A NEW, SOLVABLE, PRIMAL RELAXATION FOR NONLINEAR INTEGER PROGRAMMING PROBLEMS WITH LINEAR CONSTRAINTS

by

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Abstract

This paper describes a new *primal relaxation* for nonlinear integer programming problems with linear constraints. This relaxation, contrary to the standard Lagrangean relaxation, can be solved efficiently. It requires the solution of a nonlinear penalized problem whose linear constraint set is known only implicitly, but whose solution is made possible by the use of a linearization method (see for instance Gunn and Rai [12]). It can be solved iteratively by a sequence of linear integer programming problems and nonlinear problems over a simplex. The relaxation should be designed so that the linear integer programming problems are relatively easy to solve. These subproblems yield Lagrangean-like integer solutions that can be used as starting points for Lagrangean heuristics. We also describe a primal decomposition, similar in spirit to Lagrangean decomposition, for problems with several structured subsets of constraints. A small example is solved explicitly in each case by an extension of the linearization method of

Frank and Wolfe [7], similar in spirit to Simplicial Decomposition (von Hohenbalken, [17]).

The primal relaxation was introduced in Guignard [9], and described briefly in Guignard [10]. Improved solution methods are studied in Contesse and Guignard [5] and [6], and successful implementations are described in [1], [2], [3] and [4]. This paper by contrast concentrates on the concept of primal relaxation and its properties.

In the linear case, the primal relaxation is equivalent to Lagrangean relaxation.

Introduction

Lagrangean relaxation [14], [15], has been used for decades as a helpful tool in solving difficult integer programming problems. Its main advantages over the continuous relaxation are that (1) it may yield a tighter bound than the continuous relaxation if the subproblems do not have the Integrality Property, and (2) it produces integer, rather than fractional, solutions that are often only mildly infeasible, and therefore can be used as good starting points for Lagrangean heuristics. While one usually solves the Lagrangean dual in the dual space by searching for a best set of Lagrangean multipliers, this is not the only method possible. Michelon and Maculan [16] showed that one can also solve the primal equivalent of the Lagrangean dual by placing the relaxed constraints into the objective function with a large penalty coefficient, and then using a linearization method such as Frank and Wolfe to solve the resulting nonlinear problem. A key realization here is an idea that had already been used in particular by Geoffrion [8]: when one maximizes a linear function over the integer points of a bounded polyhedron, one optimal solution at least is an extreme point of the convex hull of these integer points. Once a nonlinear objective function is linearized, one can therefore equivalently optimize it over the integer points of a bounded polyhedron, or over the convex hull of these integer points, whichever is easier. In the case of Michelon and Maculan's approach, then, each iteration of Frank and Wolfe involves solving a linear integer Lagrangean-like subproblem and performing a nonlinear line search.

Consider now the case of an integer problem with linear constraints and a nonlinear convex objective function. It is usually very difficult to obtain strong bounds for such problems. Indeed it is not easy to use standard Lagrangean relaxation in this case, as the Lagrangean subproblems are still nonlinear integer problems, and a priori not easier to solve than the original problem. We introduced in 1994 in an unpublished report [9] and describe again here a novel relaxation, which is primal in nature, and can be used with nonlinear objective functions and linear constraints[1]. It coincides with the standard Lagrangean relaxation in the linear case, but it is new for the nonlinear case, and it is computationally feasible. It can be solved for instance by penalizing some constraints in the objective function, and then using a linearization method (see Gunn and Rai [12] and Michelon and Maculan [16]). At each iteration of the algorithm, one solves a linear integer programming problem over the remaining constraints. and one performs either a simple line search if using Frank and Wolfe [7], or a search over a simplex, in the case of simplicial decomposition [17]. There are no more *nonlinear integer* subproblems to solve. We show that the bound obtained in this manner is at least as good as the continuous relaxation bound, and may be substantially stronger.

This relaxation is very attractive, as its implementation requires solving integer subproblems that are *linear* and for which one can select a good (or several for primal decomposition) structured subset(s) of constraints, exactly as in Lagrangean relaxation for *linear* integer programming problems. Finally there are better choices than a penalty method for solving the relaxation, and Contesse and Guignard [5],[6], propose instead to

[1] It is also briefly described in [10].

use a (Proximal) Augmented Lagrangean (PAL) scheme, for its improved convergence and conditioning properties.

Notation

For an optimization problem (P), FS(P) denotes the feasible set, V(P) the optimal value and OS(P) the optimal set of (P). If (P) is a (mixed-)integer programming problem, CR(P) (or (CR) if it is not ambiguous) denotes the continuous relaxation of (P). If K is a set in $\Re^n$, Co(K) denotes the convex hull of K. If x is a vector of $\Re^n$, $|x|$ denotes a norm of x.

1. Primal Equivalent of Lagrangean Relaxation for Linear Integer Problems

We shall first recall Michelon and Maculan's approach [16] for solving Lagrangean duals in the linear integer problem case. Consider a *linear* integer programming problem

(LIP) $\quad \text{Min}_x \{ fx \mid Ax=b,\ Cx\le d,\ x\in X\}$

where X specifies in particular the integrality requirements on x, and a Lagrangean relaxation of (LIP) :

LR(u) $\quad \text{Min}_x \{fx+u(Ax-b) \mid Cx\le d,\ x\in X\}$

with the corresponding Lagrangean dual

(LR) $\quad \text{Max}_u\ \text{Min}_x\ \{fx+u(Ax-b) \mid Cx\le d,\ x\in X\}$

and its primal equivalent problem (Geoffrion [8])

(PLR) $\quad \text{Min}_x\ \{fx \mid Ax=b,\ x\in \text{Co}\{x|Cx\le d,\ x\in X\}\}$.

As ρ approaches infinity, (PLR) becomes equivalent to the penalized problem

(PP) $\quad \text{Min}_x\ \{\varphi(x) = fx + (½)\,\rho\,|Ax-b|^2 \mid x\in\text{Co}\{ x|Cx\le d,\ x\in X\}\}$.

Notice that $\varphi(x)$ is a convex function. (PP) can be solved by a linearization method such as the method of Frank and Wolfe or, even better, simplicial decomposition. For simplicity, let us describe the approach using Frank and Wolfe. At iteration k, one has a current iterate $x(k)$ in whose vicinity one creates a linearization of the function $\varphi(x)$:

$\psi_k \,.\, x = \varphi\,[\,x(k)] + \nabla\varphi\,[x(k)].[x-x(k)]$.

One solves the linearized problem

(LPP$_k$) $\quad$ $\text{Min}_x \,\{\psi_k\, x \mid x \in \text{Co}\{\, x | Cx \leq d,\, x \in X\}\}$

or equivalently, because the objective function is linear,

(LPP$_k$) $\quad$ $\text{Min}_x \,\{\psi_k\, x \mid Cx \leq d,\, x \in X\}$.

Let $y(k)$ be its optimal solution. Then $x(k+1)$ is obtained by minimizing $\varphi(x)$ on the half-line $x=x(k)+\lambda\,[y(k)-x(k)]$, $\lambda \geq 0$. The process is repeated until either a convergence criterion is satisfied or a limit on the iteration number is reached.

The idea is attractive because

(1) while one cannot eliminate the convex hull in (PP), one can do so after the linearization, i.e., the convex hull computation is not necessary any more after (PP) has been transformed into a sequence of problems (LPP$_k$). Notice too that (LPP$_k$) has the same constraint set as LR(u), i.e., it must be solvable if (LR(u)) is.

(2) even in case (LR(u)) decomposes into a family of smaller subproblems, this is usually not the case for (PP). (LPP$_k$), though, will also decompose, and the primal approach is fully as attractive as the original Lagrangean relaxation.

The slow convergence of Frank and Wolfe's algorithm, however, may make one prefer a faster linearization method, such as simplicial decomposition [17], or restricted simplicial decomposition [13].

2.Primal Relaxation for Nonlinear Integer Programming Problems

Consider now an integer programming problem with a nonlinear convex objective function and linear constraints

(IP) $\quad \text{Min}_x \ \{f(x) \mid Ax=b,\ Cx \le d,\ x \in X\}$.

We could try to solve (IP) directly by noticing that as ρ goes to infinity, (IP) becomes equivalent to

(P1) $\quad \text{Min}_x \ \{f(x) + (½)\,\rho\, \|Ax-b\|^2 \mid Cx \le d,\ x \in X\}$.

Unfortunately (P1) is almost always as difficult to solve as (IP). The constraint set of (P1) is not a polygon, and the objective function of (P1) is still nonlinear. We could consider problem (P2) $\quad \text{Min}_x \ \{ f(x) + (½)\,\rho\, \|Ax-b\|^2 \mid x \in \text{Co}\{x \mid Cx \le d,\ x \in X\}\}$.

which is a relaxation of (P1), but in general is not equivalent to (P1) because the optimal solution of (P2) is not necessarily an extreme point of FS(P2) and thus not necessarily a point in FS(P1).

Since in any case neither (P1) nor (IP) is easy to solve, we will build a *new* primal relaxation of (IP) which will use (P2) as a subproblem. We will then show in detail how the relaxed problem can actually be solved.

We will more specifically show that if a linear integer programming problem of the form

(LIP) $\quad \text{Min}_x \ \{gx \mid Cx \le d,\ x \in X\}$

can be solved relatively easily, then we can design a relaxation approach similar to the one described above for the integer *linear* case.

2.1.Definition of the Primal Relaxation.

We now formally define the new relaxation.

Definition 1.

We define the *Primal Relaxation* problem as the problem

(PR) $\quad \text{Min}_x \ \{f(x) \mid Ax=b,\ x\in Co\{x \mid Cx\leq d,\ x\in X\}\}.$

See Figure 1.

(PR) is indeed a relaxation of (IP):

$\{x \mid Ax=b,\ x\in \text{Co}\{x \mid Cx\leq d,\ x\in X\}\} \supseteq \{x \mid Ax=b,\ Cx\leq d,\ x\in X\},$

and the so-called *continuous* relaxation of (IP), (CR) $\text{Min}_x \ \{f(x) \mid Ax=b,\ Cx\leq d,\ x\in \text{Co}(X)\},$

is itself a relaxation of (PR), since

$\{ x\in \text{Co}(X) \mid Ax=b,\ Cx\leq d \} \supseteq \{ x\in \text{Co}\{x\in X \mid Cx\leq d \} \mid Ax=b\}.$

(PR) cannot in general be solved directly, since Co{ $x\in X \mid Cx\leq d$} is usually not known explicitly, and even if it were, (PR) would probably be of the same level of difficulty as (IP).

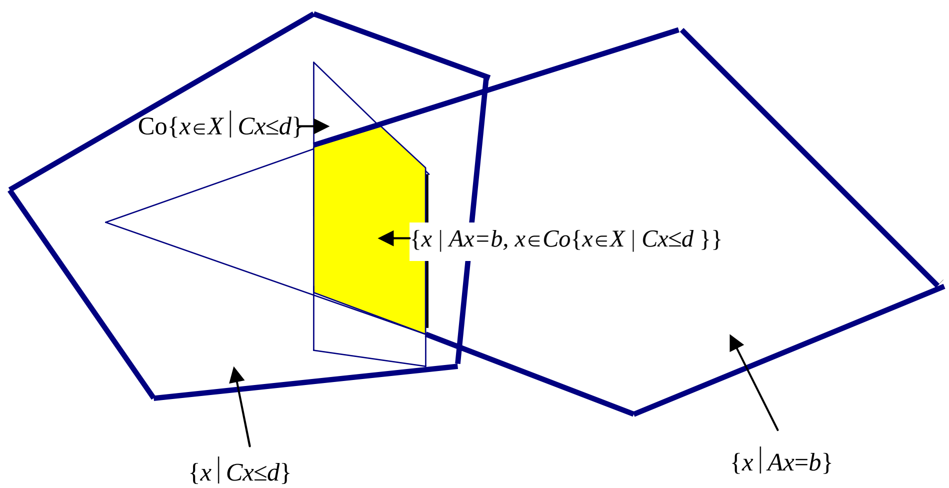


Figure 1

Roughly speaking, though, for ρ large enough, (PR) is equivalent to the penalized problem

(PP) $\quad \text{Min}_x \{\varphi(x) = f(x)+\rho||Ax-b||^2 \mid x\in \text{Co}\{x\in X \mid Cx\le d\}\}$,

where $\varphi(x)$ is a convex function. (PP) can be solved by a linearization method such as Frank and Wolfe. This method unfortunately is known to converge rather slowly. Another linearization method, called Simplicial Decomposition, could be used as well, and the overall convergence would be improved further if one used an augmented Lagrangean method instead of the penalization method described above. Such an approach is studied in Contesse and Guignard [5],[6], and successful implementations are described in [1] and [2], and in recent papers by Ahn, Contesse and Guignard [3] and Ahlatcioglu and Guignard [4].

2.2 Properties of the Primal Relaxation.

We concentrate in this paper on the characteristics of the primal relaxation and not on algorithmic details or on obtaining an efficient implementation. This is why we choose to describe the approach based on Frank and Wolfe's linearization method, to illustrate the relaxation and the general idea of its solution, rather than a more efficient linearization method, such as simplicial decomposition.

At iteration k, one has a current iterate $x(k)$ in whose vicinity one creates a linearization of the function $\varphi(x)$:

$$\psi_k . x = \varphi[x(k)]+ \nabla\varphi\, [x(k)][x-x(k)].$$

One solves the linearized problem

(LP_k) $\quad \text{Min}_x \{\psi_k.x \mid x\in \text{Co}\{x \mid Cx \le d, x\in X\}\}$

or equivalently, because the objective function is linear,

(LP_k) $\quad \text{Min}_x \{\psi_k.x \mid Cx \le d, x\in X\}$.

Let $y(k)$ be its optimal solution. Then $x(k+1)$, the new linearization point, is obtained by minimizing $\varphi(x)$ on the half-line $x=x(k) + \lambda\, [y(k)-x(k)]$, $\lambda\ge 0$. The process is repeated until either a convergence criterion is satisfied or a limit on the iteration number is reached. This process has roughly the same advantages as in the linear case:

(1) while one cannot eliminate the convex hull in (PP), one can do it for (LP_k). We made the assumption earlier that a problem with a structure such as (LP_k) is solvable.

(2) in case the constraints of (IP) decompose into a family of smaller subproblems if the constraints $Ax=b$ are removed, this property allows (LP_k) to decompose as well, even though this is not the case for (PP). The linearization of the objective function thus allows one to solve the problem via a sequence of decomposable linear integer programs and line searches. This is very attractive if it reduces substantially the size of the integer problems one has to solve. It is usually much easier to solve ten problems with thirty 0-1 variables each than a single problem with three hundred 0-1 variables.

One can also handle the case of inequality constraints $Ax\leq b$ with some minor modification (see [16]).

2.3. A special case.

As in standard Lagrangean relaxation, the "extreme" case of subproblems with the Integrality Property will not yield any improvement over the continuous nonlinear programming relaxation.

Proposition 1.

If in the *Primal Relaxation* problem

(PR) $\quad \text{Min}_x \ \{f(x) \mid Ax=b,\ x\in Co\{x \mid Cx\leq d,\ x\in X\}\}$,

the polyhedron P = $Co\{x \mid Cx\leq d,\ x\in X\}$ coincides with the set $\{x \mid Cx\leq d,\ x\in X\}$, then

v(PR) = v(CR).

Although, as in the linear case, one will not be able then to use primal relaxation to obtain a stronger bound than (CR), one might still want to use it if solving (CR) requires using an exponential number of constraints. One such example is the TSP, for which Held and Karp [14], [15], showed that Lagrangean relaxation was nevertheless an attractive option.

2.4. An Example.

The following example illustrates that the bound v(PR) can be anywhere between V(IP) and V(CR), depending on the problem parameters, as happens for standard Lagrangean relaxation bounds.

Consider the following very simple 2-dimensional problem (see Figure 2). One wants to minimize the distance to the point A(1,1) subject to the constraints $x_1 = 2x_2$ and $ax_1 + bx_2 \le c$, where x_1 and x_2 are (0-1) variables. We will write z(M) to denote the value of the objective function at the point M(x_1,x_2). The problems under consideration are:

(IP) Min $(1-x_1)^2 + (1-x_2)^2$	(PR) Min $(1-x_1)^2 + (1-x_2)^2$	(CR) Min $(1-x_1)^2 + (1-x_2)^2$
s.t. $x_1 - 2x_2 = 0$	s.t. $x_1 - 2x_2 = 0$	s.t. $x_1 - 2x_2 = 0$
$ax_1 + bx_2 \le c$	$x \in \mathrm{Co}\{x \mid ax_1 + bx_2 \le c,$	$ax_1 + bx_2 \le c$
$x_1, x_2 \in \{0,1\}$	$x_1, x_2 \in \{0,1\}\}$	$x_1, x_2 \in [0,1]$

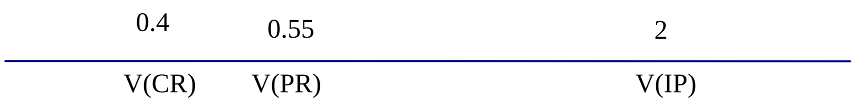
We will place $x_1 = 2x_2$ in the objective function as a penalty term. We will consider several cases :

1. a=10, b=1, c=9. Then Co $\{x \mid 10x_1 + x_2 \le 9,\ x_1, x_2 \in \{0, 1\}\}$ is OD, and $\{x \mid Ax=b,\ x \in \mathrm{Co}\{x \mid Cx \le d,\ x \in X\}\}$ is O. Thus V(PR) = V(IP) = z(O) = $(1-0)^2 + (1-0)^2 = 2$, while V(CR) is reached at P(18/21, 9/21) and is equal to z(P) = 0.35.

0.35 — V(CR)

2 — V(PR)=V(IP)

2. a=2, b=1, c=2. Then Co $\{x \mid 2x_1 + x_2 \le 2,\ x_1, x_2 \in \{0,1\}\}$ is ODF, and $\{x \mid Ax=b,\ x \in \text{Co}\{x \mid Cx \le d,\ x \in X\}\}$ is OS. Thus V(IP) = $(1-0)^2 + (1-0)^2 = 2$, while V(PR) = z(S) = $(1-2/3)^2+(1-1/3)^2$ and V(CR) is reached at Q (2/5, 4/5) and is equal to z(Q) = $(1-4/5)^2 + (1-2/5)^2$ =0.4.



3. a=1, b=1, c=1. Then Co $\{x \mid x_1 + x_2 \le 1,\ x_1, x_2 \in \{0, 1\}\}$ is ODF, and $\{x \mid Ax=b,\ x \in \text{Co}\{x \mid Cx \le d,\ x \in X\}\}$ is OS. Thus V(IP) = $(1-0)^2 + (1-0)^2 = 2$, while V(PR) = V(CR) = z(S) = $(1-2/3)^2 +(1-1/3)^2$ = 0.55.

It can be seen on the above examples that the value of V(PR) can be arbitrarily close to either the integer optimum or the continuous optimum. This is rather similar to what happens for Lagrangean relaxation bounds in linear integer programming.

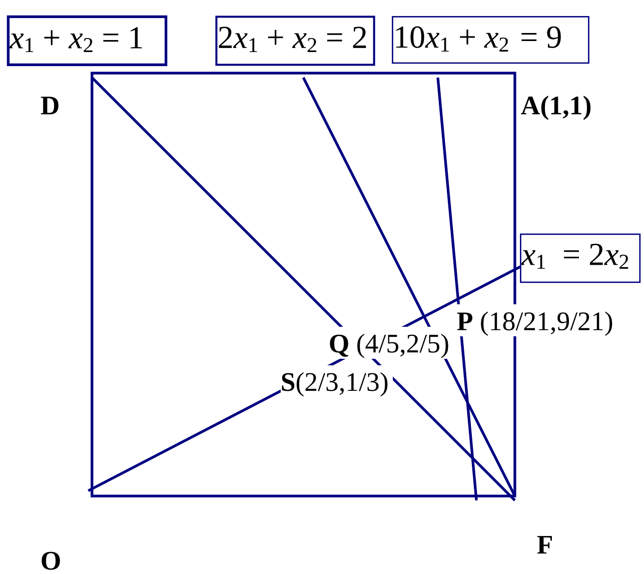


Figure 2

3. Primal Decomposition for Nonlinear Integer Programming Problems

We will now show that one can similarly define a primal decomposition, similar in spirit to that described for instance in Guignard and Kim [11].

Consider an integer programming problem with a nonlinear objective function and linear constraints in which one has replaced x by y in some of the constraints, after adding the copy constraint $x=y$:

(IP) $\quad \mathrm{Min}_x \ \{f(x) \mid Ay \le b,\ y \in X,\ Cx \le d,\ x \in X,\ x=y\}$.

We will show that if *linear* integer programming problems of the form

(LIP_x) $\quad \mathrm{Min}_x \ \{gx \mid Cx \le d,\ x \in X\}$

and

(LIP_y) $\quad \mathrm{Min}_y \ \{hy \mid Ay \le b,\ y \in X\}$

can be solved relatively easily, then we can design a primal decomposition approach similar to the primal relaxation approach described above. A related decomposition idea for continuous problems can be found in [12], in which equality constraints linking variables are placed in the objective function to form an augmented Lagrangean, and the resulting problem is solved by the Frank and Wolfe algorithm, allowing the objective function to decompose thanks to the linearization. In our case, the linearization procedure allows us in addition to replace *linear programs* with *implicitly* defined polyhedral constraint sets by *linear integer programs* with well structured discrete constraint sets. If we applied the decomposition idea directly to (IP), we would obtain a nonlinear integer program for which the Frank and Wolfe algorithm would be meaningless. This is why we consider a convex hull relaxation of the constraint set *first* before introducing a penalty function.

3.1. Definition of Primal Decomposition.

We define the **primal decomposition** of problem (IP) to be problem

(PD) $\quad \text{Min}_x \ \{f(x) \mid x \in \text{Co}\{x \mid Ax \le b, x \in X\} \cap \text{Co}\{x \mid Cx \le d, x \in X\}\}$.

See Figure 3.

Problem (PD) is indeed a relaxation of (IP), since

$\text{Co}\{x \mid Ax \le b, x \in X\} \cap \text{Co}\{x \mid Cx \le d, x \in X\} \supseteq \{x \mid Ax \le b, Cx \le d, x \in X\}$.

At the same time, problem

(PR) $\quad \text{Min}_x \ \{f(x) \mid Ax \le b, x \in Co\{x \mid Cx \le d, x \in X\}\}$.

is a relaxation of (PD), since

$\{x \mid Ax \le b, x \in Co\{x \mid Cx \le d, x \in X\} \supseteq \text{Co}\{x \mid Ax \le b, x \in X\} \cap \text{Co}\{x \mid Cx \le d, x \in X\}$

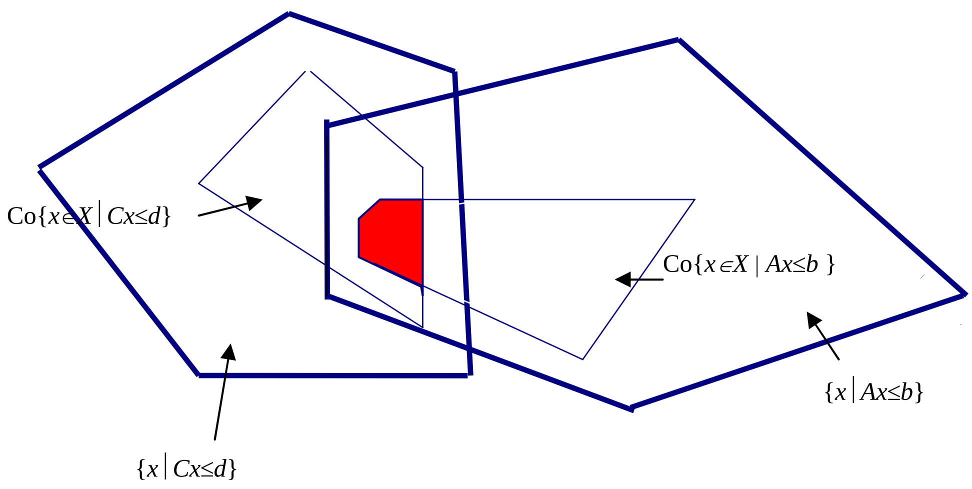


Figure 3

and finally problem

(CR) $\quad \text{Min}_x \ \{ f(x) \mid Ax \le b,\ Cx \le d,\ x \in \text{Co}(X)\}$,

the so-called *continuous* relaxation of (IP), is itself a relaxation of (PD), since

$\{x \mid Ax \le b,\ Cx \le d,\ x \in \text{Co}(X)\} \supseteq \text{Co}\{x \mid Ax \le b,\ x \in X\} \cap \text{Co}\{x \mid Cx \le d,\ x \in X\}$

(PD) cannot in general be solved directly, since on the one hand $\text{Co}\{x \mid Cx \le d,\ x \in X\}$ and $\text{Co}\{x \mid Ax \le b,\ x \in X\}$ are usually not known explicitly, and on the other hand, even if they were, (PD) would probably be of the same level of difficulty as (IP).

Again, roughly speaking, for ρ large enough, (PD) is equivalent to the penalized problem

(PP) $\quad \text{Min}_x \ \{\varphi(x,y) = f(x) + \rho \,|x-y|^2 \ \mid\ y \in \text{Co}\{y \mid Ay \le b,\ y \in X\},\ x \in \text{Co}\{x \mid Cx \le d,\ x \in X\}\}$.

(PP) can be solved by a linearization method. We describe here the approach based on Frank and Wolfe. At iteration k, one has a current iterate $(x(k),y(k))$ in whose vicinity one creates a linearization of the function $\varphi(x,y)$:

$$\psi_k \,.\, (x,y) = \varphi[x(k),y(k)] + \nabla\varphi[x(k),y(k)].[x-x(k),y-y(k)].$$

One solves the linearized problem

(LP_k) $\quad \text{Min}_{x,y} \ \{\psi_k(x,y) \mid x \in \text{Co}\{x \mid Cx \le d,\ x \in X\},\ y \in \text{Co}\{y \mid Ay \le b,\ y \in X\}\}$

which separates as follows, because the objective function is linear:

(LP_k) $\quad \text{Min}_x \ \{\psi^{x}{}_{k} . x \mid Cx \le d,\ x \in X\} + \text{Min}_y \ \{\psi^{y}{}_{k} . y \mid Ay \le b,\ y \in X\}$.

and again the relaxed problem separates into two linear subproblems of a type which we assumed we can solve. Decomposition in this case is achieved at each iteration of Frank and Wolfe where LP's with implicit constraints are replaced by IP's with a good structure.

3.2. An Example

Consider again the very simple example considered earlier. One wants to minimize the distance to the point A (1,1) subject to the constraints $x_1 = 2x_2$ and $ax_1 + bx_2 \le c$, where x_1 and x_2 are (0-1) variables. The problems under consideration are:

(IP)	(PD)	(CR)
$\text{Min } (1-x_1)^2 + (1-x_2)^2$	$\text{Min } (1-x_1)^2 + (1-x_2)^2$	$\text{Min } (1-x_1)^2 + (1-x_2)^2$
s.t. $x_1 - 2x_2 = 0$	s.t. $x \in \text{Co}\{x \mid x_1 - 2x_2 = 0$	s.t. $x_1 - 2x_2 = 0$
$ax_1 + bx_2 \le c$	$x_1, x_2 \in \{0,1\}\}$	$ax_1 + bx_2 \le c$
$x_1, x_2 \in \{0,1\}$	$x \in \text{Co}\{x \mid ax_1 + bx_2 \le c,$	$x_1, x_2 \in [0,1]$
	$x_1, x_2 \in \{0,1\}\}$	

We will call z(M) the value of the objective function at M(x_1,x_2).

We will reformulate (PD), creating a copy y of the variable x and adding the constraint $x = y$.

We will place $x = y$ in the objective function as a penalty term. We will consider several cases :

1. a=10, b=1, c=9. Then Co $\{x \mid 10x_1 + x_2 \le 9, x_1, x_2 \in \{0, 1\}\}$ is OD, and Co$\{x \mid Ax \le b , x \in X\}$ is O. Thus V(PR) = V(IP) = z(O) = $(1-0)^2 + (1-0)^2 = 2$, while V(CR) is reached at P(18/21, 9/21) and is equal to z(P) = 0.35.

0.35	2
V(CR)	V(PR)=V(IP)=V(PD)

2. a=2, b=1, c=2. Then Co $\{x \mid 2x_1 + x_2 \le 2, x_1, x_2 \in \{0,1\}\}$ is ODF, and Co$\{x \mid Ax \le b, x \in X\}$ is O. Thus V(IP) = $(1-0)^2 + (1-0)^2 = 2$ = z(O) = V(PD), while V(PR) = z(S) = $(1-2/3)^2+(1-1/3)^2$ and V(CR) is reached at Q (2/5, 4/5) and is equal to z(Q) = $(1-4/5)^2 + (1-2/5)^2$ =0.4.

0.4	0.55	2
V(CR)	V(PR)	V(IP)=V(PD)

3. a=1, b=1, c=1. Then Co $\{x \mid x_1 + x_2 \leq 1, x_1, x_2 \in \{0, 1\}\}$ is ODF, and Co$\{x \mid Ax \leq b, x \in X\}$ is O. Thus V(IP) = $(1\text{-}0)^2 + (1\text{-}0)^2$ = 2=V(PD), while V(PR) = V(CR) = z(S) = $(1\text{-}2/3)^2 + (1\text{-}1/3)^2$ = 0.55.

. 55 2

V(CR)=V(PR) V(IP)=V(PD)

It can be seen on the above examples that the value of V(PD) can be equal to the integer optimum, even when V(PR) is equal to the continuous optimum, V(CR) is always weaker than V(PR) which is itself weaker than V(PD), given than FS(CR) contains FS(PR) which in turn contains FS(PR) which itself contains FS(PD).

4. Bound computation: an example

We will now consider a three dimensional example on which we will demonstrate what bound computation involves. We shall use a slight modification of the algorithm of Frank and Wolfe, in which instead of a one-dimensional line search one performs a 2-dimensional triangular search in the triangle formed by the current linearization point and the last two solutions of linearized subproblems. It is actually almost a form of restricted simplicial decomposition [13].

The problem, represented in figure 4, is as follows:

Min $\{(2-x_2)^2 \mid x_1 - 2x_2 + x_3 = 0,\ 10x_1 + x_2 - x_3 \leq 9,\ x_1, x_2, x_3 \in \{0,1\}\}$.

In (PR) and (PD), we let $x_1 - 2x_2 + x_3 = 0$ stand for $Ax \leq b$, and $10x_1 + x_2 - x_3 \leq 9$ for $Cx \leq d$.

That is,

(IP) Min $(2-x_2)^2$

$$\text{s.t. } x_1 - 2x_2 + x_3 = 0$$
$$10x_1 + x_2 - x_3 \leq 9$$
$$x_1, x_2, x_3 \in \{0,1\}$$

(PD) Min $(2-x_2)^2$

$$\text{s.t. } x \in \text{Co}\{x \mid x_1 - 2x_2 + x_3 = 0, \text{ and } x_1, x_2, x_3 \in \{0,1\}\}$$
$$x \in \text{Co}\{x \mid 10x_1 + x_2 - x_3 \leq 9, \text{ and } x_1, x_2, x_3 \in \{0,1\}\}$$

(CR) Min $(2-x_2)^2$

$$\text{s.t. } x_1 - 2x_2 + x_3 = 0$$
$$10x_1 + x_2 - x_3 \leq 9$$
$$x_1, x_2, x_3 \in [0,1]$$

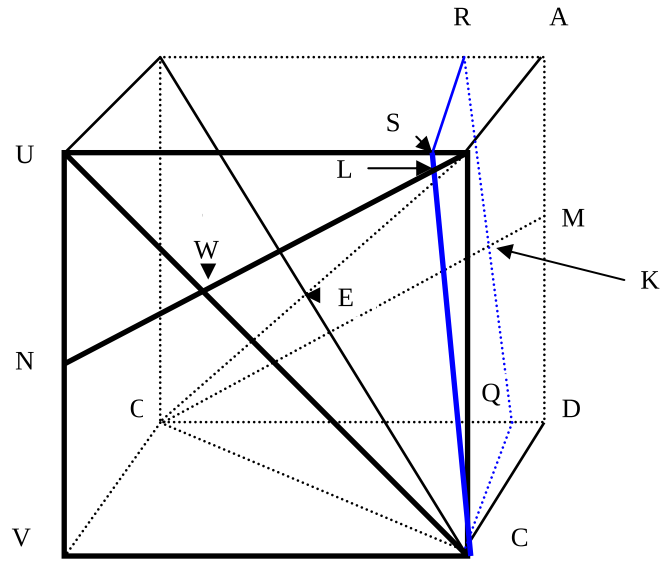


Figure 4

and (PR) Min $(2-x_2)^2$

$$\text{s.t. } x_1 - 2x_2 + x_3 = 0$$
$$x \in \text{Co}\{x \mid 10x_1 + x_2 - x_3 \leq 9, x_1, x_2, x_3 \in \{0,1\}\}.$$

Then FS(CR) = OKLN, FS(PR) = OEWN, FS(PD) = OE, and FS(IP)=O, and

V(CR) = z (L) = 1.1, V(PR) = z (W) =1.7, V(PD) = z (E) =2.25, V(IP) = z (O) =4.

V(CR)	V(PR)	V(PD)	V(IP)
1.1	1.7	2.25	4

We will show the computation for (PR):

(PR) Min $(2-x_2)^2$

s.t. $x_1 - 2x_2 + x_3 = 0$

$x \in \mathrm{Co}\{x \mid 10x_1 + x_2 - x_3 \le 9,\ x_1, x_2, x_3 \in \{0,1\}\}$.

(PR) is asymptotically equivalent, as ρ goes to infinity, to

Min $\varphi(x) = (2-x_2)^2 + \rho\,(x_1 - 2x_2 + x_3)^2$

s.t. $x \in \mathrm{Co}\,\{x \mid 10x_1 + x_2 - x_3 \le 9,\ x_1, x_2, x_3 \in \{0,1\}\}$.

The linearization of the objective function at $x^{(0)}$ yields the function

$[2\rho\,(x_1 - 2x_2 + x_3),\ -2\,(2-x_2) - 4\rho\,(x_1 - 2x_2 + x_3),\ 2\rho\,(x_1 - 2x_2 + x_3)]_{x=x0}\ [x_1, x_2, x_3]$

The initial point, $x^{(1)} = (0.5, 1, 0)$, is chosen arbitrarily. The slack in the equality constraint at $x(1)$, i.e., the amount of violation in the penalized constraint, is $s(1) = -1.5$. The first linearized problem is

Min $-3\rho\, x_1 + (-2 + 6\rho)\, x_2 - 3\rho\, x_3$

s.t. $10x_1 + x_2 - x_3 \le 9$,

$x_1, x_2, x_3 \in \{0,1\}$.

We choose to take $\rho = 5000$.

Iteration 1

The gradient at $x(1)$ is (-15000, 29998, -15000). The solution of the linearized problem is $y(1)$= (1, 0, 1). Since this is the first iteration, one only does a line search, in the direction $da(1) = y(1) - x(1) = (0.5, -1, 1)$. The line search yields a stepsize of 0.429. The corresponding solution is $x(2)$ =(0.714, 0.571, 0.429). The slack in the equality constraint at $x(2)$ is $s(2)$ = -8.16313E-5. The nonlinear objective function value is 2.041, and the penalty term is 6.66367E-9.

Iteration 2

The current linearization point is $x(2)$ = (0.714, 0.571, 0.429). The gradient at $x(2)$ is (-0.816,1.224, -0.81). The solution of the linearized problem is $y(2)$ = (0, 1, 1). The directions of triangular search are $da(2)$ = $y(2)$ - $x(2)$ =(-0.714, 0.429, 0.571) and $db(2)$ = $y(2)$ - $x(2)$ = (0.286, -0.571, 0.571). The search is over the triangle formed by $x(2)$, $y(1)$ and $y(2)$, with sides $da(2)$ and $db(2)$. The stepsizes are $step_a$ = 0.667 in the direction $da(2)$ and $step_b$ = 0.333 in the direction $db(2)$. The sum of the stepsizes must be less than or equal to 1 if one wants to stay within the triangle. The solution of the search is $x(3)$ = (0.333, 0.667, 1), and the slack in the equality constraint at $x(3)$ is $s(3)$ = 8.88869E-5.

The nonlinear objective function value is 1.778 and the penalty term value is 7.90088E-9.

Iteration 3

The current linearization point $x(3)$ is (0.333, 0.667, 1). The gradient at $x(3)$ is (-0.889, -0.889, -0.889), and the solution $y(3)$ of the linearized problem is (1, 0, 1). The directions of triangular search $da(3)$ and $db(3)$ are respectively (0.667, 0.667, 0) and (-0.333, 0.333, 0). The stepsizes are respectively 0.282 in the direction $da(3)$ and 0.564 in the direction $db(3)$. The solution is $x(4)$ = (0.333, 0667, 1), and the slack in the equality constraint at $x(4)$ is $s(4)$ = -8.88869E-5. The nonlinear objective function value is 1.778 and the penalty value is 7.900883E-9. Since $x(3)$ and $x(4)$ are identical, the algorithm stops. Since the penalty value does not affect the objective function value any more, we can consider that problem (PR) is solved, with V(PR) = 1.778.

Conclusion

Even though in case of a nonlinear objective function the primal relaxation proposed above may not always be equivalent to a Lagrangean relaxation, it will work in a manner quite similar to Lagrangean relaxation. The subproblems solved in the linearization steps have the same constraints one would have chosen in the Lagrangean relaxation. If the constraints are separable, so will be the subproblems. The relaxation proposed here is always at least as good as the continuous relaxation. and possibly much stronger as demonstrated by some of the examples presented.

The same idea can be applied to yield relaxations akin, but not necessarily equivalent, to Lagrangean decompositions or substitutions.

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