. The intuition behind this behavior is that for S sufficiently below M, not enough time (probabilistically) remains to establish a new maximum and to then establish a larger $(M - S)$. Thus, increases in S lead to smaller expected $[M(T) - S(T)]$. However, for S sufficiently close to M, enough time remains to establish a new maximum and to then establish a larger expected $[M(T) - S(T)]$. In other words, at inception and at all other times when $M = S$, and throughout the life of the option whenever $[M - S]$ is sufficiently small, the purchaser of the option first hopes that the stock will go up (as he would if he held a call) and establish a new high. He then hopes that the stock will go down (as he would if he held a put). Analogously, when θ is sufficiently small (i.e., a new maximum is unlikely) $P_{\max}$ behaves as an ordinary put (i.e., a decreasing function of S) and when θ is sufficiently large (i.e., a new maximum is likely) $P_{\max}$ behaves as an ordinary call (i.e., an increasing function of S).

GSG also illustrate that $P_{\max}$ first decreases and then increases as a function of time to maturity. Finally they illustrate the somewhat amusing result that as a function of θ, for any S, the value of $P_{\max}$ is uniquely maximized with either $\theta = 0$ or $\theta = \infty$ implying that an American $P_{\max}$ would be more valuable than its European counterpart.

Turning now to the new options we have the following theorems.

THEOREM *1:* Given the barriers $m(\tau)$, $\hat{m}(\tau)$, and $n(\tau)$; V_{bs}, V_{sb}, and V_{bsb} are monotone increasing in S. P_{bs}, P_{sb}, P_{bsb} are decreasing functions of S for small S and increasing functions of S for large S.

Proof: Increases in S obviously cause a first order stochastic dominance shift in the distribution of outcomes for V_{bs}, V_{sb}, and V_{bsb} which yields the monotonicity result.

[3] Like an ordinary call, $C_{\min}$ is an increasing-monotone function of S and θ.

Note that,

$$\lim_{\ln(S)\to n(\tau)} \frac{\partial V_{bs}}{\partial S} = \lim_{\ln(S)\to n(\tau)} \frac{\partial V_{bsb}}{\partial S} = \lim_{S\to 0} \frac{\partial V_{sb}}{\partial S} = 0.$$

Hence, for small S (either $\exp(n(\tau))$ or 0) the derivatives of these puts with respect to S are -1.

Further,

$$\lim_{S\to\infty} \frac{\partial V_{bs}}{\partial S} = \exp(-n(\tau)) > 1,$$

$$\lim_{\ln S\to m(\tau)} \frac{\partial V_{sb}}{\partial S} = \int_0^\infty \exp(z) f_\theta(z)\, dz > 1, \text{ and}$$

$$\lim_{\ln(S)\to n(\tau)+\hat{m}(\tau)} \frac{\partial V_{bsb}}{\partial S} = E_\theta[\exp(\hat{m}(\theta)]\exp(-n(\tau) - m(\tau)) > 1.$$

Hence, for the largest S the puts have positive slope. Q.E.D.

Theorem 1 implies that the new puts, like $P_{\max}$, first decrease and then increase as a function of S. This behavior derives from recognizing that for small S optimal *BS*, *SB*, and *BSB* portfolios would, with high probability, be in bonds. Thus, for small S, these complex puts are approximately composed of bonds short one share of stock—very much akin to regular puts—with the characteristic inverse relation between stock price and put value. On the other hand, for large S, optimal *BS*, *SB*, *BSB* portfolios are with high probability heavily in stock—creating a call-like direct relation between stock price and put value.

THEOREM 2: V_{bs}, V_{sb}, V_{bsb}, P_{bs}, P_{sb}, and P_{bsb} are all increasing functions of θ, the time to expiration.

Proof: Since the payouts for all sample paths increase by at least the time value of money, for V_{sb} and V_{bsb} as T increases, V_{sb} and V_{bsb} must be increasing functions of θ. S is on average invariant w.r.t. θ (on a discounted basis) so, $P_{sb} \equiv V_{sb} - S$, and $P_{bsb} \equiv V_{bsb} - S$ must also be increasing in θ.

For V_{bs} and hence P_{bs}, an increment of θ will add (in the limit) a truncated normally distributed increment in $\hat{y}$ (where the left tail is truncated). The untruncated density of the increment has mean $(-\sigma^2/2)\,d\theta$ and variance $\sigma^2 d\theta$. Without truncation such an increment would leave V_{bs} invariant w.r.t. changes in θ. The truncation increases the expected value by limiting losses of V_{bs} through the stretching of θ, (i.e., $E[\exp(\Delta\hat{y})] > 1$). Q.E.D.

The intuition behind this theorem is as follows. For P_{sb} and P_{bsb} any profit that accrues can be "stashed" away That is, the switch from stock to bonds can be made thus locking in a profit. If for some subsequent period a larger profit occurs then the timing of the switch to bonds can be altered. Since any locked-in profit position will grow at rate r and since increasing θ may bring about a larger profit, clearly P_{sb} and P_{bsb} benefit by increases in the term of the option.

For P_{bs} the story is not as transparent. Suppose that favorable terms of trade between bonds and stock have been established and that the stock price then rises. A "paper" profit occurs. However, since the profit can't be locked in by a switch into bonds, it may either increase or be wiped out by a subsequent decline in the stock. It is also possible that a later rise in the stock price will generate an even larger "paper" profit, but nothing guarantees this. However, the last part of Theorem 2 establishes that, in terms of expected value, the good effects of increasing θ dominate the bad.

Theorem 2 has the obvious corollary that if these options were American (i.e., could be exercised at any time up to and including the termination date) with an exercise price computed using the current parameters, they would never be exercised early. This conclusion is a direct result of having these options account for the time value of money and should be contrasted with the discussion of $P_{\max}$ where an American feature would be of value.

In addition to theorems concerning the value of these options as a function of the state variable, we have the following theorems concerning the relative valuation of these options.

THEOREM *3:* At initial time ($\tau = 0, m(0) = 0, n(0) = 0$) $V_{bs} = V_{sb}$, and $P_{bs} = P_{sb}$.

Proof: The reader should convince himself that the payout patterns are equivalent because of the symmetry of the independent increment sample paths. Equivalently, the densities of payouts for *SB* and *BS* are identical (see section 2). Q.E.D.

THEOREM *4:* At the initial time, V_{bs}, V_{sb}, V_{bsb}, P_{bs}, P_{sb}, and P_{bsb} are invariant to shifts in r, the interest rate.

Proof: Inspection of formulae for initial value exhibit no dependence whatsoever on r. Q.E.D.

In addition, the following obvious relations can be established by simple dominance arguments.

(1) For $r > 0$ and for all $\theta > 0$
$P_{sb} \geqslant P_{\max}$.

(2) For $r = 0$ and for all $\theta > 0$
$P_{sb} = P_{\max}$.

(3) For all $\theta > 0$
$P_{bsb} \geqslant P_{sb}$, and $P_{bsb} \geqslant P_{bs}$.

We can also investigate when, ex-post, the switches from bonds to stock (or vice versa) were made by V_{sb}, V_{bs} and V_{bsb} (and hence by P_{sb}, P_{bs} and P_{bsb}).

(1) V_{bs} does not switch to stock before S achieves its minimum. (In terms of Figure 1 this statement implies that $\tau_b \geqslant \tau_1$).

(2) V_{sb} does not switch to bonds after S achieves its maximum ($\tau_2 \leqslant \tau_5$).

These two statements are simply the result of the time value of money.

Since V_{bsb} (and hence P_{bsb}) may take advantage of any increasing interval in the stock price, its purchase and sale of stock cannot, in general, be related to the occurrence of the minimum or the maximum, or to points where P_{sb} and P_{bs} switch investment policies. However, a wee bit of mental gymnastics establishes the following statements:

(1) If, ex-post, P_{bsb} and P_{bs} are both in stock at any point in time, then they will have started in stock at the same time.

(2) If, ex-post, P_{bsb} and P_{sb} are both in stock at any point in time, then they will both switch to bonds at the same time.

Finally, we note that should the investor be able to go into and out of the market N times (that is, $N = 1$ is equivalent to BSB), then the value of optimally doing so is an increasing, concave function of N.

4. Conclusion

Our casual empiricism makes compelling the demand for these new assets. These new assets in a loose but intuitive sense minimize investor regret. These options allow a *direct* and *effective* speculative instrument using the typical forecasts of the share price distribution.

The value of these options can be used in computing an upper bound for the value of management services in achieving optimal timing. Of course the valuation formulae of this paper are only guidelines to the true values of such options since the assumptions of our model are but a rough description of reality. In fact, it is these very market imperfections (the deviations from our model's perfect market context) that give meaning to the new assets. The information heterogeneity of investors and the costliness of creating perfect hedges of the new assets make the options particularly desirable.

If these options are desirable then why don't they already exist? We believe that markets inherently take advantage of scale economies and attempt to internalize various externalities. The creation of a market is frought with danger—sufficient scale may not be immediately achieved, the benefits of creation may in large part not be capturable by the creators, legal impediments may prove overly burdensome to the creators, etc. Accordingly, a desirable and viable security may not currently exist in the market. The test of a security's viability is not its existence but rather its capacity to survive in a fully developed market. Notice that the now flourishing CBOE bears little resemblance to the OTC options market of the preceding era.

REFERENCES

1. Black, F., and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 1973, 81, 637–659.
2. Cox, D. R., and H. D. Miller, *Theory of Stochastic Processes*, Methuen & Co., Ltd., London, 1965.
3. Cox, J. C., and S. A. Ross. "The Valuation of Options For Alternative Stochastic Processes," *Journal of Financial Economics*, 1976, 3, 145–166.
4. Goldman, M., H. Sosin, and M. Gatto, "Path Dependent Options, Buy at the Low, Sell at the High," BTL memorandum, 1978.
5. Harrison, M., and D. Kreps, "Martingales and the Pricing of Contingent Claims," Stanford University, Research paper #444, May, 1978.
6. Merton, R. C., "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 1973, 4, 141–183.
7. Page, E. S. "A Test for a Change in a Parameter Occurring at an Unknown Point," *Biometrika*, 1955, 42, 523–527.
8. Robbins, N. B., "Detecting a Change in the Mean of a Distribution with Applications to Queueing Theory," BTL memorandum, 1972.

9. Shepp, L. A., "The Joint Density of the Maximum and its Location for a Wiener Process with Drift," BTL memorandum January, 1978.
10. Sweet, A. L. and J. C. Hardin, "Solution for Some Diffusion Processes with Two Barriers," *Journal of Applied Probability*, 1970, 7, 423–431.

DISCUSSION

SIMON BENNINGA*: Professors Bierwag and Khang's paper contains three principal results:

1. For the future value at time m of an income stream to be immune to an instantaneous shock in the term structure, it is necessary and sufficient that the Macauley-Hicks duration of the income stream equal m.
2. An investor who chooses a portfolio so as to immunize the future value in the method prescribed above guarantees himself at a minimum the yield h(o, m) which spanned the time period *m* in the original (pre-shock) term structure. The immunization strategy is thus a maximum strategy.
3. Suppose the investor ranks multi-period investments by their mean return and their lower parital moment (this is a framework recently discussed by both Fishburn (4) and Bawa and Lindenberg (1)). Then any immunized portfolio of the type proposed by Bierwag and Khang is a zero-beta portfolio; additional expected return must be purchased at the cost of additional lower partial moment (risk).

The Bierwag-Khang result is an interesting one, especially in light of the recent interest in alternatives to the mean-variance capital asset pricing models. If an investor has a time horizon of *m* and associates risk only with below-target returns, then the largest return he can expect with no risk whatsoever is $h(o, m)$—the yield in the original term structure—and the portfolio which gives this yield is one which has Macauley-Hicks duration *m*.

We should be careful to note that Bierwag and Khang are not advocating that every investor immunize his portfolio. As Grove (5) has shown, such behavior will not, in most cases be consistent with expected utility maximization. Instead, the Bierwag-Khang results pin down the characteristics of one end of the market line in a mean-lower partial moment framework.

I am not sure that the Macauley-Hicks concept of duration is as important here as the fact that, as Bierwag (3) himself has shown, different kinds of shock to the term-structure demand different kinds of immunizing strategies. It seems that each different kind of shock leads to a different concept of duration. In the area of security markets (one need not talk in terms of bonds, since the concept of duration can apply to any income stream), it seems to me that the relevant question is: What kind of diversity do we need in obtainable returns in order to guarantee that immunizing strategies exist? A sufficient answer to this is already in hand: Any return over a planning period m can be immunized if there exists a pure discount *m*-period bond. I suggest that we ought to start looking for a

* University of Pennsylvania.