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Closed-Form Characteristic Functions for Certain Random Variables Related to Brownian Motion

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Abstract

In what follows let $W(t) = (W_1(t), W_2(t), W_3(t))$ be a three-dimensional Wiener process (Brownian motion) starting at the origin. P. Lévy has shown that the characteristic function of the random variable $X = \int_0^1 [W_1(t) - tW_1(1)]\, d[W_2(t) - tW_2(1)]$, which measures the signed area of a random planar loop, is $Ee^{izX} = (z/2)\,\mathrm{csch}(z/2)$. (The corresponding density, of importance in polymer physics, is $(\pi/2)\,\mathrm{sech}^2 \pi x$). Lévy's derivation uses N. Wiener's simple sine-cosine Fourier expansion of white Gaussian noise. This expansion is used here as the starting point in showing that the six-dimensional vector $(W(1), \int_0^1 W(t) \times dW(t))$, needed to characterize polarization dispersion effects for high speed communications over optical fibers, also has an elementary closed form characteristic function.

Employing the same Fourier expansion we prove the following result, generalizing the two examples: Let V be any finite dimensional vector whose components are arbitrary linear combinations of random variables of the form $W_k(1)$, $W_k(1)W_l(1)$, $W_k(1)\int_0^1 W_l(t)\,dt$, $\int_0^1 W_k(t)\,dW_l(t)$ $(1 \le k \le 3,\ 1 \le l \le 3)$. The joint characteristic function of V can be expressed in closed form using elementary functions.

ISBN 0-12-481005-5

1 Introduction

Let $W(t) = (W_1(t), W_2(t), W_3(t))$ be a standardized Wiener process (Brownian motion) on [0,1]. By standardized, we mean that for each t, $W(t)$ is distributed as $N(0,tI)$. Every time we write an integral sign in this paper the integral is understood to be over the unit interval. In the next section we derive characteristic functions as noted in the following two examples, the first of which uses only two dimensions of $W(t)$ and is a known result.

Example I: Signed Area of a Random Planar Loop, X. Let $\overline{W}_m(t)$ denote the pinned Wiener process [1-3] (also called the Brownian bridge)

$$\overline{W}_m(t) \triangleq W_m(t) - tW_m(1) . \tag{1}$$

Since $\overline{W}_m(t)$ starts and ends at 0, the curve $\{(\overline{W}_1(t), \overline{W}_2(t))\}_{0 \le t \le 1}$ is a random planar loop. Using Green's Theorem (see for example, [4]) the signed loop area is given by

$$X = \int \overline{W}_1(t) \cdot d\overline{W}_2(t) . \tag{2}$$

In this case the characteristic function is

$$Ee^{izX} = (z/2)/\sinh(z/2) . \tag{3}$$

(The corresponding density is $p(x) = \frac{\pi}{2}\frac{1}{\cosh^2 \pi x}$.) This was first obtained by P. Lévy [5] using a series representation for $W_i(t)$ that was introduced by Wiener in [6] (see equation (7) below). Sparked by interest for studying topologically constrained polymers, (3) was recently rediscovered using different methods, in [7]. Results related to (3) appear in [8-13].

Example II: Fiber Optical Polarization Dispersion Vector Y. Let $\times$ denote vector cross-product

$$Y \triangleq (W(1), \int W(t) \times dW(t)) . \tag{4}$$

Let $z = (z_1, z_2, z_3)$, $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ be the transform variates with denoting transpose and vector multiplication such as $z'\zeta$ denoting the scalar product. We have

$$Ee^{iz'W(1)+i\zeta'\int W(t)\times dW(t)} =$$

$$\operatorname{sech}|\zeta|\exp-1/2\left\{|z|^2\frac{\tanh|\zeta|}{|\zeta|}+\frac{(z'\zeta)^2}{|\zeta|^2}\left(1-\frac{\tanh|\zeta|}{|\zeta|}\right)\right\}. \quad (5)$$

See [14] for a discussion of the importance of (5) for the theory of polarization dispersion in single mode fibers.

As we shall see, the following simple Fourier expansions [1,3] of the white Gaussian noise processes $\{dW_m(t)/dt\}_{m=1}^{3}$ are useful in deriving the characteristic functions for the examples

$$\dot{W}_m(t) = \xi_0^m + 2^{1/2}\sum_{j=1}^{\infty}(\xi_j^m \cos 2\pi jt + \eta_j^m \sin 2\pi jt). \quad (6)$$

Each element of $\{\xi_k^m, 0 \le k < \infty, 1 \le m \le 3\} \cup \{\eta_k^m, 1 \le k \le \infty, 1 \le m \le 3\}$ is distributed as $N(0,1)$ and statistically independent of all the other elements. As remarked, this was first used by Wiener [6]. A similar expansion holds for any complete orthonormal sequence [1]. The sines and cosines have a crucial advantage here in that the expansion is orthogonal in more than one way as we shall see in (9) below. In particular, the Karhunen-Loeve expansion of the Wiener process is not the right expansion to use.

Employing the formal expansion (6) we prove the following:

Theorem. *For $W(t)$ a three-dimensional Wiener Process, the joint characteristic function of any random variables of the form*

$$W_k(1),\ W_k(1)W_l(1),\ W_k(1)\int W_l(t)dt,\ \int W_k(t)dW_l(t)$$

can be expressed in closed form using elementary functions.

The theorem is valid for either Ito or Stratonovich [15,16] integrals.

The above result, has the immediate implication that any finite dimensional vector, V, whose components, $(V_1, V_2, \dots, V_K)$, are linear combinations of the random variables mentioned in the theorem, also has a joint characteristic function expressible in closed form using elementary functions. So this generalization covers the second example. To see that the theorem itself covers the first example, use integration by parts on the $-\int tW_1(1)dW_2(t)$ term that appears when expanding $\int \overline{W}_1(t)d\overline{W}_2(t)$ out into the four integrals. The curve $\{\overline{W}_1(t),\overline{W}_2(t),\overline{W}_3(t)\}$ $0 \le t \le 1$ is a three dimensional loop. One could use the constructive proof of the theorem to find the joint characteristic function of the signed areas of linear mappings of this loop onto a finite set of planes.

The very basic example $\int W_1(t)dW_2(t)$ is clearly covered by the theorem. The characteristic function turns out to be $\operatorname{sech}^{1/2} z$ (the corresponding density is $\left((2^{3/2}\pi)^{-1}\left|\Gamma\left(\frac{1}{4}+i\frac{x}{2}\right)\right|^2\right)$. We will not derive the characteristic function explicitly in this case since the derivation is very much along the lines of the derivation of Example II, but simpler. The last component of Y in Example II is $\int W_1(t)dW_2(t) - \int W_2(t)dW_1(t)$. From (5) the characteristic function is $\operatorname{sech} z$, so $\int W_1(t)dW_2(t)$ and $-\int W_2(t)dW_1(t)$ add like independent random variables. (The probability density function corresponding to $\operatorname{sech} z$ is $1/2\operatorname{sech}(\pi x/2)$).

All densities mentioned thus far can be obtained from their characteristic function using the tables [17]. As of this writing (5) has not been inverted in closed form. We note that the six univariate marginals densities of Y have precisely the same functional form as their characteristic functions.

Lévy's formula ((1.3.4) of [5]) is more general than (3). It is a formula for the characteristic function of the area included by the loop defined by a planar Brownian motion and its (origin-to-endpoint) chord, conditioned on the chord length. This celebrated formula is a straight-forward consequence of our equation (5).

2 Derivation of Characteristic Functions for the Examples

The expansion (6) is formal, the integrated form

$$W_m(t)=\xi_0^m t+2^{1/2}\sum_{j=1}^{\infty}(2\pi j)^{-1}(\xi_j^m \sin 2\pi jt+\eta_j^m(1-\cos 2\pi jt)) \quad (7)$$

converges uniformly with probability one [1,3,18]. Note $W_m(1) = \xi_0^m$.

Although the theorem generalizes the examples, the derivation of Example I is short and serves to very simply illustrate the use of (6) and (7). We will also derive Example II as it provides a concrete illustration of a somewhat elaborate case and provides much of what is needed for the theorem.

The following expansions will be needed. They follow using (6) and (7)

$$W_k(1)\int_0^1 W_l(t)dt = \xi_0^k\Big((\xi_0^l/2)+2^{½}\sum_{j=1}^{\infty}(2\pi j)^{-1}\eta_j^l\Big) \tag{8a}$$

$$\int_0^1 W_k(t)dW_l(t) = \\ (\xi_0^k\xi_0^l/2)+\sum_{j=1}^{\infty}(2\pi j)^{-1}[\eta_j^l(\xi_j^k-2^{½}\xi_0)^k-\eta_j^k(\xi_j^l-2^{½}\xi_0^l)]\,. \tag{8b}$$

Example I. Using the definitions of $\overline{W}_1(t)$ and $\overline{W}_2(t)$ in $X=\int\overline{W}_1(t)d\overline{W}_2(t)$, and then substituting using (8a) and (8b), we obtain on account of the orthogonality of the terms in the expansion (7) *and their derivatives*,

$$Ee^{izX} = \prod_{j=1}^{\infty} E\exp\Big\{iz(\xi_j^1\eta_j^2-\xi_j^2\eta_j^1)/(2\pi j)\Big\} \tag{9a}$$

$$= \prod_{j=1}^{\infty} E_{\xi_j}(e^{-½(z\xi_j/(2\pi j))})^2 = \prod_{j=1}^{\infty}\Big(1+(z/(2\pi j))^2\Big) \tag{9b}$$

$$= (z/2)\,\sinh\,(z/2)\,. \tag{9c}$$

See [19] for the last equality.

Example II. In deriving the characteristic function of the six dimensional random vector $(W(1),\int W(\tau)\times dW(\tau))$ we will use the following representation of the last three components constituting the integrated vector cross product

$$\begin{aligned}\int W(\tau) \times dW(\tau) = (2\int W_2(\tau)dW_3(\tau) - W_2(1)W_3(1),\\ 2\int W_3(\tau)dW_1(\tau) - W_3(1)W_1(1),\\ 2\int W_1(\tau)dW_2(\tau) - W_1(1)W_2(1)) . \qquad (10)\end{aligned}$$

We have used integration by parts, so one integral, not two, appears in each component.

Let (p,q,r) index the cyclic permutations $\{(1,2,3), (3,1,2), (2,3,1)\}$. Form the six dimensional characteristic function as a product over the three cycles

$$\phi(z_1, z_2, z_3, \zeta_{23}, \zeta_{13}, \zeta_{12}) =$$
$$E \prod_{\{(p,q,r)\}} e^{i\{W_p(1)z_p + [2\int W_q(\tau)dW_r(\tau) - W_q(1)W_r(1)]\zeta_{qr}\}} . \qquad (11)$$

The dual subscripting of the last three transform variables is a temporary notational convenience. Drawing on (6) and (7) and doing the elementary integrals we rewrite (11) as

$$\phi(z_1, z_2, z_3, \zeta_{23}, \zeta_{13}, \zeta_{12}) =$$
$$E \prod_{\{p,q,r\}} e^{iz_p\xi_o^p + i\zeta_{qr}\sum_{j=1}^{\infty}\left[\frac{\eta_j^r(\xi_j^q - \sqrt{2}\,\xi_o^q) - \eta_j^q(\xi_j^r - \sqrt{2}\,\xi_o^r)}{\pi j}\right]} \qquad (12)$$

where the $\zeta_{qr} W_q(1) W_r(1)$ terms cancelled out of each sum. We will next proceed to take the expectation in three stages: first with respect to the $\eta_j^{(\cdot)}$ variates, second with respect to the $\xi_j^{(\cdot)}$ $(j \geq 1)$ variates, and last with respect to the three $\xi_o^{(\cdot)}$ variates.

Expectation over the η Variates

In preparation for taking the expectation over the η_j^p, η_j^q, η_j^r variates we collect $\eta_j^{(\cdot)}$ terms with the same superscript. Collecting and taking the expectation

$$\phi(z_1, z_2, z_3, \zeta_{23}, \zeta_{13}, \zeta_{12})$$

$$= Ee^{z'\xi_0} \prod_{j=1}^{\infty} \prod_{\{r,q,p\}} e^{\frac{i}{\pi j}\left\{\eta_j^r [z_{qr}(\xi_j^q - \sqrt{2}\,\xi_0^q) - z_{rp}(\xi_j^p - \sqrt{2}\,\xi_0^p)]\right\}} \tag{13a}$$

$$= Ee^{z'\xi_0} \prod_{j=1}^{\infty} \prod_{\{r,q,p\}} e^{-(\frac{1}{\pi j})^2 [z_{qr}(\xi_j^q - \sqrt{2}\,\xi_0^q) - z_{rp}(\xi_j^p - \sqrt{2}\,\xi_0^p)]^2} . \tag{13b}$$

For this example dual subscripting on the three ζ variables is no longer useful. We will employ the following notation

$$z \triangleq (z_1, z_2, z_3) \tag{14a}$$

$$\zeta \triangleq (\zeta_1, \zeta_2, \zeta_3) \text{ in place of } (\zeta_{23}, \zeta_{31}, \zeta_{12}) \tag{14b}$$

$$|\zeta| \triangleq \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2} \tag{14c}$$

$$\xi_j \triangleq (\xi_j^1, \xi_j^2, \xi_j^3) \quad j \geq 0 . \tag{14d}$$

The next step is, for each $j \geq 1$, to collect the quadratic forms in each vector $\xi_j - 2^{1/2}\xi_0$. We express $\phi(z,\zeta)$ using an infinite product of exponentials of quadratic forms

$$\phi(z,\zeta) = Ee^{iz'\xi_0} \prod_{j=1}^{\infty} e^{-1/2(\xi_j - 2^{1/2}\xi_0)' A_j (\xi_j - 2^{1/2}\xi_0)} \tag{15}$$

where A_j is the 3×3 matrix

$$A_j = (\pi j)^{-2} [|\zeta|^2 I - \zeta\zeta'] \tag{16a}$$

$$= (|\zeta| / \pi j)^2 \, (I - \zeta\zeta' / |\zeta|^2) . \tag{16b}$$

For short we are writing A_j instead of $A_j(\zeta)$. Notice

$$A_j^k = (|\zeta| / \pi j)^{2k} \, (I - \zeta\zeta' / |\zeta|^2) . \tag{17}$$

Expectation over the ξ_j Variates

To take the expectation over each ξ_j ($j \geq 1$), a triple integral is required where the integrand of the exponent is

$$-1/2[(\xi_j - 2^{1/2}\xi_0)' A_j(\xi_j - 2^{1/2}\xi_0) + \xi_j' I \xi_j] . \tag{18}$$

To evaluate the triple integral we rewrite (17) as a single quadratic form in ξ_j. Then the triple integral, with ξ_0 fixed, can be viewed as the integral over all of three dimensional space of an *unnormalized* trivariate Gaussian density. The triple integral can be easily evaluated, by comparing the integrand with the corresponding trivariate Gaussian probability density (same mean and variance-covariance matrix) that is properly normalized (integrates to unity).

We begin this evaluation process by rewriting (17) as

$$-1/2[(\xi_j - \mu_j)'(A_j + I)(\xi_j - \mu_j) + 2\xi_0' A_j \xi_0 - \mu_j'(A_j + I)\mu_j] \tag{19}$$

where we solve for μ_j so (17) and (18) are equal. The solution is

$$\mu_j = 2^{1/2}(A_j + I)^{-1} A_j \xi_0 \tag{20}$$

which, when used in the last term of (19), gives

$$-1/2\left\{(\xi_j - \mu_j)'(A_j + I)(\xi_j - \mu_j) + 2\xi_0'[A_j - A_j(I + A_j)^{-1}A_j]\xi_0\right\}. \tag{21}$$

Conditional on the value of ξ_0 we use (21) to get

$$Ee^{-1/2(\xi_j - 2^{1/2}\xi_0)' A_j(\xi_j - 2^{1/2}\xi_0)} = |I + A_j|^{-1/2} e^{-1/2\xi_0' 2[A_j - A_j(I + A_j)^{-1}A_j]\xi_0} . \tag{22}$$

Here the $|I + A_j|$ means the determinant of $I + A_j$.

The formula for $\phi(z, \zeta)$ can now be written

$$\phi(z, \zeta) = Ee^{iz\xi_0 - 1/2\xi_0' B \xi_0} \prod_{j=1}^{\infty} |I + A_j|^{-1/2} \tag{23}$$

where

$$B(\zeta) = 2\sum_{j=1}^{\infty} [A_j - A_j(I + A_j)^{-1}A_j] . \tag{24}$$

For short we will write B instead of $B(\zeta)$.

Closed Form Representation for Infinite Product and Infinite Sum

To express (23) is closed form we need to evaluate an infinite product and an infinite sum. We do the infinite product first. The matrix $I + A_j$ has eigenvalues of 1 with multiplicity one and $1 + (|\zeta| / \pi j)^2$ of multiplicity 2. Therefore

$$\prod_{j=1}^{\infty} |I + A_j|^{-1/2} = \prod_{j=1}^{\infty} (1 + (|\zeta| / \pi j)^2)^{-1} \tag{25a}$$

$$= |\zeta| / \sinh |\zeta| . \tag{25b}$$

Now we turn our attention to finding B in closed form. We evaluate B by rewriting (24) as

$$B = 2 \sum_{j=1}^{\infty} \{A_j - A_j^2 + A_j^3 - A_j^4 + \cdots\} \tag{26a}$$

$$= 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k+1} (|\zeta| / \pi j)^{2k} [I - (\zeta\zeta' / |\zeta|^2)] \tag{26b}$$

$$= 2 \sum_{j=1}^{\infty} (|\zeta| / \pi j)^2 \frac{1}{1 + (|\zeta| / \pi j)^2} [I - (\zeta\zeta' / |\zeta|^2)] \tag{26c}$$

$$= 2 \sum_{j=1}^{\infty} \frac{|\zeta / \pi|^2}{j^2 + |\zeta / \pi|^2} [I - (\zeta\zeta' / |\zeta|^2)] \tag{26d}$$

$$= \left(\frac{|\zeta|}{\tanh |\zeta|} - 1 \right) [I - (\zeta\zeta' / |\zeta|^2)] . \tag{26e}$$

The passage from (26d) to (26e) uses the following well known series representation which appears in reference [20]

$$2 |\zeta / \pi|^2 \sum_{j=1}^{\infty} \frac{1}{j^2 + |\zeta / \pi|^2} = \frac{|\zeta|}{\tanh |\zeta|} - 1 . \tag{27}$$

Expectation over ξ_0

We use the same sort of trick to evaluate the last expectation as we used to evaluate the previous expectation. Here the evaluation is much easier since we have no need to complete the square. We get

$$Ee^{iz'\xi_0 - \frac{1}{2}\xi_0' B \xi_0} = |B+I|^{-\frac{1}{2}} e^{-\frac{1}{2}z'(I+B)^{-1}z} \tag{28}$$

From (26e)

$$I + B = \frac{|\zeta|}{\tanh|\zeta|} I + \left(1 - \frac{|\zeta|}{\tanh|\zeta|}\right) \frac{\zeta\zeta'}{|\zeta|^2}. \tag{29}$$

It is straightforward verification to show

$$(I+B)^{-1} = \frac{\tanh|\zeta|}{|\zeta|} I + \left(1 - \frac{\tanh|\zeta|}{|\zeta|}\right)\left(\frac{\zeta\zeta'}{|\zeta|^2}\right). \tag{30}$$

This inverse has eigenvalues 1, with multiplicity one, and $|\zeta|^{-1}\tanh|\zeta|$, with multiplicity two, so

$$|B+I|^{-\frac{1}{2}} = |\zeta|^{-1}\tanh|\zeta|. \tag{31}$$

Therefore

$$\phi(z,\zeta) = \frac{|\zeta|}{\sinh|\zeta|} \cdot \frac{\tanh|\zeta|}{|\zeta|}\, e^{-\frac{1}{2}\left\{|z|^2 \frac{\tanh|\zeta|}{|\zeta|} + \frac{(z'\zeta)^2}{|\zeta|^2}\left(1 - \frac{\tanh|\zeta|}{|\zeta|}\right)\right\}}. \tag{32}$$

Finally

$$\phi(z,\zeta) = \operatorname{sech}|\zeta| e^{-\frac{1}{2}\left\{|z|^2 \frac{\tanh|\zeta|}{|\zeta|} + \frac{(z'\zeta)^2}{|\zeta|^2}\left(1 - \frac{\tanh|\zeta|}{|\zeta|}\right)\right\}}. \tag{33}$$

3 Proof of the Theorem

3.1 Paring the Listed Variates

It is easy to see that if the theorem is true, it remains true for new variates defined in terms of the original variates V_j listed in the theorem,

$$\tilde{V}_j = \alpha_j V_j + \beta_j \quad (\alpha_j \text{ and } \beta_j \text{ constants}) . \tag{34}$$

So if the theorem is true for either of the Ito or Stratonovich interpretation, it is true for the other, since the difference in interpretation amounts to a constant offset. The only terms sensitive to the choice of interpretation, are the $\int W_k(t)dW_k(t)$ terms, and since linear combinations of 1 and $W_k^2(1)$, span both interpretations it suffices to prove the theorem with $\int_0^1 W_k(t)dW_k(t)$ omitted from the variates listed in the theorem. For $l \neq k$, it follows from the integration by parts formula

$$\int W_l(t)dW_k(t) = W_l(1)W_k(1) - \int W_k(t)dW_l(t) \tag{35}$$

that terms of the form $\int W_l(t)dW_k(t)$ can also be dropped from the list. So it suffices to prove the theorem with all terms of the form $\int W_l(t)dW_k(t)$ with $k \geq l$ omitted from the list. It is therefore enough to prove the theorem with only the 21 variates $\{W_k(1)\ 1 \leq k \leq 3;\ W_l(1)W_k(1)\quad 1 \leq l \leq k \leq 3;\quad W_l(1)\int W_k(t)dt \quad 1 \leq l,\quad k \leq 3;\ \int W_l(t)dW_k(t)\ 1 \leq l < k \leq 3\}$ on the list.

Let $\hat{V}$ be a 21 dimensional random vector with each of the 21 listed variates the sole occupant of one component. Let U be the 12 dimensional vector with the $W_k(t)\int W_l(t)dt$ variates occupying the first nine positions in the order $(k,l)=(1,1), (1,2) \cdots (3,3)$ and the $\int_0^1 W_k(t)dW_l(t)$ variates occupying the last three coordinates in the order $(k,l)=(2,3), (1,3), (1,2)$. We shall see that we can essentially limit our consideration to U.

Before we show how to use U to complete the proof of the theorem we need the following notation:

The twelve dimensional vector transform variate:

$$Z = (z_{11}, z_{12}, \ldots, z_{33}, \zeta_{23}, \zeta_{13}, \zeta_{23}) \tag{36a}$$

and random vectors, the first three dimensional, and the last two infinite dimensional with all components i.i.d. standard normal:

$$\xi_0 = (\xi_0^1, \xi_0^2, \xi_0^3) \tag{37b}$$

$$\xi = (\xi_1^1, \xi_1^2, \xi_1^3, \xi_2^1, \xi_2^2, \xi_2^3, \cdots) \tag{37c}$$

$$\eta = (\eta_1^1, \eta_1^2, \eta_1^3, \eta_2^1, \eta_2^2, \eta_2^3, \cdots). \tag{37d}$$

Suppose, upon taking the two inner expectations on the right-hand side of

$$E^{iZU} = E_{\xi_0} E_\xi E_\eta e^{iZU} \tag{38}$$

we have an equation of the form

$$Ee^{iZU} = E_{\xi_0} e^{\xi_0' C \xi_0 + c' \xi_0 + \gamma} \tag{39}$$

with the entries of the matrix C, the vector c, and the scalar γ are deterministic closed form expressions involving the transform variables. Then the expectation in (39) can clearly be carried out to express Ee^{iZU} in closed form. Consequently, the characteristic function of $\hat{V}$ is also expressible in closed form. So we need only establish the form of (39) and that C,c and γ are expressible in closed form. We refer to the exponent on the right-hand-side of (39) as a quadratic form in ξ_0 (meaning second order *and* lower).

Expectation With Respect to η

We proceed now to take the expectation with respect to η and then with respect to ξ. To begin this process for η, we look at iZU and use (8) and the (p,q,r) cycle notation of Example II to express the η_j^r term

$$i\left\{\frac{\eta_j^r}{2\pi j}\,[\zeta_p(\xi_j^q - 2^{1/2}\xi_0^q) - \zeta_q(\xi_j^p - 2^{1/2}\xi_0^p) \right.$$
$$\left. + 2^{1/2}(z_{1r}\xi_0^1 + z_{2r}\xi_0^2 + z_{3r}\xi_0^3)]\right\}. \tag{40}$$

For $s = 1,2,3$ let

$$\hat{\xi}_j^s \triangleq (\xi_j^s - 2^{1/2}\xi_0^s) \tag{41a}$$

$$\hat{\xi}_j \triangleq (\hat{\xi}_j^1, \hat{\xi}_j^2, \hat{\xi}_j^3) \tag{41b}$$

$$z_s \triangleq (z_{1s}, z_{2s}, z_{3s})' \tag{41c}$$

$$\Theta_j^r \triangleq \text{term in } \{\} \text{ in equation (40)} \tag{41d}$$

$$\mu_s \triangleq 2^{1/2} z_s' \xi_0 \tag{41e}$$

$$\mu^2 \triangleq \mu_1^2 + \mu_2^2 + \mu_3^2 \,. \tag{41f}$$

Using (41d), independence of the η_j, and the characteristic function formula for a univariate Gaussian we get

$$E_\eta e^{iZU} = E_\eta e^{i\sum_{j=1}^{\infty}(\Theta_j^1 + \Theta_j^2 + \Theta_j^3)} \tag{42a}$$

$$= \prod_{j=1}^{\infty} E_{\eta_j} e^{i(\Theta_j^1 + \Theta_j^2 + \Theta_j^3)} \tag{42b}$$

$$= \prod_{j=1}^{\infty} \exp - 1/2 (2\pi j)^{-2} \{(\Theta_j^1)^2 + (\Theta_j^2)^2 + (\Theta_j^3)^2\} \,. \tag{42c}$$

Working out this exponent in detail, we have

$$-1/2(2\pi j)^{-2}\{(\Theta_j^1)^2 + (\Theta_j^2)^2 + (\Theta_j^3)^2\} = \tag{43a}$$

$$-1/2(2\pi j)^{-2} \sum_{\{(p,q,r)\}} \left\{ \zeta_p \hat{\xi}_j^q - \zeta_q \hat{\xi}_j^p + 2^{1/2}(z_r' \xi_0) \right\}^2 = \tag{43b}$$

$$-1/2(2\pi j)^{-2} \sum_{\{(p,q,r)\}} \Big[\zeta_p^2 (\hat{\xi}_j^q)^2 - 2\zeta_p \zeta_q \hat{\xi}_j^q \hat{\xi}_j^p$$

$$+ \zeta_q^2 (\hat{\xi}_j^p)^2 + 2\zeta_p \hat{\xi}_j^q \mu_r - 2\zeta_q \hat{\xi}_j^p \mu_r + \mu_r^2 \Big] \,. \tag{43c}$$

Expectation With Respect to ξ

To proceed with the proof that the form of $E_\xi E_\eta e^{iZU}$ is as indicated in (39), collect terms in (43c) that involve ξ_j with j an arbitrary fixed index strictly greater than zero. In carrying out the integration required for taking the expectation with respect to ξ_j the exponent of the integrand is obtained by subtracting $-1/2(\xi_j' I \xi_j)$ from (43c). Employing A_j from Example II, we rewrite the integrand exponent as

$$-1/2\Big\{(\xi_j-2^{½}\xi_0)'A_j(\xi_j-2^{½}\xi_0) + \xi_j'\xi_j +$$

$$(2\pi j)^{-2}[2(\zeta_3\mu_2-\zeta_2\mu_3,\ \zeta_1\mu_3-\zeta_3\mu_1,\ \zeta_2\mu_1-\zeta_1\mu_2)'(\xi_j-2^{½}\xi_0)]$$

$$+|\mu/2\pi j|^{-2}\Big\}. \tag{44}$$

Let

$$\alpha = 2(\zeta_3\mu_2-\zeta_2\mu_3,\zeta_1\mu_3-\zeta_3\mu_1,\zeta_2\mu_1-\zeta_1\mu_2)\,. \tag{45}$$

We complete the square for (44) rewriting it as

$$-1/2\Big\{(\xi_j-\omega_j)'(A_j+I)(\xi_j-\omega_j)-\omega_j'(A_j+I)\,\omega_j$$

$$+\ 2\xi_0'A_j\xi_0-(2\pi j)^{-2}2^{½}\alpha'\xi_0+(2\pi j)^{-2}\,|\mu|^2\Big\} \tag{46}$$

where ω_j is defined to be the vector required for (44) and (46) to be equal.

If we sum the last three terms in (46)

$$-1/2\sum_{j=1}^{\infty}[2\xi_0'A_j\xi_0-(2\pi j)^{-2}2^{½}\alpha'\xi_0+(2\pi j)^{-2}\,|\mu(\xi_0)|^2] \tag{47}$$

it is evident that we obtain a closed form quadratic form in ξ_0. Summing the first term in (46) over $j=1, 2, \ldots$ gives rise to a closed form multiplier that does not depend on ξ_0 in the expression for the characteristic function (just as in the examples).

Checking for Closed Form

From the above paragraph, the only term we need to consider further to see if it is a closed form quadratic form in ξ_0 is

$$-1/2\sum_{j=1}^{\infty} \omega_j'(A_j+I)\omega_j \tag{48}$$

where the vector ω_j', which gives equality between (44) and (46), is

$$\omega_j' = (2^{1/2}\xi_0' A_j - 1/2(2\pi j)^{-2}\alpha'(I+A_j)^{-1}\ . \tag{49}$$

Writing out (46), with the ω_j expression from (49) substituted, gives

$$-1/2\left\{(2^{1/2}A_j\xi_0 - 1/2(2\pi j)^{-2}\alpha)'(I+A_j)^{-1}(2^{1/2}A_j\xi_0 - 1/2(2\pi j)^{-2}\alpha)\right\}. \tag{50}$$

Multiplying out, we get four terms in the brackets, two of which are the same. The distinct terms are

$$2\,\xi_0^1 A_j(I+A_j)^{-1}A_j\,\xi_0 \tag{51a}$$

$$-2^{1/2}(2\pi j)^{-2}\alpha'(I+A_j)^{-1}A_j\,\xi_0 =$$

$$-2^{1/2}(2\pi j)^{-2}\alpha'[I-(I+A_j)^{-1}]\xi_0 \tag{51b}$$

$$1/4(2\pi j)^{-4}\alpha^1(I+A_j)^{-1}\ . \tag{51c}$$

The middle equality comes from replacing A_j by $I+A_j-I$. From Example II, upon summing over $j \geq 1$, expression (51a) leads to a closed form expression.

Looking at (51b) and (51c) it is enough to check that the following two matrices can be expressed in closed form

$$\sum_{j=1}^{\infty} (2\pi j)^{-2}(I+A_j)^{-1} \tag{52a}$$

and

$$\sum_{j=1}^{\infty} (2\pi j)^{-4}(I+A_j)^{-1}\ . \tag{52b}$$

A simple check shows

$$(I+A_j)^{-1} = \frac{I}{1+|\zeta/\pi j|^2} + \frac{\zeta\zeta'}{(\pi j)^2(1+|\zeta/\pi j|^2)}\,. \tag{53}$$

From (53), closed form for (52a) and (52b) turns on closed form for the left-hand sides of

$$\sum_{j=1}^{\infty}(|\zeta/\pi|^2+j^2)^{-1} = 1/2|\pi/\zeta|^2\left(\frac{|\zeta|}{\tanh|\zeta|}-1\right) \tag{54a}$$

$$\sum_{j=1}^{\infty} j^{-2}(|\zeta/\pi|^2+j^2)^{-1} = |\pi/\zeta|^2[(\pi^2/6)+1/2|\pi/\zeta|^2] - (\pi/2)|\pi/\zeta|^3\coth|\zeta| \tag{54b}$$

$$\sum_{j=1}^{\infty} j^{-4}(|\zeta/\pi|^2+j^2)^{-1} = (\pi/|\zeta|)^2\sum_{j=1}^{\infty}(j^{-4}-j^{-2}((\zeta/\pi)^2+j^2)^{-1})\,. \tag{54c}$$

The right-hand side of (54a) comes from (27), which comes from reference [20] as does (54b). Equation (54c) is just elementary algebra. The right-hand side of (54c) is closed form from (54b) and the result that $\sum_{j=1}^{\infty} j^{-4} = (\pi^4/90)$ as given in reference [19].

4 Closing Remarks

By generalizing beyond the examples, we demonstrated that closed form joint characteristic functions for certain Wiener functionals are not just a fluke. No attempt was made to find a maximal list in the statement of the theorem and we would be extremely surprised if the list could not be greatly expanded. From [21], we see that the joint characteristic function of $W_m(1)$, $\int W_m(t)dt$, $\int W_m^2 dt$ can be expressed in closed form. While there are many ways to seek to generalize the theorem, perhaps the next logical step is to check and see if the theorem remains true if all random variables of the form

$$\int W_k(t)\,dt,\ \int W_k^2(t)\,dt,\ \int W_k(t)\,dt \int W_l(t)\,dt$$

are added to the list.

Acknowledgements

J. M. Steele posed the question of finding the distribution of the area integral solved in (3) to one of us. C. L. Mallows has given a different derivation of (3) as a corollary of other results. Neal Madras and Jim Pitman pointed out some of the references. The problem of finding the characteristic function in Example II arose in (joint) work with Craig Poole [14].

Bibliography

[1] L. A. Shepp, "Radon-Nikodym derivatives of Gaussian measures," *Ann. Math. Statist.*, vol. 37, pp. 321-354, 1966.

[2] S. Karlin and H. M. Taylor, "A Second Course in Stochastic Process," New York: Academic, ch. 15, 1981.

[3] F. B. Knight, "Essentials of Brownian Motion and diffusion," *American Mathematical Society*, pp. 12-20, 1981.

[4] W. Kaplan, "Advanced Calculus," *Addison-Wesley*, Massachusetts, Chapter IV, 1959.

[5] P. Lévy, Wiener's random function and other Laplacian random functions, Proceedings Second Berkeley Symposium, Mathematical Statistics, Probability Volume II, University of California, 171-186, 1950.

[6] N. Wiener, "Une probleme de probabilities denombrables," Bullitin Societe Mathematique France, Vol 52, pp. 569-578, 1924.

[7] D. C. Khandekar and F. W. Wiegel, Distribution of the area enclosed by a plane random walk, J. Phys. A: Math. Gen. 21 (1988), L 563-566.

[8] Jim Pitman and Marc Yor, A decomposition of Bessel bridges, Z. Wahr. verw. Gebiete 59, 425-457, 1982.

[9] J. F. Le Gall and Marc Yor, Enlacements du mouvement Brownian autour des combes de l'espace, Trans. AMS, 317, 687-722, 1990.

[10] P. Biane and M. Yor, Variations sur une formule de Paul Lévy, Annals Institute Henri Poincare 23, 359-377, 1987.

[11] M. Yor, On stochastic areas and averages of planar Brownian motion, J. Phys. A: Math. Gen. 22, 3049-3057, 1989.

[12] B. Duplantier, Areas of planar Brownian curves, J. Phys. A: Math. Gen. 22, 3033-3048, 1989.

[13] P. Biane and M. Yor, A relation between Lévy's stochastic area formula, Legendre polynomials and some continued fractions of Gauss, Technical Report no. 74, Department of Statistics, Berkeley, 1986.

[14] G. J. Foschini and C. D. Poole, "Statistical theory of polarization dispersion in single mode fibers," to be published.

[15] A. H. Jazwinski, "Stochastic processes and filtering theory," *Academic Press*, New York, Chapter IV, 1970.

[16] L. Arnold, "Stochastic differential equations theory and application," John Wiley and Sons, New York, Chapters 4 and 10, 1974.

[17] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, "Tables of integral transforms," *Bateman Manuscript Project*, vol. I, McGraw-Hill, 1954.

[18] J. Delporte, "Fonctions aléatories presque sûrement continues sur un intervalle fermé," *Ann. Inst. Henri Poincare´ Sec. B1* 111-215,1964.

[19] I. S. Gradshteyn and I. M. Ryzhik, "A table of integrals series and products," *Academic Press*, 1980.

[20] A. D. Wheelon, "Tables of summable series and integrals involving Bessel functions," *Holden-Day Inc.*, Chapters 2 and 5, 1968.