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A FRICTION MODEL FOR DESCRIBING AND FORECASTING PRICE CHANGES

WAYNE S. DESARBO, VITHALA R. RAO, JOEL H. STECKEL, JERRY WIND AND RICHARD COLOMBO
Southern Methodist University
Cornell University
Columbia University
University of Pennsylvania
New York University

This paper presents a new friction model for describing the price changes of a product or brand over time and for forecasting both the timing and magnitude of such changes from one period to the next. After a review of the related pricing literature, we present our model and a modified controlled random search procedure for estimating its parameters. The model is applied to describe and forecast the weekly mortgage interest rates for a local bank in the Philadelphia area. Finally, limitations and several potential applications of this methodology are discussed.
(Friction Model; Price Changes; Price Reaction; Description; Forecasting)

1. Introduction

There is no concept more central to the study of economic activity than price. Although economists and marketers have expended substantial effort on studying price, their perspectives differ significantly (see the recent reviews by Nagle 1984 and Rao 1984). Economists develop models which explain the general principles to which optimal pricing strategies conform. Their goal is to explain why certain strategies persist. Even though marketers recognize the importance of price determination by firms, research on the descriptive aspects of price setting is quite limited. Recent modeling efforts have produced both normative decision models (e.g., Robinson and Lakhani 1975) for determining optimal prices for new products over time and individual choice models which incorporate price (e.g., Rao and Gautschi 1982; DeSarbo and Hoffman 1986, 1987), thereby allowing the manager to examine the effect of price on choice, and consequently on sales and market share. It is fair to say that both marketers and economists have, for the most part, neglected the *descriptive* question of the process by which prices are set in a given period. In particular, as Monroe and Della Bitta (1978) note, knowledge is sparse concerning price changes of an existing product. One thing is sure however. Prices do not change simply in response to the forces of supply and demand (Carlton 1986).

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This paper presents a descriptive "friction" model for predicting price changes. The model provides a paramorphic description of the process; that is, we assume that management behaves as if it follows the postulated model. Details of the actual process are in practice unobservable. Thus, we proceed from a set of three reasonable premises. First, stochastic, latent (unobservable) variables are defined to summarize the existing internal and external pressures which would tend to push price up or down. Second, the model describes price changes as occurring only when these latent variables move beyond sufficiently high/low threshold levels. This "friction" premise corresponds to Carlton's (1986) "price rigidity" and Liebenstein's (1980) "inert area" concepts. Inert areas refer to those actions which would raise profits, but are deemed not to be worth it either because of actual implementation costs or management effort. Third, the strength of the pressures captured by these latent variables directly relates to the magnitude of the price change.

Models which describe price changes are important for several reasons. First, predictions derived from well-formulated descriptive models may outperform the decision maker himself (cf. Bowman 1963; Kunreuther 1969), particularly when such formal models are developed from the decision maker's own rules and behavior. Second, such models when appropriately (empirically) estimated could be used to analyze competition, forecast their price movements, and consequently help a firm plan its own tactical decisions on prices. Third, successful development and implementation of normative models depend on the congruence of the normative model with the organizational context (Ein-Dor and Segev 1978, Schultz and Henry 1981) and the decision maker's style (Huysmans 1970). Therefore, descriptive models of such decisions could indirectly aid the implementation of normative models.

The paper proceeds as follows. The next two sections review relevant background material concerning previous research on pricing. We then present the friction model for describing and forecasting price changes and discuss estimation issues. The model and algorithm are illustrated next with a synthetic data set. We then provide an application to weekly home mortgage rates (fixed rate-conventional mortgages) set by a local Philadelphia area bank. Finally, we discuss limitations, extensions to the methodology, and other potential applications.

2. Background

Several streams of research[1] are related to the descriptive modeling of price changes. These include: (1) univariate time-series models, (2) experience curve effects, and (3) information processing research on price determination. We review each of these briefly.

Time series analyses of price movements attempt to relate price in a given period to those in previous periods under the assumption that the mechanisms by which price was established in the past shall continue. Economists have applied a number of these methods to the study of commodity prices and price indices (Nerlove et al. 1979). In marketing, the method most often used has been that of Box and Jenkins (1976). They propose the ARIMA (Autoregressive Integrated Moving Average) class of models which can represent a wide variety of time series behaviors. See Geurts and Ibrihim (1975) and Doyle and Saunders (1985) for marketing applications.

As with most time series methods, Box-Jenkins analysis offers little insight on the underlying decision processes behind price changes. Furthermore, time-series analyses

[1] Hedonic price functions (Griliches 1971) relate prices to product attributes and are useful in examining price changes adjusted for corresponding quality changes. See Rosen (1974) and Ratchford (1980). However, this approach is not considered here. In many cases, short-term price changes generally do not arise due to product (attribute) changes but are instead a response to the competitive environment or internal factors.

involving price changes as the dependent variable are subject to what we call the "zero-price change" effect. Zero price change plays a special role. It would take a conscious decision on the part of management to have anything but a zero price change; i.e., there is a "friction" at zero. Models, such as those in typical single and multiple time series analyses,[2] which treat all possible values of the criterion variable alike, do not recognize this friction. For example, in a multiple regression of time series data, an infinitesimal change in an independent variable would imply a change in price. This implication does not appear to be realistic. The model developed in this paper accounts for this problem.

Some authors have attempted to relate price changes to experience curve effects. The rationale for this approach is that as cumulative production (experience) increases, unit costs fall in real terms and therefore so may prices. Stobaugh and Townsend (1975) and Lieberman (1984) provide evidence that this phenomenon dominates the chemical industry. Fogg and Kohnken (1978) propose a seven-step procedure which uses cost forecasts obtained from extrapolating the experience curve to guide marketing decisions. Sallenave (1976) suggests comparing actual prices with those based on costs predicted from the experience curve to detect gross discrepancies that might signal conditions favorable to a price "war." While in the same spirit as our work, this literature focuses on only one of the many factors that can influence price changes.

The stream of research that views the executive as an information processor (Newell and Simon 1972) could be of significant help in building a descriptive model. Hulbert (1981) provides a review of this literature. Researchers in this stream, similar to those in consumer decision making (Bettman 1979), develop a flow chart for a given decision making unit that shows how various relevant factors are included in the decision process. The flow charts lay out the criteria for evaluating an action and the operational procedures employed in the organization for the decision. Howard and Morgenroth (1968) have applied this approach to price changes. Their study presents an example of the roles competitive price changes, customer requests for discounts, and unsatisfactory sales levels play in triggering price changes.

The research based on information processing presents an important first step. However, each flow chart is idiosyncratic to the decision making unit examined. This lack of generalizability has also impeded the development of a mathematical theory. On the other hand, this literature does point out that the price change decision can be fairly complex and several factors may contribute. The model presented in this paper recognizes this point and shows how various factors can be incorporated into a descriptive model for price changes by a firm. A discussion of these factors is contained in the next section.

3. Factors That Affect Price Changes

A model which describes how prices are changed must account for the factors which affect those changes. Economic theory provides an explicit framework from which we can discern what those factors might be. We begin with the theory of the profit maximizing monopolist (Varian 1984, pp. 79–81). As Hay and Morris (1979) argue, this is an appropriate point of departure since all firms which practice product differentiation are necessarily partly "monopolistic" in nature.

The profit maximizing monopolist has to decide how much to produce and at what price to sell it. Of course, these decisions are not independent. Denoting the demand function by Q and its inverse (price) function by P and the average cost function by AC,

[2] Marketers have also been interested in multiple time-series models designed to examine the interactions between two or more series (e.g., Hanssens 1980).

the optimal price, P^*, at which marginal revenue equals marginal cost ($MR = MC$) is the solution:

$$P^* = AC + Q\left(\frac{d(AC)}{dQ} - \frac{dP}{dQ}\right),$$

where all terms on the right-hand side are evaluated at the optimal price, P^*, and quantity Q^*. The values of AC and $d(AC)/dQ$ depend on the cost structure of the firm, while Q and dP/dQ depend on the demand conditions it faces. Thus, *changes* in optimal prices can be induced by *changes* in either the cost or the demand structure.

Changes in the cost structure. A change in fixed costs will not affect the $MR = MC$ relationship and therefore will not alter the optimal price. On the other hand, an increase (decrease) in variable production costs increases (decreases) marginal costs. To compensate, marginal revenue must increase (decrease). If we assume a downward sloping demand curve (i.e., dP/dQ is negative), P^* must increase.

Changes in the demand structure. We can examine changes in demand as originating in either of two ways: translation or rotation of the demand curve. The first change might occur for example when a step up in a brand's marketing effort stimulates only primary demand leaving response to price (dP/dQ) unchanged. In the second case, the response to price (dP/dQ) could change leaving the original P^* and Q^* intact. The second situation could occur when conditions such as new competition or increased differentiation change the price sensitivity of the market.

In the case of an upward shift in the demand curve, marginal revenue increases for all Q. The change in market price, however, depends on the shape of the marginal cost curve. More often than not, however, P^* will increase.[3] Similarly, P^* will usually decrease with a downward shift in demand.

In the case of a demand curve rotation, if dP/dQ becomes more negative (i.e., the market becomes less price sensitive), marginal revenue decreases for any given Q. Consequently, the optimal output will fall. This is accompanied by a price increase for a negatively sloped demand curve with and increasing marginal costs. Similar conclusions can be drawn for decreasing marginal costs. The reverse holds if dP/dQ becomes less negative.

Related Factors

Competition and the effects of marketing mix are two factors important for understanding price movements which are not often explicitly incorporated in the model of the profit-maximizing monopolist. However, as we shall argue presently, they are implicit in the determinants of price changes outlined above.

Competitors directly affect the demand structure a firm faces. Their mere presence implies that a firm's market shrinks (i.e., the demand curve shifts downward). Furthermore, a competitor's strategies (marketing or other) attempt to differentiate their offering from the firm in question thereby altering the price sensitivity (dP/dQ) of the firm's customers. For example, consider the effect of a competitor's price change on the prices

[3] Suppose that both the inverse demand and marginal cost functions were linear and represented by $P(Q) = \lambda - \mu Q$; and $MC(Q) = \nu + \phi Q$ where $\mu > 0$ and ϕ is unrestricted. These are probably reasonable approximations if we restrict ourselves to studying small changes. ϕ could be negative if a learning curve phenomenon is operating; or positive if a capacity constraint is impending.

From the equation for $P(Q)$, it can be shown that the marginal revenue curve is $\lambda - 2\mu Q$. Equating this to $MC(Q)$ yields an optimal quantity of $((\lambda - \phi)/(2\mu + \phi))$. The optimal price for these linear approximations is then given by $P^* = \lambda - \mu(\lambda - \phi)/(2\mu + \phi)$ where $\phi > -2\mu$ in order to ensure a maximum. If we represent a shift in the demand curve by a change in λ, the impact on price is $dP^*/d\lambda = 1 - (\mu/(2\mu + \phi))$. This is positive if $\phi > -\mu$. Thus, unless $-2\mu < \phi < -\mu$, an upward shift in the demand curve implies a price increase and a downward shift implies a price decrease.

of the firm in question. A competitor's price increase will shift the demand curve facing the firm upward with a possible change in its slope; and, by the previous discussion, the firm's price will increase.

Marketing efforts enter on both the demand and cost sides. Dorfman and Steiner (1954) incorporated marketing expenditures into the model of the profit-maximizing monopolist in order to derive the optimal relationship between profit, price, marketing expenditures (all of which they classify as advertising), and product quality. In any case, demand is a function of not only price, but marketing expenditures as well. Marketing expenditures can shift the firm's demand curve by expanding the market. They can also change price sensitivity by communicating particular critical benefits, building image, or simply creating awareness. Using a model of search behavior, Ehrlich and Fisher (1982) show that marketing expenditures can either increase or decrease price elasticity. They will therefore have some impact on optimal prices, but the direction of the impact is ambiguous. Looking at it from the cost side, such expenditures cannot be clearly classified as either fixed or variable. In breakeven analyses, they are usually seen as part of the fixed costs which must be recovered (Shapiro, Dolan, and Quelch 1985). On the other hand, they are discretionary expenditures which do create value by making the product more desirable to the customer. Thus, it can be reasonably argued within the economic framework that marketing expenses are variable costs and have the corresponding effect on prices.

Miscellaneous

A significant factor in the decision to raise or lower the price of a product is the managerial objective set for the product. For example, an objective of harvesting monetary benefits in a short time period at the expense of a long-term market share gain often suggests a price increase. Similarly, the *speed* with which a management group can process and react to new information from the market is critical in the price change decision. If the decision making process has more inertia, one can expect the reaction to be much slower. Other environmental factors include the level of general inflation in the economy and in the markets that supply inputs to the firm's production process.

Table 1 provides a summary of the discussion in this section. It lists and classifies the effects of the factors outlined above which impact price changes. In addition, it presents examples of firm-related or environmental events which could trigger these factors. The specific factors under each category could vary across industry type, market structure, and market position of the firm. A list of these factors can be generated using many methods including a survey of key decision makers of the firm in question.

4. The Friction Model of Price Movements

The essential point of departure for our model is that the factors discussed in the previous section exert "pressures" on a firm's price. These pressures can work in either direction. For example, consider the variable cost of production; increases in this variable will exert an upward pressure on the prices of a firm operating in a mature market. Decreases in such costs will exert a downward pressure on price. Furthermore, the pressures exerted by a single factor need not be symmetric. For example, prices must exceed variable production costs or the firm will be out of business in the long run. Therefore, a cost increase could exert substantial upward pressure on price. Conversely, a cost decrease does not always carry such severe potential consequences (unless competitors also incur lower costs and start price cutting). In principle, it simply increases the range of feasible prices and consequently will not exert a downward pressure of equal magnitude. Carlton (1986) demonstrates this empirically.

The factors which impact the pressures to raise and lower price in period t can be summarized by two sets of variables $\mathbf{X}_t$ and $\mathbf{Z}_t$ respectively. Some factors, such as change in market share, may be common. Other factors, such as inflation, may be

TABLE 1
Factors Influencing the Price Change of a Product

I.	*Cost Related Factors*	*Change in Factor*	Resultant *Change in Price*	*Possible Examples of Triggering Mechanisms*
A.	Fixed Costs	+	0	–
		–	0	–
B.	Variable Costs	+	+	Increases in costs of raw materials, labor, production, or marketing
		–	–	Learning effects and other decreases in production costs
II.	*Demand Related Factors*			
A.	Position of Demand Curve	Up	+ (Usually)	Market growth (possibly due to marketing effort)
		Down	– (Usually)	Increase in competitors' marketing efforts
B.	Slope of Inverse Demand Curve $\left(\frac{dP}{dQ}\right)$	More Negative (Less Sensitive to Price)	+	Product differentiation, increase in competitors' price
		Less Negative (More Sensitive to Price)	–	Competitive entry, decrease in competitors' prices, inflation, increase in competitors' marketing expenditures
III.	*Company Factors*			
A.	Managerial Objectives	Harvest to Maximize Profits	+	Progression through product life cycle, cash needs for other products
		Growth to Maximize Share	–	Market growth, high corporate earnings

included in only one of the two sets (here $\mathbf{X}_t$) because they usually only exert pressure one way (typically up).

We denote the total impact of pressures to raise and lower price in a time period, t, by two latent (unobservable) variables r_t and l_t. The latent variables will be modeled[4] as follows:

[4] We implicitly impose the restriction of linearity in the parameters of the models for r_t and l_t. However, this formulation is not excessively restrictive since many types of nonlinear effects of the variables can be represented adequately via such a specification by appropriately adding such terms as squares or cross products of variables.

$$r_t = \mathbf{X}_t\beta + e_{1t} \qquad \text{and} \tag{1}$$

$$l_t = \mathbf{Z}_t\gamma + e_{2t}, \tag{2}$$

where β and γ are vectors of unknown parameters with dimensionalities equal to the numbers of factors exerting upward and downward pressure respectively on price. We assume that the error terms, e_{1t} and e_{2t} for a given period t, are independent and normally distributed with mean of zero and variances equal to σ_1^2 and σ_2^2; further, we assume that these errors are homoscedastic and uncorrelated over time.[5]

Price movements will be dictated by the relative values of r_t and l_t. If $r_t(l_t)$ is *sufficiently* greater than $l_t(r_t)$, price will move in a positive (negative) direction. Toward this end, we consider the difference,[6] $f_t = r_t - l_t$. The net effect, f_t, although unobservable, depends on the two sets of factors impinging on price. Since we assume that e_{1t} and e_{2t} are uncorrelated, f_t is normally distributed with mean $\mathbf{X}_t\beta - \mathbf{Z}_t\gamma$ and variance, $\sigma^2 = \sigma_1^2 + \sigma_2^2$.

We conceptualize that the decision maker integrates the impact of the two sets of pressures on price and adopts a triggering mechanism in making a decision either to raise or lower price or to keep it the same. The triggering mechanisms are related to the psychological process of discrimination in which a decision maker would not react unless the perceived change is above a just noticeable difference (JND); see Luce and Edwards (1958). Furthermore, since our model calls for a decision to change in either direction, we could employ concepts analogous to the latitudes of acceptance, non-commitment, and rejection used in the social judgment theory (Sheriff and Howland 1961). Thus, the price change decision is equivalent to the net effect, f_t, described above to be beyond two limits.

Given these premises, we model the relationship between f_t and the observed price changes as follows:

If $f_t > k_1$, then price will increase by $\alpha_t = f_t - k_1$ $(\alpha_t > 0)$;

If $f_t < k_2$, then price will decrease by $|\delta_t|$, where $\delta_t = f_t - k_2$ $(\delta_t < 0)$;

If $k_2 \le f_t \le k_1$, then price will not change.

In this model, k_1 and k_2 are constants or threshold values (to be estimated), with $k_2 < 0 < k_1$, which define the two latitudes of acceptance and rejection mentioned above.

Econometricians have labeled the general class of models, to which the above model belongs, as friction models. Rosett (1959) coined the term while investigating the effects of changes in asset yields on changes in asset holdings. Here, due to transaction costs, individuals do not alter their portfolios of stocks when small changes in yields occur. Other examples are found in Dagenais (1975), Fishe and Lahiri (1981), and Maddala et al. (1982). Our proposed friction model is different from these other approaches in that it allows the factors which exert upward pressures on price to be different from those factors exerting downward pressures on price; i.e., the variables in $\mathbf{X}_t$ and $\mathbf{Z}_t$ are not necessarily the same.

Our friction model is illustrated in Figure 1 where a two dimensional portrayal of actual price changes (ΔP_t) and f_t is presented. As can be seen, the friction model predicts no price change unless the latent variable f_t is beyond the range of theshold values (k_2, k_1). Thus, there is "friction" or resistance to change price until some threshold is passed. In this model, k_1 is not in general equal to $-k_2$. Further, there is an implicit constraint of equal slopes to the right of k_1 and to the left of k_2, although we shall discuss an extension of the methodology to accommodate different slopes later in the paper.

[5] While we acknowledge possible autocorrelations, we make this assumption to keep the estimation complexities at a manageable level. The likelihood function for correlated errors is quite complicated.

[6] We might just as easily represent the force to change price by the ratio r_t/l_t (rather than $r_t - l_t$). Such a representation would be mathematically intractable given our formulations of the functions for r_t and l_t. However, if r_t and l_t are specified as multiplicative functions (e.g. $\Pi_j x_{jt}^{\beta_j}$) the results will be equivalent.

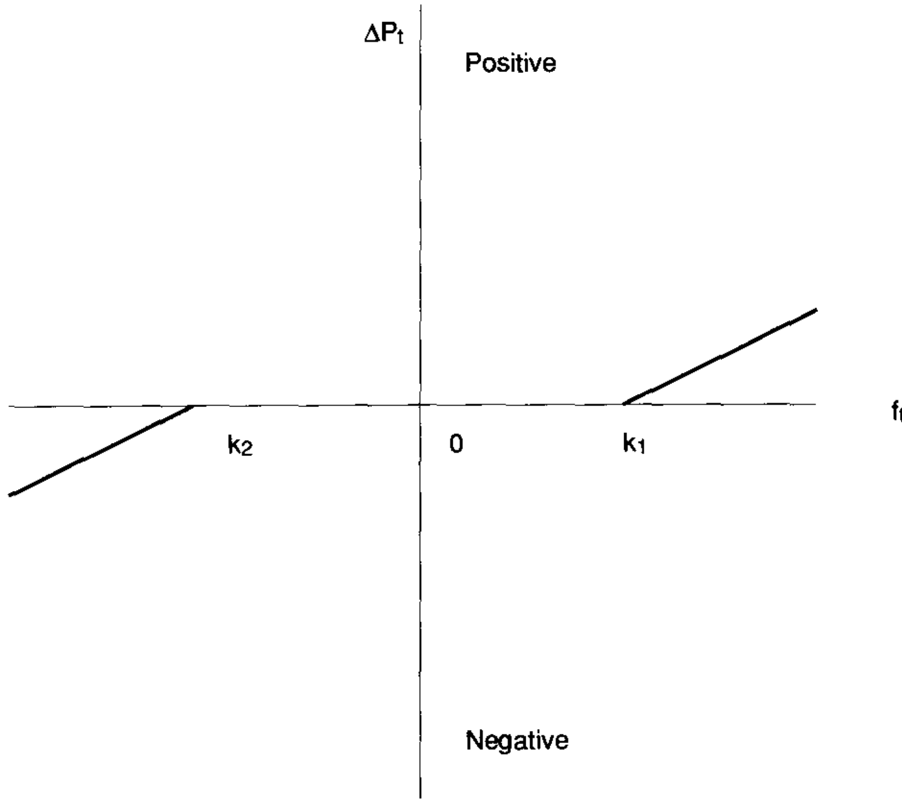

FIGURE 1. Pictorial Description of the Friction Model.

With appropriate data on historical movements in prices and the impact variables ($\mathbf{X}_t$, $\mathbf{Z}_t$), the model can be estimated by maximum likelihood methods. Before addressing this, however, a few comments are in order. First, the model is potentially underidentified. For an exogenous variable common to both $\mathbf{X}_t$ and $\mathbf{Z}_t$, we cannot separate its β component from its γ component, but can only estimate the difference between the two impact coefficients.

Second, it is not clear what a negative pressure to increase price really means. It might be to decrease, stay put, or something else entirely. Up to this point, we have not restricted r_t and l_t to be nonnegative; such restrictions can be included in the computational algorithm discussed later. Doing so would change the distributions of e_{1t} and e_{2t} to incomplete normals. The distribution of f_t would no longer be simple and would render the resultant likelihood function mathematically intractable. One way to minimize the possibility of negative pressures would be to express the $\mathbf{X}_t$ and $\mathbf{Z}_t$ as positive quantities. For example, instead of the single factor-change in market share, one could use two independent factors increase and decrease in market share with both expressed positively. One could then constrain the β and γ coefficients to be nonnegative ensuring that r_t and l_t are non-negative. In fact, such an option is present in our methodology. This could also help mitigate the identification problem.

The Likelihood Function

The likelihood function has three components; one for increases, one for decreases, and one for no changes:

$$L = \prod_{t \epsilon T_1} h_1(\alpha_t) \prod_{t \epsilon T_2} h_2(\delta_t) \prod_{t \epsilon T_3} \text{Prob(no change)}_t, \tag{3}$$

where T_1, T_2, and T_3 denote respectively the sets of periods where price increases, decreases, and remains the same. Here, $h_1(\cdot)$ and $h_2(\cdot)$ denote the density functions for increases and decreases respectively, and Prob(no change)$_t$ is the probability of no change in period t. The time period in this model is assumed to correspond to the smallest possible decision period by the firm in question. We do not allow for two decisions to be made in a single period.

Let $g(\cdot)$ represent the probability density function of f_t. Recalling that g is assumed to be normal, we can write

$$\text{Prob(no change)}_t = \int_{k_2}^{k_1} g(f_t)df_t$$
$$= \Phi\left(\frac{k_1 - \mathbf{X}_t\boldsymbol{\beta} + \mathbf{Z}_t\boldsymbol{\gamma}}{\sigma}\right) - \Phi\left(\frac{k_2 - \mathbf{X}_t\boldsymbol{\beta} + \mathbf{Z}_t\boldsymbol{\gamma}}{\sigma}\right), \quad (4)$$

where $\Phi(\cdot)$ is the standard normal distribution function.

In addition, the likelihood that the price increase will be α_t is equal to

$$h_1(\alpha_t) = g(\alpha_t + k_1|\alpha_t > 0)\cdot\text{Prob}(\alpha_t > 0)$$
$$= \frac{g(\alpha_t + k_1)}{\text{Prob}(\alpha_t > 0)}\cdot\text{Prob}(\alpha_t > 0)$$
$$= g(\alpha_t + k_1). \quad (5)$$

Similarly,

$$h_2(\delta_t) = g(k_2 + \delta_t). \quad (6)$$

The likelihood function can now be written as:

$$L = (2\pi\sigma^2)^{-T/2}\,[\exp(\sum_{t\epsilon T_1} A_t + \sum_{t\epsilon T_2} B_t)]\cdot\prod_{t\epsilon T_3}[\Phi(C_t) - \Phi(D_t)], \quad \text{where} \quad (7)$$

$T = |T_1| + |T_2|$; the total number of periods that there is some price change,

$$A_t = -\frac{1}{2}\left(\frac{\alpha_t + k_1 - \mathbf{X}_t\boldsymbol{\beta} + \mathbf{Z}_t\boldsymbol{\gamma}}{\sigma}\right)^2; \quad (8)$$

$$B_t = -\frac{1}{2}\left(\frac{\delta_t + k_2 - \mathbf{X}_t\boldsymbol{\beta} + \mathbf{Z}_t\boldsymbol{\gamma}}{\sigma}\right)^2; \quad (9)$$

$$C_t = \frac{k_1 - \mathbf{X}_t\boldsymbol{\beta} + \mathbf{Z}_t\boldsymbol{\gamma}}{\sigma}; \quad \text{and} \quad (10)$$

$$D_t = \frac{k_2 - \mathbf{X}_t\boldsymbol{\beta} + \mathbf{Z}_t\boldsymbol{\gamma}}{\sigma}. \quad (11)$$

The logarithm of the likelihood function in (7) is:

$$\ln L = -\frac{T}{2}\ln(2\pi) - T\ln\sigma + \sum_{t\epsilon T_1} A_t + \sum_{t\epsilon T_2} B_t + \sum_{t\epsilon T_3}\ln[\Phi(C_t) - \Phi(D_t)]. \quad (12)$$

5. Estimation

We estimate σ, k_1, k_2, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ by maximizing the log likelihood in expression (12) (or by minimizing $-\ln L$) given the data on $\mathbf{X}_t$, $\mathbf{Z}_t$, and the corresponding price change (α_t, δ_t or 0) for the T periods of time. A number of specific concerns may be noted with respect to this problem of optimization/estimation. First, expression (12) is mathematically complicated rendering the partial derivatives of the log likelihood function "messy", and not enabling a closed-form analytical solution. Even when finite differencing methods are employed to approximate the likelihood equations, computations would be infeasible owing to the large number of evaluations of the objective function required for each element of the vector of partial derivatives.

The second important issue relevant to the estimation problem is that the log likelihood is not defined over particular regions of the solution space. For example, unless the constraints of $\sigma > 0$ and $k_1 > k_2$ are placed, portions of the log likelihood function in (12) would involve taking logs of a negative number.

The third problem results from the possibility that the log likelihood function in (12) may not behave regularly at k_1 and k_2 (see Figure 1). Therefore, partial derivatives may not exist at these points and it is not clear what effect this discontinuity would have on a conventional gradient based procedure.

Finally, as recently demonstrated by Dharmadhikari and Joag-dev (1985), several cases arise in maximum likelihood estimation where the resulting estimators are non-unique. The log likelihood function (12) may be multimodal with more than one set of estimates.

In order to deal with these concerns, we chose to utilize a polytope random search procedure. A modified version of the controlled random search (CRS) procedure proposed by Price (1976) was programmed to estimate the desired set of parameters. This method is effective in searching for global minima of a possibly multimodal function, with or without constraints (e.g., β, $\gamma \geq 0$). It is a direct random search method which does not require the objective function to be differentiable or the variables to be continuous. The details of the Controlled Random Search Algorithm are described in Appendix I.

We have tested the ability of the CRS algorithm to accurately estimate parameters via a Monte Carlo analysis using two sets of synthetic data. For each data set, we first prespecified a set of values for the parameters of the model and generated synthetic data for analysis. The CRS algorithm was used to recover the original parameters. For a comparison, we applied OLS to the same data set. The estimates obtained with the CRS algorithm were naturally much closer to the true values used in the creation of the data than the OLS ones. Various details of this simulation are provided in Appendix II.

6. An Application

We applied the model to weekly data on mortgage interest rates for conventional home mortgages published weekly by the *Philadelphia Inquirer* for some 15 different major Philadelphia banks. The additional "points" charges (if any) are not included since the within and across bank variation of the points among the banks included in our analysis is very small during the time period of this analysis. Clearly, an interest rate can be considered as a price charged by the bank for a mortgage loan. Mortgage rates serve as a good illustration since data on the major factors which affect them can be obtained from public sources.

Given some personal contacts we had with the mortgage department there, we selected one particular bank in the area. These personal contacts enabled us to talk at length to the chief mortgage officer and associated officials about how these rates were set each week by that bank for the following week taking into account information available on the competitor's rates and other data. It turns out that the chief mortgage officer plays a major role in the mortgage interest rate determination.

Based on these discussions and literature search,[7] we identified four sets of major factors (a through d below) which are considered by this bank in its determination of a

[7] The review of this literature is not necessary here. The sources consulted include: Schafer and Ladd (1981), Meador (1982) and Marlow (1982, 1984), Burke and Rhoades (1984) and Curry and Rose (1984). We thank Louis K. Chan for his help in identifying certain literature sources and Mark Longworth for his assistance.

mortgage interest rate. These factors generally span the categories of influences shown in Table 1; in particular, factors a and b affect the variable cost of a loan, while c represents the impact of competition and demand. These are:

a. *Federal Reserve Bank's Policy.* The Fed. publishes weekly data about the prime rate, the federal fund rate, the discount rate, various treasury bill rates, etc. which affect the cost of borrowing from the Fed. by member banks.

b. *Secondary Mortgage Markets.* Many banks are too small to process and keep all accepted mortgages. They resell their mortgages to secondary federal agencies (e.g., Freddie Mac, Fanny Mae, R.F.C.). The rates published by these secondary institutions often play a major role in mortgage rate setting for a bank involved in this process.

c. *Competing Banks in Area.* A bank in a large area such as Philadelphia must compete for mortgages with other local banks. Our discussions with various mortgage officials at this one (smaller size) bank revealed a tendency for smaller banks to follow what larger banks do with respect to mortgage rates. So in addition to being directly competitive with other local area banks, market leader-follower relationships often come into play.

d. *Internal Bank Factors.* The complete financial picture of a bank also affects the weekly determination of mortgage interest rates. Such internal factors as savings balances, loan delinquency, checking accounts status, etc. have a major impact on these decisions at this particular bank.

Variable selection. Further discussions helped us specify **X** and **Z**. For this bank, the secondary mortgage market rates appeared to be the most significant factor as conveyed by management. The cost of money from the Fed. was important in decisions to increase the mortgage rate. This one bank also tended to monitor the rates of the two larger competitive banks (denoted as banks A and B). Finally, internal bank factors were also very important, especially concerning decisions to raise mortgage rates—*although these data were not made available to us.* An interesting aspect of exclusion of internal factors is that this model could, in principle, be developed by a competing bank to predict the behavior of the bank in question. Such a view expands the applicability of our model when estimated with publicly available data. We do, however, conjecture that the friction model would perform better if variables measuring the internal factors for the bank could have been added.

Based on these discussions with this bank's mortgage officials, we decided to attempt to model/forecast their mortgage interest rates using the following variables in $\mathbf{X}_t$ (i.e., those variables that will have an impact on the price increase decision):

$$X_{1t} = \max(0, (\text{discount rate}_{t-1} - \text{discount rate}_{t-2})),$$

$$X_{2t} = \max(0, (\text{prime rate}_{t-1} - \text{prime rate}_{t-2})),$$

$$X_{3t} = \max(0, (\text{bank } A_{t-1} \text{ rate} - \text{bank } A_{t-2} \text{ rate})),$$

$$X_{4t} = \max(0, (\text{bank } B_{t-1} \text{ rate} - \text{bank } B_{t-2} \text{ rate})),$$

$$X_{5t} = \max(0, (\text{Fanny Mae}_{t-1} \text{ rate} - \text{Fanny Mae}_{t-2} \text{ rate})),$$

and the following variables in $\mathbf{Z}_t$ (i.e., those variables that will have an impact on the price reduction decision):

$$Z_{1t} = \min(0, (\text{bank } A_{t-1} \text{ rate} - \text{bank } A_{t-2} \text{ rate})),$$

$$Z_{2t} = \min(0, (\text{bank } B_{t-1} \text{ rate} - \text{bank } B_{t-2} \text{ rate})),$$

$$Z_{3t} = \min(0, (\text{Fanny Mae}_{t-1} \text{ rate} - \text{Fanny Mae}_{t-2} \text{ rate})).$$

The particular lags were selected on the basis of discussions with the relevant bank officials. They were also supported by preliminary correlational analyses. We obtained weekly data on ΔP_t, $\mathbf{X}_t$, and $\mathbf{Z}_t$ for 82 weeks, representing 81 potential changes. We will compare the results from the constrained search algorithm for the friction model with those of ordinary least squares (OLS) estimates of a linear model (which, as described in Appendix II, is used to generate the initial starting estimates in the CRS method), and those obtained from a Box-Jenkins (1976) time series analysis of an ARIMA model. Note that these comparisons cut across models as well as estimation techniques.

OLS results. Table 2 presents the summary results for OLS utilized to initiate the CRS algorithm. Here X_2, X_4, and X_5 are significant factors in $\mathbf{X}$, while Z_3 is significant in $\mathbf{Z}$. The Fanny Mae factors (X_5, Z_3) dominate the analysis. The sum of squares between actual and predicted values in the OLS is 1.196. The bottom panel of Table 2 also presents the prediction results for OLS when utilized to forecast the 81 actual P_t data values from which the parameters were estimated. Here, we tally the increase, decrease, and stay-the-same predictions with the actual data. If one were to sum the entries on the main diagonal and divide by the total number of entries, the prediction accuracy = 35 ÷ 81 or 43.2%. In some sense, this may not be a fair treatment for OLS since it is structurally incapable of predicting *exactly* no change (unless all predictor variables equal zero with a zero intercept). We thus examined 95% confidence intervals for these predictions on historical data and regarded those intervals containing zero as predicting "no change." These prediction results are also shown in the bottom panel of Table 2. The prediction accuracy now jumps to 47 ÷ 81 or 59.2%. Note how poor the

TABLE 2
OLS Results on Mortgage Interest Rates

Variable	Parameter	Estimate
Intercept	β_0	−0.034
Positive		
Discount Rate	β_1	0.236
Prime Rate	β_2	0.391*
Bank *A*'s Rate	β_3	0.223
Bank *B*'s Rate	β_4	0.275*
Fanny Mae Rate	β_5	0.505**
Negative[1]		
Bank *A*'s Rate	γ_1	0.034
Bank *B*'s Rate	γ_2	−0.031
Fanny Mae Rate	γ_3	0.508***

S.E.	0.129	Adj. R^2	0.509
R^2	0.558	$F(8,72)$	11.354***

Note: *$p \le 0.10$, **$p \le 0.05$, ***$p \le 0.01$

Actual	Predicted From Regression: Raise	Stay the Same	Lower	Total
Raise	19; 9[a]	0; 11	1; 0	20; 20
Stay the Same	18; 5	0; 25	23; 11	41; 41
Lower	4; 0	0; 7	16; 13	20; 20
Total	41; 14	0; 43	40; 24	81

[a] The second number in each cell shows the predictions when 95% confidence interval is used.
[1] In order to be consistent with the notation of the friction model, we show the estimates of γ's, which are the negatives of the computed regression coefficients in the OLS model.

TABLE 3

Box and Jenkins ARMA (2,2) Model Results

Residual Sum of Squares 2.38384

SUMMARY OF FITTED MODEL

Parameter	Estimate	Standard Error	t-value	Prob (>"t")
AR (1)	0.00637	0.09398	0.06775	0.94615
AR (2)	0.74559	0.07543	9.88392	0.00000
MA (1)	−0.23707	0.04927	−4.81142	0.00001
MA (2)	0.83476	0.02650	31.50365	0.00000

Model fitted to differences of order 1.

Estimated white noise variance = 0.0291923 with 81 degrees of freedom.

Chi-square test statistic on first 20 residual autocorrelations = 9.06436 with probability of a larger value given white noise = 0.91074.

predictions have become in the upper left and lower right cells. However, this modification has improved the predictive accuracy of the middle cell and the entire table.

Time series results. Potential mortgage rate (price) forecasting was also examined with the Box and Jenkins (1976) ARIMA methodology. Initially, the raw level mortgage rates (P_t) were input as a series of 82 observations. Based on the results from autocorrelation and partial autocorrelation analyses, and trial and error, a nonseasonal ARIMA (2, 1, 2) was identified as being a parsimonious representation of the time series. In order to analyze such a process, we model the one-period differences as an ARMA (2, 2) process. Table 3 presents the statistical details of this estimated ARMA (2, 2) model. As shown, the AR(2), MA(1), and MA(2) terms are quite significant. Also, the appropriate χ^2 test indicates that the residuals follow a white noise or random process. This is also supported by the residual autocorrelations, partial correlations, and integrated periodogram (not shown here). Each of these indicate a random-like residual pattern with no evidence of further significant AR or MA terms. Table 3 indicates[8] an error sum of squares of 2.383 compared to the total sum of squares of 2.711 (or reduction of about 12 percent for this model).

Friction model results. Table 4 presents the parameter estimates of the friction model obtained with the CRS algorithm. Estimates of the σ, k_1, and k_2 parameters are all significantly different from zero. We must hasten to add that these significance tests utilize asymptotic maximum likelihood theory where the standard deviations are derived from the approximated information matrix. Since the ratio of data points to parameters is not very large in this application, it may not be wise to place too much emphasis on these tests.

The estimates of the k_1 and k_2 parameters lend credence to the latitude concepts in our friction model. The Fanny Mae variables (β_5, γ_3) again dominate this analysis as in the OLS, with X_2 (prime rate) and X_4 (Bank B) also significant. These estimates also lend the notion of an asymmetric impact of factors in the raise/lower decisions. These results seem to indicate that pressures to raise mortgage interest rates may involve more diverse factors than pressures to lower them. Since we were unable to obtain data on internal variables specific to this bank, the set of variables included in the model is necessarily incomplete.

[8] A back-forecast of the data utilized to calibrate the ARMA (2, 2) model was not available in the computer program by period. Thus, we cannot present the analog to the bottom of Table 2.

TABLE 4
Friction Model Results

Variable	Parameter	Estimate
Standard Deviation	σ	0.254***
Upper Threshold	k_1	0.295***
Lower Threshold	k_2	−0.225***
Positive Impact		
Discount Rate	β_1	0.531
Prime Rate	β_2	0.739*
Bank *A*'s Rate	β_3	0.305
Bank *B*'s Rate	β_4	0.699**
Fanny Mae Rate	β_5	0.715*
Negative Impact		
Bank *A*'s Rate	γ_1	0.021
Bank *B*'s Rate	γ_2	−0.133
Fanny Mae Rate	γ_3	1.220***

Note. *$p \le 0.10$, **$p \le 0.05$, ***$p \le 0.01$

	Predicted Using Thresholds			
Actual	Raise	Stay the Same	Lower	Total
Raise	8	12	0	20
Stay the Same	1	34	6	41
Lower	0	11	9	20
Total	9	57	15	81

The predictive accuracy results for the 81 observations from the friction model are also shown in Table 4. The sum of squares between actual and predicted was 1.136—slightly lower than the OLS case and less than 50% of that for Box and Jenkins case. The predictive accuracy was 51 ÷ 81 or 63% which again is superior to the OLS results. However, the difficulty in matching particular cells with this model indicates that we perhaps do not have all the factors specified in our model that affect raise and lower decisions (again, this may be due to the lack of information on this bank's internal data). The value of the $-\ln L$ was 30.835; the OLS results produced a value of 42.761 with $k_1 = k_2 = 0$. Utilizing the Akaike Information Criterion (AIC) model selection, the friction model produced a value of 83.67 while OLS produced an AIC of 103.52 indicating the superiority of the friction model according to the minimum AIC rule. However, the R^2 from the friction model (0.538) was a bit lower than that for OLS (0.558); this result is possible since the two procedures optimize different objective functions, which are not necessarily jointly monotonic.

Comparative predictive validation. To further compare the three procedures, predictions were made for eight weeks beyond the data[9] for OLS, the ARMA (2, 2) model, and the friction model. The sum of squares between actual and predicted for the OLS

[9] A two-month period is considered by the management of this bank to be the *maximum* forecast period of interest to them given their planning horizon and the economic uncertainties that are certainly unpredictable for longer term forecasts.

TABLE 5

Comparative Validation Results

(a) OLS

Actual	Predicted: Raise	Stay the Same	Lower	Total
Raise	0	0	1	1
Stay the Same	0	0	3	3
Lower	1	0	3	4
Total	1	0	7	8

(b) OLS with 95% prediction interval

Actual	Predicted: Raise	Stay the Same	Lower	Total
Raise	0	1	0	1
Stay the Same	0	3	0	3
Lower	0	3	1	4
Total	0	7	1	8

(c) ARMA (2,2) Model

	Predicted			
Actual	Raise	Stay the Same	Lower	Total
Raise	1	0	0	1
Stay the Same	0	0	3	3
Lower	3	0	1	4
Total	4	0	4	8

(d) Friction Model

Actual	Predicted: Raise	Stay the Same	Lower	Total
Raise	0	0	1	1
Stay the Same	0	3	0	3
Lower	1	0	3	4
Total	1	3	4	8

and ARMA predictions were 0.298 and 0.292 as compared to 0.277 for the friction model. Table 5 shows the prediction accuracy for OLS, OLS with 95% prediction intervals, ARMA, and the friction model. The prediction accuracy rates are 0.375, 0.500, 0.250, and 0.75 respectively, the highest being that of the friction model.

If we also examine the 95% prediction intervals, as was done in the case of OLS, the prediction accuracy for the ARMA model increases to 100%, as shown in Table 6. However, the size of these predictions are exceedingly large and can accommodate increases, staying the same, and decreases simultaneously. The average length of such an interval exceeds a whole percent which is much larger than the largest change

TABLE 6

ARMA (2, 2) with 95% Prediction Intervals

Time	Upper Limit	Actual	Lower Limit
1	13.81	13.38	13.14
2	14.06	13.25	12.99
3	14.17	13.25	12.85
4	14.34	12.88	12.75
5	14.44	12.88	12.63
6	14.58	13.00	12.54
7	14.67	13.00	12.43
8	14.79	12.75	12.35

witnessed in the entire time series! Thus, these prediction intervals for the ARMA (2, 2) model are excessively wide to be managerially useful.[10]

7. Discussion and Future Research

Our methodology to describe a firm's (a bank) decisions on price changes enables the analyst to incorporate relevant variables that would have an impact on the decision to raise or lower decision by the firm. The inertia that would naturally exist in a price change decision is captured in our model by the two thresholds (i.e., latitudes of acceptance and rejection).

In our empirical application to the movements of secondary mortgage rates for a local bank in the Philadelphia area, the estimated friction model performed somewhat better than the OLS and ARIMA models both in terms of fit and predictions for a holdout sample.

The implementation of the friction model for price changes is significantly facilitated by the existence of a price monitoring system in the firm of the type suggested by Oxenfeldt (1973). Such a monitoring system will keep track of various factors (some of which are shown in Table 1) that could influence a price change decision. Many factors (e.g., competitor marketing effort) may not always be objectively determined and consequently must be either forecasted separately or subjectively estimated. It should be noted that the performance of the friction model is, of course, limited by the quality of these inputs. In cases where the model is being used to analyze a competitor,[11] this problem, as well as that of omitted variables, becomes more extreme. The advantage of the friction model, however, is that it summarizes several influences into a single index and offers a prediction of the future price change.

[10] Note that a variety of other competitive models were also compared with the results of the three models. In particular, we fitted three models: Brown's simple, linear, and quadratic exponential smoothing procedure, Holt's two-parameter linear exponential smoothing procedure, and the following alternative OLS model suggested by a reviewer:

$$P_t \cong a + b_1[DR_{t-1} - DR_{t-2}] + b_2[PR_{t-1} - PR_{t-2}] + b_3[A_{t-1} - A_{t-2}] + b_4[B_{t-1} - B_{t-2}] + b_5[FM_{t-1} - FM_{t-2}].$$

Each of these alternative models was clearly dominated by the performance of each of the three models discussed in the text, and thus were not mentioned. For example, the best fitting (MSE criterion) Brown model predicted a constant value of 13.489 for each of the eight forecast periods. The best (MSE criterion) Holt model ($\alpha = 0.9, \beta = 0.1$) predicted increasing interest rates for every forecast period. The OLS model proposed above had an $R^2 = 0.165$ (adj. $R^2 = 0.109$) with poor forecast accuracy.

[11] Although our methodology was not developed for competitive analysis, our approach can be potentially valuable in modeling competitive behavior, where competitors are assumed to behave according to their historical decision rules. One could substitute the empirical decision rule for the possibly simplifying optimizing rule in building competitive models. The result would likely be something more readily applicable than the game theory models currently in vogue.

Various directions for further work can be identified. First, extensive tests on the behavior of the CRS algorithm will be useful in examining the technical aspects of the constrained random search method. An extension of this model would involve the provision for distinct and unequal slopes to the latent variables representing raise and lower decisions. We believe that while this extension is realistic in capturing the sticky nature of market prices, it is likely to create estimation problems due to the larger number of parameters to be estimated.

A further extension of the model discussed here is incorporation of autocorrelation between successive periods of the error terms of the model. This extension would be useful in modeling those situations where managers are constantly on the alert of competition and use price as a significant competitive tool; an illustration is the price war in the airlines.

Another way of dealing with the issue of autocorrelation is by considering the parameters k_2 and k_1 (and hence the range) to be changing over time for continuous segments of time periods. This situation can occur when the change decisions are essentially adaptive in nature. Our model can be extended to deal with this problem.

We have implicitly specified that the decision period is the same as the smallest time unit for which data are available. Such a specification afforded us to simplify the estimation problem. However, situations do arise when competitors make simultaneous change decisions and where only data aggregated over longer time intervals are available. Development of appropriate extensions to our model to handle these cases should prove to be a fruitful research avenue of the future.

In addition to these directions, we believe that the friction model can be employed in various situations in marketing (as well as other functional areas) where the decision outcome involves a conscious change from the current situation. Illustrations include wholesale and retail price changes, changes in media budgets for a given brand, changes in marketing budgets across brands in a division, changes in allocation of sales effort across territories, changes in allocation of corporate funds across alternative financial instruments and the like. In each specific case, a separate case of factors which impact such changes needs to be developed. Thus, the independent variables, estimation, and testing requirements are application dependent.[12]

[12] This paper was received in December 1985 and has been with the authors for 2 revisions.

Appendix I. Details of the CRS Algorithm

Figure 2 provides a flow chart summarizing the essential features of this modified CRS algorithm. An initial region for search, V, is defined by the user by specifying upper and lower limits to the domain of each of the parameters to be estimated that are consistent with any user specified constraints. These limits are then "sharpened" for the β and γ parameters by first running OLS and taking $(\beta, \gamma)_{OLS} \pm L$ (where the default value of L is 2). We find that using OLS estimates as a method of specifying the search region accelerates convergence of the method. The user also specifies the number of trial points, N (default is $N = 100$), that are chosen at random over the search region, V, consistent with the constraints. The particular constraints that are enforced in this algorithm are: (a) $\sigma > 0$; (b) $k_1 \geq 0$; (c) $k_2 \leq 0$. Options exist in the algorithm to constrain $\beta \geq 0$ and/or $\gamma \geq 0$ given the potential problem of a negative "raise" or "lower" latent variables.

For each of the N random trial points, the negative of the log likelihood function value (we are minimizing the negative of expression (12)) is calculated and stored with these N points in an array, A. We use Hasting's (1955) approximation for the normal cumulative distribution function. At each iteration, a new trial point, P, is selected/calculated randomly from a set of possible trial points. To calculate P, $n + 1$ distinct points (n = the total number of parameters to be estimated) $R_1, R_2, \ldots, R_{n+1}$ are chosen at random from the N ($N \gg n$) points in storage; this constitutes a simplex of points in n-space. The point R_{n+1} is arbitrarily taken as the designated vertex or pole of the simplex and the next trial point, P, is defined as $P = 2 \times G - R_{n+1}$, where G is the centroid of the remaining n points; P then becomes the image point of the pole with respect to the centroid. If P does not satisfy the designated set of constraints, a new simplex is calculated via another random set of $n + 1$ points and a new value of P is calculated according to the above expression. Assuming P satisfies all the constraints,

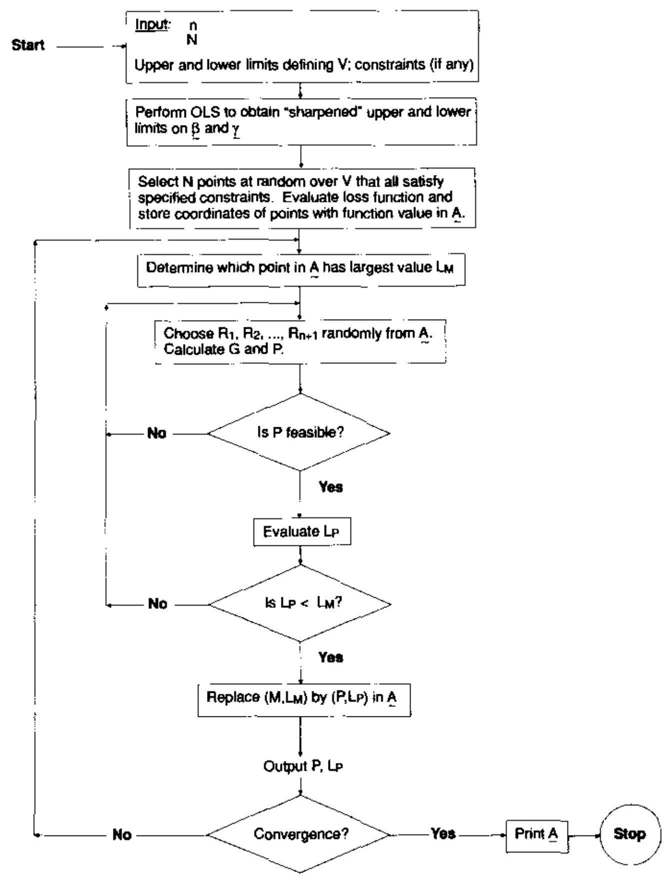

FIGURE 2. Flow Chart of the CRS Algorithm.

the negative log likelihood function is evaluated at P and the function value, L_p, is compared with the L_M, M being the point which has the largest negative log likelihood function value of N points presently stored in **A**. If $L_P < L_M$ then M is replaced, in A, by P. If either P fails to satisfy the constraints or $L_P > L_M$, then P is discarded and a fresh point is calculated from the potential trial set. Thus, as the algorithm proceeds, the current set of N stored points tend to cluster around minima which are lower than the current value of L_M. Convergence occurs when the variance of the N values of the loss function stored in **A** are below some critical value (e.g., 1% of the mean value) assuming that the N points remaining all fall within a sufficiently small region of the n-space around the global minima.

Appendix II. Monte Carlo Analysis of the CRS Algorithm

In order to test the performance of the CRS algorithm, a synthetic data set was created according to the friction model. **X** and **Z** were randomly generated with 50 observations and 3 variables in **X** and 3 in **Z**. Values for the model parameters, namely, σ, k_1, k_2, γ, and β were also randomly generated. These are shown in Table 7.

Values of r_t and l_t were then created from the **X** and **Z** variables and the parameter values in γ and β. The latent variable $f_t = r_t - l_t$ was calculated, and then the vector of price changes, $\Delta\mathbf{P}_t$, was created by comparing f_t with k_1 and k_2. Table 7 also presents the value of the $-$log likelihood, R^2, and sum of squares using the "true" values of the parameters used to generate the data, $\Delta\mathbf{P}_t$.

The CRS algorithm was applied to the data on $\Delta\mathbf{P}_t$, **X** and **Z** to estimate σ, k_1, k_2, β and γ. The ordinary least squares procedure (OLS) was first used to generate the initial search region V (see Appendix I). Table 7 also shows the estimates obtained via OLS. There is a wide disagreement between the OLS estimates and the true values; this result might be expected since these data were purposely generated from the structure of a friction model. (Note that OLS method assumes that $k_1 = k_2 = 0$.)

TABLE 7

A Comparison of Estimates for OLS and CRS Algorithms for Synthetic Data

Parameter of the Model	True Values Used for Synthetic Data	OLS Estimates[1]	CRS Estimates
β_1	0.508	0.178	0.513
β_2	0.158	0.032	0.134
β_3	1.018	0.198	1.067
γ_1	0.492	0.101	0.552
γ_2	0.594	0.215	0.627
γ_3	0.952	0.186	0.893
k_1	1.00	—	1.023
k_2	−1.25	—	−1.272
σ	0.10	0.210	0.047
Intercept	—	0.23	—
$-\ln L$	−23.021	—	−31.026
R^2	0.999	0.572	0.995
F	—	9.563	—
Sum of Squares	0.036	2.18	0.152

[1] In order to be consistent with the notation of the friction model, we show the estimates of γ's, which are the negatives of the computed regression coefficients in the OLS model.

The CRS estimates of the parameters also shown in the table are quite close, but not exactly the same, as the true values. (This divergence is to be expected since we are dealing with a stochastic model.) It is interesting to note that the CRS procedure actually converged on a solution with smaller $-\ln L$ value than the one computed using the true values of parameter. In the CRS algorithm, the statistics of $-\ln L$ and R^2 are not jointly inversely monotonic; this feature can also be seen in Table 7 by comparing these statistics for the true parameters and for CRS estimates.

This same procedure was repeated on another data set where an error term was added to f_t prior to generating the values of ΔP_t; the error was assumed to be distributed according to a standard normal density function. The results (not shown here) also indicate that the friction model outperformed OLS in reproducing the "true" estimates of parameters. While these two small tests give us some confidence in applying the algorithm to real problems, we believe that extensive Monte Carlo analyses are required in order to rigorously explore the performance of the algorithm as a number of data points, model, and algorithm factors are systematically varied. For example, an experimental design could be devised which varied such factors as: the number of observations, the number of parameters to be estimated, constrained vs. unconstrained specifications, the size of N in the algorithm, and various convergence rules.

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